Multiple binomial sums

Alin Bostan, Pierre Lairez, Bruno Salvy

MULTIPLE BINOMIAL SUMS

ALIN BOSTAN, PIERRE LAIREZ, AND BRUNO SALVY

Abstract. Multiple binomial sums form a large class of multi-indexed sequences, closed under partial summation, which contains most of the sequences obtained by multiple summation of products of binomial coefficients and also all the sequences with algebraic generating function. We study the representation of the generating functions of binomial sums by integrals of rational functions. The outcome is twofold. Firstly, we show that a univariate sequence is a multiple binomial sum if and only if its generating function is the diagonal of a rational function. Secondly, we propose algorithms that decide the equality of multiple binomial sums and that compute recurrence relations for them. In conjunction with geometric simplifications of the integral representations, this approach behaves well in practice. The process avoids the computation of certificates and the problem of the appearance of spurious singularities that afflicts discrete creative telescoping, both in theory and in practice.

Introduction

The computation of definite sums in computer algebra is classically handled by the method of *creative telescoping* initiated in the 1990s by Zeilberger (Zeilberger 1990, 1991a; Wilf and Zeilberger 1992). For example, it applies to sums like

$$(1)\qquad \sum_{k=0}^{n} \frac{4^k}{\binom{2k}{k}}, \quad \sum_{k=0}^{n} \left(\sum_{j=0}^{k} \binom{n}{j} \right)^3 \quad \text{or} \quad \sum_{i=0}^{n} \sum_{j=0}^{n} \binom{i+j}{j}^2 \binom{4n-2i-2j}{2n-2i}.$$

In order to compute a sum $\sum_k u(n,k)$ of a bivariate sequence u, this method computes an identity of the form

$$a_p(n)u(n+p,k) + \cdots + a_0(n)u(n,k) = v(n,k+1) - v(n,k).$$

Provided that it is possible to sum both sides over k and that the sequence v vanishes at the endpoints of the domain of summation, the left-hand side—called a *telescoper*—gives a recurrence for the sum. The sequence v is then called the *certificate* of the identity.

In the case of multiple sums, this idea leads to searching for a telescoping identity of the form

$$(2)\quad \begin{aligned} a_p(n)u(n+p,k_1,\dots,k_m) + \cdots + a_0(n)u(n,k_1,\dots,k_m) = \\ \big(v_1(n,k_1+1,k_2,\dots,k_m) - v_1(n,k_1,\dots,k_m)\big) + \cdots \\ + \big(v_m(n,k_1,\dots,k_m+1) - v_m(n,k_1,\dots,k_m)\big). \end{aligned}$$

Again, under favorable circumstances the sums of the sequences on the right-hand side telescope, leaving a recurrence for the sum on the left-hand side.

Date: June 15, 2016.

2010 *Mathematics Subject Classification.* 05A10 (33F10 68W30).

Key words and phrases. Binomial sum, multiple sum, symbolic computation, diagonal, integral representation.

This high-level presentation hides practical difficulties. It is important to check that the sequences on both sides of the identities above are defined over the whole range of summation (Abramov 2006; Abramov and Petkovšek 2005). More often than not, singularities do appear. To the best of our knowledge, no algorithm based on creative telescoping manages to work around this difficulty; they all let the user handle it. As a consequence, computing the certificate is not merely a useful by-product of the algorithm, but indeed a necessary part of the computation. Unfortunately, the size of the certificate may be much larger than that of the final recurrence and thus costly in terms of computational complexity.

The computation of multiple integrals of rational functions has some similarities with the computation of discrete sums and the method of creative telescoping applies there too. It may also produce extra singularities in the certificate, but in the differential setting this is not an issue anymore: for the integrals we are interested in, the integration path can always be moved to get around any extra singularity. Moreover, we have showed (Bostan et al. 2013a; Lairez 2016) that integration of multivariate rational functions over cycles can be achieved efficiently without computing the corresponding certificate and without introducing spurious singularities. In that case, the algorithm computes a linear differential equation for the parameterized integral. It turns out that numerous multiple sums can be cast into problems of rational integration by passing to generating functions. The algorithmic consequences of this observation form the object of the present work.

Content. In §1, we define a class of multivariate sequences, called *(multiple) binomial sums*, that contains the binomial coefficient sequence and that is closed under pointwise addition, pointwise multiplication, linear change of variables and partial summation. Not every sum that creative telescoping can handle is a binomial sum: for example, among the three sums in Eq. (1), the second one and the third one are binomial sums but the first one is not, since it contains the inverse of a binomial coefficient; moreover, it cannot be rewritten as a binomial sum (see §1.2). Yet many sums coming from combinatorics and number theory are binomial sums. In §2, we explain how to compute *integral representations* of the generating function of a binomial sum in an automated way. The outcome is twofold. Firstly, in §3, we work further on these integral representations to show that the generating functions of *univariate* binomial sums are exactly the diagonals of rational power series. This equivalence characterizes binomial sums in an intrinsic way which dismisses the arbitrariness of the definition. All the theory of diagonals transfers to univariate binomial sums and gives many interesting arithmetic properties. Secondly, in §4, we show how to use integral representations to actually compute with binomial sums (e.g. find recurrence relations or prove identities automatically) *via* the computation of Picard-Fuchs equations. The direct approach leads to integral representations that involve far too many variables to be efficiently handled. In §5, we propose a general method, that we call *geometric reduction*, to reduce tremendously the number of variables in practice. In §6, we describe some variants with the purpose of implementing the algorithms, and finally, in §7, we show how the method applies to some classical identities and more recent ones that were conjectural so far.

All the algorithms that are presented here are implemented in Maple and are available at `https://github.com/lairez/binomsums`.

Example. The following proof of an identity of Dixon (1891),

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^3=(-1)^n\frac{(3n)!}{n!^3},\tag{3}$$

illustrates well the main points of the method. The strategy is as follows: find an integral representation of the generating function of the left-hand side; simplify this integral representation using partial integration; use the simplified integral representation to compute a differential equation of which the generating function is solution; transform this equation into a recurrence relation; solve this recurrence relation.

First of all, the binomial coefficient $\binom{n}{k}$ is the coefficient of x^k in $(1+x)^n$. Cauchy's integral formula ensures that

$$\binom{n}{k}=\frac{1}{2\pi i}\oint_\gamma\frac{(1+x)^n}{x^k}\frac{\mathrm{d}x}{x},$$

where γ is the circle $\left\{x\in\mathbb{C}\;\middle|\;|x|=\frac12\right\}$. Therefore, the cube of a binomial coefficient can be represented as a triple integral

$$\binom{2n}{k}^3=\frac{1}{(2\pi i)^3}\oint_{\gamma\times\gamma\times\gamma}\frac{(1+x_1)^{2n}}{x_1^k}\frac{(1+x_2)^{2n}}{x_2^k}\frac{(1+x_3)^{2n}}{x_3^k}\frac{\mathrm{d}x_1}{x_1}\frac{\mathrm{d}x_2}{x_2}\frac{\mathrm{d}x_3}{x_3}.$$

As a result, the generating function of the left-hand side of Equation (3) is

$$\begin{aligned}
y(t)&\overset{\text{def}}{=}\sum_{n=0}^{\infty}t^n\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^3\\
&=\frac{1}{(2i\pi)^3}\oint_{\gamma^3}\sum_{n=0}^{\infty}\sum_{k=0}^{2n}\left(t\prod_{i=1}^{3}(1+x_i)^2\right)^n\left(\frac{-1}{x_1x_2x_3}\right)^k\frac{\mathrm{d}x_1}{x_1}\frac{\mathrm{d}x_2}{x_2}\frac{\mathrm{d}x_3}{x_3}\\
&=\frac{1}{(2i\pi)^3}\oint_{\gamma^3}\sum_{n=0}^{\infty}\left(t\prod_{i=1}^{3}(1+x_i)^2\right)^n\frac{1-\left(\frac{-1}{x_1x_2x_3}\right)^{2n+1}}{1+\frac{1}{x_1x_2x_3}}\frac{\mathrm{d}x_1}{x_1}\frac{\mathrm{d}x_2}{x_2}\frac{\mathrm{d}x_3}{x_3}\\
&=\frac{1}{(2i\pi)^3}\oint_{\gamma^3}\frac{\left(x_1x_2x_3-t\prod_{i=1}^3(1+x_i)^2\right)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3}{\left(x_1^2x_2^2x_3^2-t\prod_{i=1}^3(1+x_i)^2\right)\left(1-t\prod_{i=1}^3(1+x_i)^2\right)}.
\end{aligned}$$

The partial integral with respect to x_3 along the circle $|x_3|=\frac12$ is the sum of the residues of the rational function being integrated at the poles whose modulus is less than $\frac12$. When $|t|$ is small and $|x_1|=|x_2|=\frac12$, the poles coming from the factor $x_1^2x_2^2x_3^2-t\prod_{i=1}^3(1+x_i)^2$ all have a modulus that is smaller than $\frac12$: they are asymptotically proportional to $|t|^{1/2}$. In contrast, the poles coming from the factor $1-t\prod_{i=1}^3(1+x_i)^2$ behave like $|t|^{-1/2}$ and have all a modulus that is bigger than $\frac12$. In particular, any two poles that come from the same factor are either both asymptotically small or both asymptotically large. This implies that the partial integral is a rational function of t, x_1 and x_2; and we compute that

$$y(t)=\frac{1}{(2i\pi)^2}\oint_{\gamma\times\gamma}\frac{x_1x_2\mathrm{d}x_1\mathrm{d}x_2}{x_1^2x_2^2-t(1+x_1)^2(1+x_2)^2(1-x_1x_2)^2}.$$

This formula echoes the original proof of Dixon (1891) in which the left-hand side of (3) is expressed as the coefficient of $(xy)^{4n}$ in $((1-y^2)(1-z^2)(1-y^2z^2))^{2n}$. Using any algorithm that performs definite integration of rational functions (Chyzak

2000; Koutschan 2010; Lairez 2016) reveals a differential equation satisfied by $y(t)$:

$$t(27t+1)y'' + (54t+1)y' + 6y = 0.$$

Looking at the coefficient of t^n in this equality leads to the recurrence relation

$$3(3n+2)(3n+1)u_n + (n+1)^2 u_{n+1} = 0,$$

where $u_n = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3$. Since $u_0 = 1$, it leads to a proof of Dixon's identity by induction on n. The treatment above differs in one important aspect from what follows: the use of genuine integrals and explicit integration paths rather than *formal residues* that will be introduced in §2.

Comparison with creative telescoping. As mentioned above, the computation of multiple binomial sums can be handled by the method of creative telescoping. The amount of work in this direction is considerable and we refer the reader to surveys (Chyzak 2014; Koutschan 2013). In the specific context of multiple sums, the most relevant works are those of Wegschaider (1997), Chyzak (2000), Apagodu and Zeilberger (2006), and Garoufalidis and Sun (2010). We show on the example of Dixon's identity how the method of creative telescoping and the method of generating functions differ fundamentally even on a single sum.

Let $u_{n,k} = (-1)^k \binom{2n}{k}^3$. This bivariate sequence satisfies the recurrence relations

$$\begin{aligned} &(2n+2-k)^3(2n+1-k)^3 u_{n+1,k} - 8(1+n)^3(1+2n)^3 u_{n,k} = 0 \\ &\qquad\text{and}\quad (k+1)^3 u_{n,k+1} + (2n-k)^3 u_{n,k} = 0. \end{aligned} \tag{4}$$

With these relations as input, Zeilberger's algorithm finds the sequence

$$\begin{aligned} v_{n,k} = &\frac{P(n,k)}{2(2n+2-k)^3(2n+1-k)^3} u_{n,k}, \\ &\text{where}\quad P(n,k) = k^3(9k^4n - 90k^3n^2 + 348k^2n^3 - 624kn^4 + 448n^5 \\ &+ 6k^4 - 132k^3n + 792k^2n^2 - 1932kn^3 + 1760n^4 - 48k^3 + 594k^2n \\ &- 2214kn^2 + 2728n^3 + 147k^2 - 1113kn + 2084n^2 - 207k + 784n + 116), \end{aligned}$$

that satisfies

$$3(3n+2)(3n+1)u_{n,k} + (n+1)^2 u_{n+1,k} = v_{n,k+1} - v_{n,k}. \tag{5}$$

Whatever the way the sequence $v_{n,k}$ is found, it is easy to check the telescopic relation (5): using the recurrence relations for $u_{n,k}$, each of the four terms in (5) rewrites in the form $R(n,k)u_{n,k}$, for some rational function $R(n,k)$. However, for some specific values of n and k, the sequence $v_{n,k}$ is not defined, due to the denominator.

To deduce a recurrence relation for $a_n = \sum_{k=0}^{2n} u_{n,k}$, it is desirable to sum the telescopic relation (5), over k, from 0 to $2n+2$. Unfortunately, that would hit the forbidden set where $v_{n,k}$ is not defined. We can only safely sum up to $k = 2n-1$. Doing so, we obtain that

$$3(3n+2)(3n+1)\sum_{k=0}^{2n-1} u_{n,k} + (n+1)^2 \sum_{k=0}^{2n-1} u_{n+1,k} = v_{n,2n} - v_{n,0},$$

and then

$$\begin{aligned} 3(3n+2)(3n+1)(a_n - u_{n,2n}) + (n+1)^2(a_{n+1} - u_{n+1,2n+2} - u_{n+1,2n+1} - u_{n+1,2n}) \\ = n^3(8n^5 + 52n^4 + 146n^3 + 223n^2 + 185n + 58). \end{aligned}$$

It turns out that $3(3n+2)(3n+1)u_{n,2n} + (n+1)^2(u_{n+1,2n+2} + u_{n+1,2n+1} + u_{n+1,2n})$ evaluates exactly to the right-hand side of the above identity, and this leads to Dixon's identity.

In this example, spurious singularities clearly appear in the range of summation. Thus, deriving an identity such as Dixon's from a telescopic identity such as (5) is not straightforward and involves the certificate. A few works address this issue for single sums (Abramov and Petkovšek 2005; Abramov 2006), but none for the case of multiple sums: existing algorithms (Wegschaider 1997; Chyzak 2000; Garoufalidis and Sun 2010) only give the telescopic identity without performing the summation. A recent attempt by Chyzak et al. (2014) to check the recurrence satisfied by Apéry's sequence in the proof assistant Coq has shown how difficult it is to formalize this summation step. Note that because of this issue, even the *existence* of a linear recurrence for such sums can hardly be inferred from the fact that the algorithm of creative telescoping always terminates with success.

