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# HYPOCOERCIVITY FOR A LINEARIZED MULTI-SPECIES BOLTZMANN SYSTEM

ESTHER S. DAUS, ANSGAR JÜNGEL, CLÉMENT MOUHOT, AND NICOLA ZAMPONI

ABSTRACT. A new coercivity estimate on the spectral gap of the linearized Boltzmann collision operator for multiple species is proved. The assumptions on the collision kernels include hard and Maxwellian potentials under Grad's angular cut-off condition. Two proofs are given: a non-constructive one, based on the decomposition of the collision operator into a compact and a coercive part, and a constructive one, which exploits the "cross-effects" coming from collisions between different species and which yields explicit constants. Furthermore, the essential spectra of the linearized collision operator and the linearized Boltzmann operator are calculated. Based on the spectral-gap estimate, the exponential convergence towards global equilibrium with explicit rate is shown for solutions to the linearized multi-species Boltzmann system on the torus. The convergence is achieved by the interplay between the dissipative collision operator and the conservative transport operator and is proved by using the hypocoercivity method of Mouhot and Neumann.

## 1. INTRODUCTION

This paper is concerned with the proof of (explicit) spectral-gap estimates of the linearized Boltzmann operator for gas mixtures in the case of hard and Maxwellian potentials as well as the exponential decay of solutions to a *multi-species* Boltzmann system. Spectral-gap estimates and the large-time behavior of the *mono-species* Boltzmann equations were intensively studied in the literature, but are unknown for *multi-species* systems. First, we review the literature for the mono-species case.

The study of the linearized collision operator, in the spatially homogeneous and hard-potential case, goes back to Hilbert [23]. For this operator, Carleman [8] proved the existence of a spectral gap. The results were extended by Grad [17] for hard potentials with cut-off. Baranger and Mouhot [2] derived constructive estimates in the hard-sphere case. For Maxwell molecules, Fourier transform methods were employed in [37] to achieve explicit spectral properties. A spectral-gap estimate for the linearized Boltzmann operator, consisting of the sum of the linearized collision operator and the transport operator, was

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first shown by Ukai [34]. Improved estimates (in smaller spaces of Sobolev type), still for hard potentials, were established in [29]. In [30], spectral-gap estimates for moderately soft potentials (without angular cut-off) were proved, improving and extending previous results by Pao [31]. Hypocoelliptic estimates for the linearized operator without cut-off can be found in [1] and references therein. A spectral analysis with relaxed tail decay and regularity conditions on the solutions was performed recently in an abstract framework [19]. Dolbeault et al. [13] derived exponential decay rates in weighted  $L^2$  spaces, which improves previous Sobolev estimates. For further references, we refer to [30, Section 1.5].

Spectral properties of the linearized Boltzmann operator were already investigated by Grad [18]. Based on these results, Schechter [32] located the essential spectrum of the classical collision operator in  $L^2$ . The spectrum of the Boltzmann operator for hard spheres was also analyzed in  $L^p$  for  $p \neq 2$ ; see [25]. We refer to the recent work [15] for further results in  $L^p$  for  $1 \leq p \leq \infty$  and more references. A detailed analysis of the resolvent and spectrum of the linearized Boltzmann operator can be found in [35, Section 2.2]. A complete analysis for the essential and discrete spectra for the linearized collision operator with hard potentials was performed in [28].

All these results are valid for the linearized *mono-species* collision operator. Our aim is to extend the spectral-gap analysis to the case of the linearized *multi-species* Boltzmann system modeling an ideal gas mixture. This is achieved by generalizing the coercivity method of [29], including quantitative estimates on the spectral gap for the multi-species collision operator. A crucial step of our analysis is the observation that the multi-species version of the  $H$ -theorem implies conservation of mass for each species but conservation of momentum and energy only for the sum of all species. As a consequence, we need to study carefully the “cross-effects” of the collisions, i.e., how collisions between different species act on distribution functions which are elements of the nullspace of the mono-species collision operator. The crucial step is to relate these “cross-effects” to the differences of momentum and energy. Before stating the main results, we introduce the kinetic setting.

**1.1. The Boltzmann equation.** The evolution of a dilute ideal gas composed of  $n \geq 2$  different species of chemically non-interacting mono-atomic particles (see [11] for chemically reacting gases) with the same particle mass can be modeled by the following system of Boltzmann equations, stated on the three-dimensional torus  $\mathbb{T}^3$ ,

$$(1) \quad \partial_t F_i + v \cdot \nabla_x F_i = Q_i(F), \quad t > 0, \quad F_i(x, v, 0) = F_{I,i}(x, v), \quad (x, v) \in \mathbb{T}^3 \times \mathbb{R}^3,$$

where  $1 \leq i \leq n$ . The vector  $F = (F_1, \dots, F_n)$  is the distribution function of the system, with  $F_i$  describing the  $i$ th species. The variables are the position  $x \in \mathbb{T}^3$ , the velocity  $v \in \mathbb{R}^3$ , and the time  $t \geq 0$ . The right-hand side of the kinetic equation in (1) is the  $i$ th component of the nonlinear collision operator, defined by

$$Q_i(F) = \sum_{j=1}^n Q_{ij}(F_i, F_j), \quad 1 \leq i \leq n,$$

where  $Q_{ij}$  models interactions between particles of the same ( $i = j$ ) or of different species ( $i \neq j$ ),

$$Q_{ij}(F_i, F_j)(v) = \int_{\mathbb{R}^3 \times \mathbb{S}^2} B_{ij}(|v - v^*|, \cos \vartheta) (F'_i F'_j - F_i F_j) dv^* d\sigma,$$

with the abbreviations  $F'_i = F_i(v')$ ,  $F_i^* = F_i(v^*)$ ,  $F'_i = F_i(v')$ , the three-dimensional unit sphere  $\mathbb{S}^2$ , and

$$(2) \quad v' = \frac{v + v^*}{2} + \frac{|v - v^*|}{2} \sigma, \quad v'^* = \frac{v + v^*}{2} - \frac{|v - v^*|}{2} \sigma$$

are the pre-collisional velocities depending on the post-collisional velocities  $(v, v^*)$ . These expressions follow from the fact that we assume the collisions to be elastic, i.e., the momentum and kinetic energy are conserved on the microscopic level:

$$(3) \quad v' + v'^* = v + v^*, \quad \frac{1}{2}|v'|^2 + \frac{1}{2}|v'^*|^2 = \frac{1}{2}|v|^2 + \frac{1}{2}|v^*|^2.$$

The collision kernels  $B_{ij}$  are nonnegative functions of the modulus  $|v - v^*|$  and the cosine of the deviation angle  $\vartheta \in [0, \pi]$ , defined by  $\cos \vartheta = \sigma \cdot (v - v^*)/|v - v^*|$ .

Although we will analyze a linearized version of  $Q_i$ , let us recall the main properties of the nonlinear operator  $Q_i$ . Using the techniques from [10, pp. 36-42], it is not difficult to see that  $Q := (Q_1, \dots, Q_n)$  conserves the mass of each species but only total momentum and energy, i.e.

$$\int_{\mathbb{R}^3} \sum_{i,j=1}^n Q_{ij}(F_i, F_j) \psi_i(v) dv = 0$$

if and only if  $\psi(v) \in \text{span}\{e^{(1)}, \dots, e^{(n)}, v_1 \mathbf{1}, v_2 \mathbf{1}, v_3 \mathbf{1}, |v|^2 \mathbf{1}\}$ , where  $e^{(i)}$  is the  $i$ th unit vector in  $\mathbb{R}^n$  and  $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^n$ . It is shown in [11] that  $Q$  satisfies a multi-species version of the  $H$ -theorem which implies that any local equilibrium, i.e. any function  $F$  being the maximum of the Boltzmann entropy, has the form of a local Maxwellian  $M_{\text{loc}} = (M_{\text{loc},1}, \dots, M_{\text{loc},n})$  with

$$F_i(x, v, t) = M_{\text{loc},i}(x, v, t) = \frac{\rho_{\text{loc},i}(x, t)}{(2\pi\theta_{\text{loc}}(x, t))^{3/2}} \exp\left(-\frac{|v - u_{\text{loc}}(x, t)|^2}{2\theta_{\text{loc}}(x, t)}\right),$$

where, introducing the total local density  $\rho_{\text{loc}} = \sum_{i=1}^n \rho_{\text{loc},i}$ ,

$$\rho_{\text{loc},i} = \int_{\mathbb{R}^3} F_i dv, \quad u_{\text{loc}} = \frac{1}{\rho_{\text{loc}}} \sum_{i=1}^n \int_{\mathbb{R}^3} F_i v dv, \quad \theta_{\text{loc}} = \frac{1}{3\rho_{\text{loc}}} \sum_{i=1}^n \int_{\mathbb{R}^3} F_i |v - u|^2 dv$$

are the (local) masses of the species, the total momentum and total energy, respectively.

On the other hand, the global equilibrium, which is the unique stationary solution  $F$  to (1), is given by  $M = (M_1, \dots, M_n)$  with

$$F_i(x, v) = M_i(v) = \frac{\rho_{\infty,i}}{(2\pi\theta_{\infty})^{3/2}} \exp\left(-\frac{|v - u_{\infty}|^2}{2\theta_{\infty}}\right),$$

where now, setting  $\rho_\infty = \sum_{i=1}^n \rho_{\infty,i}$ ,

$$\rho_{\infty,i} = \int_{\mathbb{T}^3 \times \mathbb{R}^3} F_i dx dv, \quad u_\infty = \frac{1}{\rho_\infty} \int_{\mathbb{T}^3 \times \mathbb{R}^3} F_i v dx dv, \quad \theta_\infty = \frac{1}{3\rho_\infty} \int_{\mathbb{T}^3 \times \mathbb{R}^3} F_i |v - u|^2 dx dv$$

do not depend on  $(x, t)$ . By translating and scaling the coordinate system, we may assume that  $u_\infty = 0$  and  $\theta_\infty = 1$  such that the global equilibrium becomes

$$(4) \quad M_i(v) = \frac{\rho_{\infty,i}}{(2\pi)^{3/2}} e^{-|v|^2/2}, \quad 1 \leq i \leq n.$$

**1.2. Linearized Boltzmann collision operator.** We assume that the distribution function  $F_i$  is close to the global equilibrium such that we can write  $F_i = M_i + M_i^{1/2} f_i$  for some small perturbation  $f_i$ , where  $M_i$  is given by (4). Then, dropping the small nonlinear remaining term,  $f_i$  satisfies the linearized equation

$$(5) \quad \partial_t f_i + v \cdot \nabla_x f_i = L_i(f), \quad t > 0, \quad f_i(x, v, 0) = f_{L,i}(x, v), \quad (x, v) \in \mathbb{T}^3 \times \mathbb{R}^3,$$

for  $1 \leq i \leq n$ , where  $f = (f_1, \dots, f_n)$  and the  $i$ th component of the linearized collision operator  $L = (L_1, \dots, L_n)$  is given by

$$L_i(f) = \sum_{j=1}^n L_{ij}(f_i, f_j), \quad 1 \leq i \leq n,$$

with

$$(6) \quad \begin{aligned} L_{ij}(f_i, f_j) &= M_i^{-1/2} (Q_{ij}(M_i, M_j^{1/2} f_j) + Q_{ij}(M_i^{1/2} f_i, M_j)) \\ &= \int_{\mathbb{R}^3 \times \mathbb{S}^2} B_{ij} M_i^{1/2} M_j^* (h'_i + h'_j - h_i - h_j) dv^* d\sigma, \quad h_i := M_i^{-1/2} f_i. \end{aligned}$$

Here, we have used  $M_i^* M_j = M_i M_j^*$  for any  $i, j$ , which follows from (3). Notice that we have chosen the linearization considered in, e.g., [29, 35]. Another linearization is given by  $F_i = M_i + M_i g_i$  (see, e.g., [27]), namely  $\tilde{L}_{ij}(g_i, g_j) = M_i^{-1} (Q_{ij}(M_i, M_i g_i) + Q_{ij}(M_i g_i, M_i))$ . This choice gives the same results as with the linearization (6) since both linearizations correspond to the same space of solutions, but it turned out that the computations are easier using (6).

The linearized Boltzmann system satisfies an  $H$ -theorem with the linearized entropy  $H(f) = \frac{1}{2} \sum_{i=1}^n \int_{\mathbb{R}^3} f_i^2 dv$ ,

$$-\frac{dH}{dt} = - \sum_{i=1}^n \int_{\mathbb{R}^3} f_i L_i(f) dv =: -(f, L(f))_{L_v^2} \geq 0,$$

where  $(\cdot, \cdot)_{L_v^2}$  is the scalar product on  $L_v^2 := L^2(\mathbb{R}^3; \mathbb{R}^n)$ . We will prove in Lemma 5 that  $(f, L(f))_{L_v^2} = 0$  if and only if  $M_i^{-1/2} f_i$  lies in  $\text{span}\{e^{(1)}, \dots, e^{(n)}, v_1 \mathbf{1}, v_2 \mathbf{1}, v_3 \mathbf{1}, |v|^2 \mathbf{1}\}$ , which is the null space  $\mathcal{N}(L)$  of the linear operator  $L$ . The main aim of this paper is to show that, under suitable assumptions on the collision kernels, there exists a constant  $\lambda > 0$ , which can be computed explicitly, such that for all suitable functions  $f$ ,  $-(f, L(f))_{L_v^2} \geq \lambda \|f - \Pi^L(f)\|_{\mathcal{H}}^2$ , where  $\Pi^L$  is the projection onto  $\mathcal{N}(L)$  and  $\mathcal{H}$  is a subset of  $L_v^2$  (see Theorem

3 for the precise statement). This spectral-gap estimate, together with hypocoercivity techniques, allows us to conclude that exponential decay of the solutions  $f(t)$  towards the global equilibrium holds (see Theorem 4).

**1.3. Assumptions on the collision kernels.** We impose the following assumptions on the collision kernels  $B_{ij}$  arising in (6).

(A1) The collision kernels satisfy

$$B_{ij}(|v - v^*|, \cos \vartheta) = B_{ji}(|v - v^*|, \cos \vartheta) \quad \text{for } 1 \leq i, j \leq n.$$

(A2) The collision kernels decompose in the kinetic part  $\Phi_{ij} \geq 0$  and the angular part  $b_{ij} \geq 0$  according to

$$B_{ij}(|v - v^*|, \cos \vartheta) = \Phi_{ij}(|v - v^*|)b_{ij}(\cos \vartheta), \quad 1 \leq i, j \leq n.$$

(A3) For the kinetic part, there exist constants  $C_1, C_2 > 0$ ,  $\gamma \in [0, 1]$ , and  $\delta \in (0, 1)$  such that for all  $1 \leq i, j \leq n$  and  $r > 0$ ,

$$C_1 r^\gamma \leq \Phi_{ij}(r) \leq C_2 (r + r^{-\delta}).$$

(A4) For the angular part, there exist constants  $C_3, C_4 > 0$  such that for all  $1 \leq i, j \leq n$  and  $\vartheta \in [0, \pi]$ ,

$$0 < b_{ij}(\cos \vartheta) \leq C_3 |\sin \vartheta| |\cos \vartheta|, \quad b'_{ij}(\cos \vartheta) \leq C_4.$$

Furthermore,

$$C^b := \min_{1 \leq i \leq n} \inf_{\sigma_1, \sigma_2 \in \mathbb{S}^2} \int_{\mathbb{S}^2} \min \{b_{ii}(\sigma_1 \cdot \sigma_3), b_{ii}(\sigma_2 \cdot \sigma_3)\} d\sigma_3 > 0.$$

(A5) For all  $1 \leq i, j \leq n$ ,  $b_{ij}$  is even in  $[-1, 1]$  and the mapping  $v \mapsto \Phi'_{ij}(|v|)$  on  $\mathbb{R}^3$  is locally integrable on  $\mathbb{R}^3$  and bounded as  $|v| \rightarrow \infty$ .

(A6) There exists  $\beta > 0$  such that for all  $1 \leq i, j \leq n$ ,  $s > 0$ , and  $\sigma \in [-1, 1]$ , we have  $B_{ij}(s, \sigma) \leq \beta B_{ii}(s, \sigma)$ .