This issue is rooted in the method of creative telescoping by the fact that sequences are *represented* through the linear recurrence relations that they satisfy. Unfortunately, this representation is not very faithful when the leading terms of the relations vanish for some values of the indices. The method of generating functions avoids this issue. For example, the binomial coefficient $\binom{n}{k}$ is represented unambiguously as the coefficient of x^k in $(1+x)^n$ (to be understood as a power series when $n < 0$), rather than as a solution to the recurrence relations $(n-k)\binom{n}{k} = n\binom{n-1}{k}$ and $k\binom{n}{k} = n\binom{n-1}{k-1}$.

Related work. The method of generating functions is classical and has been largely studied, in particular for the approach described here by Egorychev (1984), see also (Egorychev and Zima 2008). Egorychev's method is a general approach to summation, but not quite an algorithm. In this work, we make it completely effective and practical for the class of binomial sums.

The special case when the generating functions are differentially finite (D-finite) has been studied by Lipshitz (1989). From the effectivity point of view, the starting point is his proof that diagonals of D-finite power series are D-finite (Lipshitz 1988). The argument, based on linear algebra, is constructive but does not translate into an efficient algorithm because of the large dimensions involved. This led Wilf and Zeilberger (1992, p. 596) to comment that *"This approach, while it is explicit in principle, in fact yields an infeasible algorithm."* Still, using this construction of diagonals, many closure properties of the sequences under consideration (called P-recursive) can be proved (and, in principle, computed). Then, the representation of a convergent definite sum amounts to evaluating a generating series at 1 and this proves the existence of linear recurrences for the definite sums of all P-recursive sequences. Abramov and Petkovšek (2002) showed that in particular the so-called *proper hypergeometric sequences* are P-recursive in the sense of Lipshitz. The proof is also constructive, relying on Lipshitz's construction of diagonals to perform products of sequences.

While we are close to Lipshitz's approach, three enhancements make the method of generating functions presented here efficient: we use more efficient algorithms for computing multiple integrals and diagonals that have appeared in the last twenty years (Chyzak 2000; Koutschan 2010; Bostan et al. 2013a; Lairez 2016); we restrict ourselves to binomial sums, which makes it possible to manipulate the generating functions through rational integral representations (see §2.2 and §2.3); and a third decisive improvement comes with the geometric reduction procedure for simplifying integral representations (see §5).

Creative telescoping is another summation algorithm developed by Zeilberger (1991b) and proved to work for all proper hypergeometric sequences (Wilf and Zeilberger 1992). This method has the advantage of being applicable and often efficient in practice. However, as already mentioned, it relies on certificates whose size grows fast with the number of variables (Bostan et al. 2013a) and, more importantly, whose summation is not straightforward, making the complete automation of the method problematic. For proper hypergeometric sums, a different effective approach developed by Takayama (1995) does not suffer from the certificate problem. It consists in expressing the sum as the evaluation of a hypergeometric series and reducing its shifts with respect to a non-commutative Gröbner basis of the contiguity relations of the series, reducing the question to linear algebra in the finite-dimensional quotient.

The class of sums we consider is a subclass of the sums of proper hypergeometric sequences. We give an algorithm that avoids the computation of certificates in that case, and relies on an efficient method to deal with the integral representations of sums. The same approach has been recently used by Bostan et al. (2013b) on various examples, though in a less systematic manner. Examples in §7 give an idea of the extent of the class we are dealing with. It is a subclass of the *balanced multisums*, shown by Garoufalidis (2009) to possess nice asymptotic properties. More recently, a smaller family of *binomial multisums* was studied by Garrabrant and Pak (2014): they are further constrained to be diagonals of $\mathbb{N}$-rational power series.

Acknowledgments. We thank Marko Petkovšek for orienting us in some of the literature. This work has been supported in part by FastRelax ANR-14-CE25-0018-01 and by the research grant BU 1371/2-2 of the Deutsche Forschungsgemeinschaft.

1. The algebra of binomial sums

1.1. **Basic objects.** For all $n, k \in \mathbb{Z}$, the binomial coefficient $\binom{n}{k}$ is considered in this work as *defined* to be the coefficient of x^k in the formal power series $(1+x)^n$. In other words,

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} \text{ for } k \geqslant 0 \quad \text{and} \quad \binom{n}{k} = 0 \text{ for } k < 0.$$

For all $a, b \in \mathbb{Z}$, we define the *directed sum* ${\sum'}_{k=a}^{b}$ as

$$\sideset{}{'}\sum_{k=a}^{b} u_k \stackrel{\text{def}}{=} \begin{cases} \sum_{k=a}^{b} u_k & \text{if } a \leqslant b, \\ 0 & \text{if } a = b+1, \\ -\sum_{k=b+1}^{a-1} u_k & \text{if } a > b+1, \end{cases}$$

in contrast with the usual convention that $\sum_{k=a}^{b} u_k = 0$ when $a > b$. This implies the following flexible summation rule for directed sums:

$$\sideset{}{'}\sum_{k=a}^{b} u_k + \sideset{}{'}\sum_{k=b+1}^{c} u_k = \sideset{}{'}\sum_{k=a}^{c} u_k, \quad \text{for all} \quad a, b, c \in \mathbb{Z},$$

and also the geometric summation formula

$$\sideset{}{'}\sum_{k=a}^{b} r^k = \frac{r^a - r^{b+1}}{1-r} \quad \text{for any } r \neq 1,$$

valid independently of the relative position of a and b.

Let $\mathbb{K}$ be a field of characteristic zero, and let $d \geqslant 1$. We denote by $\mathcal{S}_d$ the $\mathbb{K}$-algebra of sequences $\mathbb{Z}^d \to \mathbb{K}$, where addition and multiplication are performed component-wise. Elements of $\mathbb{Z}^d$ are denoted using underlined lower case letters, such as $\underline{n}$. The algebra $\mathcal{S}_d$ may be embedded in the algebra of all functions $\mathbb{Z}^{\mathbb{N}} \to \mathbb{K}$ by sending a sequence $u : \mathbb{Z}^d \to \mathbb{K}$ to the function $\tilde{u}$ defined by

$$\tilde{u}(n_1, n_2, \dots) = u(n_1, \dots, n_d).$$

Let $\mathcal{S}$ be the union of the $\mathcal{S}_d$'s in the set of all functions $\mathbb{Z}^{\mathbb{N}} \to \mathbb{K}$. For $u \in \mathcal{S}$ and $\underline{n} \in \mathbb{Z}^d$, the notation $u_{\underline{n}}$ represents $u_{\underline{n},0,0,\dots}$. These conventions let us restrict or extend the number of indices as needed without keeping track of it.

Definition 1.1. The algebra of *binomial sums* over the field $\mathbb{K}$, denoted $\mathcal{B}$, is the smallest subalgebra of $\mathcal{S}$ such that:

(a) The Kronecker delta sequence $n \in \mathbb{Z} \mapsto \delta_n$, defined by $\delta_0 = 1$ and $\delta_n = 0$ if $n \neq 0$, is in $\mathcal{B}$.
(b) The geometric sequences $n \in \mathbb{Z} \mapsto C^n$, for all $C \in \mathbb{K} \setminus \{0\}$, are in $\mathcal{B}$.
(c) The binomial sequence $(n, k) \mapsto \binom{n}{k}$ (an element of $\mathcal{S}_2$) is in $\mathcal{B}$.
(d) If $\lambda : \mathbb{Z}^d \to \mathbb{Z}^e$ is an affine map and if $u \in \mathcal{B}$, then $\underline{n} \in \mathbb{Z}^d \mapsto u_{\lambda(\underline{n})}$ is in $\mathcal{B}$.
(e) If $u \in \mathcal{B}$, then the following directed indefinite sum is in $\mathcal{B}$:

$$(\underline{n}, m) \in \mathbb{Z}^d \times \mathbb{Z} \mapsto \sideset{}{'}\sum_{k=0}^{m} u_{\underline{n},k}.$$

Let us give a few useful examples. All polynomial sequences $\mathbb{Z}^d \to \mathbb{K}$ are in $\mathcal{B}$, because $\mathcal{B}$ is an algebra and because the sequence $n \mapsto \binom{n}{1} = n$ is in $\mathcal{B}$, thanks to points (c) and (d) of the definition.

Let $(H_n)_{n\in\mathbb{Z}}$ be the sequence defined by $H_n = 1$ if $n \geqslant 0$ and $H_n = 0$ if $n < 0$. It is a binomial sum since $H_n = \sideset{}{'}\sum_{k=0}^{n} \delta_k$. As a consequence, we obtain the closure of $\mathcal{B}$ by usual (indefinite) sums since

$$\sum_{k=0}^{m} u_{\underline{n},k} = H_m \sideset{}{'}\sum_{k=0}^{m} u_{\underline{n},k}.$$

By combining the rules (d) and (e) of the definition, we also obtain the closure of $\mathcal{B}$ under sums whose bounds depend linearly on the parameter: if $u \in \mathcal{B}$ and if λ and μ are affine maps $: \mathbb{Z}^d \to \mathbb{Z}$, then the sequence

$$\underline{n} \in \mathbb{Z}^d \mapsto \sideset{}{'}\sum_{k=\lambda(\underline{n})}^{\mu(\underline{n})} u_{\underline{n},k} \tag{6}$$

is a binomial sum.

See also Figure 1 for an example of a classical binomial sum.

1.2. **Characterization of binomial sums.** A few simple criteria make it possible to prove that a given sequence is not a binomial sum. For example:

- The sequence $(n!)_{n\geqslant 0}$ is not a binomial sum. Indeed, the set of sequences that grow at most exponentially is closed under the rules that define binomial sums. Since $\binom{n}{k} \leqslant 2^{|n|+|k|}$, every binomial sum grows at most exponentially; but this is not the case for the sequence $(n!)_{n\geqslant 0}$.

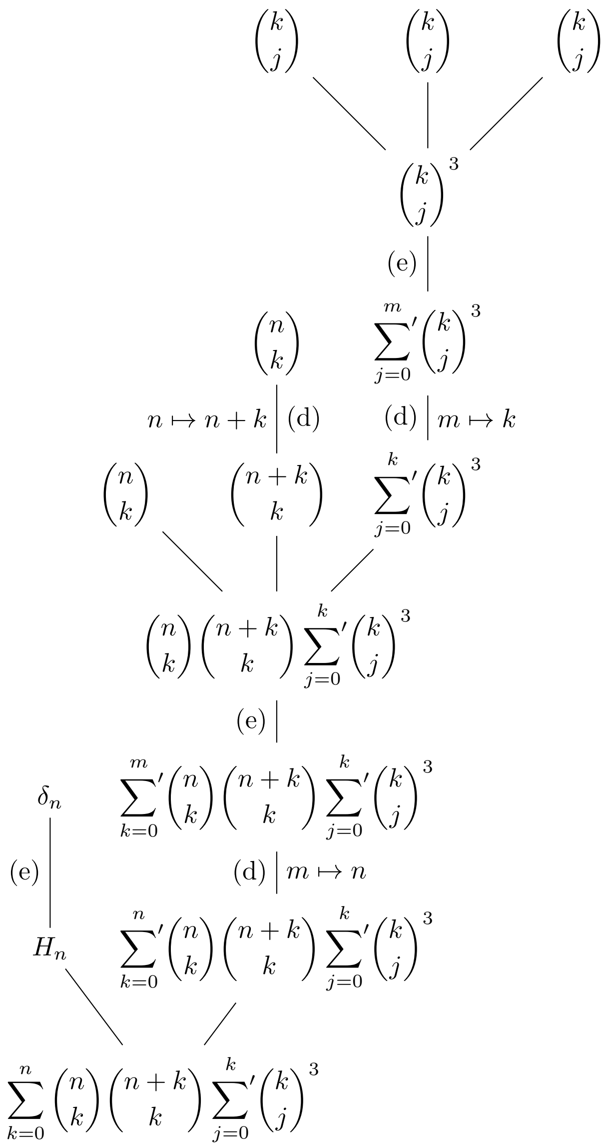


Figure 1. Construction of a binomial sum. The last step replaces the directed sum by a usual sum in order to have nonzero values for $n \geqslant 0$ only.

- The sequence of all prime integers is not a binomial sum because it does not satisfy any nonzero linear recurrence relation with polynomial coefficients (Flajolet et al. 2005), whereas every binomial sum does, see Corollary 3.6.
- The sequence $(1/n)_{n\geqslant 1}$ is not a binomial sum. To prove this, we can easily reduce to the case where $\mathbb{K}$ is a number field and then study the denominators that may appear in the elements of a binomial sum. One may introduce new prime divisors in the denominators only by multiplying with a scalar or with rule (b), so that the denominators of the elements of a given binomial sum contain only finitely many prime divisors. This is clearly not the case for the sequence $(1/n)_{n\geqslant 1}$.

By the same argument, the first sum of Eq. (1) is not a binomial sum. Indeed, by creative telescoping, it can be shown to equal $(2n+1)4^{n+1}/\binom{2n+2}{n+1} + 1/3$ and thus all prime numbers appear as denominators.

- The sequence $(u_n)_{n\geqslant 0}$ defined by $u_0 = 0, u_1 = 1$ and by the recurrence $(2n+1)u_{n+2} - (7n+11)u_{n+1} + (2n+1)u_n = 0$ is not a binomial sum. This follows from the asymptotic estimate $u_n \sim C \cdot (4/(7-\sqrt{33}))^n \cdot n^{\sqrt{75/44}}$, with $C \approx 0.56$, and the fact that $\sqrt{75/44}$ is not a rational number (Garoufalidis 2009, Theorem 5).

These criteria are, in substance, the only ones that we know to prove that a given sequence is not a binomial sequence. Conjecturally, they characterize univariate binomial sums. Indeed, we will see that the equivalence between univariate binomial sums and diagonals of rational functions (Theorem 3.5) leads, among many interesting corollaries, to the following reformulation of a conjecture due to Christol (1990, Conjecture 4): "any sequence $(u_n)_{n\geqslant 0}$ of integers that grows at most exponentially and that is solution of a linear recurrence equation with polynomial coefficients is a binomial sum."

2. Generating functions

To go back and forth between sequences and power series, one can use convergent power series and Cauchy's integrals to extract coefficients, as shown in the introduction with Dixon's identity, or one can use formal power series. Doing so, the theory keeps close to the actual algorithms and we avoid the tedious tracking of convergence radii. However, this requires the introduction of a *field* of multivariate formal power series that makes it possible, by embedding the rational functions into it, to define what the coefficient of a given monomial is in the power series expansion of an arbitrary multivariate rational function. We choose here to use the field of *iterated Laurent series*. It is an instance of the classical field of Hahn series when the value group is $\mathbb{Z}^n$ with the lexicographic order. We refer to Xin (2004) for a complete treatment of this field and we simply gather here the main definitions and results.

2.1. **Iterated Laurent series.** For a field $\mathbb{A}$, let $\mathbb{A}(\!(t)\!)$ be the field of univariate formal Laurent power series with coefficients in $\mathbb{A}$. For $d \geqslant 0$, let $\mathbb{L}_d$ be the field of iterated formal Laurent power series $\mathbb{K}(\!(z_d)\!)(\!(z_{d-1})\!)\cdots(\!(z_1)\!)$. It naturally contains the field of rational functions in $z_1, \dots, z_d$. For $\underline{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$, let $\underline{z}^{\underline{n}}$ denote the monomial $z_1^{n_1} \cdots z_d^{n_d}$, and for $f \in \mathbb{L}_d$, let $[\underline{z}^{\underline{n}}]f$ denote the coefficient of $\underline{z}^{\underline{n}}$ in f, that is the coefficient of $z_d^{n_d}$ in the coefficient of $z_{d-1}^{n_{d-1}}$ in $[\dots]$ in the coefficient of $z_1^{n_1}$ of f. An element of $f \in \mathbb{L}_d$ is entirely characterized by its coefficient function $\underline{n} \in \mathbb{Z}^d \mapsto [\underline{z}^{\underline{n}}]f$. On occasion, we will write $[z_1^n]f$, this means $[z_1^n z_2^0 \cdots z_d^0]f$, e.g., $[1]f$ means $[z_1^0 \cdots z_d^0]f$; more generally we write $[m]f$, where m is a monomial. In any case, the bracket notation always yields an element of $\mathbb{K}$.

Let $\prec$ denote the lexicographic ordering on $\mathbb{Z}^d$. For $\underline{n}, \underline{m} \in \mathbb{Z}^d$ we write $\underline{z}^{\underline{n}} \prec \underline{z}^{\underline{m}}$ if $\underline{m} \prec \underline{n}$ (mind the inversion: a monomial is larger when its exponent is smaller). In particular $z_1 \prec \cdots \prec z_d \prec 1$. With an analytic point of view, we work in an infinitesimal region where z_d is infinitesimally small and where z_i is infinitesimally small compared to z_{i+1}. The relation $\underline{z}^{\underline{n}} \prec \underline{z}^{\underline{m}}$ means that $|\underline{z}^{\underline{n}}|$ is smaller than $|\underline{z}^{\underline{m}}|$ in this region.

For $f \in \mathbb{L}_d$, the *support* of f, denoted $\operatorname{supp}(f)$, is the set of all $\underline{n} \in \mathbb{Z}_d$ such that $[\underline{z}^{\underline{n}}]f$ is not zero. It is well-known (e.g. Xin 2004, Prop. 2-1.2) that a function φ :

$\mathbb{Z}^d \to \mathbb{K}$ is the coefficient function of an element of $\mathbb{L}_d$ if and only if the support of φ is well-ordered for the order $\prec$ (that is to say every subset of the support of φ has a least element). The *valuation* of a non zero $f \in \mathbb{L}_d$, denoted $v(f)$, is the smallest element of $\operatorname{supp}(f) \subset \mathbb{Z}^d$ for the ordering $\prec$. Since $\operatorname{supp}(f)$ is well-ordered, $v(f)$ does exist. The *leading monomial* of f, denoted $\operatorname{lm}(f)$, is $\underline{z}^{v(f)}$; it is the largest monomial that appears in f.

For $1 \leqslant i \leqslant d$, the partial derivative $\partial_i = \partial/\partial z_i$ with respect to the variable z_i, defined for rational functions, extends to a derivation in $\mathbb{L}_d$ such that $[\underline{z}^{\underline{n}}]\partial_i f = (n_i + 1)[\underline{z}^{\underline{n}}](f/z_i)$ for any $f \in \mathbb{L}_d$.

Let $w_1, \dots, w_d$ be monomials in the variables $z_1, \dots, z_e$. When f is a rational function, the substitution $f(w_1, \dots, w_d)$ is defined in a simple way. For elements of $\mathbb{L}_d$, it is slightly more technical. Let $\varphi : \mathbb{Z}^d \to \mathbb{Z}^e$ be the additive map defined by $\varphi(\underline{n}) = v(\underline{w}^{\underline{n}})$ (recall that v denotes the valuation). Conversely, this map entirely determines the monomials $w_1, \dots, w_d$. When φ is strictly increasing (and thus injective) with respect to the lexicographic ordering, we define f^φ as the unique element of $\mathbb{L}_e$ such that

$$[\underline{z}^{\underline{n}}]f^\varphi = \begin{cases} [\underline{z}^{\varphi^{-1}(\underline{n})}]f & \text{if } \underline{n} \in \varphi(\mathbb{Z}^d), \\ 0 & \text{otherwise.} \end{cases}$$

The map $f \in \mathbb{L}_d \mapsto f^\varphi \in \mathbb{L}_e$ is a field morphism. In particular, for a monomial $\underline{z}^{\underline{n}}$, we check that $(\underline{z}^{\underline{n}})^\varphi = \underline{z}^{\varphi(\underline{n})}$. When f is a rational function, f^φ coincides with the usual substitution $f(w_1, \dots, w_d)$.

Another important construction is the sum of geometric sequences. Let $f \in \mathbb{L}_d$ be a Laurent power series with $\operatorname{lm}(f) \prec 1$. The set of all $\underline{n} \in \mathbb{Z}^d$ such that there is at least one $k \in \mathbb{N}$ such that $[\underline{z}^{\underline{n}}]f^k \neq 0$ is well-ordered (Neumann 1949, Theorem 3.4). Moreover, for any $\underline{n} \in \mathbb{Z}^d$, the coefficient $[\underline{z}^{\underline{n}}]f^k$ vanishes for all but finitely many $k \in \mathbb{N}$ (Neumann 1949, Theorem 3.5). The following result is easily deduced.

Lemma 2.1. *Let $f \in \mathbb{L}_d$ and let $\underline{z}^{\underline{n}}$ be a monomial. If $\operatorname{lm}(f) \prec 1$, then $[\underline{z}^{\underline{n}}]f^k = 0$ for all but finitely many $k \in \mathbb{N}$ and moreover*

$$[\underline{z}^{\underline{n}}]\frac{1}{1-f} = \sum_{k \geqslant 0} [\underline{z}^{\underline{n}}]f^k.$$

In what follows, there will be variables $z_1, \dots, z_d$, denoted $z_{1 \smile d}$ (and sometimes under different names) and we will denote $z_1 \prec \dots \prec z_d$ the fact that we consider the field $\mathbb{L}_d$ with this ordering to define coefficients, residues, etc. The variables are always ordered by increasing index, and $t_{1\smile d} \prec z_{1 \smile e}$ denotes $t_1 \prec \dots \prec t_d \prec z_1 \prec \dots \prec z_e$. An element of $\mathbb{L}_d \cap \mathbb{K}(z_{1\smile d})$ is called a *rational Laurent series*, and an element of $\mathbb{K}[\![z_{1\smile d}]\!] \cap \mathbb{K}(z_{1\smile d})$ is called a *rational power series*.

Example 1. Since $\mathbb{L}_d$ is a field containing all the rational functions in the variables $z_1, \dots, z_d$, one may define the *coefficient of a monomial in a rational function*. However, it strongly depends on the ordering of the variables. For example, in $\mathbb{L}_2 = \mathbb{K}(\!(z_2)\!)(\!(z_1)\!)$, the coefficient of 1 in $z_2/(z_1 + z_2)$ is 1 because

$$\frac{z_2}{z_1 + z_2} = 1 - \frac{1}{z_2} z_1 + \mathcal{O}(z_1^2),$$

whereas the coefficient of 1 in $z_1/(z_1 + z_2)$ is 0 because

$$\frac{z_1}{z_1 + z_2} = \frac{1}{z_2} z_1 + \mathcal{O}(z_1^2).$$

2.2. **Binomial sums and coefficients of rational functions.** Binomial sums can be obtained as certain extractions of coefficients of rational functions. We will make use of sequences of the form $u : \underline{n} \in \mathbb{Z}^d \mapsto [1]\left(R_0 R_1^{n_1} \cdots R_d^{n_d}\right)$, where $R_{0\smile d}$ are rational functions of ordered variables $z_{1\smile r}$. They generalize several notions. For example if $R_0 \in \mathbb{K}(z_1, \dots, z_r)$ and $R_i = 1/z_i$, then u is simply the coefficient function of R_0; and if $d = 1$, $R_0 \in \mathbb{K}(z_1, \dots, z_r)$ and $R_1 = 1/(z_1 \cdots z_r)$, then u is the sequence of the diagonal coefficients of R_0.

Theorem 2.2. *Every binomial sum is a linear combination of finitely many sequences of the form $\underline{n} \in \mathbb{Z}^d \mapsto [1]\left(R_0 R_1^{n_1} \cdots R_d^{n_d}\right)$, where $R_{0\smile d}$ are rational functions of ordered variables $z_{1\smile r}$, for some $r \in \mathbb{N}$.*

Proof. It is clear that $\delta_n = [1]z^n$, that $C^n = [1]C^n$ and that $\binom{n}{k} = [1](1+z)^n z^{-k}$, for all $n, k \in \mathbb{Z}$. Thus, it is enough to prove that the vector space generated by the sequences of the form $[1]\left(R_0 R_1^{n_1} \cdots R_d^{n_d}\right)$ is a subalgebra of $\mathcal{S}$ which is closed by the rules defining binomial sums, see §1.1.

First, it is closed under product: if $R_{0\smile d}$ and $R'_{0\smile d}$ are rational functions in the variables $z_{1\smile r}$ and $z'_{1\smile r'}$ respectively, then

$$[1]\left(R_0 \prod_{i=1}^d R_i^{n_i}\right)[1]\left(R'_0 \prod_{i=1}^d {R'_i}^{n_i}\right) = [1]\left(R_0 R'_0 \prod_{i=1}^d (R_i R'_i)^{n_i}\right),$$

with the ordering $z_{1\smile r} \prec z'_{1\smile r'}$, for example. Then, to prove the closure under change of variables, it is enough to reorder the factors:

$$[1]\left(R_0 \prod_{i=1}^d R_i^{\sum_{j=1}^e a_{ij} n_j + b_i}\right) = [1]\left(\left(R_0 \prod_{i=1}^d R_i^{b_i}\right) \prod_{j=1}^e \left(\prod_{i=1}^d R_i^{a_{ij}}\right)^{n_j}\right).$$

Only the closure under partial sum remains. Let u be a sequence of the form

$$(\underline{n}, k) \in \mathbb{Z}^d \times \mathbb{Z} \mapsto [1]\left(T^k R_0 \prod_{i=1}^d R_i^{n_i}\right),$$

where T and $R_{0\smile d}$ are rational functions. If $T = 1$, then

$$\sum_{k=0}^{m}{}' u_{\underline{n},k} = [1]\left((m+1) R_0 \prod_{i=1}^d R_i^{n_i}\right) = [1]\left(\frac{(1+v)R_0}{v}(1+v)^m \prod_{i=1}^d R_i^{n_i}\right),$$

where v is a new variable, because $[v](1+v)^{m+1} = m+1$. If $T \neq 1$, then

$$\sum_{k=0}^{m}{}' u_{\underline{n},k} = [1]\left(\frac{R_0}{1-T} \prod_{i=1}^d R_i^{n_i}\right) - [1]\left(\frac{R_0 T}{1-T} T^m \prod_{i=1}^d R_i^{n_i}\right),$$

which concludes the proof. ☐

The previous proof is algorithmic and Algorithm 1 p. 14 proposes a variant which is more suitable for actual computations—see §2.4. It is a straightforward rewriting procedure and often uses more terms and more variables than necessary. The geometric reduction procedure, see §5, handles this issue.

Example 2. Following step by step the proof of Theorem 2.2, we obtain that

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k} = [1]\frac{1+z_2}{1+z_1-z_1z_2}\left(\frac{(1+z_1)(1+z_2)^2}{z_1z_2}\right)^n - [1]\frac{z_1z_2}{1+z_1-z_1z_2}\left((1+z_1)(1-z_2)\right)^n.$$

We notice that the second term is always zero.

Corollary 2.3. *For any binomial sum $u : \mathbb{Z}^d \to \mathbb{K}$, there exist $r \in \mathbb{N}$ and a rational function $R(t_{1\smile d}, z_{1\smile r})$, with $t_{1\smile d} \prec z_{1\smile r}$, such that for any $\underline{n} \in \mathbb{Z}^d$,*

$$[\underline{t}^{\underline{n}}\underline{z}^{\underline{0}}]R = \begin{cases} u_{\underline{n}} & \textit{if } n_1, \dots, n_d \geqslant 0, \\ 0 & \textit{otherwise.} \end{cases}$$

(See Proposition 3.10 below for a converse.)