Following [29], since the functions  $b_{ij}$  are integrable, we define

$$(7) \quad \ell^b := \min_{1 \leq i, j \leq n} \int_0^\pi b_{ij}(\cos \theta) \sin \theta d\theta > 0.$$

Let us discuss these assumptions. The first hypothesis (A1) means that the collisions are micro-reversible. Assumption (A2) is satisfied, for instance, for collision kernels derived from interaction potentials behaving like inverse-power laws. The lower bound in hypothesis (A3) includes power-law functions  $\Phi_{ij}(r) = r^\gamma$  with  $\gamma > 0$  (hard potential) and  $\gamma = 0$  (Maxwellian molecules). The assumption  $\gamma \geq 0$  is crucial since the linearized collision operator in the mono-species case for soft potentials ( $\gamma < 0$ ) with angular cut-off has no spectral gap [2]; however, degenerate spectral-gap estimates are possible [16, 26]. The upper bound in (A3) means that the kinetic part is of restricted growth for both small and large values of  $|v - v^*|$ . In hypothesis (A4), the upper bound for  $b_{ij}$  implies Grad's cut-off assumption. The positivity of  $C^b$  in Assumption (A4) is used in the constructive proof of the multi-species spectral-gap estimate (Theorem 3) via the *mono-species* spectral-gap

estimate which depends on  $C^b$ ; see also the proofs of Theorem 1.1 in [2] and Theorem 6.1 in [26]. The positivity of  $C^b$  is satisfied for the main physical case of a collision kernel satisfying Grad's cut-off, i.e. for hard spheres with  $B_{ij}(|v - v^*|, \cos \vartheta) = |v - v^*|$ . Conditions (A1)-(A4) are also imposed in [2, 26, 27] for the linearized mono-species Boltzmann operator.

Assumption (A5) imposes technical conditions needed to verify the abstract hypotheses in [29]. More precisely, the evenness of  $b_{ij}$  is employed to show hypothesis (H2) (see section 5) and the properties on  $\Phi'_{ij}$  are used to verify (51) in hypothesis (H1). The conditions on  $\Phi'_{ij}$  are satisfied for hard and Maxwellian power-law potentials  $\Phi_{ij}(r) = r^\gamma$  with exponent  $\gamma \in [0, 1]$ , for instance. Finally, condition (A6) states that the ratio of the off-diagonal and diagonal collision kernels can be bounded uniformly from above by a constant  $\beta > 0$ . This hypothesis will be needed for the explicit computation of the constants in Theorems 3 and 4. More precisely, (A6) allows us to estimate the mono-species part of the collision operator using the computation of [26]; see Lemma 11.

**1.4. Notation and definitions.** We call  $\text{Dom}(F)$  the domain of an operator  $F$  and  $\text{Im}(f)$  the image of a mapping  $f$ . We introduce the spaces  $L_v^2 = L^2(\mathbb{R}^3; \mathbb{R}^n)$ ,  $L_{x,v}^2 = L^2(\mathbb{T}^3 \times \mathbb{R}^3; \mathbb{R}^n)$ ,  $H_{x,v}^1 = H^1(\mathbb{T}^3 \times \mathbb{R}^3; \mathbb{R}^n)$ , and

$$(8) \quad \begin{aligned} \mathcal{H} &= \left\{ f \in L_v^2 : \|f\|_{\mathcal{H}}^2 = \sum_{i=1}^n \int_{\mathbb{R}^3} f_i^2 \nu_i dv < \infty \right\}, \\ \mathcal{D} &= \left\{ f \in L_v^2 : \|f\|_{\mathcal{H}}^2 = \sum_{i=1}^n \int_{\mathbb{R}^3} f_i^2 \nu_i^2 dv < \infty \right\}. \end{aligned}$$

Here,  $\nu_i$  is the collision frequency, given by

$$(9) \quad \nu_i(v) = \sum_{j=1}^n \int_{\mathbb{R}^3 \times \mathbb{S}^2} B_{ij}(|v^* - v|, \cos \vartheta) M_j^* dv^* d\sigma, \quad i = 1, \dots, n,$$

For Maxwellian modelcules  $\Phi_{ij}(r) = \text{const.}$ , the collision frequency is constant but for strictly hard potentials  $\Phi_{ij}(r) = r^\gamma$  with  $0 < \gamma \leq 1$ ,  $\nu_i$  is unbounded. In fact, it satisfies  $\nu_0(1 + |v|)^\gamma \leq \nu_i(v) \leq \nu_1(1 + |v|)^\gamma$  for some constants  $\nu_1 \geq \nu_0 > 0$  [29, p. 991]. In the physically most relevant case of hard spheres ( $\gamma = 1$ ,  $b_{ij} = 1$ ), the collision frequency can be computed explicitly, see formula (2.13) in [10, Section 7.2]. For more properties of the collision frequencies, we refer to [9, Section III.3]. If the collision frequencies are bounded,  $\mathcal{H} = L_v^2$ . Generally,  $\nu_i$  is unbounded and so,  $\mathcal{H}$  is a proper subset of  $L_v^2$ . The norm on  $L_v^2$  (and similarly for the other spaces) is defined by

$$\|f\|_{L_v^2}^2 = \sum_{i=1}^n \int_{\mathbb{R}^3} f_i^2 dv \quad \text{for } f = (f_1, \dots, f_n) \in L_v^2.$$

We distinguish the following linear operators. We define the operator  $\Lambda = (\Lambda_1, \dots, \Lambda_n) : \text{Dom}(\Lambda) \rightarrow L_v^2$  by

$$\Lambda_i(f) = \nu_i f_i, \quad i = 1, \dots, n,$$



where  $\text{Dom}(\Lambda) = \{f \in L_v^2 : \Lambda f \in L_v^2\} = \mathcal{D}$ . It is closed, densely defined, selfadjoint and, by Lemma 7 below, coercive. The linearized collision operator  $L : \text{Dom}(L) \rightarrow L_v^2$ , introduced in section 1.2, can be written as  $L = K - \Lambda$ , where  $K := L + \Lambda$ , or, more explicitly,

$$K_i(f) = \sum_{j=1}^n \int_{\mathbb{R}^3 \times \mathbb{S}^2} B_{ij} M_i^{1/2} M_j^* (h'_i + h'_j - h_j^*) dv^* d\sigma, \quad i = 1, \dots, n.$$

It was shown in [4] that  $K$  is a compact operator in  $L_v^2$ . Thus,  $\text{Dom}(L) = \text{Dom}(\Lambda) = \mathcal{D}$  and  $L$  is closed and densely defined. Furthermore,  $L$  is nonpositive and selfadjoint on  $L_v^2$ . We define the transport operator

$$T = v \cdot \nabla_x : \text{Dom}(T) \rightarrow L_v^2,$$

where  $\text{Dom}(T) = \{f \in L_{x,v}^2 : v \cdot \nabla_x f \in L_{x,v}^2\}$ . Finally, we consider the linearized Boltzmann operator

$$B = L - T : \text{Dom}(B) \rightarrow L_v^2,$$

which is unbounded, closed, and densely defined with  $\text{Dom}(B) = \text{Dom}(L) \cap \text{Dom}(T)$ .

We denote by  $\mathcal{N}(A)$  and  $\mathcal{R}(A)$  the kernel and range of a linear operator  $A$ , respectively. Its resolvent set is denoted by  $\rho(A)$  and its spectrum by  $\sigma(A) = \mathbb{C} \setminus \rho(A)$ . For a linear unbounded operator  $A$  with  $\sigma(A) \subset (-\infty, 0]$ , we say that  $A$  has a spectral gap when the distance between 0 and  $\sigma(A) \setminus \{0\}$  is positive. Finally, the essential spectrum of  $A$  is defined as the set of all complex numbers  $\lambda \in \mathbb{C}$  such that  $A - \lambda I$  is not Fredholm, where  $I$  is the identity operator. We refer to section 3 for details regarding this definition.

**1.5. Main results.** In this subsection, we state the main results of the paper. The first result is a geometric property of the essential spectrum of the linearized collision operator  $L$  and the linearized Boltzmann operator  $B = L - T$ .

**Theorem 1** (Essential spectrum of  $L$  and  $L - T$ ). *Let the collision kernels  $B_{ij}$  satisfy assumptions (A1)-(A4) and set  $J = \cup_{i=1}^n \text{Im}(\nu_i) \subset [\nu_0, \infty)$ , where  $\nu_0 = \min_{i=1, \dots, n} \sup_{v \in \mathbb{R}^3} \nu_i(v) > 0$  (see Lemma 7). Then*

$$\sigma_{\text{ess}}(L) = -J, \quad \sigma_{\text{ess}}(L - T) = \{\lambda \in \mathbb{C} : \Re(\lambda) \in -J\}.$$

**Remark 2.** We observe that if  $\lim_{|v| \rightarrow \infty} \nu_i(v) = \infty$  for  $i = 1, \dots, n$  then

$$\sigma_{\text{ess}}(L) = (-\infty, -\nu_0], \quad \sigma_{\text{ess}}(L - T) = \{\lambda \in \mathbb{C} : \Re(\lambda) \leq -\nu_0\}.$$

Indeed, under the assumption  $\nu_i(v) \rightarrow \infty$  as  $|v| \rightarrow \infty$ , the continuity of  $\nu_i$ , and the Weierstraß theorem show that  $J = [\nu_0, \infty)$ . Thus, the essential spectrum of the linearized *multi-species* collision operator is very similar to the *mono-species* operator, where  $\nu_0$  corresponds to the infimum in  $\mathbb{R}^3$  of the single collision frequency; see [27, Section 3] and [28, Prop. 3.1].  $\square$

The proof of Theorem 1 is based on perturbation theory [24, Chap. IV] and is similar to the proof for the mono-species collision operator [35], but we show new explicit spectral-gap estimates related to the particular structure of the kernel in the multi-species case. More precisely, we write  $L = K - \Lambda$  as described in section 1.4. It turns out that  $K = L + \Lambda$



is compact on  $L_v^2$  (see section 3 for details). Weyl's theorem [20, Theorem S] states that the essential spectrum of  $L = K - \Lambda$  coincides with that of  $-\Lambda$ . Thus it remains to show that  $\sigma_{\text{ess}}(\Lambda) = J$ . This is done by using Weyl's singular sequences, which allow for a sufficient and necessary condition for  $\lambda \in \mathbb{C}$  being an element of the essential spectrum of the selfadjoint operator  $\Lambda$ .

The proof of the second statement in Theorem 1 is more involved since  $K$  is not compact on  $L_{x,v}^2$  and hence, Weyl's theorem cannot be applied directly. The idea is to employ an extended Weyl theorem, which states that the essential spectrum is conserved under a *relatively compact* perturbation [24, Section IV.5.6, Theorem 5.35]. Indeed, if  $K$  is relatively compact with respect to  $\Lambda + T$  then  $\sigma_{\text{ess}}(L - T) = \sigma_{\text{ess}}(K - (\Lambda + T)) = -\sigma_{\text{ess}}(\Lambda + T)$ , and it remains to compute the essential spectrum of  $\Lambda + T$ .

The next theorem concerns an explicit spectral-gap estimate. It is the main result of the paper.

**Theorem 3** (Explicit spectral-gap estimate). *Let the collision kernels  $B_{ij}$  satisfy assumptions (A1)-(A4). Then there exists a constant  $\lambda > 0$  such that*

$$(10) \quad -(f, L(f))_{L_v^2} \geq \lambda \|f - \Pi^L(f)\|_{\mathcal{H}}^2 \quad \text{for all } f \in \mathcal{D},$$

where  $\Pi^L$  is the projection onto the null space  $\mathcal{N}(L)$ . If additionally hypothesis (A6) holds, the constant  $\lambda$  can be computed explicitly:

$$\lambda = \frac{\eta D^b}{8C_k}, \quad \eta = \min \left\{ 1, \frac{4C^m C_k}{16C_k + D^b} \right\},$$

where  $C^m$ ,  $D^b$ , and  $C_k$  are defined in (38), (43), and (45), respectively.

Note that the constant  $C^m$  depends on the mono-species spectral-gap constant  $C^b$  via (38) below. We present two proofs of this theorem. The first proof is non-constructive and relies on an abstract functional theoretical argument, based on the decomposition  $L = K - \Lambda$  and Weyl's perturbation theorem. This abstract spectral-gap estimate is proved in Lemma 10. The second proof provides a constructive spectral-gap estimate, generalizing the result in [27] (also see [26, Theorem 6.1]) from the mono-species to the multi-species case. For this, we split the operator  $L = L^m + L^b$  in the mono-species part  $L^m = (L_1^m, \dots, L_n^m)$  and the bi-species part  $L^b = (L_1^b, \dots, L_n^b)$ ,

$$L_i^m(f_i) = L_{ii}(f_i, f_i), \quad L_i^b(f) = \sum_{j \neq i} L_{ij}(f_i, f_j).$$

The proof consists of four main steps.

*Step 1: Coercivity of the mono-species operator  $L^m$ .* The bi-species part of  $L$  satisfies  $-(f, L^b(f))_{L_v^2} \geq 0$  for all  $f \in \mathcal{D}$ . Furthermore, the results of [26, Theorem 6.1] show that for the mono-species part,

$$(11) \quad -(f, L^m(f))_{L_v^2} \geq C^m \|f - \Pi^m(f)\|_{\mathcal{H}}^2 \quad \text{for } f \in \text{Dom}(L^m),$$

where the constant  $C^m > 0$  can be computed explicitly and  $\Pi^m$  is the projection onto  $\mathcal{N}(L^m)$  (see Lemma 11). Inequality (11) may be interpreted as a coercivity estimate for

$L^m$  on  $\mathcal{N}(L^m)^\perp$ . It is related to the ‘‘microscopic coercivity’’ in [13, Section 1.3] for the mono-species setting. Hence, we obtain the ‘‘naive’’ spectral-gap estimate

$$-(f, L(f))_{L_v^2} \geq C^m \|f - \Pi^m(f)\|_{\mathcal{H}}^2 \quad \text{for } f \in \mathcal{D}.$$

This estimate is not sharp enough for the multi-species case since we need an inequality for all  $f - \Pi^L(f) \in \mathcal{N}(L)^\perp$  and not only for  $f - \Pi^m(f) \in \mathcal{N}(L^m)^\perp \subset \mathcal{N}(L)^\perp$ . By projecting onto  $\mathcal{N}(L^m)^\perp$  only, we neglect the ‘‘cross-effects’’ coming from the bi-species part of the collision operator. Thus, we need a better estimate for  $-(f, L(f))_{L_v^2}$ , which is achieved as follows.

*Step 2: Absorption of the orthogonal parts.* The contribution  $f^\perp := f - \Pi^m(f)$  in  $-(f, L^b(f))_{L_v^2}$  can be absorbed by the  $\mathcal{H}$  norm of  $f^\perp$  (see Lemma 12), giving for a certain  $\eta > 0$ ,

$$-(f, L(f))_{L_v^2} \geq (C^m - 4\eta) \|f^\perp\|_{\mathcal{H}}^2 - \frac{\eta}{2} (\Pi^m(f), L^b(\Pi^m(f)))_{L_v^2}.$$

*Step 3: Coercivity of the bi-species operator  $L^b$ .* The projection  $\Pi^m(f)$  depends on the velocities  $u_i$  and energies  $e_i$  of the  $i$ th species, and thus, the cross terms can be bounded by the differences of momentum and differences of energies,

$$(12) \quad -(\Pi^m(f), L^b(\Pi^m(f)))_{L_v^2} \geq C \sum_{i,j=1}^n (|u_i - u_j|^2 + |e_i - e_j|^2),$$

for some constant  $C > 0$ . This is the key step of the proof. The inequality may be considered as a coercivity estimate for the bi-species operator. A key observation is that the differences of momenta and energies converge to zero as  $f$  approaches the global equilibrium. Whereas (11) acts on  $\mathcal{N}(L^m)^\perp$ , (12) gives an estimate on the orthogonal complement  $\mathcal{N}(L^m)$ .

*Step 4: Lower bound for the differences of momenta and energy.* The last step consists in estimating the differences  $|u_i - u_j|$  and  $|e_i - e_j|$  from below by the error made by projecting onto  $\mathcal{N}(L^m)^\perp$  instead of  $\mathcal{N}(L)^\perp$ :

$$\sum_{i,j=1}^n (|u_i - u_j|^2 + |e_i - e_j|^2) \geq C \|f - \Pi^L(f)\|_{\mathcal{H}}^2 - 2C \|f - \Pi^m(f)\|_{\mathcal{H}}^2.$$

Putting together the above inequalities, Theorem 3 follows; we refer to section 4 for details.