Proof. It is enough to prove the claim for a generating set of the vector space of all binomial sums. In accordance with Theorem 2.2, let $u : \mathbb{Z}^d \to \mathbb{K}$ be a sequence of the form $\underline{n} \in \mathbb{Z}^d \mapsto [1]\,(S_0 S_1^{n_1} \cdots S_d^{n_d})$, where $S_{0\smile d}$ are rational functions of ordered variables $z_{1\smile r}$. Let $R(t_{1\smile d}, z_{1\smile r})$ be the rational function

$$R = S_0 \prod_{i=1}^{d} \frac{1}{1 - t_i S_i}.$$

Lemma 2.1 implies that for any $\underline{n} \in \mathbb{Z}^d$

$$[\underline{t}^{\underline{n}}\underline{z}^{\underline{0}}]R = \sum_{k_1,\dots,k_d \geqslant 0} [\underline{t}^{\underline{n}}\underline{z}^{\underline{0}}]\left(S_0 \cdot (t_1S_1)^{k_1} \cdots (t_dS_d)^{k_d}\right).$$

Since the variables t_i do not appear in the S_i's, the coefficient in the sum is not zero only if $\underline{n} = \underline{k}$. In particular, if $\underline{n}$ is not in $\mathbb{N}^d$, then $[\underline{t}^{\underline{n}}\underline{z}^{\underline{0}}]R = 0$. And if $\underline{n} \in \mathbb{N}^d$, then

$$[\underline{t}^{\underline{n}}\underline{z}^{\underline{0}}]R = [\underline{t}^{\underline{n}}\underline{z}^{\underline{0}}]\left(S_0 \cdot (t_1S_1)^{n_1} \cdots (t_dS_d)^{n_d}\right) = [1]\left(S_0 S_1^{n_1} \cdots S_d^{n_d}\right) = u_{\underline{n}},$$

which concludes the proof. ☐

2.3. **Residues.** The notion of residue makes it possible to represent the full generating function of a binomial sum. It is a key step toward their computation. For $f \in \mathbb{L}_d$ and $1 \leqslant i \leqslant d$, the *formal residue* of f with respect to z_i, denoted $\operatorname{res}_{z_i} f$, is the unique element of $\mathbb{L}_d$ such that

$$[\underline{z}^{\underline{n}}](\operatorname*{res}_{z_i} f) = \begin{cases} [\underline{z}^{\underline{n}}](z_i f) & \text{if } n_i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

This is somehow the coefficient of $1/z_i$ in f, considered as a power series in z_i (though f is not a Laurent series in z_i since it may contain infinitely many negative powers of z_i). When f is a rational function, care should be taken not to confuse the formal residue with the *classical residue* at $z_i = 0$, which is the coefficient of $1/z_i$ in the partial fraction decomposition of f with respect to z_i. However, the former can be expressed in terms of classical residues, see §5, and like the classical residues, it vanishes on derivatives:

Lemma 2.4. $\operatorname{res}_{z_i} \partial_i f = 0$ *for all* $f \in \mathbb{L}_d$.

If α is a set of variables $\{z_{i_1}, \dots, z_{i_r}\}$, let $\operatorname{res}_\alpha f$ denote the iterated residue $\operatorname{res}_{z_{i_1}} \cdots \operatorname{res}_{z_{i_r}} f$. It is easily checked that this definition does not depend on the order in which the variables appear. This implies, together with Lemma 2.4, the following lemma:

Lemma 2.5. $\operatorname{res}_\alpha \left(\sum_{v\in\alpha} \partial_v f_v\right) = 0$, *for any family* $(f_v)_{v\in\alpha}$ *of elements of* $\mathbb{L}_d$.

The following lemma will also be useful:

Lemma 2.6. *Let* $\alpha, \beta \subset \{z_{1\smile d}\}$ *be disjoint sets of variables. If* $f \in \mathbb{L}_d$ *does not depend on the variables in* β *and if* $g \in \mathbb{L}_d$ *does not depend on the variables in* α, *then*

$$\big(\operatorname*{res}_\alpha f\big)\big(\operatorname*{res}_\beta g\big) = \operatorname*{res}_{\alpha\cup\beta}(fg).$$

Proof. $\operatorname{res}_{\alpha\cup\beta} fg = \operatorname{res}_\alpha \operatorname{res}_\beta fg = \operatorname{res}_\alpha \left(f \operatorname{res}_\beta g\right) = \left(\operatorname{res}_\alpha f\right)\left(\operatorname{res}_\beta g\right)$. □

The order on the variables matters. For example, let $F(x, y)$ be a rational function, with $x \prec y$. The residue $\operatorname{res}_x F$ is a rational function of y. Indeed, if $-n$ is the exponent of x in $\operatorname{lm}(F)$, which we assume to be negative, then

$$\operatorname*{res}_x F = \frac{1}{(n-1)!}\left(\frac{\partial^{n-1}}{\partial x^{n-1}} x^n F\right)\bigg|_{x=0}.$$

In contrast, the residue with respect to y (or any other variable which is not the smallest) is not, in general, a rational function. It is an important point because we will represent generating series of binomial sums—which need be neither rational nor algebraic—as residues of rational functions.

Example 3. Let $F = \frac{1}{xy(y^2+y-x)}$. Assume that $y \prec x$, that is to say, we work in the field $\mathbb{K}(\!(x)\!)(\!(y)\!)$. We compute that $\operatorname{res}_y F = -\frac{1}{x^2}$ because

$$F = -\frac{1}{x^2 y} + \frac{(y+1)}{x^2(y^2+y-x)}$$

and the second term does not contain any negative power of y.

On the other hand, if we assume that $x \prec y$, then $\operatorname{res}_y F$ is not a rational function anymore. Using Proposition 5.2, we can compute that

$$\operatorname*{res}_y F = -\frac{1}{x^2} + \frac{2}{x + 4x^2 - x(1+4x)^{1/2}} = -\frac{1}{x} + 3 - 10x + 35x^2 + \mathcal{O}(x^3).$$

Residues of rational functions can be used to represent any binomial sum; it is the main point of the method and it is the last corollary of Theorem 2.2. Following Egorychev we call them *integral representations*, or *formal* integral representations, to emphasize the use of formal power series and residues rather than of analytic objects.

Corollary 2.7 (Integral representations)**.** *For any binomial sum* $u : \mathbb{Z}^d \to \mathbb{K}$, *there exist* $r \in \mathbb{N}$ *and a rational function* $R(t_{1\smile d}, z_{1\smile r})$, *with* $t_{1\smile d} \prec z_{1\smile r}$, *such that*

$$\sum_{\underline{n}\in\mathbb{N}^d} u_{\underline{n}} \underline{t}^{\underline{n}} = \operatorname*{res}_{z_{1\smile r}} R.$$

In other words, the generating function of the restriction to $\mathbb{N}^d$ of a binomial sum is a residue of a rational function.

Proof. By Corollary 2.3, there exist $r \in \mathbb{N}$ and a rational function $\tilde{R}(t_{1\smile d}, z_{1\smile r})$ such that $u_{\underline{n}} = [\underline{t}^{\underline{n}} \underline{z}^{\underline{0}}]\tilde{R}$ for all $\underline{n} \in \mathbb{N}^d$. Let R be $\tilde{R}/(z_1 \cdots z_r)$. Then

$$\sum_{\underline{n}\in\mathbb{N}^d} u_{\underline{n}} \underline{t}^{\underline{n}} = \sum_{\underline{n}\in\mathbb{N}^d} \left([\underline{t}^{\underline{n}} \underline{z}^{\underline{0}}]\tilde{R}\right) \underline{t}^{\underline{n}} = \operatorname*{res}_{z_{1\smile r}} R.$$ □

Algorithm 1. Constant term representation of a binomial sum

Input: A binomial sum $u : \mathbb{Z}^d \to \mathbb{K}$ given as an abstract syntax tree

Output: $P(n_{1\smile d}; z_{1\smile r})$ a linear combination of expressions of the form $\underline{n}^{\underline{\alpha}} R_0 R_1^{n_1} \cdots R_d^{n_d}$ where $\underline{\alpha} \in \mathbb{N}^d$ and $R_{0\smile d}$ are rational functions of $z_{1\smile r}$.

Specification: $u_{\underline{n}} = [1]P(n_{1\smile d}; z_{1\smile r})$ for any $\underline{n} \in \mathbb{Z}^d$.

function SumToCT(u)
  if $u_{\underline{n}} = \delta_{n_1}$ **then return** $z_1^{n_1}$
  else if $u_{\underline{n}} = a^{n_1}$ for some $a \in \mathbb{K}$ **then return** a^{n_1}
  else if $u_{\underline{n}} = \binom{n_1}{n_2}$ **then return** $(1+z_1)^{n_1}/z_1^{n_2}$
  else if $u_{\underline{n}} = v_{\underline{n}} + w_{\underline{n}}$ **then return** SumToCT(v) + SumToCT(w)
  else if $u_{\underline{n}} = v_{\underline{n}} w_{\underline{n}}$ **then**
    $P(n_{1\smile d}; z_{1\smile r}) \leftarrow$ SumToCT(v)
    $Q(n_{1\smile e}; z_{1\smile s}) \leftarrow$ SumToCT(w)
    return $P(n_{1\smile d}; z_{1\smile r}) Q(n_{1\smile e}; z_{r+1\smile r+s})$
  else if $u_{\underline{n}} = v_{\lambda(\underline{n})}$ for some affine map $\lambda : \mathbb{Z}^d \to \mathbb{Z}^e$ **then**
    $P(n_{1\smile d}; z_{1\smile r}) \leftarrow$ SumToCT(v)
    return $P(\lambda(n_{1\smile d}); z_{1\smile r})$
  else if $u_{n_1,n_2,\dots} = \sum_{k=0}^{\prime\, n_1} v_{k,n_2,\dots}$ **then**
    $P(n_{1\smile d}; z_{1\smile r}) \leftarrow$ SumToCT(v)
    Compute $Q(n_{1\smile d}; z_{1\smile r})$ s.t. $Q(n_1+1, n_{2\smile d}; z_{1\smile r}) - Q(n_{1\smile d}; z_{1\smile r}) = P$ ▷ See Lemma 2.8.
    return $Q(n_1+1, n_{2\smile d}; z_{1\smile r}) - Q(0, n_{2\smile d}; z_{1\smile r})$

In §3, we prove an equivalence between univariate binomial sums and *diagonals of rational functions* that are a special kind of residues.

2.4. **Algorithms.** The proofs of Theorem 2.2 and its corollaries are constructive, but in order to implement them, it is useful to consider not only sequences of the form $[1](R_0 \underline{R}^{\underline{n}})$ but also sequences of the form $[1](\underline{n}^{\underline{\alpha}} R_0 \underline{R}^{\underline{n}})$, for some rational functions $R_{0\smile d}$ and $\underline{\alpha} \in \mathbb{N}^d$. They bring two benefits. First, polynomial factors often appear in binomial sums and it is convenient to be able to represent them without adding new variables. Second, we can always perform sums without adding new variables, contrary to what is done in the case $T = 1$ of the proof of Theorem 2.2.

Lemma 2.8. *For any $A \neq 0$ in some field k, and any $\alpha \in \mathbb{N}$, there exists a polynomial $P \in k[n]$ of degree at most $\alpha + 1$ such that for all $n \in \mathbb{Z}$,*

$$n^\alpha A^n = P(n+1)A^{n+1} - P(n)A^n.$$

Proof. It is a linear system of $\alpha+2$ equations, one for each monomial $1, n, \dots, n^{\alpha+1}$, in the $\alpha+2$ coefficients of P. If $A \neq 1$, the homogeneous equation $P(n+1)A = P(n)$ has no solutions, so the system is invertible and has a solution. If $A = 1$, then the equation $P(n+1)A = P(n)$ has a one-dimensional space of solutions but the equation corresponding to the monomial $n^{\alpha+1}$ reduces to $0 = 0$, so the system again has a solution. □

Algorithm 1 implements Theorem 2.2 in this generalized setting. The input data, a binomial sum, is given as an expression considered as an abstract syntax tree, as depicted in Figure 1. The implementation of Corollaries 2.3 and 2.7 must also be adjusted to handle the monomials in $\underline{n}$.

Algorithm 2. Integral representation of a binomial sum

Input: A binomial sum $u : \mathbb{Z}^d \to \mathbb{K}$ given as an abstract syntax tree
Output: A rational function $R(t_{1\smile d}, z_{1\smile r})$
Specification: $\sum_{\underline{n}\in\mathbb{N}^d} u_{\underline{n}} t^{\underline{n}} = \operatorname{res}_{z_{1\smile r}} R$, where $t_{1\smile d} \prec z_{1\smile r}$

function SumToRes(u)

$$\sum_{k=1}^{m} \underline{n}^{\alpha_k} R_{k,0} R_{k,1}^{n_1} \cdots R_{k,d}^{n_d} \leftarrow \text{SumToCT}(u)$$

return $\displaystyle \frac{1}{z_1 \ldots z_r} \sum_{k=1}^{m} R_{k,0} F_{\alpha_1}(t_1 R_{k,1}) \cdots F_{\alpha_d}(t_d R_{k,d})$

where $F_0(z) = \frac{1}{1-z}$ and $F_{\alpha+1}(z) = zF'_\alpha(z)$

To this effect, for $\alpha \in \mathbb{N}$, let

$$F_\alpha(z) \stackrel{\text{def}}{=} \sum_{n\geqslant 0} n^\alpha z^n = \left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^\alpha \cdot \frac{1}{1-z}.$$

Now let us consider a sequence $u : \underline{n} \in \mathbb{Z}^d \mapsto [1]\,(\underline{n}^{\underline{\alpha}} S_0 S_1^{n_1} \cdots S_d^{n_d})$, where $S_{0\smile d}$ are rational functions of ordered variables $z_{1\smile r}$ and $\underline{\alpha} \in \mathbb{N}^d$ Let

$$\tilde{R} = S_0 \prod_{i=1}^{d} \frac{1}{1 - t_i S_i} \quad \text{and} \quad R = S_0 \prod_{i=1}^{d} F_{\alpha_i}(t_i S_i),$$

where $t_{1\smile d} \prec z_{1\smile r}$. By definition of the F_α's, we check that

$$R = \left(t_1 \frac{\partial}{\partial t_1}\right)^{\alpha_1} \cdots \left(t_d \frac{\partial}{\partial t_d}\right)^{\alpha_d} \cdot \tilde{R},$$

so that $[\underline{t}^{\underline{n}} \underline{z}^{\underline{0}}]R = \underline{n}^{\underline{\alpha}} [\underline{t}^{\underline{n}} \underline{z}^{\underline{0}}]\tilde{R}$, for $\underline{n} \in \mathbb{Z}^d$, by definition of the differentiation in $\mathbb{L}_r$. In the proof of Corollary 2.3, we checked that $[\underline{t}^{\underline{n}} \underline{z}^{\underline{0}}]\tilde{R} = [1] S_0 S_1^{n_1} \cdots S_d^{n_d}$, for $\underline{n} \in \mathbb{N}^d$, and 0 otherwise. Therefore, $u_{\underline{n}} = [\underline{t}^{\underline{n}} \underline{z}^{\underline{0}}]R$, for $n \in \mathbb{N}^d$. This gives an implementation of Corollaries 2.3 and 2.7 in the generalized setting. Algorithm 2 sums up the procedure for computing integral representations.

2.5. **Analytic integral representations.** When $\mathbb{K}$ is a subfield of $\mathbb{C}$, then formal residues can be written as integrals.

Proposition 2.9. *Let $R(t_{1\smile d}, z_{1\smile e})$ be a rational function whose denominator does not vanish when $t_1 = \cdots = t_d = 0$. There exist positive real numbers $s_{1\smile d}$ and $r_{1\smile e}$ such that on the set $\left\{(t_{1\smile d}) \in \mathbb{C}^d \;\middle|\; \forall i, |t_i| \leqslant s_i\right\}$, the power series $\operatorname{res}_{z_{1\smile e}} R \in \mathbb{C}[\![t_{1\smile d}]\!]$ converges and*

$$\operatorname*{res}_{z_{1\smile e}} R = \frac{1}{(2\pi i)^e} \oint_\gamma R(t_{1\smile d}, z_{1\smile e}) \mathrm{d}z_{1\smile e},$$

where $\gamma = \{z \in \mathbb{C}^e \mid \forall 1 \leqslant i \leqslant e,\ |z_i| = r_i\}$.

Proof. When R is a Laurent monomial, the equality follows from Cauchy's integral formula. By linearity, it still holds when R is a Laurent polynomial.

In the general case, let R be written as a/f, where a and f are polynomials. We may assume that the leading coefficient of f is 1 and so f decomposes as $\operatorname{lm}(f)(1-g)$ where g is a Laurent polynomial with monomials $\prec 1$. The hypothesis that f does not vanish when $t_1 = \cdots = t_d = 0$ implies that $\operatorname{lm}(f)$ depends only on the variables $z_{1\smile e}$ and that g contains no negative power of the variables $t_{1\smile d}$. This and the fact that all monomials of g are $\prec 1$ imply that there exist positive real numbers $s_{1\smile d}$

and $r_{1\smile e}$ such that $|g(t_{1\smile d}, z_{1\smile e})| \leqslant \frac{1}{2}$ if $|t_i| \leqslant s_i$ and $|z_i| = r_i$. For example, we can take $s_i = \exp(-\exp(N/i))$ and $r_i = \exp(-\exp(N/(d+i)))$, for some large enough N, because

$$\exp(-\exp(N/i))^p = o\left(\exp(-\exp(N/j))^q\right), \quad N \to \infty,$$

for any $p, q > 0$ and $i < j$. On the one hand

$$\operatorname*{res}_{z_{1\smile e}} R = \sum_{k\geqslant 0} \operatorname*{res}_{z_{1\smile e}} \left(\frac{ag^k}{\operatorname{lm}(f)}\right),$$

by Lemma 2.1, and on the other hand, if $|t_i| \leqslant s_i$, for $1 \leqslant i \leqslant d$ then

$$\oint_\gamma R(t_{1\smile d}, z_{1\smile e}) \mathrm{d}z_{1\smile e} = \sum_{k\geqslant 0} \oint_\gamma \frac{ag^k}{\operatorname{lm}(f)} \mathrm{d}z_{1\smile e}.$$

where $\gamma = \{z \in \mathbb{C}^e \mid \forall 1 \leqslant i \leqslant e,\ |z_i| = r_i\}$, because the sum $\sum_{k\geqslant 0} g^k$ converges uniformly on γ, since $|g| \leqslant \frac{1}{2}$. And the lemma follows from the case where R is a Laurent polynomial. □

3. Diagonals

Let $R(z_{1\smile d}) = \sum_{\underline{n}\in\mathbb{N}^d} a_{\underline{n}} \underline{z}^{\underline{n}}$ be a rational power series in $\mathbb{K}(z_1, \dots, z_d)$. The *diagonal* of R is the univariate power series

$$\operatorname{diag} R \stackrel{\text{def}}{=} \sum_{n\geqslant 0} a_{n,\dots,n} t^n.$$

Diagonals have be introduced to study properties of the Hadamard product of power series (Cameron and Martin 1938; Furstenberg 1967). They also appear in the theory of G-functions (Christol 1988). They can be written as residues: with $t \prec z_{2\smile d}$, it is easy to check that

$$\operatorname{diag} R = \operatorname*{res}_{z_2,\dots,z_d} \frac{1}{z_2 \cdots z_d} R\left(\frac{t}{z_2 \cdots z_d}, z_2, \dots, z_d\right). \tag{7}$$

Despite their simplistic appearance, diagonals have very strong properties, the first of which is *differential finiteness*:

Theorem 3.1 (Christol 1985, Lipshitz 1988)**.** *Let $R(z_{1\smile d})$ be a rational power series in $\mathbb{K}(z_1, \dots, z_d)$. There exist polynomials $p_{0\smile r} \in \mathbb{K}[t]$, not all zero, such that $p_r f^{(r)} + \cdots + p_1 f' + p_0 f = 0$, where $f = \operatorname{diag} R$.*

We recall that a power series $f \in \mathbb{K}[\![t]\!]$ is called *algebraic* if there exists a nonzero polynomial $P \in \mathbb{K}[x, y]$ such that $P(t, f) = 0$.

Theorem 3.2 (Furstenberg 1967)**.** *A power series $f \in \mathbb{K}[\![t]\!]$ is algebraic if and only if f is the diagonal of a* bivariate *rational power series.*

Diagonals of rational power series in more than two variables need not be algebraic, as shown by

$$\sum_{n\geqslant 0} \frac{(3n)!}{n!^3} t^n = \operatorname{diag}\left(\frac{1}{1 - z_1 - z_2 - z_3}\right).$$

However, the situation is simpler modulo any prime number p. The following result is stated by Furstenberg (1967, Theorem 1) over a field of finite characteristic; since reduction modulo p and diagonal extraction commute, we obtain the following statement:

Theorem 3.3 (Furstenberg 1967)**.** *Let $f \in \mathbb{Q}[\![t]\!]$ be the diagonal of a rational power series with rational coefficients. Finitely many primes may divide the denominator of a coefficient of f. For all prime p except those, the power series $f \pmod p \in \mathbb{F}_p[\![t]\!]$ is algebraic.*

Example 4. It is easy to check that the series $f = \sum_{n\geqslant 0} \frac{(3n)!}{n!^3} t^n$ satisfies

$$\begin{aligned} f &\equiv (1+t)^{-\frac{1}{4}} \mod 5, \\ f &\equiv \left(1+6t+6t^2\right)^{-\frac{1}{6}} \mod 7, \\ f &\equiv \left(1+6t+2t^2+8t^3\right)^{-\frac{1}{10}} \mod 11, \text{ etc.} \end{aligned}$$

The characterization of diagonals of rational power series remains largely an open problem, despite very recent attempts (Christol 2015). A natural measure of the complexity of a power series $f \in \mathbb{Q}[\![t]\!]$ is the least integer n, if any, such that f is the diagonal of a rational power series in n variables. Let $N(f)$ be this number, or $N(f) = \infty$ if there is no such n.[1] It is clear that $N(f) = 1$ if and only if f is rational. According to Theorem 3.2, $N(f) = 2$ if and only if f is algebraic and not rational. It is easy to find power series f with $N(f) = 3$, for example $N\left(\sum_n \frac{(3n)!}{n!^3} t^n\right) = 3$ because it is the diagonal of a trivariate rational power series but it is not algebraic. However, we do not know of any power series f such that $N(f) > 3$.

Conjecture 3.4 (Christol 1990)**.** *Let $f \in \mathbb{Z}[\![t]\!]$ be a power series with integer coefficients, positive radius of convergence and such that $p_r f^{(r)} + \dots + p_1 f' + p_0 f = 0$ for some polynomials $p_{0\smile r} \in \mathbb{Q}[t]$, not all zero. Then f is the diagonal of a rational power series.*

In this section, we prove the following equivalence:

Theorem 3.5. *A sequence $u : \mathbb{N} \to \mathbb{K}$ is a binomial sum if and only if the generating function $\sum_{n\geqslant 0} u_n t^n$ is the diagonal of a rational power series.*

In §3.1, we prove that the generating function of a binomial sum is a diagonal and in §3.3, we prove the converse. When writing that a sequence $u : \mathbb{N} \to \mathbb{K}$ is a binomial sum, we mean that there exists a binomial sum $v : \mathbb{Z} \to \mathbb{K}$ whose support in contained in $\mathbb{N}$ and which coincides with u on $\mathbb{N}$. Theorem 3.5 can be stated equivalently without restriction on the support: a sequence $u : \mathbb{Z} \to \mathbb{K}$ is a binomial sum if and only if the generating functions $\sum_{n\geqslant 0} u_n t^n$ and $\sum_{n\geqslant 0} u_{-n} t^n$ are diagonals of rational power series.

Note that Garrabrant and Pak (2014, Theorem 1.3) proved a similar, although essentially different, result: the subclass of binomial multisums they consider corresponds to the subclass of diagonals of $\mathbb{N}$-rational power series.

Theorem 3.5 and the theory of diagonals of rational functions imply right away interesting corollaries.

Corollary 3.6. *If $u : \mathbb{Z} \to \mathbb{K}$ is a binomial sum, then there exist polynomials $p_{0\smile r}$ in $\mathbb{K}[t]$, not all zero, such that $p_r(n) u_{n+r} + \dots + p_1(n) u_{n+1} + p_0(n) u_n = 0$ for all $n \in \mathbb{Z}$.*

As mentioned in the introduction, this is a special case of a more general result for proper hypergeometric sums that can alternately be obtained as a consequence of the results of Lipshitz (1989) and Abramov and Petkovšek (2002).

[1] This notion is closely related to the *grade* of a power series (Allouche and Mendès France 2011).

Corollary 3.7 (Flajolet and Soria 1998)**.** *If the generating function $\sum_n u_n t^n$ of a sequence $u : \mathbb{N} \to \mathbb{K}$ is algebraic, then u is a binomial sum.*

Corollary 3.8. *Let $u : \mathbb{N} \to \mathbb{Q}$ be a binomial sum. Finitely many primes may divide the denominators of values of u. For all primes p except those, the generating function of a binomial sum is algebraic modulo p.*

Moreover, Christol's conjecture is equivalent to the following:

Conjecture 3.9. *If an integer sequence $u : \mathbb{N} \to \mathbb{Z}$ grows at most exponentially and satisfies a recurrence $p_r(n)u_{n+r} + \cdots + p_0(n)u_n = 0$, for some polynomials $p_{0\smile r}$ with integer coefficients, not all zero, then u is a binomial sum.*

The proof of Theorem 3.5 also gives information on binomial sums depending on several indices, in the form of a converse of Corollary 2.3.

Proposition 3.10. *A sequence $\mathbb{N}^d \to \mathbb{K}$ is a binomial sum if and only if there exists a rational function $R(t_{1\smile d}, z_{1\smile r})$, with $t_{1\smile d} \prec z_{1\smile r}$, such that $u_{\underline{n}} = [\underline{t}^{\underline{n}} \underline{z}^{\underline{0}}]R$ for all $\underline{n} \in \mathbb{Z}^d$.*

Sketch of the proof. The "only if" part is Corollary 2.3. Conversely, let $R(t_{1\smile d}, z_{1\smile r})$ be a rational function and let $u_{\underline{n}} = [\underline{t}^{\underline{n}} \underline{z}^{\underline{0}}]R$, for $\underline{n} \in \mathbb{Z}^d$. We assume that $u_{\underline{n}} = 0$ if $\underline{n} \notin \mathbb{N}^d$. Using the same technique as in §3.1, we show that there exist a rational power series $S(z_{1\smile d+r})$ and monomials $w_{1\smile d}$ in the variables $z_{1\smile d+r}$ such that $u_{\underline{n}} = [\underline{w}^{\underline{n}}]S$. Then, with the same technique as in §3.3, we write $u_{\underline{n}}$ as a binomial sum. □

In other words, binomial sums are exactly the constant terms of rational power series, where the largest variables, for the order $\prec$, are eliminated.

3.1. **Binomial sums as diagonals.** Corollary 2.7 and Equation (7) provide an expression of the generating function of a binomial sum (of one free variable) as the diagonal of a rational *Laurent* series, but more is needed to obtain it as the diagonal of a rational *power* series and obtain the "only if" part of Theorem 3.5.

Let $u : \mathbb{N} \to \mathbb{K}$ be a binomial sum. In this section, we aim at constructing a rational power series S such that $\sum_{n \geqslant 0} u_n t^n = \operatorname{diag} S$. By Corollary 2.3, there exists a rational function $\tilde{R}(z_{0\smile r}) = \frac{a}{f}$ such that $u_n = [z_0^n z_1^0 \cdots z_r^0]\tilde{R}$ for all $n \in \mathbb{Z}$.

Recall the notation f^φ, for $f \in \mathbb{L}_{r+1}$ and φ an increasing endomorphism of $\mathbb{Z}^{r+1}$, introduced in §2.1. For example, if $f = z_1 + z_2 \in \mathbb{L}_2$ and $\varphi(n_1, n_2) = (n_1, n_1 + n_2)$ then $f^\varphi = z_1 z_2 + z_2 = z_2(1 + z_1)$.