As a consequence of the spectral-gap estimate, we are able to prove the exponential decay of the solution  $f(t)$  to (5) to the global equilibrium with an explicit decay rate.

**Theorem 4** (Convergence to equilibrium). *Let the collision kernels  $B_{ij}$  satisfy assumptions (A1)-(A5) and let  $f_I \in H_{x,v}^1$ . Then the linearized Boltzmann operator  $B = L - T$  generates a strongly continuous semigroup  $e^{tB}$  on  $H_{x,v}^1$ , which satisfies*

$$(13) \quad \|e^{tB}(I - \Pi^B)\|_{H_{x,v}^1} \leq Ce^{-\tau t}, \quad t \geq 0,$$

for some constants  $C, \tau > 0$ . In particular, the solution  $f(t) = e^{tB} f_I$  to (5) satisfies

$$(14) \quad \|f(t) - f_\infty\|_{H_{x,v}^1} \leq Ce^{-\tau t} \|f_I - f_\infty\|_{H_{x,v}^1}, \quad t \geq 0,$$

where  $f_\infty := \Pi^B(f_I)$  is the global equilibrium of (5). Moreover, under the additional assumption (A6) and lower bound in (A4), the constants  $C$  and  $\tau$  depend only on the constants appearing in hypotheses (H1)-(H3) in section 5 and in particular on  $\lambda$  defined in Theorem 3.

The idea of the proof is to employ the hypocoercivity of the linearized Boltzmann operator  $L - T$ , using the interplay between the degenerate-dissipative properties of  $L$  and the conservative properties of  $T$ . The aim is to find a functional  $G[f]$  which is equivalent to the square of the norm of a Banach space (here,  $H_{x,v}^1$ ),

$$\kappa_1 \|f\|_{H_{x,v}^1}^2 \leq G[f] \leq \kappa_2 \|f\|_{H_{x,v}^1}^2 \quad \text{for } f \in H_{x,v}^1,$$

leading to

$$(15) \quad \frac{d}{dt} G[f(t)] \leq -\kappa \|f(t)\|_{H_{x,v}^1}^2, \quad t > 0,$$

where  $\kappa_1, \kappa_2, \kappa > 0$  and  $f(t) = e^{tB} f_I$ . These two estimates yield exponential convergence of  $f(t)$  in  $H_{x,v}^1$ . It turns out that the obvious choice  $G[f] = c_1 \|f\|_{L_{x,v}^2} + c_2 \|\nabla_x f\|_{L_{x,v}^2} + c_3 \|\nabla_v f\|_{L_{x,v}^2}$  does not lead to a closed estimate. The key idea, inspired from [36] and worked out in [29], is to add the ‘‘mixed term’’  $c_4 (\nabla_x f, \nabla_v f)_{L_{x,v}^2}$  to the definition of  $G[f]$ . Then

$$\frac{d}{dt} (\nabla_x f, \nabla_v f)_{L_{x,v}^2} = -\|\nabla_x f\|_{L_{x,v}^2}^2 + 2(\nabla_x L(f), \nabla_v f)_{L_{x,v}^2},$$

and the last term can be estimated in terms of expressions arising from the time derivative of the other norms in  $G[f(t)]$ . Thus, choosing  $c_i > 0$  in a suitable way, one may conclude that (15) holds.

In [29], the calculation of (15) is reduced to the validity of certain abstract conditions on the operators  $K$  and  $\Lambda$  (see section 5). These conditions state that  $\Lambda$  is coercive in a certain sense,  $K$  has a regularizing effect, and  $L = K - \Lambda$  has a local spectral gap. The last condition is proved in Theorem 3, while the other conditions follow from direct calculations, since the operators  $K$  and  $\Lambda$  are given explicitly. As a consequence, the proof of Theorem 4 essentially consists in verifying the abstract conditions stated in [29]. In contrast to the estimate of Theorem 3, where the multi-species character plays a role in the spectral-gap estimate, there are no ‘‘cross-effects’’ here and the same modified functional  $G[f]$  as above, including the mixed term, can be used. However, the decay rate  $\tau$  changes, since the constant in hypothesis (H3) (see section 5) differs in the mono- and multi-species case and  $\tau$  depends also on that constant.

We finish the introduction by commenting possible generalizations. First, the convergence result based on hypocoercivity requires some regularity on the initial data, namely  $f_I \in H_{x,v}^1$ . The extension of the exponential decay to initial data from  $L_{x,v}^2$  might be done by using the method of [19], which is based on a high-order factorization argument on the resolvents and semigroups. The proof of exponential decay is expected to be constructive and to preserve the optimal rate.

Second, it seems to be not trivial to extend the results to the whole-space case. The problem is that one loses the compactness in the  $x$ -space. One possibility is to assume some

confinement potential which, under some appropriate weighted Poincaré inequality, can yield compactness of the resolvent and hence a spectral gap. For instance, Duan [14] used non-constructive techniques to prove decay rates for the mono-species linearized Boltzmann equation. Still in the mono-species case, with one-dimensional collisional invariants and using constructive methods, the decay is investigated by, e.g., Hérau and Nier [22] and Villani [36], working in the space  $H_{x,v}^1$ , and by Hérau [21] and Dolbeault, Mouhot, Schmeiser [13], working in the space  $L_{x,v}^2$ . The tasks in the multi-species case are first to extend the non-constructive methods, which probably does not contain new difficulties, and second to devise a constructive method, which is more involved and work in progress; see [12].

Third, a Cauchy theory for the full nonlinear multi-species Boltzmann equation in a perturbative regime is work in progress [7].

The paper is organized as follows. In section 2, some properties of the linearized collision operator (6) are collected. Theorem 1 on the essential spectrum of  $L$  and  $L - T$  and the abstract spectral-gap estimate in Theorem 3 are proved in section 3. We present a second proof of Theorem 3 in section 4, by exploiting the conservation properties and leading to explicit constants. Finally, Theorem 4 is shown in section 5.

## 2. PROPERTIES OF THE KINETIC MODEL

We show some properties of the linearized collision operator (6) and the collision frequencies (9). Let assumptions (A1)-(A4) hold. First we prove an  $H$ -theorem for (6).

**Lemma 5** ( $H$ -theorem for the linearized collision operator). *It holds  $(f, L(f))_{L_v^2} \leq 0$  for all  $f \in \mathcal{D}$  and  $(f, L(f))_{L_v^2} = 0$  if and only if  $f \in \mathcal{N}(L)$ , where*

$$\mathcal{N}(L) = \left\{ f \in L_v^2 : \exists \alpha_1, \dots, \alpha_n, e \in \mathbb{R}, u \in \mathbb{R}^3, \forall 1 \leq i \leq n, \right. \\ \left. f_i = M_i^{1/2}(\alpha_i + u \cdot v + e|v|^2) \right\},$$

and  $M_i$  is given by (4).

The proof is similar to the mono-species case except that the elements of the null space of  $L$  depend on the total mean velocity  $u$  and total energy  $e$  instead of the individual velocities and energies. Therefore, we give a complete proof. We note that an  $H$ -theorem for the nonlinear Boltzmann operator for a mixture of reactive gases was proved in [11].

*Proof.* By the change of variables  $(v, v^*) \mapsto (v^*, v)$  and  $(v, v^*) \mapsto (v', v'^*)$  and the symmetry of  $B_{ij}$  (assumption (A1)), we can write for  $f \in L_v^2$ ,

$$(f, L(f))_{L_v^2} = -\frac{1}{4} \sum_{i,j=1}^n \int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} M_i M_j^* (h'_i + h'_j - h_i - h_j^*)^2 dv^* dv d\sigma,$$

where we recall that  $h_i = M_i^{-1/2} f_i$ . This shows that  $(f, L(f))_{L_v^2} \leq 0$  for all  $f \in \mathcal{D}$ . Moreover,  $(f, L(f))_{L_v^2} = 0$  if and only if

$$(16) \quad h'_i + h'_j - h_i - h_j^* = 0 \quad \text{for all } (v, v^*) \in \mathbb{R}^3 \times \mathbb{R}^3, 1 \leq i, j \leq n.$$

It is shown in [10, pp. 36-42] that (16) for  $i = j$  implies that  $h_i$  has the form  $h_i(v) = \alpha_i + u_i \cdot v + e_i |v|^2$  for suitable constants  $\alpha_i, e_i \in \mathbb{R}$  and  $u_i \in \mathbb{R}^3$ . Inserting this expression into (16) leads to

$$(17) \quad u_i \cdot (v' - v) + u_j \cdot (v'^* - v^*) + e_i(|v'|^2 - |v|^2) + e_j(|v'^*|^2 - |v^*|^2) = 0$$

for  $1 \leq i, j \leq n$ . We consider the particular type of collisions with  $v' = v^*, v'^* = v$ , and  $|v| = |v'|$ . For such collisions,  $\sigma = (v^* - v)/|v^* - v|$ . Then the above equation becomes

$$(u_i - u_j) \cdot (v' - v) = 0, \quad 1 \leq i, j \leq n.$$

By rotating the velocities  $v, v'$  in all possible ways, we deduce that  $(u_i - u_j) \cdot w = 0$  for all  $w \in \mathbb{R}^3$  and thus,  $u_i = u_j$  for all  $1 \leq i, j \leq n$ . We set  $u := u_1$ . This fact, together with the conservation of momentum  $v' - v + v'^* - v^* = 0$ , implies that (17) becomes

$$e_i(|v'|^2 - |v|^2) + e_j(|v'^*|^2 - |v^*|^2) = 0, \quad 1 \leq i, j \leq n.$$

Taking into account the conservation of energy  $|v'|^2 - |v|^2 + |v'^*|^2 - |v^*|^2 = 0$ , we infer that  $(e_i - e_j)(|v'|^2 - |v|^2) = 0$  and consequently,  $e_i = e_j$  for all  $1 \leq i, j \leq n$ . Set  $e := e_1$ . We have shown that  $(f, L(f))_{L^2} = 0$  if and only if there exist  $\alpha_1, \dots, \alpha_n, e \in \mathbb{R}$  and  $u \in \mathbb{R}^3$  such that for  $1 \leq i \leq n$ ,  $f_i(v) = M_i^{1/2}(\alpha_i + u \cdot v + e|v|^2)$ . These functions clearly belong to  $\mathcal{N}(L)$ , which finishes the proof.  $\square$

The next result is concerned with the stationary solutions of (5).

**Lemma 6.** *The global equilibrium  $f_\infty = (f_{\infty,1}, \dots, f_{\infty,n})$  of (5), i.e. the unique stationary solution, is given by*

$$f_{\infty,i}(v) = M_i^{1/2}(\alpha_i + u \cdot v + e|v|^2), \quad 1 \leq i \leq n,$$

where  $\alpha_i, e \in \mathbb{R}$  and  $u \in \mathbb{R}^3$  are uniquely determined by the global conservation laws of mass, momentum, and energy, i.e. by the equations

$$\int_{\mathbb{R}^3} M_i^{1/2}(f_{\infty,i} - f_{I,i})\psi(v)dv = 0, \quad 1 \leq i \leq n,$$

for  $\psi(v) = 1, v_1, v_2, v_3, |v|^2$ , where  $f_{I,i}$  are the initial data.

*Proof.* First, we claim that  $\mathcal{N}(B) = \mathcal{N}(L) \cap \mathcal{N}(T)$ , where  $B = L - T$  and  $T = v \cdot \nabla_x$  are considered on  $\mathbb{T}^3 \times \mathbb{R}^3$ . The inclusion  $\mathcal{N}(L) \cap \mathcal{N}(T) \subset \mathcal{N}(B)$  being trivial, let  $f \in \mathcal{N}(B)$ . Then, using the skew-symmetry of  $T$ ,

$$0 = (f, B(f))_{L^2_{x,v}} = (f, L(f))_{L^2_{x,v}} - (f, T(f))_{L^2_{x,v}} = (f, L(f))_{L^2_{x,v}}.$$

Lemma 5 shows that  $f \in \mathcal{N}(L)$ . But this implies that  $T(f) = L(f) - B(f) = 0$  and hence  $f \in \mathcal{N}(T)$ . This shows the claim. Let  $f_\infty$  be a stationary solution. Then  $f_\infty \in \mathcal{N}(B)$  and by our claim,  $f_\infty \in \mathcal{N}(L) \cap \mathcal{N}(T)$ . Since  $\mathcal{N}(T) = \{f \in L^2_{x,v} : \nabla_x f = 0\}$  [6, Lemma B.2],  $f_\infty$  does not depend on  $x$ . Because of  $f_\infty \in \mathcal{N}(L)$ , Lemma 5 shows the result.  $\square$

Finally, we prove that the collision frequencies (9) are strictly positive with bounded derivative.

**Lemma 7.** *Let Assumptions (A2)-(A4) hold. The collision frequencies (9) satisfy*

$$(18) \quad \min_{1 \leq i \leq n} \inf_{v \in \mathbb{R}^3} \nu_i(v) \geq \nu_0 := 2^{3\gamma/2} \frac{C_1 \ell^b \rho_\infty}{\sqrt{\pi}} \Gamma\left(\frac{\gamma+3}{2}\right) > 0,$$

where  $C_1 > 0$  is given by assumption (A3),  $\ell^b > 0$  is defined in (7),  $\rho_\infty := \sum_{j=1}^n \rho_{j,\infty}$  (see (4)), and  $\Gamma$  is the Gamma function. Furthermore, if additionally (A5) holds, then  $\nabla_v \nu_i \in L_v^\infty(\mathbb{R}^3)$ , implying that  $|\nu_i(v)| \leq C_\nu(1 + |v|)$  for some  $C_\nu > 0$  and for all  $v \in \mathbb{R}^3$ ,  $i = 1, \dots, n$ .

*Proof.* The decomposition of  $B_{ij}$ , according to assumption (A2), implies that

$$\nu_i(v) = \sum_{j=1}^n \int_{\mathbb{R}^3} \Phi_{ij}(|v - v^*|) M_j^* dv^* \int_{\mathbb{S}^2} b_{ij}(\cos \vartheta) d\sigma.$$

The integral

$$c_{ij} := \int_{\mathbb{S}^2} b_{ij}(\cos \vartheta) d\sigma = 2\pi \int_0^\pi b_{ij}(\cos \vartheta) \sin \vartheta d\vartheta$$

does not depend on  $v$  or  $v^*$ . We conclude from (A2)-(A4) that

$$(19) \quad \begin{aligned} \nu_i(v) &= (2\pi)^{-3/2} \sum_{j=1}^n c_{ij} \rho_{\infty,j} \int_{\mathbb{R}^3} \Phi_{ij}(|v - v^*|) e^{-|v^*|^2/2} dv^* \\ &\geq \frac{C_1 \ell^b \rho_\infty}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} |v - v^*|^\gamma e^{-|v^*|^2/2} dv^*. \end{aligned}$$

Observe that the function

$$G(v) := \int_{\mathbb{R}^3} |v - v^*|^\gamma e^{-|v^*|^2/2} dv^*$$

is uniformly positive since the transformation  $v^* \mapsto -v^*$  and the elementary inequality

$$|v - v^*|^\gamma + |v + v^*|^\gamma \geq |(v - v^*) + (v + v^*)|^\gamma = 2^\gamma |v^*|^\gamma$$

for  $\gamma \in [0, 1]$  lead to

$$G(v) = \frac{1}{2} \int_{\mathbb{R}^3} (|v - v^*|^\gamma + |v + v^*|^\gamma) e^{-|v^*|^2/2} dv^* \geq 2^{\gamma-1} \int_{\mathbb{R}^3} |v^*|^\gamma e^{-|v^*|^2/2} dv^* = 2^{\gamma-1} G(0).$$

Actually, using spherical coordinates and the change of unknowns  $s = r^2/2$ ,

$$G(0) = 4\pi \int_0^\infty r^{\gamma+2} e^{-r^2/2} dr = 2^{(\gamma+5)/2} \pi \int_0^\infty s^{(\gamma+1)/2} e^{-s} ds = 2^{(\gamma+5)/2} \pi \Gamma\left(\frac{\gamma+3}{2}\right).$$

Inserting the above estimate on  $G(v)$  into (19) shows (18).