Lemma 3.11. *For any polynomial $f \in \mathbb{K}[z_{0\smile r}]$ there exists an increasing endomorphism φ of $\mathbb{Z}^{r+1}$ such that $f^\varphi = C\underline{z}^{\underline{m}}(1 + g)$, for some $\underline{m} \in \mathbb{N}^{r+1}$, $C \in \mathbb{K} \setminus \{0\}$ and $g \in \mathbb{K}[z_{0\smile r}]$ with $g(0, \ldots, 0) = 0$.*

Proof. Let $\underline{z}^{\underline{a}}$ and $\underline{z}^{\underline{b}}$ be monomials of f such that $\underline{z}^{\underline{a}} \prec \underline{z}^{\underline{b}}$. Let i be the smallest integer such that $a_i \neq b_i$. By definition $a_i > b_i$. For $k \in \mathbb{N}$, let $\varphi_k : \mathbb{Z}^{r+1} \to \mathbb{Z}^{r+1}$ be defined by

$$\varphi_k : (n_0, \ldots, n_r) \in \mathbb{Z}^{r+1} \mapsto (n_0, \ldots, n_i, n_{i+1} + kn_i, \ldots, n_r + kn_i) \in \mathbb{Z}^{r+1}.$$

It is strictly increasing and if k is large enough then $\varphi_k(\underline{a}) \geqslant \varphi_k(\underline{b})$ componentwise, that is $(\underline{z}^{\underline{b}})^{\varphi_k}$ divides $(\underline{z}^{\underline{a}})^{\varphi_k}$. We may apply repeatedly this argument to construct an increasing endomorphism φ of $\mathbb{Z}^{r+1}$ such that the leading monomial of f^φ divides all the monomials of f^φ, which proves the Lemma. □

Let φ be the endomorphism given by the lemma above applied to the denominator of $\tilde{R}$. For $0 \leqslant 0 \leqslant r$, let $w_i = z_i^\varphi$. Then

$$\tilde{R}(w_{0\smile r}) = \tilde{R}^\varphi = \frac{a^\varphi}{C\underline{z}^{\underline{m}}(1+g)},$$

because $\cdot^\varphi$ is a field morphism, as explained in §2.1. And by definition of $\tilde{R}$, we have $u_n = [w_0^n]\tilde{R}(w_{0\smile r})$ for all $n \in \mathbb{Z}$. Let R be the rational power series $\frac{a^\varphi}{C(1+g)}$, so that $u_n = [w_0^n](R/\underline{z}^{\underline{m}})$. We now prove that we can reduce to the case where $\underline{m} = 0$. If $\underline{m} \neq 0$, let i be the smallest integer such that $m_i \neq 0$. The specialization $R|_{z_i=0}$ is a rational power series and $R - R_{|z_i=0} = z_i T$ for some rational power series T. For all $n \geqslant 0$, the coefficient of w_0^n in $R|_{z_i=0}/\underline{z}^{\underline{m}}$ is zero because the exponent of z_i in w_0^n is nonnegative while the exponent of z_i in every monomial in $R|_{z_i=0}/\underline{z}^{\underline{m}}$ is $-m_i < 0$. Thus $u_n = [w_0^n]\frac{T}{\underline{z}^{\underline{m}}/z_i}$, and we can replace R by T and subtract 1 to the first nonzero coordinate of $\underline{m}$, which makes $\underline{m}$ decrease for the lexicographic ordering. Iterating this procedure leads to $\underline{m} = 0$ and thus $u_n = [w_0^n]T$ for some rational power series T.

Let us write w_0 as $z_0^{a_0} \cdots z_r^{a_r}$, with $a_{0\smile r} \in \mathbb{N}^{r+1}$. If all the a_i's were equal to 1, then $\sum_n u_n t^n$ would be the diagonal of T. First, we reduce to the case where all the a_i's are positive. If some a_i is zero, then

$$[w_0^n]T = [z_0^{a_0} \cdots z_{i-1}^{a_{i-1}} z_{i+1}^{a_{i+1}} \cdots z_r^{a_r}]T|_{z_i=0}$$

and we can simply remove the variable z_i.

Second, we reduce further to the case where all the a_i's are 1. Let us consider the following rational power series:

$$U = \frac{1}{a_0 \cdots a_r} \sum_{\varepsilon_0^{a_0}=1} \cdots \sum_{\varepsilon_r^{a_r}=1} T(\varepsilon_0 z_0, \dots, \varepsilon_r z_r),$$

where the ε_i ranges over the a_i-th roots of unity. By construction, if m is a monomial in the $z_i^{a_i}$, then $[m]U = [m]T$. In particular $u_n = [w_0^n]U$.

We may consider T and U as elements of the extension of the field $\mathbb{K}(z_0^{a_0}, \dots, z_r^{a_r})$ by the roots of the polynomials $X^{a_i} - z_i$, for $0 \leqslant i \leqslant r$. By construction, the rational function U is left invariant by the automorphisms of this extension. Thus $U \in \mathbb{K}(z_0^{a_0}, \dots, z_r^{a_r})$. Let $S(z_{0\smile r})$ be the unique rational function such that $U = S(z_0^{a_0}, \dots, z_r^{a_r})$. It is a rational power series and $u_n = [z_0^n \cdots z_r^n]S$. Thus $\sum_{n\geqslant 0} u_n t^n = \operatorname{diag} S$, which concludes the proof that binomial sums are diagonals of rational functions, the "only if" part of Theorem 3.5.

3.2. **Summation over a polyhedron.** To prove that, conversely, diagonals of rational power series are generating functions of binomial sums, we prove a property of closure of binomial sums under certain summations that generalize the indefinite summation of rule (e) of Definition 1.1. Let $u : \mathbb{Z}^{d+e} \to \mathbb{K}$ be a binomial sum and let $\Gamma \subset \mathbb{R}^{d+e}$ be a rational polyhedron, that is to say

$$\Gamma = \bigcap_{\lambda \in \Lambda} \left\{ x \in \mathbb{R}^{d+e} \;\middle|\; \lambda(x) \geqslant 0 \right\},$$

for a finite set Λ of linear maps $\mathbb{R}^{d+e} \to \mathbb{R}$ with integer coefficients in the canonical bases. This section is dedicated to the proof of the following:

Proposition 3.12. *If for all $\underline{n} \in \mathbb{Z}^d$ the set $\{\underline{m} \in \mathbb{R}^e \mid (\underline{n}, \underline{m}) \in \Gamma\}$ is bounded, then the sequence*

$$v : \underline{n} \in \mathbb{Z}^d \mapsto \sum_{\underline{m} \in \mathbb{Z}^e} u_{\underline{n},\underline{m}} \mathbf{1}_\Gamma(\underline{n}, \underline{m})$$

is a binomial sum, where $\mathbf{1}_\Gamma(\underline{n}, \underline{m}) = 1$ if $(\underline{n}, \underline{m}) \in \Gamma$ and 0 otherwise.

Recall that $H : \mathbb{Z} \to \mathbb{K}$ is the sequence defined by $H_n = 1$ if $n \geqslant 0$ and 0 otherwise; it is a binomial sum, see §1.1. Thus $\mathbf{1}_\Gamma$ is a binomial sum, because $\mathbf{1}_\Gamma(\underline{n}) = \prod_{\lambda \in \Lambda} H_{\lambda(\underline{n})}$ and each factor of the product is a binomial sum, thanks to rule (d) of the definition of binomial sums. However, the summation defining v ranges over an infinite set, rule (e) is not enough to conclude directly, neither is the generalized summation of Equation (6).

For $\underline{m} \in \mathbb{R}^e$, let $|\underline{m}|_\infty$ denote $\max_{1 \leqslant i \leqslant e} |m_i|$. For $\underline{n} \in \mathbb{Z}^d$, let $B(\underline{n})$ be

$$B(\underline{n}) \stackrel{\text{def}}{=} \sup \{|\underline{m}|_\infty \mid \underline{m} \in \mathbb{R}^e \text{ and } (\underline{n}, \underline{m}) \in \Gamma\} \cup \{-\infty\}.$$

By hypothesis on Γ, $B(\underline{n}) < \infty$ for all $\underline{n} \in \mathbb{Z}^d$.

Lemma 3.13. *There exists $C > 0$ such that $B(\underline{n}) \leqslant C(1 + |n|_\infty)$, for all $\underline{n} \in \mathbb{Z}^d$.*

Proof. By contradiction, let us assume that such a C does not exist, that is there exists a sequence $\underline{p}_k = (\underline{n}_k, \underline{m}_k)$ of elements of Γ such that $|\underline{m}_k|_\infty / |\underline{n}_k|_\infty$ and $|\underline{n}_k|_\infty$ tend to ∞. Up to considering a subsequence, we may assume that $\underline{m}_k / |\underline{m}_k|_\infty$ has a limit $\ell \in \mathbb{R}^e$, which is nonzero. For all $\alpha \in [0, 1]$ and $k \geqslant 0$, the point $\underline{p}_0 + \alpha(\underline{p}_k - \underline{p}_0)$ is in Γ, because Γ is convex. Let $u > 0$ and let $\alpha_k = u / |\underline{m}_k|_\infty$. If k is large enough, then $0 \leqslant \alpha_k \leqslant 1$. Moreover

$$\underline{p}_0 + \alpha_k(\underline{p}_k - \underline{p}_0) \underset{k \to \infty}{\to} (\underline{n}_0, \underline{m}_0 + u\ell).$$

Yet Γ is closed so $(\underline{n}_0, \underline{m}_0 + u\ell) \in \Gamma$. By definition $|\underline{m}_0 + u\ell|_\infty \leqslant B(\underline{n}_0) < \infty$. This is a contradiction because $\ell \neq 0$ and u is arbitrarily large. □

Thus, for all $\underline{n} \in \mathbb{Z}^d$,

$$v_{\underline{n}} = \sum_{\substack{\underline{m} \in \mathbb{Z}^e \\ |\underline{m}| \leqslant C(1 + |\underline{n}|_\infty)}} u_{\underline{n},\underline{m}} \mathbf{1}_\Gamma(\underline{n}, \underline{m}). \tag{8}$$

For $1 \leqslant i \leqslant d$, let $w^{i,+}$ be the sequence

$$w^{i,+}_{\underline{n}} \stackrel{\text{def}}{=} H_{n_i} \prod_{j=1}^{i-1} (H_{n_i - n_j - 1} H_{n_i + n_j - 1}) \prod_{j=i+1}^{d} (H_{n_i - n_j} H_{n_i + n_j}).$$

After checking that for any $a, b \in \mathbb{Z}$, $H_{a+b} H_{a-b} = 1$ if $a \geqslant |b|$ and 0 otherwise, we see that $w^{i,+}_{\underline{n}} = 1$ if $n_i = |n|_\infty$ and $|n_j| < |n_i|$ for all $j < i$; and 0 otherwise. Likewise, let $w^{i,-}$ be the sequence

$$w^{i,-}_{\underline{n}} \stackrel{\text{def}}{=} H_{-n_i - 1} \prod_{j=1}^{i-1} (H_{-n_i - n_j - 1} H_{-n_i + n_j - 1}) \prod_{j=i+1}^{d} (H_{-n_i - n_j} H_{-n_i + n_j}),$$

so that $w^{i,-}_{\underline{n}} = 1$ if $n_i = -|n|_\infty < 0$ and $|n_j| < |n_i|$ for all $j < i$; and 0 otherwise. The $2d$ sequences $w^{i,+}$ and $w^{i,-}$ are binomial sums that partition 1: for all $\underline{n} \in \mathbb{Z}^d$

$$1 = \sum_{i=1}^{d} w^{i,+}_{\underline{n}} + \sum_{i=1}^{d} w^{i,-}_{\underline{n}}.$$

By design, the sum in Equation (8) rewrites as

$$v_{\underline{n}} = \sum_{i=1}^{d} w_{\underline{n}}^{i,+} \sum_{m_1=-C(1+n_i)}^{C(1+n_i)} \cdots \sum_{m_e=-C(1+n_i)}^{C(1+n_i)} u_{\underline{n},\underline{m}} \mathbf{1}_\Gamma(\underline{n},\underline{m})$$
$$+ \sum_{i=1}^{d} w_{\underline{n}}^{i,-} \sum_{m_1=-C(1-n_i)}^{C(1-n_i)} \cdots \sum_{m_e=-C(1-n_i)}^{C(1-n_i)} u_{\underline{n},\underline{m}} \mathbf{1}_\Gamma(\underline{n},\underline{m}),$$

which concludes the proof of Proposition 3.12, because now the summation bounds are affine functions with integer coefficients of $\underline{n}$.

From a practical point of view, summations over polyhedra can be handled in a very different way, see §6.3.

3.3. **Diagonals as binomial sums.** We now prove the second part of Theorem 3.5: the diagonal of a rational function is the generating function of a binomial sum. Let $R(z_{1\smile d})$ be a rational power series and let us prove that the sequence defined by $u_n = [z_1^n \cdots z_d^n]R$ is a binomial sum. Since the binomial sums are closed under linear combinations, it is enough to consider the case where the numerator of R is a monomial and where the constant term of its denominator is 1. (It cannot be zero since R is a power series.) Thus R has the form

$$R = \frac{\underline{z}^{\underline{m}_0}}{1 + \sum_{k=1}^{e} a_k \underline{z}^{\underline{m}_k}},$$

where the $\underline{m}_k$'s have nonnegative coordinates and $\underline{m}_k \neq 0$ if $k \neq 0$. Let $y_{0\smile e}$ be new variables and let S be the rational power series

$$S(y_{0\smile e}) \stackrel{\text{def}}{=} \frac{y_0}{1 + \sum_{k=1}^{e} a_k y_k} = y_0 \sum_{\underline{k} \in \mathbb{N}^e} \underbrace{\binom{k_1 + \cdots + k_e}{k_1, \ldots, k_e} a_1^{k_1} \cdots a_e^{k_e}}_{C_{\underline{k}}} y_1^{k_1} \cdots y_e^{k_e}.$$

The coefficient sequence $C_{\underline{k}}$ of this power series is a binomial sum because the multinomial coefficient is a product of binomial coefficients:

$$\binom{k_1 + \cdots + k_e}{k_1, \ldots, k_e} = \binom{k_1 + \cdots + k_e}{k_1} \binom{k_2 + \cdots + k_e}{k_2} \cdots \binom{k_{e-1} + k_e}{k_{e-1}}.$$

Let $\Gamma \subset \mathbb{R} \times \mathbb{R}^e$ be the rational polyhedron defined by

$$\Gamma \stackrel{\text{def}}{=} \left\{ (n, \underline{k}) \in \mathbb{R} \times \mathbb{R}^e \;\middle|\; k_1 \geqslant 0, \ldots, k_e \geqslant 0 \text{ and } \underline{m}_0 + \sum_{i=1}^{e} k_i \underline{m}_i = (n, \ldots, n) \right\}.$$

By construction $R(z_{1\smile d}) = S(\underline{z}^{\underline{m}_0}, \ldots, \underline{z}^{\underline{m}_e})$, and the image of a monomial $y_0 y_1^{k_1} \cdots y_e^{k_e}$ is a diagonal monomial $z_1^n \cdots z_d^n$ if and only if $(n, \underline{k}) \in \Gamma$. Thus

$$[z_1^n \cdots z_d^n]R = \sum_{k \in \mathbb{Z}^e} C_{\underline{k}} \mathbf{1}_\Gamma(n, \underline{k}).$$

Thanks to the positivity conditions on the $\underline{m}_k$'s, it is obvious that Γ satisfies the finiteness hypothesis of Proposition 3.12. Thus the sequence u_n is a binomial sum.

4. Computing binomial sums

Computing may have different meanings. When manipulating sequences like binomial sums, one may want, for example, to compute recurrence relations that they satisfy, to decide their equality, to compute their asymptotic behavior or to find *simple* closed-form formulas for them. The representation of the generating series of binomial sums as residues or diagonals of rational functions gives an interesting tool to tackle these goals for binomial sums, though some questions remain open.

While the customary tool to compute recurrence relations satisfied by binomial sums is *creative telescoping*, integral representations of binomial sums give another approach: given a binomial sum we first compute an integral representation (this is fast and easy, see Algorithm 1), and next we compute a differential equation satisfied by the integral, which translates immediately into a recurrence relation for the binomial sum. The *decision problem* $A = B$ can be solved by computing recurrence relations for $A - B$ and checking sufficiently many initial conditions, see §4.4 for more details.

A recurrence relation is also a good starting point for deriving the *asymptotic behavior* of a univariate binomial sum (e.g. Mezzarobba and Salvy 2010). However, some constants that determine the asymptotic behavior can be computed numerically but it is not known how to decide their nullity, which makes it difficult to catch the subdominant behavior. A more direct method is possible in many cases once the generating function has been written as the diagonal of a rational power series (Pemantle and Wilson 2013; Salvy and Melczer 2016).

The problem of *simplifying* binomial sums is still largely open: for example, given the left-hand side of Strehl's identity (see §7.2.1), it is not known how to *discover* automatically the right-hand side. The only known automatic simplification procedures consist in computing a recurrence relation and applying the algorithm of Petkovšek (1992) to find hypergeometric solutions or the algorithm of Abramov and Petkovšek (1994) to find D'Alembertian solutions. See §7.2 and §7.3 for numerous examples.

The computation of a recurrence relation for a binomial sum is done in two steps. Firstly, an integral representation is found and then a Picard-Fuchs equation is computed.

4.1. **Picard-Fuchs equations.** Let $\mathbb{L}$ be a field of characteristic zero and let $R(z_{1\smile r})$ be a rational function with coefficients in $\mathbb{L}$, written as $R = \frac{P}{F}$ where P and F are polynomials. Let A_F be the localized ring $\mathbb{L}[z_{1\smile r}, F^{-1}]$. It is known that the $\mathbb{L}$-linear quotient space

$$H_F \stackrel{\text{def}}{=} A_F / \left(\tfrac{\partial}{\partial z_1} A_F + \cdots + \tfrac{\partial}{\partial z_r} A_F \right)$$

is finite-dimensional (Grothendieck 1966). Let us assume that there is a derivation ∂ defined on $\mathbb{L}$. It extends naturally, with $\partial z_i = 0$, to a derivation on A_F that commutes with the derivations $\frac{\partial}{\partial z_i}$, so that ∂ defines a derivation on the $\mathbb{L}$-linear space H_F. Since H_F is finite-dimensional, there exist $c_{0\smile m} \in \mathbb{L}$, not all zero, such that

$$c_m \partial^m R + \cdots + c_1 \partial R + c_0 R \in \tfrac{\partial}{\partial z_1} A_F + \cdots + \tfrac{\partial}{\partial z_r} A_F.$$

Now let us assume that $\mathbb{L}$ is the field of rational functions $\mathbb{K}(t_{1\smile d})$ and that ∂ is the derivation $\frac{\partial}{\partial t_i}$ for some i. Then the operator ∂ commutes with the operator $\operatorname{res}_{z_{1\smile r}}$, as do the multiplications by elements of $\mathbb{L}$, and Lemma 2.5 implies that

$$c_m \tfrac{\partial^m}{\partial t_i^m} f + \cdots + c_1 \tfrac{\partial}{\partial t_i} f + c_0 f = 0,$$

where $f = \operatorname{res}_{z_{1\smile r}} R$, with $t_{1\smile d} \prec z_{1\smile r}$. Differential equations that arise in this way are called *Picard-Fuchs equations*.

Recall that $\mathbb{L}_d$ is the field of iterated Laurent series introduced in §2.1. A series $f(t_{1\smile d}) \in \mathbb{L}_d$ is called *differentially finite* if the $\mathbb{K}(t_{1\smile d})$-linear subspace of $\mathbb{L}_d$ generated by the derivatives $\partial^{n_1+\cdots+n_d} f/\partial t_1^{n_1}\cdots\partial t_d^{n_d}$, for $n_{1\smile d}\geqslant 0$, is finite-dimensional. In particular, a univariate Laurent series $f\in\mathbb{L}_1$ is differentially finite if and only if there exist $m\geqslant 0$ and polynomials $p_{0\smile m}\in\mathbb{K}[t]$, not all zero, such that $p_m f^{(m)}+\cdots+p_1 f'+p_0 f=0$. In that case, we say that f is solution of a differential equation of *order* m and *degree* $\max_k \deg p_k$. The above argument implies the following classical theorem of which Lipshitz (1988) gave an elementary proof.

Theorem 4.1. *For any rational function $R(t_{1\smile d}, z_{1\smile r})$ with $t_{1\smile d}\prec z_{1\smile r}$, the residue $\operatorname{res}_{z_{1\smile r}} R$ is differentially finite.*

In previous work (Bostan et al. 2013a, Theorem 12; Lairez 2014, §I.35.3), we proved the following quantitative result about the size of Picard-Fuchs equations and the complexity of their computation. We also described an efficient algorithm to compute Picard-Fuchs equations (Lairez 2016).

Theorem 4.2. *Let $R\in\mathbb{K}(t,z_{1\smile r})$ be a rational function, written as $R=\frac{P}{F}$, with P and F polynomials. Let*

$$N=\max(\deg_{z_{1\smile r}} P+r+1,\deg_{z_{1\smile r}} F)\ \text{and}\ d_t=\max(\deg_t P,\deg_t F).$$

Then $\operatorname{res}_{z_{1\smile r}} R$, with $t\prec z_{1\smile r}$, is solution of a linear differential equation of order at most N^r and degree at most $(\frac{5}{8}N^{3r}+N^r)\exp(r)d_t$. Moreover, this differential equation can be computed with $\mathcal{O}(\exp(5r)N^{8r}d_t)$ arithmetic operations in $\mathbb{K}$, uniformly in all the parameters.

4.2. **Power series solutions of differential equations.** When a power series satisfies a given linear differential equation with polynomial coefficients, one only needs a few initial conditions to determine entirely the power series.

Let $\mathcal{L}\in\mathbb{K}[t]\langle\partial_t\rangle$ be a linear differential operator in ∂_t with polynomial coefficients in t. There exists a unique $n\in\mathbb{Z}$ and a unique $b_{\mathcal{L}}\in\mathbb{K}[a]$ such that $\mathcal{L}(t^a)=b_{\mathcal{L}}(a)t^{a+n}+o(t^{a+n})$ for all $a\in\mathbb{Z}$ and $t\to 0$. The polynomial $b_{\mathcal{L}}$ is the *indicial polynomial* of $\mathcal{L}$ at $t=0$. For more details about the indicial polynomial, see *Ordinary Differential Equations* (Ince 1944). It is easy to check that if $f\in\mathbb{K}[\![t]\!]$ is annihilated by $\mathcal{L}$, then its leading monomial t^n satisfies $b_{\mathcal{L}}(n)=0$.

Proposition 4.3. *Let $f\in\mathbb{K}[\![t]\!]$. If $\mathcal{L}(f)=0$ and if $[t^n]f=0$ for all $n\in\mathbb{N}$ such that $b_{\mathcal{L}}(n)=0$, then $f=0$.*

It is worth noting that the indicial equation of a Picard-Fuchs equation is very special:

Theorem 4.4 (Katz 1970)**.** *If $\mathcal{L}$ is a Picard-Fuchs equation, then the degree of $b_{\mathcal{L}}$ equals the order of $\mathcal{L}$ and all the roots of $b_{\mathcal{L}}$ are rational.*

The data of a differential operator $\mathcal{L}$ and elements of $\mathbb{K}$ for each nonnegative integer root of $b_{\mathcal{L}}$ (that we will call here *sufficient initial conditions*) determines entirely an element of $\mathbb{K}[\![t]\!]$. It is an excellent data structure for manipulating power series (Salvy and Zimmermann 1994). For example, it lets one compute efficiently the coefficients of the power series $\sum_n u_n t^n$: the differential equation translates into a recurrence relation

$$p_r(n)u_{n+r}+\ldots+p_1(n)u_{n+1}+p_0(n)u_n=0$$

for some polynomials $p_{0\smile r}\in\mathbb{K}[n]$ and the sufficient initial conditions translate into initial conditions for the recurrence, exactly where we need them.

4.3. **Equality test for univariate differentially finite power series.** Let $f \in \mathbb{K}[\![t]\!]$ be a power series given by a differential operator $\mathcal{L} \in \mathbb{K}[t]\langle\partial_t\rangle$ such that $\mathcal{L}(f) = 0$, and by sufficient initial conditions. Let $\mathcal{M}$ be another differential operator. We may decide if $\mathcal{M}(f) = 0$ in the following way. Firstly, we compute the right g.c.d. of $\mathcal{M}$ and $\mathcal{L}$: this is the operator $\mathcal{D}$ of the largest order such that $\mathcal{M} = \mathcal{M}'\mathcal{D}$ and $\mathcal{L} = \mathcal{L}'\mathcal{D}$ for some operators $\mathcal{M}'$ and $\mathcal{L}'$ in $\mathbb{K}(t)\langle\partial_t\rangle$. Then, it is enough to compute the indicial equation $b_{\mathcal{L}'}$ and to compute $[t^n]\mathcal{D}(f)$ for each nonnegative integer root n of $b_{\mathcal{L}'}$. We will find only zeros if and only if $\mathcal{M}(f) = 0$. Indeed, $\mathcal{M}(f) = 0$ if and only if $\mathcal{D}(f) = 0$ and since $\mathcal{L}'(\mathcal{D}(f)) = 0$, we may apply Proposition 4.3 to check whether $\mathcal{D}(f) = 0$ or not.

Now, let $g \in \mathbb{K}[\![t]\!]$ be another power series given by a differential operator $\mathcal{M} \in \mathbb{K}[t]\langle\partial_t\rangle$ and sufficient initial conditions. To decide if $f = g$, it is enough to check that $\mathcal{M}(f) = 0$, with the above method, and to check that the coefficients of f and g corresponding to the nonnegative integer roots of $b_{\mathcal{M}}$ are the same.

4.4. **Equality test for binomial sums.** Let $u : \mathbb{Z} \to \mathbb{K}$ be a binomial sum. Up to considering separately the binomial sums $H_n u_n$ and $H_n u_{-n}$, it is enough to look at the case where $u_n = 0$ for $n < 0$. In this case u is entirely determined by its generating function $f = \sum_{n\geqslant 0} u_n t^n$. Using Algorithm 1 we obtain an integral representation of f, and then, as explained in §4.1, we obtain a differential operator $\mathcal{L}$ that annihilates f. Since u is a binomial sum given explicitly, we can compute sufficient initial conditions. Given another binomial sum $v : \mathbb{N} \to \mathbb{K}$, we can check that $u = v$ by computing a differential operator annihilating the generating function g of v together with sufficient initial conditions and by checking that $f = g$ with the method explained in §4.3.

Let us now consider the multivariate case. As above, we can reduce the equality test for binomial sums $\mathbb{Z}^d \to \mathbb{K}$ to the equality test for binomial sums $\mathbb{N}^d \to \mathbb{K}$. And this equality test can be reduced to the equality test for binomial sums $\mathbb{N}^{d-1} \to \mathbb{K}$ as follows. Let $u : \mathbb{N}^d \to \mathbb{K}$ be a binomial sum. It is determined by its generating function $f(t_{1\smile d}) = \sum_{\underline{n}\in\mathbb{N}^d} u_{\underline{n}} \underline{t}^{\underline{n}} \in \mathbb{L}_d$. With Algorithm 1, we compute a rational function $R(t_{1\smile d}, z_{1\smile r})$ such that $f = \operatorname{res}_{z_{1\smile r}} f$. Let $\mathbb{K}'$ be the field $\mathbb{K}(t_{1\smile d-1})$. Following §4.1, we can compute an operator $\mathcal{L} \in \mathbb{K}'[t_d]\langle\partial_{t_d}\rangle$ such that $\mathcal{L}(f) = 0$. This gives a differential equation for f considered as a power series in t_d with coefficients in $\mathbb{L}_{d-1}$. The sufficient initial conditions to determine f are given by the power series $\sum_{\underline{n}\in\mathbb{N}^{d-1}} u_{\underline{n},k} \underline{t}^{\underline{n}}$, where k ranges over the nonnegative integer roots of $b_{\mathcal{L}}$. These power series are the generating functions of binomial sums in $d-1$ variables. This reduces the equality test for binomial sums in d variables to the equality test for binomial sums in $d-1$ variables.

5. Geometric reduction

Putting into practice the computation of binomial sums through integral representations shows immediately that the number of integration variables is high and makes the computation of the Picard-Fuchs equations slow. However, the rational functions obtained this way are very peculiar. For example, their denominator often factors into small pieces. This section presents a sufficient condition under which a residue $\operatorname{res}_v F$ of a rational function F is rational. This leads to rewriting an iterated residue of a rational function, like the ones given by Corollary 2.7, into another one with one or several variables less. This simplification procedure is very efficient on the residues we are interested in and reduces the number of variables significantly.

Conceptually the simplification procedure is simple: in terms of integrals it boils down to computing partial integrals in specific cases where we know that they are rational. The rational nature of a partial integral depends on the integration cycle and integration algorithms usually forget about this cycle. Instead, they compute the Picard-Fuchs equations—see §4.1—that annihilate a given integral for *any* integration cycle. In our setting, the integration cycle underlies the notion of residue—see §2.5. This provides a symbolic treatment that we call *geometric reduction* since it takes into account the geometry of the cycle and decreases the number of variables for which the general integration algorithm above is actually needed. The time required by the computation is dramatically reduced by this symbolic treatment.

5.1. **Rational poles.** Let us consider variables v and $z_{1\smile d}$, where v can appear anywhere in the variable ordering. Let $F(v, z_{1\smile d}) = a/f$ be a rational function. In general, $\operatorname{res}_v F$ is not a rational function—except if v is the smallest variable, see §2.3.