It remains to prove that  $\nabla_v \nu_i \in L_v^\infty(\mathbb{R}^3)$ . To this end, we compute

$$|\nabla \nu_i(v)| = (2\pi)^{-3/2} \left| \sum_{j=1}^n c_{ij} \rho_{\infty,j} \int_{\mathbb{R}^3} \Phi'_{ij}(|v - v^*|) \cdot \frac{v - v^*}{|v - v^*|} e^{-|v^*|^2/2} dv^* \right|$$

$$(20) \quad \leq (2\pi)^{-3/2} \sum_{j=1}^n c_{ij} \rho_{\infty,j} \int_{\mathbb{R}^3} |\Phi'_{ij}(|v - v^*|)| e^{-|v^*|^2/2} dv^*.$$

For given  $R > 0$ , we decompose

$$|\Phi'_{ij}(|v|)| = |\Phi'_{ij}(|v|)| \chi_{\{|v| < R\}}(v) + |\Phi'_{ij}(|v|)| \chi_{\{|v| \geq R\}}(v).$$

Assumption (A5) means that there exists  $R > 0$  such that

$$|\Phi'_{ij}(|\cdot|)| \chi_{\{|\cdot| < R\}} \in L^1_v(\mathbb{R}^3) \quad \text{and} \quad |\Phi'_{ij}(|\cdot|)| \chi_{\{|\cdot| \geq R\}} \in L^\infty_v(\mathbb{R}^3).$$

Thus, the right-hand side of (20) is bounded since it can be written as the sum of two terms, each of which is the convolution of an  $L^1$  and an  $L^\infty$  function. This shows that  $\nabla_v \nu_i \in L^\infty_v(\mathbb{R}^3)$ .  $\square$

**Remark 8.** We observe that  $\nu_i$  is generally not bounded since the kinetic part  $\Phi_{ij}(r)$  may grow like  $r$  as  $r \rightarrow \infty$ . It is possible to show that  $\nu_i$  is bounded if  $\Phi_{ij}$  is bounded. The unboundedness of  $\nu_i$  implies that the spaces  $L^2_v$  and  $\mathcal{H}$  are not isomorphic.  $\square$

### 3. GEOMETRIC PROPERTIES OF THE SPECTRUM

In this section, we prove Theorem 1 and the spectral-gap estimate (10) in Theorem 3 by using arguments from functional analysis.

First, we study the essential spectrum of  $L$  and  $L - T$ . There exist several definitions of the essential spectrum of a linear operator. Given a linear, closed and densely defined operator  $A : \text{Dom}(A) \subset X \rightarrow X$  on a Banach space  $X$ , we define

$$\sigma_{\text{ess}}(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not Fredholm}\}.$$

We recall that a linear, closed, and densely defined operator  $A$  is Fredholm if its range  $\mathcal{R}(A)$  is closed and both its kernel and cokernel are finite-dimensional. For other definitions of the essential spectrum, we refer to [20]. The essential spectrum is closed and conserved under compact perturbations, i.e., the bounded operators  $A$  and  $B$  have the same essential spectrum if  $A - B$  is compact (Weyl's theorem; see [20, Theorem S]).

If  $X$  is a Hilbert space and  $A$  is selfadjoint, it holds  $\sigma_{\text{ess}}(A) \subset \mathbb{R}$  and for given  $\lambda \in \mathbb{R}$ , we have  $\lambda \in \sigma_{\text{ess}}(A)$  if and only if  $A - \lambda I$  is not closed or the kernel of  $A - \lambda I$  is infinite dimensional. (This follows from the fact that  $\mathcal{R}(A - \lambda I)^\perp = \mathcal{N}(A - \lambda I)$  for closed, selfadjoint operators  $A$  [24, Chap. V.3.1].) Moreover, Weyl's criterion holds [33, Lemma 5.17]:  $\lambda \in \sigma_{\text{ess}}(A)$  if and only if  $A - \lambda I$  admits a *singular sequence*, i.e. a sequence  $(f_k) \subset \text{Dom}(A)$  such that (i)  $\|f_k\|_X = 1$  for all  $k \in \mathbb{N}$ ; (ii)  $\|(A - \lambda I)f_k\|_X \rightarrow 0$  as  $k \rightarrow \infty$ ; and (iii)  $(f_k)$  has no convergent subsequences in  $X$ .

We decompose  $L$  as  $L = K - \Lambda$ , where  $K = (K_1, \dots, K_n)$ ,  $\Lambda = (\Lambda_1, \dots, \Lambda_n)$ , and

$$(21) \quad \begin{aligned} K_i(f) &= \sum_{j=1}^n \int_{\mathbb{R}^3 \times \mathbb{S}^2} B_{ij} M_i^{1/2} M_j^* (h'_i + h'_j - h_j^*) dv^* d\sigma, \\ \Lambda_i(f) &= \nu_i f_i, \quad 1 \leq i \leq n, \end{aligned}$$



and the collision frequencies  $\nu_i$  are defined in (9). We recall from Lemma 7 that they satisfy  $\nu_i(v) \geq \nu_0 > 0$  and  $|\nu_i(v)| \leq C_\nu(1 + |v|)$  for all  $i = 1, \dots, n$  and  $v \in \mathbb{R}^3$ .

*Proof of Theorem 1.* Since  $K$  is compact on  $X = L_v^2$  [4, Prop. 2], it follows that  $\sigma_{\text{ess}}(L) = \sigma_{\text{ess}}(-\Lambda) = -\sigma_{\text{ess}}(\Lambda)$ . Thus, we will first study the essential spectrum of  $\Lambda$ . The proof is divided into several steps. Recall that  $J = \cup_{i=1}^n \text{Im}(\nu_i) \subset [\nu_0, \infty)$ .

*Step 1:*  $\overline{J} \subset \sigma_{\text{ess}}(\Lambda)$ . Let  $\lambda \in J$ . There exists  $j \in \{1, \dots, n\}$  and  $\widehat{v} \in \mathbb{R}^3$  such that  $\lambda = \nu_j(\widehat{v})$ . We define the sequence  $(f_k) \subset \mathcal{D}$  by

$$f_{k,i}(v) = (2\pi\sigma_k)^{-3/4} \exp\left(-\frac{|v - \widehat{v}|^2}{4\sigma_k}\right) \quad \text{if } i = j, \quad f_{k,i}(v) = 0 \quad \text{if } i \neq j,$$

where  $\sigma_k = 1/k$ ,  $k \in \mathbb{N}$ . Clearly, condition (i) for the singular sequence is satisfied. Furthermore,

$$\begin{aligned} \|(\Lambda - \lambda I)f_k\|_{L_v^2}^2 &= \sum_{i=1}^n \int_{\mathbb{R}^3} (\nu_i(v) - \lambda)^2 f_{k,i}(v)^2 dv \\ &= (2\pi\sigma_k)^{-3/2} \int_{\mathbb{R}^3} (\nu_i(v) - \nu_j(\widehat{v}))^2 \exp\left(-\frac{|v - \widehat{v}|^2}{2\sigma_k}\right) dv. \end{aligned}$$

The limit of a sequence of Gaussians with variance tending to zero converges to the delta distribution  $\delta_{\widehat{v}}$  (in the sense of distributions), which means that

$$(2\pi\sigma_k)^{-3/2} \int_{\mathbb{R}^3} u(v) \exp\left(-\frac{|v - \widehat{v}|^2}{2\sigma_k}\right) dv \rightarrow u(\widehat{v}) \quad \text{as } k \rightarrow \infty$$

for all functions  $u \in C^0(\mathbb{R}^3)$  with polynomial growth at infinity. Since  $|\nu_i(v)| \leq C_\nu(1 + |v|)$ , this condition is satisfied and we conclude that  $\|(\Lambda - \lambda I)f_k\|_{L_v^2} \rightarrow 0$  as  $k \rightarrow \infty$ , showing that condition (ii) holds.

Let us assume by contradiction that condition (iii) does not hold. Then there exists a subsequence  $(f_{k_\ell})$  of  $(f_k)$  that converges in  $L_v^2$  to some function  $f \in L_v^2$ . As a consequence,  $|f_{k_\ell}|^2 \rightarrow |f|^2$  in  $L_v^1$  as  $\ell \rightarrow \infty$ . In particular,  $f \in L_v^2$ . However, the distributional limit  $|f_{k_\ell}|^2 \rightarrow \delta_{\widehat{v}}$  and the uniqueness of the limit imply that  $\delta_{\widehat{v}} = |f|^2 \in L_v^1$ , which is absurd. Thus, condition (iii) holds, and we infer that  $\lambda \in \sigma_{\text{ess}}(\Lambda)$ . Then, since  $\sigma_{\text{ess}}(\Lambda)$  is closed,  $\overline{J} \subset \sigma_{\text{ess}}(\Lambda)$ .

*Step 2:*  $\sigma_{\text{ess}}(\Lambda) \subset \overline{J}$ . Let  $\lambda \in \mathbb{R} \setminus \overline{J}$ . Then there exists a constant  $c > 0$  such that for all  $v \in \mathbb{R}^3$  and  $i = 1, \dots, n$ ,  $|\nu_i(v) - \lambda| \geq c$ . If  $(f_k) \subset \mathcal{D}$  with  $\|f_k\|_{L_v^2} = 1$  for all  $k \in \mathbb{N}$ , we have

$$\|(\Lambda - \lambda I)f_k\|_{L_v^2}^2 = \sum_{i=1}^n \int_{\mathbb{R}^3} (\nu_i(v) - \lambda)^2 f_{k,i}(v)^2 dv \geq c^2 \sum_{i=1}^n \int_{\mathbb{R}^3} f_{k,i}(v)^2 dv = c^2 > 0$$

for all  $k \in \mathbb{N}$ . Thus, condition (ii) cannot hold which implies that  $\lambda \notin \sigma_{\text{ess}}(\Lambda)$ .

Steps 1 and 2 imply that  $\sigma_{\text{ess}}(\Lambda) = \overline{J}$ .

*Step 3:*  $\{\lambda \in \mathbb{C} : \Re(\lambda) \in J\} \subset \sigma_{\text{ess}}(\Lambda + T)$ . Let  $\lambda \in \mathbb{C}$  be such that  $\Re(\lambda) \in J$ . It follows from Step 1 that  $\Re(\lambda) \in \sigma_{\text{ess}}(\Lambda)$ . Since  $\Lambda$  is selfadjoint on the Hilbert space  $L_v^2$ ,  $\Lambda - \Re(\lambda)I$  is not closed or the kernel of  $\Lambda - \Re(\lambda)I$  is infinite dimensional. As the operator

$\Lambda - \Re(\lambda)I$  is closed, its kernel must be infinite dimensional. Therefore, there exists a sequence  $(f_k) \subset L_v^2$  such that  $\Lambda(f_k) - \Re(\lambda)f_k = 0$  and  $(f_k, f_\ell)_{L_v^2} = \delta_{k\ell}$  for  $k, \ell \in \mathbb{N}$ . Let us define  $\phi(x, v) = \exp(i\Im(\lambda)x \cdot v/|v|^2)$  and  $g_k = \phi f_k \in L_{x,v}^2$ . Since  $|\phi| = 1$ , we have

$$(22) \quad (g_k, g_\ell)_{L_{x,v}^2} = (f_k, f_\ell)_{L_v^2} = \delta_{k\ell} \quad \text{for } k, \ell \in \mathbb{N}.$$

Furthermore,  $\phi \in \text{Dom}(T)$  and  $T(\phi) = i\Im(\lambda)\phi$  for  $v \neq 0$ , and thus,

$$(\Lambda + T - \lambda I)g_k = \phi(\Lambda - \Re(\lambda)I)f_k + f_k(T - i\Im(\lambda)I)\phi = 0,$$

which shows that  $g_k \in \mathcal{N}(\Lambda + T - \lambda I)$  for  $k \in \mathbb{N}$ . This fact, together with relation (22), implies that  $\mathcal{N}(\Lambda + T - \lambda I)$  is infinite dimensional. As a consequence,  $\Lambda + T - \lambda I$  is not Fredholm and  $\lambda \in \sigma_{\text{ess}}(\Lambda + T)$ , which proves the claim.

*Step 4:*  $\{\lambda \in \mathbb{C} : \Re(\lambda) \notin \overline{\mathcal{J}}\} \subset \rho(\Lambda + T)$ . Clearly, this gives

$$\sigma_{\text{ess}}(\Lambda + T) \subset \sigma(\Lambda + T) \subset \{\lambda \in \mathbb{C} : \Re(\lambda) \in \overline{\mathcal{J}}\}.$$

Let  $\lambda \in \mathbb{C}$  be such that  $\Re(\lambda) \in \mathbb{R} \setminus \overline{\mathcal{J}}$ . We show first that  $\mathcal{N}(\Lambda + T - \lambda I) = \{0\}$ . We assume by contradiction that there exists  $f \in \text{Dom}(\Lambda + T)$  satisfying  $\|f\|_{L_{x,v}^2} > 0$  and  $(\Lambda + T - \lambda I)f = 0$ . In particular, there is an index  $\ell \in \{1, \dots, n\}$  such that  $\int_{\mathbb{R}^3} \int_{\mathbb{T}^3} f_\ell^2 dx dv > 0$ . Then, multiplying  $\nu_\ell f_\ell + T(f_\ell) = \lambda f_\ell$  by  $\overline{f}_\ell$  (the complex conjugate of  $f_\ell$ ) and integrating in  $\mathbb{T}^3 \times \mathbb{R}^3$ , we obtain

$$(23) \quad \int_{\mathbb{R}^3} \int_{\mathbb{T}^3} \nu_\ell |f_\ell|^2 dx dv + \int_{\mathbb{R}^3} \int_{\mathbb{T}^3} \overline{f}_\ell v \cdot \nabla_x f_\ell dx dv = \lambda \int_{\mathbb{R}^3} \int_{\mathbb{T}^3} |f_\ell|^2 dx dv.$$

By the divergence theorem, the real part of the second integral vanishes,

$$(24) \quad \begin{aligned} 2\Re \int_{\mathbb{R}^3} \int_{\mathbb{T}^3} \overline{f}_\ell v \cdot \nabla_x f_\ell dx dv &= \int_{\mathbb{R}^3} \int_{\mathbb{T}^3} (\overline{f}_\ell v \cdot \nabla_x f_\ell + f_\ell v \cdot \nabla_x \overline{f}_\ell) dx dv \\ &= \int_{\mathbb{R}^3} \int_{\mathbb{T}^3} v \cdot \nabla_x |f_\ell|^2 dx dv = 0. \end{aligned}$$

Then, taking the real part of (23), we infer that

$$\Re(\lambda) = \frac{\int_{\mathbb{R}^3} \int_{\mathbb{T}^3} \nu_\ell |f_\ell|^2 dx dv}{\int_{\mathbb{R}^3} \int_{\mathbb{T}^3} |f_\ell|^2 dx dv}.$$

Consequently,  $\inf_{\mathbb{R}^3} \nu_\ell \leq \Re(\lambda) \leq \sup_{\mathbb{R}^3} \nu_\ell$  and, thanks to the continuity of  $\nu_\ell$ ,  $\Re(\lambda) \in \overline{\text{Im}(\nu_\ell)} \subset \overline{\mathcal{J}}$ , which is a contradiction. Thus,  $\mathcal{N}(\Lambda + T - \lambda I) = \{0\}$ . Similarly, we can show that  $\mathcal{N}((\Lambda + T - \lambda I)^*) = \mathcal{N}(\Lambda + T^* - \overline{\lambda}I) = \{0\}$  as well.

The operator  $L - T$  is closed [35, Theorems 2.2.1 and 2.2.2]. Thus, the boundedness of the compact operator  $K$  and the stability of closedness under bounded perturbations [24, Chap. III, Problem 5.6] imply that  $\Lambda + T = K - (L - T)$  is closed (and also densely defined). Hence,  $\mathcal{R}(\Lambda + T - \lambda I)^\perp = \mathcal{N}((\Lambda + T - \lambda I)^*) = \{0\}$ , meaning that  $\Lambda + T - \lambda I$  is invertible. If  $f \in L_{x,v}^2$  is given, there exists  $u \in \text{Dom}(\Lambda + T)$  such that  $(\Lambda + T - \lambda I)u = f$ , which translates into

$$(25) \quad (\nu_j - \Re(\lambda))u_j + (T - i\Im(\lambda))u_j = f, \quad j = 1, \dots, n.$$

We point out that, since  $\nu_j$  is continuous,  $\text{Im}(\nu_j)$  is an interval (or a point, in case that  $\nu_j$  is constant). This fact and the assumption  $\Re(\lambda) \notin \overline{J}$  imply that either  $\nu_j - \Re(\lambda) > 0$  in  $\mathbb{R}^3$  or  $\nu_j - \Re(\lambda) < 0$  in  $\mathbb{R}^3$ . This means that the sign  $s_j$  of  $\nu_j - \Re(\lambda)$  is constant in  $\mathbb{R}^3$ , for  $j = 1, \dots, n$ . By multiplying (25) by  $s_j \overline{u}_j$ , integrating over  $\mathbb{T}^3 \times \mathbb{R}^3$ , taking the real part, and summing over  $j = 1, \dots, n$ , we find that

$$\sum_{j=1}^n \int_{\mathbb{R}^3} \int_{\mathbb{T}^3} |\nu_j - \Re(\lambda)| |u_j|^2 dx dv = \frac{1}{2} \sum_{j=1}^n s_j \int_{\mathbb{R}^3} \int_{\mathbb{T}^3} (\overline{u}_j f_j + u_j \overline{f}_j) dx dv.$$

The (real part of the) second term in (25) vanishes after integration; see (24). Since  $\lambda \in \mathbb{R} \setminus \overline{J}$ , by definition of  $J$ , there exists  $c_\lambda > 0$  such that  $|\nu_j - \Re(\lambda)| \geq c_\lambda$  in  $\mathbb{R}^3$  for all  $j = 1, \dots, n$ . Then the Cauchy-Schwarz inequality shows that  $\|u\|_{L_{x,v}^2} \leq c_\lambda^{-1} \|f\|_{L_{x,v}^2}$ . This means that  $(\Lambda + T - \lambda I)^{-1}$  is bounded, so  $\lambda \in \rho(\Lambda + T)$ .

Steps 3 and 4 show that  $\sigma_{\text{ess}}(\Lambda + T) = \{\lambda \in \mathbb{C} : \Re(\lambda) \in \overline{J}\}$ .

*Step 5:*  $\sigma_{\text{ess}}(-\Lambda - T) = \sigma_{\text{ess}}(K - \Lambda - T)$ . The operator  $K$  is compact on  $L_v^2$  but not on  $L_{x,v}^2$ , so the claim does not follow from the original form of Weyl's theorem. Instead we will employ the fact that the essential spectrum is conserved under a relatively compact perturbation [24, Section IV.5.6, Theorem 5.35]. More precisely, we prove that  $K$  is relatively compact with respect to  $\Lambda + T$ , i.e.,  $B_z := K(\Lambda + T - zI)^{-1}$  is compact on  $L_{x,v}^2$  for some  $z \in \mathbb{C}$  with  $\Re(z) \in \mathbb{R} \setminus \overline{J}$ . (Notice that by Step 4,  $z \in \rho(\Lambda + T)$ .) Then

$$(26) \quad \sigma_{\text{ess}}(K - \Lambda - T) = \sigma_{\text{ess}}(-\Lambda - T) = \{\lambda \in \mathbb{C} : \Re(\lambda) \in -J\}.$$

The second identity is a consequence of Steps 3 and 4.

To prove the compactness of  $B_z$ , we introduce the space  $W := \ell^2(\mathbb{Z}^3; L_v^2)$  of sequences  $f = (f_m) \subset L_v^2$  with the canonical norm  $\|f\|_W = (\sum_{m \in \mathbb{Z}^3} \|f_m\|_{L_v^2}^2)^{1/2}$ . Clearly,  $W$  is a Hilbert space with the scalar product  $(f, g)_W = \sum_{m \in \mathbb{Z}^3} (f_m, g_m)_{L_v^2}$ . Furthermore, we introduce the Fourier mapping  $F : L_{x,v}^2(\mathbb{T}^3 \times \mathbb{R}^3) \rightarrow W$  by

$$F(f) = (\widehat{f}_m), \quad \widehat{f}_m(v) = \int_{\mathbb{T}^3} e^{-2\pi i m \cdot x} f(x, v) dx \quad \text{for } m \in \mathbb{Z}^3, v \in \mathbb{R}^3.$$

This mapping is bounded, invertible, and has a bounded inverse. We wish to show that  $\widehat{B}_z = FB_z F^{-1} : W \rightarrow W$  is compact. Then also  $B_z = F^{-1} \widehat{B}_z F$  is compact as a composition of a compact and two bounded operators. This idea is due to Ukai; see e.g. [35, Section 2.2.1].

Since  $K$  and  $\Lambda$  do not depend on  $x$ , it holds that  $\widehat{B}_z = K(\Lambda + \widehat{T} - z)^{-1}$ , where  $\widehat{T} = 2\pi i v \cdot m$ . Let  $(f^{(k)}) = (f_m^{(k)}) \subset W$  be a bounded sequence in  $W$ , i.e., there exists  $c_0 > 0$  such that for all  $k \in \mathbb{N}$ ,

$$(27) \quad \|f^{(k)}\|_W^2 = \sum_{m \in \mathbb{Z}^3} \|f_m^{(k)}\|_{L_v^2}^2 \leq c_0.$$

As  $\Re(z) \in \mathbb{R} \setminus \overline{J}$ , there is a constant  $c_z > 0$  such that for all  $i = 1, \dots, n$  and  $v \in \mathbb{R}^3$ ,  $|\nu_i(v) + 2\pi i v \cdot m - z| \geq c_z$ . Thus,

$$\begin{aligned} \|(\Lambda + \widehat{T} - z)^{-1} f_m^{(k)}\|_{L_v^2}^2 &= \sum_{i=1}^n \int_{\mathbb{R}^3} |(\nu_i(v) + 2\pi i v \cdot m - z)^{-1} f_{m,i}^{(k)}|^2 dv \\ &\leq c_z^{-2} \sum_{i=1}^n \int_{\mathbb{R}^3} |f_{m,i}^{(k)}|^2 dv = c_z^{-2} \|f_m^{(k)}\|_{L_v^2}^2. \end{aligned}$$

Summing these inequalities over  $m \in \mathbb{Z}^3$ , we infer that

$$\|(\Lambda + \widehat{T} - z)^{-1} f^{(k)}\|_W^2 \leq c_z^{-2} \|f^{(k)}\|_W^2 \leq c_0 c_z^{-2}.$$

Consequently, the sequence  $g^{(k)} := (\Lambda + \widehat{T} - z)^{-1} f^{(k)}$  is bounded in  $W$ . In particular, for any  $s \in \mathbb{Z}^3$ ,  $\|g_s^{(k)}\|_{L_v^2}^2 \leq \sum_{m \in \mathbb{Z}^3} \|g_m^{(k)}\|_{L_v^2}^2 = \|g^{(k)}\|_W^2 \leq c_0 c_z^{-2}$ . Hence, for any  $s \in \mathbb{Z}^3$ , the sequence  $(g_s^{(k)}) \subset L_v^2$  is bounded in  $L_v^2$ . Since  $K : L_v^2 \rightarrow L_v^2$  is compact and  $\mathbb{Z}^3$  is countable, we may apply Cantor's diagonal argument to find a subsequence  $(g^{(k_\ell)})$  of  $(g^{(k)})$  such that  $(K(g_m^{(k_\ell)}))$  is convergent in  $L_v^2$  as  $\ell \rightarrow \infty$ , for all  $m \in \mathbb{Z}^3$ .

We will show that  $(\widehat{B}_z(f^{(k_\ell)}))$  is a Cauchy sequence in  $W$ . To this end, let  $\ell, s, N \in \mathbb{N}$ . We write

$$\begin{aligned} (28) \quad \|\widehat{B}_z(f^{(k_\ell)}) - \widehat{B}_z(f^{(k_s)})\|_W^2 &= \sum_{m \in \mathbb{Z}^3} \|K(g_m^{(k_\ell)}) - K(g_m^{(k_s)})\|_{L_v^2}^2 \\ &= \sum_{|m| \leq N} \|K(g_m^{(k_\ell)}) - K(g_m^{(k_s)})\|_{L_v^2}^2 + \sum_{|m| > N} \|K(g_m^{(k_\ell)}) - K(g_m^{(k_s)})\|_{L_v^2}^2, \end{aligned}$$

where  $|m| = \sum_{i=1}^3 |m_i|$  for all  $m \in \mathbb{Z}^3$ . First, we consider the second sum on the right-hand side. Denote by  $\|\cdot\|_{\mathcal{L}(L_v^2)}$  the norm in the space of linear bounded operators on  $L_v^2$ . By the definition of  $g_m^{(k)}$ , we obtain

$$\begin{aligned} (29) \quad \sum_{|m| > N} \|K(g_m^{(k_\ell)}) - K(g_m^{(k_s)})\|_{L_v^2}^2 &= \sum_{|m| > N} \|K(\Lambda + 2\pi i v \cdot m - z)^{-1} (f_m^{(k_\ell)} - f_m^{(k_s)})\|_{L_v^2}^2 \\ &\leq 2 \sum_{|m| > N} \|K(\Lambda + 2\pi i v \cdot m - z)^{-1}\|_{\mathcal{L}(L_v^2)}^2 (\|f_m^{(k_\ell)}\|_{L_v^2}^2 + \|f_m^{(k_s)}\|_{L_v^2}^2). \end{aligned}$$

For the operator norm, we employ Prop. 2.2.6 in [35], which can be applied since  $\Re(z) \in \mathbb{R} \setminus \overline{J}$ :

$$\|K(\Lambda + 2\pi i v \cdot m - z)^{-1}\|_{\mathcal{L}(L_v^2)}^2 \leq c_1 (1 + |m|)^{-\alpha} \quad \text{for all } m \in \mathbb{Z}^3$$

for some suitable constant  $c_1 > 0$  (depending on  $z$ ) and a suitable exponent  $\alpha \in (0, 1)$  (actually,  $\alpha = 4/13$ ). Let  $0 < \beta < 2\alpha/3$ . By Hölder's inequality and (27), we estimate

$$\sum_{|m| > N} \|K(\Lambda + 2\pi i v \cdot m - z)^{-1}\|_{\mathcal{L}(L_v^2)}^2 \|f_m^{(k)}\|_{L_v^2}^2 \leq c_0^{\beta/2} c_1 \sum_{|m| > N} (1 + |m|)^{-\alpha} \|f_m^{(k)}\|_{L_v^2}^{2-\beta}$$

$$\begin{aligned}
&\leq c_0^{\beta/2} c_1 \left( \sum_{|m|>N} (1+|m|)^{-2\alpha/\beta} \right)^{\beta/2} \left( \sum_{|m|>N} \|f_m^{(k)}\|_{L_v^2}^2 \right)^{1-\beta/2} \\
&\leq c_0 c_1 \left( \sum_{|m|>N} (1+|m|)^{-2\alpha/\beta} \right)^{\beta/2}.
\end{aligned}$$

Using this estimate in (29), it follows that

$$\sup_{\ell, s \in \mathbb{N}} \sum_{|m|>N} \|K(g_m^{(k_\ell)}) - K(g_m^{(k_s)})\|_{L_v^2}^2 \leq 2c_0 c_1 \left( \sum_{|m|>N} (1+|m|)^{-2\alpha/\beta} \right)^{\beta/2}.$$

The choice of  $\beta$  implies that  $2\alpha/\beta > 3$  and hence, the sum over  $|m| > N$  is finite. In particular,  $\sum_{|m|>N} (1+|m|)^{-2\alpha/\beta} \rightarrow 0$  as  $N \rightarrow \infty$ . As a consequence, for given  $\varepsilon > 0$ , there exists  $N_\varepsilon \in \mathbb{N}$  such that

$$\sup_{\ell, s \in \mathbb{N}} \sum_{|m|>N_\varepsilon} \|K(g_m^{(k_\ell)}) - K(g_m^{(k_s)})\|_{L_v^2}^2 < \frac{\varepsilon}{2}.$$

Finally, since  $(K(g_m^{(k_\ell)}))$  is convergent in  $L_v^2$  for all  $m \in \mathbb{Z}^3$ , there is a number  $\eta = \eta(\varepsilon) > 0$  such that for all  $\ell, s > \eta$ ,

$$\sum_{|m| \leq N_\varepsilon} \|K(g_m^{(k_\ell)}) - K(g_m^{(k_s)})\|_{L_v^2}^2 < \frac{\varepsilon}{2}.$$

Thus, choosing  $N = N_\varepsilon$  in (28), we deduce that  $(\widehat{B}_z(f^{(k_s)}))$  is a Cauchy sequence in the Hilbert space  $W$  and consequently, it is convergent. This shows that  $\widehat{B}_z : W \rightarrow W$  is a compact operator and (26) holds. This finishes the proof of Theorem 1.  $\square$

Next, we show the spectral-gap estimate for the linearized collision operator  $L = K - \Lambda$ , i.e. the first statement of Theorem 3. Since  $K$  is compact on  $L_v^2$ , it remains to prove that  $\Lambda : \mathcal{D} \subset L_v^2 \rightarrow L_v^2$  is coercive.

**Lemma 9.** *Let (A1)-(A4) hold. Then the embedding  $\mathcal{H} \hookrightarrow L_v^2$  is continuous and  $\Lambda : \mathcal{D} \rightarrow L_v^2$ , defined in (21), is a linear unbounded operator with the property*

$$(30) \quad (f, \Lambda(f))_{L_v^2} = \|f\|_{\mathcal{H}}^2 \geq C \|f\|_{L_v^2}^2 \quad \text{for } f \in \mathcal{H}$$

for some  $C > 0$ . Moreover,  $\Lambda$  can be extended by density to a linear bounded operator  $\Lambda : \mathcal{H} \rightarrow \mathcal{H}'$ , where  $\mathcal{H}'$  is the dual of  $\mathcal{H}$  with respect to the  $L_v^2$  scalar product. In particular, the mapping  $\mathcal{H} \rightarrow \mathbb{R}$ ,  $f \mapsto \langle \Lambda(f), f \rangle$  is continuous, where  $\langle \cdot, \cdot \rangle$  denotes the duality pairing between  $\mathcal{H}'$  and  $\mathcal{H}$ .

*Proof.* The strict positivity of  $\nu_i$  in  $\mathbb{R}^3$  (see Lemma 7) implies that the embedding  $\mathcal{H} \hookrightarrow L_v^2$  is continuous. Then the definitions of  $\Lambda_i$  and  $\mathcal{H}$  show that for all  $f \in \mathcal{H}$ , (30) holds. For given  $f \in \mathcal{H}$ , the element  $\Lambda(f) = (f_1 \nu_1, \dots, f_n \nu_n)$  can be identified with the linear bounded operator  $\mathcal{H} \rightarrow \mathbb{R}$ ,  $g \mapsto \sum_{i=1}^n \int_{\mathbb{R}^3} g_i f_i \nu_i dv$  and consequently,  $\Lambda(f) \in \mathcal{H}'$ . It is immediate to see that  $\|\Lambda(f)\|_{\mathcal{H}'} = \|f\|_{\mathcal{H}}$ , so that  $\Lambda : \mathcal{H} \rightarrow \mathcal{H}'$  is isometric and thus bounded. Moreover, it follows that  $\mathcal{H} \rightarrow \mathbb{R}$ ,  $f \mapsto \langle \Lambda(f), f \rangle$ , is continuous.  $\square$

The following result provides a spectral gap for general operators which decompose into a compact and a coercive part.

**Lemma 10.** *Let  $\mathcal{H}_0$  and  $\mathcal{H}$  be Hilbert spaces such that  $\mathcal{H} \hookrightarrow \mathcal{H}_0$  continuously and let  $L : \mathcal{H} \rightarrow \mathcal{H}'$  be a linear bounded operator such that  $L = K - \Lambda$  with linear bounded operators  $\Lambda : \mathcal{H} \rightarrow \mathcal{H}'$  and  $K : \mathcal{H}_0 \rightarrow \mathcal{H}_0$ . Furthermore, assume that*

- (i) *for all  $f \in \mathcal{H}$ ,  $\langle L(f), f \rangle \leq 0$  with equality holding if and only if  $f \in \mathcal{N}(L)$ ;*
- (ii) *the operator  $K : \mathcal{H}_0 \rightarrow \mathcal{H}_0$  is compact;*
- (iii) *there exists  $C_0 > 0$  such that for all  $f \in \mathcal{H}$ ,  $\langle \Lambda(f), f \rangle \geq C_0 \|f\|_{\mathcal{H}}^2$ .*

*Then there exists a constant  $C_1 > 0$  such that*

$$-\langle L(f), f \rangle \geq C_1 \|f\|_{\mathcal{H}}^2 \quad \text{for all } f \in \mathcal{H} \cap \mathcal{N}(L)^\perp.$$

*Proof.* We argue by contradiction. Let  $(f_n) \subset \mathcal{H} \cap \mathcal{N}(L)^\perp$  be a sequence such that  $\|f_n\|_{\mathcal{H}} = 1$  for  $n \geq 1$  but  $\langle L(f_n), f_n \rangle \rightarrow 0$  as  $n \rightarrow \infty$ . Since  $(f_n)$  is bounded in the Hilbert space  $\mathcal{H}$ , there exists a subsequence, which is not relabeled, such that  $f_n \rightharpoonup f$  weakly in  $\mathcal{H}$ . Because of the continuous embedding  $\mathcal{H} \hookrightarrow \mathcal{H}_0$ , also  $f_n \rightharpoonup f$  weakly in  $\mathcal{H}_0$ . Since  $f_n \in \mathcal{N}(L)^\perp$  and  $\mathcal{N}(L)^\perp$  is weakly closed by Mazur's lemma,  $f \in \mathcal{N}(L)^\perp$ . As the operator  $K : \mathcal{H}_0 \rightarrow \mathcal{H}_0$  is compact, by hypothesis (ii), the weak convergence of  $(f_n)$  in  $\mathcal{H}_0$  implies that  $K(f_n) \rightarrow K(f)$  strongly in  $\mathcal{H}_0$ . Hence,  $(f_n, K(f_n))_{\mathcal{H}_0} \rightarrow (f, K(f))_{\mathcal{H}_0}$ . Since  $\Lambda : \mathcal{H} \rightarrow \mathcal{H}'$  is bounded, the mapping  $G : \mathcal{H} \rightarrow \mathbb{R}$ ,  $f \mapsto \langle \Lambda(f), f \rangle$ , is continuous. The linearity of  $\Lambda$  and property (iii) imply that  $G$  is also convex. Thus,  $G$  is weakly lower semicontinuous [5, Corollary 3.9]. Therefore,

$$-\langle L(f), f \rangle = \langle \Lambda(f), f \rangle - (K(f), f)_{\mathcal{H}_0} \leq \liminf_{n \rightarrow \infty} (\langle \Lambda(f_n), f_n \rangle - (K(f_n), f_n)_{\mathcal{H}_0}) = 0,$$

because  $\langle L(f_n), f_n \rangle \rightarrow 0$  as  $n \rightarrow \infty$  by assumption. We infer from hypothesis (i) that  $f \in \mathcal{N}(L)$ . But also  $f \in \mathcal{N}(L)^\perp$ , so  $f = 0$ . Then, by hypothesis (iii),

$$0 < C_0 = C_0 \|f_n\|_{\mathcal{H}}^2 \leq \langle \Lambda(f_n), f_n \rangle = (K(f_n), f_n)_{\mathcal{H}_0} - \langle L(f_n), f_n \rangle \rightarrow 0,$$

which is a contradiction.  $\square$

Let  $\mathcal{H}_0 = L_v^2$ . By [4, Prop. 2], assumption (ii) of Lemma 10 holds. Furthermore, Lemma 9 shows that (iii) holds true. Assumption (i) is a consequence of Lemma 5. Let  $f \in \mathcal{D} \subset \mathcal{H}$  und set  $\tilde{f} = f - \Pi^L(f) \in \mathcal{N}(L)^\perp$ . Then

$$-\langle L(f), f \rangle = \langle L(\tilde{f}), \tilde{f} \rangle \geq C \|\tilde{f}\|_{\mathcal{H}}^2 = C \|f - \Pi^L(f)\|_{\mathcal{H}}^2,$$

since  $L(f) \in L_v^2$  and  $\langle L(f), f \rangle = (f, L(f))_{L_v^2}$  for  $f \in \mathcal{D}$ . This proves the first statement in Theorem 3.

#### 4. EXPLICIT SPECTRAL GAP ESTIMATE

We present a second proof of the spectral-gap estimate (10) with explicit constants. The idea is to decompose the collision operator  $L$  into a mono-species and a multi-species part and to exploit the fact that the conservation properties of  $L$  are different from those of the mono-species part  $L^m$ . Let assumptions (A1)-(A4) hold.

**4.1. Decomposition.** We decompose  $L = L^m + L^b$ , where  $L^m = (L_1^m, \dots, L_n^m)$ ,  $L^b = (L_1^b, \dots, L_n^b)$ , and

$$(31) \quad L_i^m(f_i) = L_{ii}(f_i, f_i), \quad L_i^b(f) = \sum_{j \neq i} L_{ij}(f_i, f_j).$$

Denoting by  $\Pi^m$  the orthogonal projection onto  $\mathcal{N}(L^m)$  (with respect to the scalar product in  $L_v^2$ ), we can decompose  $f$  according to

$$(32) \quad f = f^\parallel + f^\perp, \quad \text{where } f^\parallel := \Pi^m(f), \quad f^\perp := f - f^\parallel.$$

Lemma 5 shows that

$$(33) \quad f \in \mathcal{N}(L) \text{ if and only if } f_i = M_i^{1/2}(\alpha_i + u \cdot v + e|v|^2) \text{ for } \alpha_i, e \in \mathbb{R}, u \in \mathbb{R}^3,$$

$$(34) \quad f \in \mathcal{N}(L^m) \text{ if and only if } f_i = M_i^{1/2}(\alpha_i + u_i \cdot v + e_i|v|^2) \text{ for } \alpha_i, e_i \in \mathbb{R}, u_i \in \mathbb{R}^3,$$

and  $f^\parallel$  has clearly the form (34).

For later use, we define the following bilinear forms

$$(35) \quad -(f, L^m(f))_{L_v^2} = \frac{1}{4} \sum_{i=1}^n \int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ii} \Delta_i [h_i]^2 M_i M_i^* dv dv^* d\sigma,$$

$$(36) \quad -(f, L^b(f))_{L_v^2} = \frac{1}{4} \sum_{i=1}^n \sum_{j \neq i} \int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} A_{ij} [h_i, h_j]^2 M_i M_j^* dv dv^* d\sigma,$$

where  $h_i = M_i^{-1/2} f_i$  and

$$\Delta_i [h_i] := h_i' + h_i'^* - h_i - h_i^*, \quad A_{ij} [h_i, h_j] := h_i' + h_j'^* - h_i - h_j^*.$$

**4.2. Spectral-gap estimate for  $L^m$ .** Our starting point is the fact that the mono-species collision operator  $L^m$  has an explicitly computable spectral gap. A spectral-gap estimate for the linearized collision operator with  $n = 1$  was proved in [26, Theorem 6.1, Remark 1]:

$$(37) \quad \frac{1}{4} \int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ii} \Delta_i [h_i]^2 M_i M_i^* dv dv^* d\sigma \geq \frac{\lambda_m}{\rho_{\infty, i}} \int_{\mathbb{R}^3} (f_i - \Pi^m(f_i))^2 \nu_{ii} dv,$$

where  $\lambda_m = \lambda_m(\gamma, C_1, C^b) > 0$ , only depending on  $\gamma$ ,  $C_1$ , and  $C^b$  (see (A3)-(A4)), can be computed explicitly,

$$\nu_{ii}(v) := \int_{\mathbb{R}^3 \times \mathbb{S}^2} B_{ii} (|v - v^*|, \cos \vartheta) M_i^* dv^* d\sigma,$$

and  $i \in \{1, \dots, n\}$  is fixed. This yields the following estimate for  $L^m$ , where we recall that the space  $\mathcal{H}$  is defined in (8).

**Lemma 11.** *With  $L^m$  defined in (31), we have*

$$-(f, L^m(f))_{L_v^2} \geq C^m \|f - \Pi^m(f)\|_{\mathcal{H}}^2 \quad \text{for all } f \in \text{Dom}(L^m),$$



where

$$(38) \quad C^m = \frac{\lambda^m(\gamma, C_1, C^b)}{\beta\rho_\infty},$$

and  $\lambda^m = \lambda^m(\gamma, C_1, C^b)$  is given in (37).

*Proof.* We sum (37) over  $i = 1, \dots, n$  and employ (35) to obtain

$$(39) \quad -(f, L^m(f))_{L_v^2} \geq \lambda^m \sum_{i=1}^n \int_{\mathbb{R}^3} (f_i - \Pi^m(f))^2 \frac{\nu_{ii}}{\rho_{\infty,i}} dv.$$

It remains to estimate  $\nu_{ii}$  in terms of  $\nu_i$ , defined in (9). The definition of  $M_i$  implies that  $M_j = (\rho_{\infty,j}/\rho_{\infty,i})M_i$ . This fact, as well as definition (9) of  $\nu_i$ , the lower bound (18), and assumption (A6) give

$$\nu_i = \sum_{j=1}^n \frac{\rho_{\infty,j}}{\rho_{\infty,i}} \int_{\mathbb{R}^3} B_{ij} M_i^* dv^* d\sigma \leq \beta \sum_{j=1}^n \frac{\rho_{\infty,j}}{\rho_{\infty,i}} \int_{\mathbb{R}^3} B_{ii} M_i^* dv^* d\sigma = \frac{\beta\rho_\infty}{\rho_{\infty,i}} \nu_{ii}.$$

We conclude that  $\nu_{ii}/\rho_{\infty,i} \geq \nu_i/(\beta\rho_\infty)$ , and inserting this bound into (39) yields the result.  $\square$

Lemma (11) and the inequality  $-(f, L^b(f))_{L_v^2} \geq 0$  immediately show that

$$-(f, L(f))_{L_v^2} \geq C^m \|f - \Pi^m(f)\|_{\mathcal{H}}^2 \quad \text{for all } f \in \mathcal{D}.$$

However, we need the projection onto  $\mathcal{N}(L)^\perp$  instead of  $\mathcal{N}(L^m)^\perp$ , which is contained in  $\mathcal{N}(L)^\perp$ . Therefore, we will exploit the part  $-(f, L^b(f))_{L_v^2}$  to derive a sharper estimate.

**4.3. Absorption of the orthogonal parts.** We prove that the contribution  $f^\perp$  (introduced in (32)) in the term  $-(f, L^b(f))_{L_v^2} = -(f^\parallel + f^\perp, L^b(f^\parallel + f^\perp))_{L_v^2}$  can be absorbed by the  $\mathcal{H}$  norm of  $f^\perp$ .

**Lemma 12.** *Let  $\eta = \min\{1, C^m/8\}$ , where  $C^m > 0$  is given in Lemma 11. Then, for all  $f \in \mathcal{D}$ ,*

$$-(f, L(f))_{L_v^2} \geq (C^m - 4\eta) \|f - f^\parallel\|_{\mathcal{H}}^2 - \frac{\eta}{2} (f^\parallel, L^b(f^\parallel))_{L_v^2},$$

where  $f^\parallel = \Pi^m(f)$  is the projection onto  $\mathcal{N}(L^m)^\perp$ .

*Proof.* By Lemma 11, we find that

$$(40) \quad -(f, L(f))_{L_v^2} \geq C^m \|f - f^\parallel\|_{\mathcal{H}}^2 - (f, L^b(f))_{L_v^2} \geq C^m \|f - f^\parallel\|_{\mathcal{H}}^2 - \eta (f, L^b(f))_{L_v^2},$$

since  $-(1-\eta)(f, L^b(f))_{L_v^2} \geq 0$  for  $\eta \in (0, 1]$ . We estimate first the expression  $A_{ij}[h_i, h_j]$  in definition (36), writing  $h_i^\parallel = M_i^{-1/2} f_i^\parallel$  and  $h_i^\perp = M_i^{-1/2} f_i^\perp$ ,

$$\begin{aligned} A_{ij}[h_i, h_j]^2 &= (A_{ij}[h_i^\parallel, h_j^\parallel] + A_{ij}[h_i^\perp, h_j^\perp])^2 \\ &= A_{ij}[h_i^\parallel, h_j^\parallel]^2 + A_{ij}[h_i^\perp, h_j^\perp]^2 + 2A_{ij}[h_i^\parallel, h_j^\parallel]A_{ij}[h_i^\perp, h_j^\perp] \\ &\geq \frac{1}{2}A_{ij}[h_i^\parallel, h_j^\parallel]^2 - A_{ij}[h_i^\perp, h_j^\perp]^2. \end{aligned}$$

Inserting this estimate into (36) and (40) gives

$$(41) \quad \begin{aligned} -(f, L(f))_{L_v^2} &\geq C^m \|f^\perp\|_{\mathcal{H}}^2 + \frac{\eta}{8} \sum_{i=1}^n \sum_{j \neq i} \int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} A_{ij} [h_i^\parallel, h_j^\parallel]^2 M_i M_j^* dv dv^* d\sigma \\ &\quad - \frac{\eta}{4} \sum_{i=1}^n \sum_{j \neq i} \int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} A_{ij} [h_i^\perp, h_j^\perp]^2 M_i M_j^* dv dv^* d\sigma. \end{aligned}$$

We claim that the last term on the right-hand side can be estimated from below by  $\|f^\perp\|_{\mathcal{H}}^2$ , up to a small factor. For this, we employ the invariance properties of  $B_{ij}$  and the identity  $M_i M_j^* = M_i' M_j'^*$ :

$$\begin{aligned} &\int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} A_{ij} [h_i^\perp, h_j^\perp]^2 M_i M_j^* dv dv^* d\sigma \\ &\leq 4 \int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} (((h_i^\perp)')^2 + ((h_j^\perp)')^2 + (h_i^\perp)^2 + ((h_j^\perp)^*)^2) M_i M_j^* dv dv^* d\sigma \\ &\leq 16 \int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} (h_i^\perp)^2 M_i M_j^* dv dv^* d\sigma = 16 \int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} (f_i^\perp)^2 M_j^* dv dv^* d\sigma. \end{aligned}$$

Thus, the last term on the right-hand side of (41) can be estimated as

$$\begin{aligned} &-\frac{\eta}{4} \sum_{i=1}^n \sum_{j \neq i} \int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} A_{ij} [h_i^\perp, h_j^\perp]^2 M_i M_j^* dv dv^* d\sigma \\ &\geq -4\eta \sum_{i=1}^n \sum_{j \neq i} \int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} (f_i^\perp)^2 M_j^* dv dv^* d\sigma \geq -4\eta \sum_{i=1}^n \int_{\mathbb{R}^3} (f_i^\perp)^2 \nu_i dv = -4\eta \|f^\perp\|_{\mathcal{H}}^2, \end{aligned}$$

taking into account definition (9) of  $\nu_i$ . We infer from (41) that

$$-(f, L(f))_{L_v^2} \geq (C^m - 4\eta) \|f - f^\parallel\|_{\mathcal{H}}^2 + \frac{\eta}{8} \sum_{i=1}^n \sum_{j \neq i} \int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} A_{ij} [h_i^\parallel, h_j^\parallel]^2 M_i M_j^* dv dv^* d\sigma,$$

and definition (36) yields the conclusion.  $\square$

**4.4. Estimate for the remaining part.** It remains to estimate the term  $-(f^\parallel, L^b(f^\parallel))_{L_v^2}$ .

**Lemma 13.** For  $f^\parallel \in \mathcal{N}(L^m)$ , i.e.  $f_i^\parallel = M_i^{1/2}(\alpha_i + u_i \cdot v + e_i |v|^2)$  for some  $\alpha_i, e_i \in \mathbb{R}$  and  $u_i \in \mathbb{R}^3$ , we have

$$-(f^\parallel, L^b(f^\parallel))_{L_v^2} \geq \frac{D^b}{4} \sum_{i,j=1}^n (|u_i - u_j|^2 + (e_i - e_j)^2),$$

where  $D^b > 0$  is defined in (43).

*Proof.* Thanks to the momentum and energy conservation, we obtain differences of the momenta and energies, which will be crucial in the following:

$$u_i \cdot v' + u_j \cdot v'^* - u_i \cdot v - u_j \cdot v^* = (u_i - u_j) \cdot (v' - v),$$

$$e_i|v'|^2 + e_j|v'^*|^2 - e_i|v|^2 - e_j|v^*|^2 = (e_i - e_j)(|v'|^2 - |v|^2).$$

Using these identities in  $A_{ij}[h_i^\parallel, h_j^\parallel]$ , where  $h_i^\parallel = \alpha_i + u_i \cdot v + e_i|v|^2$ , we find that

$$\begin{aligned} -(f^\parallel, L^b(f^\parallel))_{L_v^2} &= \frac{1}{4} \sum_{i=1}^n \sum_{j \neq i} \int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} A_{ij}[h_i^\parallel, h_j^\parallel]^2 M_i M_j^* dv dv^* d\sigma \\ &= \frac{1}{4} \sum_{i=1}^n \sum_{j \neq i} \int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} ((u_i - u_j) \cdot (v' - v) + (e_i - e_j)(|v'|^2 - |v|^2))^2 M_i M_j^* dv dv^* d\sigma. \end{aligned}$$

Using the symmetry of  $B_{ij}$  (thanks to assumption (A1)) and of  $M_i M_j^*$  with respect to  $v$ , the function  $G(v, v^*, \sigma) = B_{ij}(u_i - u_j) \cdot (v' - v)(|v'|^2 - |v|^2)$  is odd with respect to  $(v, v^*, \sigma)$  and thus, the mixed term of the square in the above integral vanishes. Therefore, we obtain

$$\begin{aligned} -(f^\parallel, L^b(f^\parallel))_{L_v^2} &= \frac{1}{4} \sum_{i=1}^n \sum_{j \neq i} \int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} (|(u_i - u_j) \cdot (v' - v)|^2 + (e_i - e_j)^2(|v'|^2 - |v|^2)^2) \\ (42) \quad &\times M_i M_j^* dv dv^* d\sigma. \end{aligned}$$

Now, we claim that

$$\int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} ((u_i - u_j) \cdot (v' - v))^2 M_i M_j^* dv dv^* d\sigma = \frac{|u_i - u_j|^2}{3} \int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} |v - v'|^2 M_i M_j^* dv dv^* d\sigma.$$

To prove this identity, we write  $u_{i,k}$  and  $v_k$  for the  $k$ th component of the vectors  $u_i$  and  $v$ , respectively. The transformation  $(v_k, v_k^*, \sigma_k) \mapsto -(v_k, v_k^*, \sigma_k)$  for fixed  $k$  leaves  $B_{ij}$ ,  $M_i$ , and  $M_j^*$  unchanged but  $v'_k \mapsto -v'_k$  such that

$$\int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} v'_k v_\ell M_i M_j^* dv dv^* d\sigma = 0 \quad \text{for } \ell \neq k.$$

Furthermore,

$$\int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} v_k v_\ell M_i M_j^* dv dv^* d\sigma = 0 \quad \text{for } \ell \neq k,$$

since the integrand is odd. Therefore,

$$\begin{aligned} &\int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} ((u_i - u_j) \cdot (v' - v))^2 M_i M_j^* dv dv^* d\sigma \\ &= \sum_{k, \ell=1}^3 (u_{i,k} - u_{j,k})(u_{i,\ell} - u_{j,\ell}) \int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} (v'_k - v_k)(v'_\ell - v_\ell) M_i M_j^* dv dv^* d\sigma \\ &= \sum_{k=1}^3 (u_{i,k} - u_{j,k})^2 \int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} (v_k - v'_k)^2 M_i M_j^* dv dv^* d\sigma. \end{aligned}$$

In fact, we can see that the integral is independent of  $k$ , and we infer that

$$\int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} ((u_i - u_j) \cdot (v' - v))^2 M_i M_j^* dv dv^* d\sigma$$

$$= \frac{1}{3} \sum_{k=1}^3 (u_{i,k} - u_{j,k})^2 \int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} |v - v'|^2 M_i M_j^* dv dv^* d\sigma,$$

from which the claim follows.

Hence, (42) can be estimated as

$$-(f^\parallel, L^b(f^\parallel))_{L_v^2} \geq D^b \sum_{i,j=1}^3 (|u_i - u_j|^2 + (e_i - e_j)^2),$$

where

$$(43) \quad D^b = \min_{1 \leq i, j \leq n} \int_{\mathbb{R}^6 \times \mathbb{S}^2} B_{ij} \min \left\{ \frac{1}{3} |v - v'|^2, (|v'|^2 - |v|^2)^2 \right\} M_i M_j^* dv dv^* d\sigma.$$

It remains to show that  $D^b > 0$ . The integrand of (43) vanishes if and only if  $|v'| = |v|$ . However, the set

$$X = \{(v, v^*, \sigma) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 : |v'| = |v|\}$$

is closed since it is the pre-image of  $\{0\}$  of the continuous function  $F(v, v^*, \sigma) = |v'|^2 - |v|^2$ , i.e.  $X = F^{-1}(\{0\})$ , recalling that  $v'$  depends on  $(v, v^*, \sigma)$  through (2). Since  $X \neq \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2$ , its complement  $X^c$  is open and nonempty and thus has positive Lebesgue measure. Since the integrand in (43) is positive on  $X^c$ , we infer that  $D^b > 0$ . This finishes the proof.  $\square$

**4.5. Estimate for the momentum and energy differences.** The last step is to derive lower bounds for the differences  $\sum_{i,j} (|u_i - u_j|^2 + (e_i - e_j)^2)$ . First, we recall some moment identities:

$$(44) \quad \int_{\mathbb{R}^3} M_i dv = \rho_i, \quad \int_{\mathbb{R}^3} M_i v_j v_k dv = \rho_i \delta_{jk}, \quad \int_{\mathbb{R}^3} M_i |v|^4 dv = 15\rho_i$$

for all  $1 \leq i \leq n$  and  $1 \leq j, k \leq 3$ .

**Lemma 14.** *Let  $f \in L_v^2$  with  $f_i^\parallel = M_i^{1/2}(\alpha_i + u_i \cdot v + e_i |v|^2)$  for  $1 \leq i \leq n$ . Then*

$$\int_{\mathbb{R}^3} M_i^{1/2} f_i dv = \rho_i(\alpha_i + 3e_i), \quad \int_{\mathbb{R}^3} M_i^{1/2} f_i v dv = \rho_i u_i, \quad \int_{\mathbb{R}^3} M_i^{1/2} f_i |v|^2 dv = \rho_i(3\alpha_i + 15e_i).$$

*Proof.* Decomposing  $f = f^\parallel + f^\perp$ , where  $f^\parallel = \Pi^m(f)$  and  $f^\perp = f - \Pi^m(f)$ , we infer from  $M_i^{1/2} \in \mathcal{N}(L^m)$  (see (34)) that  $(M_i^{1/2}, f_i^\perp)_{L_v^2} = 0$  and hence, by (34) again,

$$(M_i^{1/2}, f_i^\parallel)_{L_v^2} = \int_{\mathbb{R}^3} M_i(\alpha_i + u_i \cdot v + e_i |v|^2) dv = \rho_i(\alpha_i + 3e_i).$$

The other identities can be shown in a similar way.  $\square$

**Lemma 15.** *For all  $f \in \mathcal{D}$ , we have*

$$\sum_{i,j=1}^n (|u_i - u_j|^2 + (e_i - e_j)^2) \geq \frac{1}{C_k} (\|f - \Pi^L(f)\|_{\mathcal{H}}^2 - 2\|f - \Pi^m(f)\|_{\mathcal{H}}^2),$$

where  $u_i, e_i$  are the coefficients of the  $i$ th component of  $\Pi^m(f)$  in (34),  $\Pi^L$  is the projection on  $\mathcal{N}(L)$ ,  $C_k > 0$  is given by

$$(45) \quad C_k = 60n\rho_\infty \max_{1 \leq k, \ell \leq 5n} \left| \sum_{i=1}^n \int_{\mathbb{R}^3} \psi_k \psi_\ell \nu_i dv \right|,$$

and  $(\psi_k)$  is an arbitrary orthonormal basis of  $\mathcal{N}(L^m)$  in  $L_v^2$ .

*Proof.* We again decompose  $f = f^\parallel + f^\perp$  with  $f^\parallel = \Pi^m(f)$  and  $f^\perp = f - f^\parallel$ . Then

$$(46) \quad \|f - \Pi^L(f)\|_{\mathcal{H}}^2 \leq 2(\|f^\perp\|_{\mathcal{H}}^2 + \|f^\parallel - \Pi^L(f)\|_{\mathcal{H}}^2).$$

We estimate first the difference  $g := f^\parallel - \Pi^L(f) = \Pi^m(f) - \Pi^L(f) \in \mathcal{N}(L^m)$  (note that  $\mathcal{N}(L) \subset \mathcal{N}(L^m)$ ). Let  $(\psi_k)$  be an arbitrary orthonormal basis of  $\mathcal{N}(L^m)$  in  $L_v^2$ . Because of (34) and  $\nabla_v \nu_i \in L^\infty(\mathbb{R}^3)$ , we have  $\psi_k \in \mathcal{H}$ . Then, by Young's inequality, we find that

$$\begin{aligned} \|g\|_{\mathcal{H}}^2 &= \sum_{i=1}^n \int_{\mathbb{R}^3} \left| \sum_{k=1}^{5n} (g, \psi_k)_{L_v^2} \psi_k \right|^2 \nu_i(v) dv = \sum_{k, \ell=1}^{5n} (g, \psi_k)_{L_v^2} (g, \psi_\ell)_{L_v^2} \sum_{i=1}^n \int_{\mathbb{R}^3} \psi_k \psi_\ell \nu_i(v) dv \\ &= \sum_{k, \ell=1}^{5n} (g, \psi_k)_{L_v^2} (g, \psi_\ell)_{L_v^2} (\psi_k, \psi_\ell)_{\mathcal{H}} \\ &\leq \frac{1}{2} \max_{1 \leq k, \ell \leq 5n} |(\psi_k, \psi_\ell)_{\mathcal{H}}| \sum_{k, \ell=1}^{5n} ((g, \psi_k)_{L_v^2}^2 + (g, \psi_\ell)_{L_v^2}^2) \\ &= 5n \max_{1 \leq k, \ell \leq 5n} |(\psi_k, \psi_\ell)_{\mathcal{H}}| \sum_{k=1}^{5n} (g, \psi_k)_{L_v^2}^2 = 5n \max_{1 \leq k, \ell \leq 5n} |(\psi_k, \psi_\ell)_{\mathcal{H}}| \|g\|_{L_v^2}^2. \end{aligned}$$

Thus, we infer from (46) that

$$\|f - \Pi^L(f)\|_{\mathcal{H}}^2 \leq 2\|f^\perp\|_{\mathcal{H}}^2 + 10n \max_{1 \leq k, \ell \leq 5n} |(\psi_k, \psi_\ell)_{\mathcal{H}}| \|f^\parallel - \Pi^L(f)\|_{L_v^2}^2.$$

Because of  $\mathcal{N}(L) \subset \mathcal{N}(L^m)$ , we have  $\Pi^m \Pi^L = \Pi^L$  and

$$\begin{aligned} \|f^\parallel - \Pi^L(f)\|_{L_v^2}^2 &= \|f^\parallel\|_{L_v^2}^2 - 2(\Pi^m(f), \Pi^L(f))_{L_v^2} + \|\Pi^L(f)\|_{L_v^2}^2 \\ &= \|f^\parallel\|_{L_v^2}^2 - 2(f, \Pi^L(f))_{L_v^2} + \|\Pi^L(f)\|_{L_v^2}^2 = \|f^\parallel\|_{L_v^2}^2 - \|\Pi^L(f)\|_{L_v^2}^2. \end{aligned}$$

Consequently, setting  $k_0 = 10n \max_{1 \leq k, \ell \leq n} |(\psi_k, \psi_\ell)_{\mathcal{H}}|$ ,

$$(47) \quad \|f - \Pi^L(f)\|_{\mathcal{H}}^2 \leq 2\|f^\perp\|_{\mathcal{H}}^2 + k_0 (\|f^\parallel\|_{L_v^2}^2 - \|\Pi^L(f)\|_{L_v^2}^2).$$

Next, we compute the  $L_v^2$  norms of  $f^\parallel$  and  $\Pi^L(f)$ . Moment identities (44) show that

$$\begin{aligned} \|f^\parallel\|_{L_v^2}^2 &= \sum_{i=1}^n \int_{\mathbb{R}^3} M_i(\alpha_i + u_i \cdot v + e_i |v|^2)^2 dv \\ &= \sum_{i=1}^n \int_{\mathbb{R}^3} M_i(\alpha_i^2 + (u_i \cdot v)^2 + e_i^2 |v|^4 + 2\alpha_i e_i |v|^2) dv \end{aligned}$$

$$= \sum_{i=1}^n \rho_{\infty,i} (\alpha_i^2 + |u_i|^2 + 15e_i^2 + 6\alpha_i e_i).$$

For the computation of the  $L_v^2$  norm of  $\Pi^L(f)$ , we choose the following orthonormal basis  $(\phi_j) = (\phi_{j,i})_{i=1,\dots,n}$  of  $\mathcal{N}(L)$  in  $L_v^2$ :

$$\phi_{j,i} = \rho_{\infty}^{-1/2} M_j^{1/2} \delta_{ij}, \quad \phi_{n+k,i} = \rho_{\infty}^{-1/2} M_i^{1/2} v_k, \quad \phi_{n+4,i} = (6\rho_{\infty})^{-1/2} M_i^{1/2} (|v|^2 - 3),$$

where  $1 \leq j \leq n$  and  $1 \leq k \leq 3$ . Then, using the moment identities of Lemma 14,

$$\|\Pi^L(f)\|_{L_v^2}^2 = \sum_{j=1}^{n+4} (f, \phi_j)_{L_v^2}^2 = \sum_{i=1}^n \rho_{\infty,i} (\alpha_i + 3e_i)^2 + \rho_{\infty} \left| \sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_{\infty}} u_i \right|^2 + 6\rho_{\infty} \left( \sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_{\infty}} e_i \right)^2.$$

Inserting the above identities for  $\|f\|_{L_v^2}^2$  and  $\|\Pi^L(f)\|_{L_v^2}^2$  into (47), we conclude that

$$\begin{aligned} \|f - \Pi^L(f)\|_{\mathcal{H}}^2 &\leq 2\|f - \Pi^m(f)\|_{\mathcal{H}}^2 + k_0 \rho_{\infty} \left( \sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_{\infty}} |u_i|^2 - \left| \sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_{\infty}} u_i \right|^2 \right) \\ &\quad + 6k_0 \rho_{\infty} \left( \sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_{\infty}} e_i^2 - \left( \sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_{\infty}} e_i \right)^2 \right). \end{aligned}$$

Then, if the inequalities

$$(48) \quad \sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_{\infty}} |u_i|^2 - \left| \sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_{\infty}} u_i \right|^2 \leq \sum_{i,j=1}^n |u_i - u_j|^2,$$

$$(49) \quad \sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_{\infty}} e_i^2 - \left( \sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_{\infty}} e_i \right)^2 \leq \sum_{i,j=1}^n (e_i - e_j)^2$$

hold, the lemma follows with  $C_k = 6k_0 \rho_{\infty}$ .

It remains to prove (48) and (49). To this end, we define the following scalar product on  $\mathbb{R}^{3n}$ :

$$(u, v)_{\rho} = \sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_{\infty}} u_i \cdot v_i, \quad u = (u_1, \dots, u_n), \quad v = (v_1, \dots, v_n) \in \mathbb{R}^{3n},$$

where  $u_i \cdot v_i$  denotes the usual scalar product in  $\mathbb{R}^3$ . The corresponding norm is  $\|u\|_{\rho} = (u, u)_{\rho}^{1/2}$ . Then  $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^{3n}$  satisfies  $\|\mathbf{1}\|_{\rho} = 1$ . The elementary identity

$$\|u\|_{\rho}^2 - (u, \mathbf{1})_{\rho}^2 = \|u - (u, \mathbf{1})_{\rho} \mathbf{1}\|_{\rho}^2$$

can be equivalently written as

$$I := \sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_{\infty}} |u_i|^2 - \left| \sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_{\infty}} u_i \right|^2 = \sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_{\infty}} \left| u_i - \sum_{j=1}^n \frac{\rho_{\infty,j}}{\rho_{\infty}} u_j \right|^2.$$

Then, using  $\sum_{j=1}^n \rho_{\infty,j} = \rho_{\infty}$ ,

$$\begin{aligned} I &= \sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_{\infty}} \left| \left(1 - \frac{\rho_{\infty,i}}{\rho_{\infty}}\right) u_i - \sum_{j \neq i} \frac{\rho_{\infty,j}}{\rho_{\infty}} u_j \right|^2 = \sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_{\infty}} \left| \sum_{j \neq i} \frac{\rho_{\infty,j}}{\rho_{\infty}} (u_i - u_j) \right|^2 \\ &= \sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_{\infty}} \left( \sum_{k \neq i} \frac{\rho_{\infty,k}}{\rho_{\infty}} \right)^2 \left| \frac{\sum_{j \neq i} (\rho_{\infty,j}/\rho_{\infty})(u_i - u_j)}{\sum_{k \neq i} \rho_{\infty,k}/\rho_{\infty}} \right|^2 \\ &= \sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_{\infty}} \left( \sum_{k \neq i} \frac{\rho_{\infty,k}}{\rho_{\infty}} \right)^2 \left| \sum_{j \neq i} \lambda_j (u_i - u_j) \right|^2, \end{aligned}$$

where  $\lambda_j = (\rho_{\infty,j}/\rho_{\infty})(\sum_{k \neq i} (\rho_{\infty,k}/\rho_{\infty}))^{-1}$ . Since  $\sum_{j \neq i} \lambda_j = 1$ , we may apply Jensen's inequality to this convex combination, leading to

$$\begin{aligned} I &\leq \sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_{\infty}} \left( \sum_{k \neq i} \frac{\rho_{\infty,k}}{\rho_{\infty}} \right)^2 \sum_{j \neq i} \lambda_j |u_i - u_j|^2 \\ &= \sum_{i=1}^n \frac{\rho_{\infty,i}}{\rho_{\infty}} \left(1 - \frac{\rho_{\infty,i}}{\rho_{\infty}}\right) \sum_{j \neq i} \frac{\rho_{\infty,j}}{\rho_{\infty}} |u_i - u_j|^2 \leq \sum_{i,j=1}^n |u_i - u_j|^2, \end{aligned}$$

since  $\rho_{\infty,j} \leq \rho_{\infty}$ . This ends the proof.  $\square$

Now, we are able to prove Theorem 3.

*Proof of Theorem 3.* By Lemmas 12, 13, and 15, we obtain

$$\begin{aligned} -(f, L(f))_{L_v^2} &\geq (C^m - 4\eta) \|f - f^{\parallel}\|_{\mathcal{H}}^2 + \frac{\eta D^b}{8} \sum_{i,j=1}^n (|u_i - u_j|^2 + (e_i - e_j)^2) \\ &\geq \left( C^m - 4\eta - \frac{\eta D^b}{4C_k} \right) \|f - f^{\parallel}\|_{\mathcal{H}}^2 + \frac{\eta D^b}{8C_k} \|f - \Pi^L(f)\|_{\mathcal{H}}^2. \end{aligned}$$

The first term on the right-hand side is nonnegative if we choose  $\eta = \min\{1, 4C^m C_k / (16C_k + D^b)\}$ , and estimate (10) follows with  $\lambda = \eta D^b / (8C_k)$ .  $\square$

## 5. CONVERGENCE TO EQUILBIUM

In this section, we prove Theorem 4. The idea of the proof is to adapt the hypocoercivity method of [29] to the multi-species setting. To this end, we need to verify the structural assumptions (H1)-(H3) in [29, Theorem 1.1]. The setting is as follows.

Let  $L$  be a closed, densely defined, and self-adjoint operator on  $\text{Dom}(L) \subset L_v^2$  such that  $L = K - \Lambda$  and the operators  $K$  and  $\Lambda$  satisfy the following assumptions:

(H1) The operator  $\Lambda$  is coercive in the following sense: There exist a norm  $\|\cdot\|_{\mathcal{H}}$  on  $\mathcal{H} \subset L_v^2$  and positive constants  $\bar{\nu}_i$  ( $0 \leq i \leq 4$ ) such that for all  $f \in \text{Dom}(L) \subset \mathcal{H}$ ,

$$(50) \quad \bar{\nu}_0 \|f\|_{L_v^2}^2 \leq \bar{\nu}_1 \|f\|_{\mathcal{H}}^2 \leq (f, \Lambda(f))_{L_v^2} \leq \bar{\nu}_2 \|f\|_{\mathcal{H}}^2,$$



$$(51) \quad (\nabla_v f, \nabla_v \Lambda(f))_{L_v^2} \geq \bar{\nu}_3 \|\nabla_v f\|_{\mathcal{H}}^2 - \bar{\nu}_4 \|f\|_{L_v^2}^2.$$

Moreover, there exists a constant  $C_L > 0$  such that for all  $f, g \in \text{Dom}(L)$ ,

$$(52) \quad (L(f), g)_{L_v^2} \leq C_L \|f\|_{\mathcal{H}} \|g\|_{\mathcal{H}}.$$

(H2) The operator  $K$  has a regularizing effect in the following sense: For all  $\varepsilon > 0$ , there exists  $C(\varepsilon) > 0$  such that for all  $f \in H_v^1$ ,

$$(\nabla_v f, \nabla_v K(f))_{L_v^2} \leq \varepsilon \|\nabla_v f\|_{L_v^2}^2 + C(\varepsilon) \|f\|_{L_v^2}^2.$$

(H3) The operator  $L$  has a finite-dimensional kernel and the following local spectral-gap assumption holds: There exists  $\lambda > 0$  such that for all  $f \in \text{Dom}(L)$ ,

$$-(f, L(f))_{L_v^2} \geq \lambda \|f - \Pi^L(f)\|_{\mathcal{H}}^2,$$

where  $\Pi^L$  is the projection on  $\mathcal{N}(L)$ .

Assumption (H3) is a consequence of Theorem 3. Next, we verify assumption (H1). Using Lemma 9 and the continuous embedding  $\mathcal{H} \hookrightarrow L_v^2$ , we see that (50) holds. For the proof of (51), we employ Young's inequality:

$$\begin{aligned} (\nabla_v f, \nabla_v \Lambda(f))_{L_v^2} &= \sum_{i=1}^n \int_{\mathbb{R}^3} \nabla_v f_i \cdot \nabla_v (\nu_i f_i) dv \\ &= \sum_{i=1}^n \left( \int_{\mathbb{R}^3} f_i \nabla_v f_i \cdot \nabla_v \nu_i dv + \int_{\mathbb{R}^3} |\nabla_v f_i|^2 \nu_i dv \right) \\ &\geq \frac{1}{2} \sum_{i=1}^n \left( - \int_{\mathbb{R}^3} \frac{|\nabla_v \nu_i|^2}{\nu_i} f_i^2 dv + \int_{\mathbb{R}^3} |\nabla_v f_i|^2 \nu_i dv \right) \\ &\geq \bar{\nu}_3 \|\nabla_v f\|_{\mathcal{H}}^2 - \bar{\nu}_4 \|f\|_{L_v^2}^2, \end{aligned}$$

where  $\bar{\nu}_3 = 1/2$  and  $\bar{\nu}_4 = \max_{1 \leq i \leq n} \sup_{v \in \mathbb{R}^3} |\nabla_v \nu_i|^2 / (2\nu_i)$ . Note that  $\bar{\nu}_4$  is finite since  $\nabla_v \nu_i$  is bounded and  $\nu_i$  is strictly positive (see Lemma 7). Finally, inequality (52) follows from the decomposition  $L = K - \Lambda$ , the compactness and hence continuity of  $K$ , the explicit expression for  $\Lambda$ , and the Cauchy-Schwarz inequality applied to  $(L(f), g)_{L_v^2}$ .

It remains to verify assumption (H2). Let  $N := \rho_{\infty, i}^{-1/2} M_i^{1/2} = (2\pi)^{-3/4} \exp(-|v|^2/4)$ . We decompose  $K = K^{(1)} - K^{(2)}$ , where  $K^{(j)} = (K_1^{(j)}, \dots, K_n^{(j)})$  and

$$\begin{aligned} K_i^{(1)} &= \sum_{j=1}^n \int_{\mathbb{R}^3 \times \mathbb{S}^2} B_{ij} M_i^{1/2} M_j^* \left( \frac{f'_i}{(M'_i)^{1/2}} + \frac{f_j^*}{(M_j^*)^{1/2}} \right) dv^* d\sigma, \\ K_i^{(2)} &= \sum_{j=1}^n \int_{\mathbb{R}^3 \times \mathbb{S}^2} B_{ij} (M_i M_j^*)^{1/2} f_j^* dv^* d\sigma \end{aligned}$$

for  $1 \leq i \leq n$ . Because of  $M'_k M_k^* = M_k M_k^*$  for all  $k$ , we find that

$$K_i^{(1)}(f) = \sum_{j=1}^n \int_{\mathbb{R}^3 \times \mathbb{S}^2} B_{ij} M_i^{1/2} M_j^* \left( \frac{(M_i^*)^{1/2} f'_i}{(M'_i M_i^*)^{1/2}} + \frac{(M_j^*)^{1/2} f_j^*}{(M_j^* M_j^*)^{1/2}} \right) dv^* d\sigma$$

$$\begin{aligned}
&= \sum_{j=1}^n \int_{\mathbb{R}^3 \times \mathbb{S}^2} B_{ij} M_i^{1/2} M_j' \left( \frac{(M_i^*)^{1/2} f_i'}{(M_i M_i^*)^{1/2}} + \frac{(M_j')^{1/2} f_j'^*}{(M_j^* M_j)^{1/2}} \right) dv^* d\sigma \\
&= \sum_{j=1}^n \int_{\mathbb{R}^3 \times \mathbb{S}^2} B_{ij} (\rho_{\infty,j}^{1/2} N'^* f_i' + \rho_{\infty,i}^{1/2} N' f_j'^*) \rho_{\infty,j}^{1/2} N^* dv^* d\sigma.
\end{aligned}$$

The transformation  $\sigma \mapsto -\sigma$  leaves  $v$  and  $v^*$  unchanged and exchanges  $v'$  and  $v'^*$ . Assumption (A5) ( $b_{ij}$  is an even function) ensures that  $B_{ij}$  is unchanged under this transformation. Therefore,

$$\int_{\mathbb{R}^3 \times \mathbb{S}^2} B_{ij} f_i' N'^* N^* dv^* d\sigma = \int_{\mathbb{R}^3 \times \mathbb{S}^2} B_{ij} f_i'^* N' N^* dv^* d\sigma,$$

and we can write  $K^{(1)}$  as

$$(53) \quad K_i^{(1)}(f) = \frac{1}{2} \sum_{j=1}^n \rho_{\infty,j}^{1/2} (\rho_{\infty,j}^{1/2} K_{ij}^{(1)}(f_i) + \rho_{\infty,i}^{1/2} K_{ij}^{(1)}(f_j)),$$

where

$$K_{ij}^{(1)}(f_k) = \int_{\mathbb{R}^3 \times \mathbb{S}^2} B_{ij} (N'^* f_k' + N' f_k'^*) N^* dv^* d\sigma, \quad 1 \leq i, j, k \leq n.$$

Note that  $K_{ij}^{(1)} = K_{ji}^{(1)}$ . In a similar way, we can decompose the operator  $K^{(2)}$ :

$$K_i^{(2)}(f) = \sum_{j=1}^n (\rho_{\infty,i} \rho_{\infty,j})^{1/2} N \int_{\mathbb{R}^3 \times \mathbb{S}^2} B_{ij} f_j^* N^* dv^* d\sigma = \sum_{j=1}^n (\rho_{\infty,i} \rho_{\infty,j})^{1/2} K_{ij}^{(2)}(f_j),$$

where

$$K_{ij}^{(2)}(f_j) = N \int_{\mathbb{R}^3 \times \mathbb{S}^2} B_{ij} N^* f_j^* dv^* d\sigma.$$

Next, we estimate the derivatives of  $K_{ij}^{(\ell)}$ . It is shown in [29, Eqs. (5.15)-(5.18)] that for all  $\varepsilon > 0$ , there exists  $C(\varepsilon) > 0$  such that for any  $f \in H_v^1$ ,  $1 \leq i, j, k \leq n$ , and  $\ell = 1, 2$ ,

$$(54) \quad \|\nabla_v K_{ij}^{(\ell)}(f_k)\|_{L_v^2}^2 \leq \varepsilon \|\nabla_v f_k\|_{L_v^2}^2 + C(\varepsilon) \|f_k\|_{L_v^2}^2.$$

Then we infer from (53) that

$$\begin{aligned}
\|\nabla_v K^{(1)}(f)\|_{L_v^2}^2 &= \sum_{i=1}^n \left\| \frac{1}{2} \sum_{j=1}^n \rho_{\infty,j}^{1/2} (\rho_{\infty,j}^{1/2} \nabla_v K_{ij}^{(1)}(f_i) + \rho_{\infty,i}^{1/2} \nabla_v K_{ij}^{(1)}(f_j)) \right\|_{L_v^2}^2 \\
&\leq \frac{n}{4} \sum_{i,j=1}^n \left\| \rho_{\infty,j}^{1/2} (\rho_{\infty,j}^{1/2} \nabla_v K_{ij}^{(1)}(f_i) + \rho_{\infty,i}^{1/2} \nabla_v K_{ij}^{(1)}(f_j)) \right\|_{L_v^2}^2 \\
&\leq n (\max_{1 \leq i \leq n} \rho_{\infty,i})^2 \sum_{i,j=1}^n \|\nabla_v K_{ij}^{(1)}(f_i)\|_{L_v^2}^2.
\end{aligned}$$

Thus, by (54), it follows that for  $\ell = 1$ ,

$$\|\nabla_v K^{(\ell)}(f)\|_{L_v^2}^2 \leq n^2 (\max_{1 \leq i \leq n} \rho_{\infty, i})^2 \sum_{i=1}^n (\varepsilon \|\nabla_v f_i\|_{L_v^2}^2 + C(\varepsilon) \|f_i\|_{L_v^2}^2).$$

A similar computation shows that this estimate also holds for  $\ell = 2$ . We infer that

$$\|\nabla K(f)\|_{L_v^2}^2 \leq 4n^2 (\max_{1 \leq i \leq n} \rho_{\infty, i})^2 \sum_{i=1}^n (\varepsilon \|\nabla_v f_i\|_{L_v^2}^2 + C(\varepsilon) \|f_i\|_{L_v^2}^2).$$

This proves assumption (H2) since  $\varepsilon > 0$  is arbitrary.

*Proof of Theorem 4.* We have verified that assumptions (H1)-(H3) are satisfied. Then, using exactly the same arguments as in the proof of Theorem 1.1 in [29], but now for the multi-species case, we conclude the exponential decay (13) of the semigroup  $e^{tB}$ , which is the first property of the theorem.

It remains to show that the decay estimate (14) follows from (13). For this, we write the initial value  $f_I$  as  $f_I = \Pi^B(f_I) + (I - \Pi^B)(f_I)$ , where  $\Pi^B$  is the projection onto  $\mathcal{N}(B)$  in  $L_{x,v}^2$ . Then the solution to (5) is given by

$$f(t) = e^{tB} f_I = e^{tB} \Pi^B(f_I) + e^{tB} (I - \Pi^B)(f_I), \quad t \geq 0.$$

We have already shown that

$$\|e^{tB} (I - \Pi^B)g\|_{H_{x,v}^1} \leq C e^{-\tau t} \|g\|_{H_{x,v}^1} \quad \text{for all } g \in H_{x,v}^1, \quad t > 0.$$

In particular, the choice  $g = (I - \Pi^B)(f_I)$  and the property  $(I - \Pi^B)^2 = I - \Pi^B$  lead to

$$\|e^{tB} (I - \Pi^B)(f_I)\|_{H_{x,v}^1} \leq C e^{-\tau t} \|(I - \Pi^B)(f_I)\|_{H_{x,v}^1}.$$

It remains to prove that  $f_\infty = \Pi^B(f_I) = e^{tB} \Pi^B(f_I)$  is the global equilibrium. Since  $B\Pi^B(f_I) = 0$  and  $\Pi^B(f_I)$  does not depend on time, the constant-in-time function  $g = \Pi^B(f_I)$  is the unique solution to the Cauchy problem

$$\partial_t g = Bg, \quad t > 0, \quad g(0) = \Pi^B(f_I).$$

This shows that  $\Pi^B(f_I) = e^{tB} g(0) = e^{tB} \Pi^B(f_I)$  and finishes the proof.  $\square$

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