Let $\rho \in \mathbb{L}_d$ be a power series in the variables $z_{1\smile d}$. The *classical* residue of F at $v = \rho$, denoted $\operatorname{Res}_{v=\rho} F$, is the coefficient of $1/(v-\rho)$ in the partial fraction decomposition of F as an element of $\mathbb{L}_d(v)$. Contrary to the *formal* residue $\operatorname{res}_v F$, the classical residue is always in the field generated by ρ and the coefficients of F. Similarly to the formal residues, the classical residues of a derivative $\partial G/\partial v$ are all zero.

Classical residues can be computed in a simple way: if ρ is not a pole of F, then $\operatorname{Res}_{v=\rho} F = 0$; if ρ is a pole of order 1, then $\operatorname{Res}_{v=\rho} F = ((v-\rho)F)\,|_{v=\rho}$; and if ρ is a pole of order $r > 1$, then

$$\operatorname*{Res}_{v=\rho} F = \frac{1}{(r-1)!} \left(\frac{\partial^{r-1}(v-\rho)^r F}{\partial^{r-1} v} \right)\Bigg|_{v=\rho}.$$

In its simplest form, the geometric reduction applies when f factors over $\mathbb{K}(z_{1\smile d})$ as a product of factors of degree 1:

$$f = C(z_{1\smile d}) \prod_{\rho \in U} (v-\rho)^{n_\rho},$$

where U is a finite subset of $\mathbb{K}(z_{1\smile d})$. Then the partial fraction decomposition of F writes as

$$F = \sum_{\rho \in U} \left(\frac{a_\rho}{v-\rho} + \sum_{k>1} \frac{b_{\rho,k}}{(v-\rho)^k} \right) + P(v),$$

where $a_\rho \in \mathbb{K}(z_{1\smile d})$ is $\operatorname{Res}_{v=\rho} F$, where $b_{\rho,k} \in \mathbb{K}(z_{1\smile d})$ and where $P \in \mathbb{K}(z_{1\smile d})[v]$. The terms with multiple poles and the polynomial term are derivatives and thus their formal residue with respect to v are zero. Hence

$$\operatorname*{res}_v F = \sum_{\rho \in U} \operatorname*{res}_v \left(\frac{a_\rho}{v-\rho} \right).$$

Let $\rho \in U$. Depending on the leading monomial $\operatorname{lm}(\rho)$ of ρ, as an element of $\mathbb{L}_d$, two situations may occur. Either $\operatorname{lm}(\rho) \prec v$, in which case

$$\frac{a_\rho}{v-\rho} = \frac{a_\rho}{v} \sum_{n=0}^{\infty} \left(\frac{\rho}{v} \right)^n,$$

and $\operatorname{res}_v \frac{a_\rho}{v-\rho} = a_\rho$; or $\operatorname{lm}(\rho) \succ v$, in which case

$$\frac{a_\rho}{v-\rho} = -\frac{a_\rho}{\rho}\sum_{n=0}^{\infty}\left(\frac{v}{\rho}\right)^n,$$

hence $\operatorname{res}_v \frac{a_\rho}{v-\rho} = 0$. Since the variable v does not appear in ρ, the equality $\operatorname{lm}(\rho) = v$ cannot happen. In the end, we obtain that

$$\operatorname*{res}_v F = \sum_{\rho\in U}[\operatorname{lm}(\rho) \prec v] \operatorname*{Res}_{v=\rho} F, \tag{9}$$

where the bracket is 1 if the inequality inside is true and 0 otherwise. In particular, the right-hand side is a rational function that we can compute.

Example 5. Let $d > 0$ be an integer and let us consider the binomial sum

$$u_n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{dk}{n}.$$

We will show that $u_n = (-d)^n$. This example is interesting because the geometric reduction procedure alone is able to compute entirely the double integral representing the generating function of u, whereas Zeilberger's algorithm finds a recurrence relation of order $d-1$ (Paule and Schorn 1995), far from the minimal recurrence relation $u_{n+1} + du_n = 0$.

Algorithm 1 computes that

$$\sum_{n\geqslant 0} u_n t^n = \operatorname*{res}_{z_1,z_2} \frac{z_2}{(z_2 - t(1+z_1))(z_1 z_2 + t(1+z_1)(1+z_2)^d)},$$

with $t \prec z_1 \prec z_2$. Let F denote the rational function on the right-hand side. Each factor of the denominator of F has degree 1 with respect to z_1. Thus Equation (9) applies. The roots of the denominator are

$$\rho_1 = \frac{z_2}{t} - 1 \quad \text{and} \quad \rho_2 = \frac{-1}{1+\frac{z_2}{t(1+z_2)^d}},$$

moreover $\operatorname{lm}(\rho_1) = z_2/t \succ z_1$ and $\operatorname{lm}(\rho_2) = t/z_2 \prec z_1$. Thus $\operatorname{res}_{z_1} F = \operatorname{Res}_{z_1=\rho_2} F$ and we obtain

$$\sum_{n\geqslant 0} u_n t^n = \operatorname*{res}_{z_2}\operatorname*{res}_{z_1} F = \operatorname*{res}_{z_2} \frac{1}{t(1+z_2)^d + z_2 - t}.$$

If $d > 2$, the denominator of the latter rational function does not split into factors of degree 1 and Equation (9) does not apply. However, the study of nonrational poles can lead to a further reduction.

5.2. **Arbitrary poles.** Equation (9) extends to the general case. To describe this generalization, we need an algebraic closure of $\mathbb{L}_d$. Let $\mathbb{L}_{d,N}$ be the field

$$\mathbb{L}_{d,N} \stackrel{\text{def}}{=} \mathbb{K}(\!(z_d^{1/N})\!)(\!(z_{d-1}^{1/N})\!)\dots(\!(z_2^{1/N})\!)(\!(z_1^{1/N})\!).$$

It is the algebraic extension of $\mathbb{L}_d$ generated by the $z_i^{1/N}$. Let $\mathbb{L}_{d,\infty}$ be the union of all $\mathbb{L}_{d,N}$, $N > 0$, and let $\overline{\mathbb{K}} \otimes_{\mathbb{K}} \mathbb{L}_{d,\infty}$ be the compositum of $\mathbb{L}_{d,\infty}$ and $\overline{\mathbb{K}}$, where $\overline{\mathbb{K}}$ is an algebraic closure of $\mathbb{K}$. The following is classical (e.g. Rayner 1974).

Lemma 5.1 (Iterated Puiseux theorem)**.** *The field $\overline{\mathbb{K}} \otimes_{\mathbb{K}} \mathbb{L}_{d,\infty}$ is an algebraic closure of $\mathbb{L}_d$.*

The field $\overline{\mathbb{K}} \otimes_{\mathbb{K}} \mathbb{L}_{d,\infty}$ is thus simply denoted $\overline{\mathbb{L}_d}$. The valuation defined on $\mathbb{L}_d$ is extended to a valuation defined on $\overline{\mathbb{L}_d}$ with values in the group $\mathbb{Q}^d$, ordered lexicographically. The leading monomial $\operatorname{lm}(\rho)$ of an $f \in \overline{\mathbb{L}_d}$ is also defined as $\underline{z}^{v(f)}$. The argument of §5.1 applies and shows that

$$\operatorname*{res}_v F = \sum_{\rho \text{ pole of } F} [\operatorname{lm}(\rho) \prec v] \operatorname*{Res}_{v=\rho} F, \tag{10}$$

where this time, the poles are in $\overline{\mathbb{L}_d}$. A root ρ is called *small* if $\rho = 0$ or $\operatorname{lm}(\rho) \prec v$ and *large* otherwise.

Example 6. Let $F = \frac{1}{xy(y^2+y-x)}$, with $x \prec y$. With respect to y, the poles of F are 0, and

$$\rho_1 = -\frac{1}{2} + \frac{\sqrt{1+4x}}{2} \quad \text{and} \quad \rho_2 = -\frac{1}{2} - \frac{\sqrt{1+4x}}{2}.$$

Only 0 and ρ_1 are small. Thus

$$\operatorname*{res}_y F = \operatorname*{Res}_{y=0} F + \operatorname*{Res}_{y=\rho_1} F = -\frac{1}{x^2} + \frac{2}{x + 4x^2 - x\sqrt{1+4x}}.$$

Equation (10) does not look as interesting as Equation (9) because the right-hand side is algebraic and need not be a rational function. However, if all roots are large, then the residue is zero, which is interesting. On the contrary, if they are all small, then $\operatorname{res}_v F$ is the sum of all the classical residues of F, which equals the residue at infinity:

$$\operatorname*{res}_v F = \operatorname*{Res}_{v=\infty} F \stackrel{\text{def}}{=} \operatorname*{Res}_{v=0} \left(-\frac{1}{v^2} F|_{v \leftarrow 1/v} \right),$$

which is a rational function. Thus, in the case where the poles are either all small or all large, Equation (10) shows that the residue $\operatorname{res}_v F$ is rational and shows how to compute it.

Actually it is enough to check that any two conjugate poles (two poles are conjugate if they annihilate the same irreducible factor of the denominator of F) are simultaneously large or small. Indeed, we can write $f = \prod_{k=1}^r f_k^{n_k}$ where $f_{1\smile r}$ are irreducible polynomials in $\mathbb{K}(z_{1\smile d})[v]$, and the partial fraction decomposition leads to

$$F = \sum_{k=1}^r \frac{a_k}{f_k^{n_k}}$$

for some polynomials $a_{1\smile r}$. Equation (10) applies to each term of this sum. If U_k denotes the set of all the roots of f_k, then

$$\operatorname*{res}_v F = \sum_{k=1}^r \sum_{\rho \in U_k} [\operatorname{lm}(\rho) \prec v] \operatorname*{Res}_{v=\rho} \frac{a_k}{f_k^{n_k}},$$

and we can apply the *all large or all small* criterion to each subsum separately.

Proposition 5.2. *With the notations above, if for all k, there exists $\varepsilon_k \in \{0,1\}$ such that $[\operatorname{lm}(\rho) \prec v] = \varepsilon_k$ for all $\rho \in U_k$, then*

$$\operatorname*{res}_v F = \sum_{k=1}^r \varepsilon_k \operatorname*{Res}_{v=\infty} \frac{a_k}{f_k^{n_k}}.$$

It remains to explain how to compute the leading monomials of the roots of a polynomial $f \in \mathbb{K}(z_{1\smile d})[v]$. If $d = 1$, then it is the classical method of Newton's polygon for the resolution of bivariate polynomial equations using Puiseux series (e.g. Walker 1950; Cutkosky 2004). When $d \geqslant 2$, we may apply this method recursively by considering $\overline{\mathbb{L}_d}$ as a subfield of the field of Puiseux series with coefficients in $\overline{\mathbb{L}_{d-1}}$. Based on this idea, Algorithm 3 proposes an implementation which avoids all

Algorithm 3. Computation of the *all large or all small* criterion

Input: $S \subset \mathbb{N}^d$ finite; an integer $1 \leqslant k \leqslant d$.

Output: A subset of $\{\text{IN}, \text{OUT}\}$.

Specification: Let $P = \sum_{\underline{n}\in\mathbb{N}^d} a_{\underline{n}} z^{\underline{n}} \in \mathbb{K}[z_{1\smile d}]$ be a polynomial. If $S = \{\underline{n} \in \mathbb{N}^d \mid a_{\underline{n}} \neq 0\}$ then $\textsc{AllLargeOrAllSmall}(S, k)$ contains OUT (resp. IN) if and only if there exists a nonzero $\rho \in \overline{\mathbb{L}_d}$ in which z_k does not appear such that $P(z_1, \dots, z_{k-1}, \rho, z_{k+1}, \dots, z_d) = 0$ and $\operatorname{lm}(\rho) \succ z_k$ (resp. $\operatorname{lm}(\rho) \prec z_k$).

function $\textsc{AllLargeOrAllSmall}(S, k)$
  if $\max_{m\in S} m_k = \min_{m\in S} m_k$ **then return** $\varnothing$
  if $k = 1$ **then return** $\{\text{OUT}\}$
  $\mu \leftarrow \min_{m\in S} m_1$
  $M \leftarrow \left\{(m_{2\smile n}) \in \mathbb{N}^{d-1} \;\middle|\; m \in S \text{ and } m_1 = \mu\right\}$
  $r \leftarrow \textsc{AllLargeOrAllSmall}(M, k-1)$
  if $\max_{m\in S} m_k > \max_{m\in M} m_k$ **then** $r \leftarrow r \cup \{\text{OUT}\}$
  if $\min_{m\in S} m_k < \min_{m\in M} m_k$ **then** $r \leftarrow r \cup \{\text{IN}\}$
  return r

technicalities. Algorithm 4 sums up the geometric reduction procedure with respect to the variable v. Of course one should apply the procedure iteratively with every variable until no further reduction is possible. The success of the reduction may depend on the order in which we eliminate all the variables. It is possible to try every possible order efficiently thanks to the following compatibility property: if both $\textsc{GeomRed}(\textsc{GeomRed}(R, u), v)$ and $\textsc{GeomRed}(\textsc{GeomRed}(R, v), u)$ do not fail, then they are both equal to $\operatorname{res}_{u,v}(R)$.

Example 7. Continuing Example 5, let us consider $\operatorname{res}_z F_1$, where $F_1 = 1/(t(1+z)^d + z - t)$, with $t \prec z$. The denominator factors as zg, where $g = 1 + t\sum_{k=0}^{d-1}\binom{d}{k+1} z^k$. Thus, the leading monomial of any root of f is $t^{-\frac{1}{d-1}}$, which it not $\prec z$. All the roots of g are large, so only the pole $z = 0$ remains and $\operatorname{res}_z F_1 = \operatorname{Res}_{z=0} F_1$, by Proposition 5.2. Thus $\operatorname{res}_z F_1 = \frac{1}{1+dt}$ and it follows that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{dk}{n} = (-d)^n.$$

If we computed the Picard-Fuchs equation of F_1, we would find a differential equation of order d because the Picard-Fuchs operator annihilates *all* the periods, and the periods associated to the roots of g are algebraic of degree $d - 1$.

6. Optimizations

6.1. **Infinite sums.** So far, we have only considered binomial sums in which the bounds of the $\sum$ symbols are finite and explicit. It is possible and desirable to consider infinite sums, or more exactly *syntactically* infinite sums which in fact reduce to finite sums but whose summation bounds are implicit. This often leads to simpler integral representations.

For example, in the sum $\sum_{k=0}^{n}\binom{n}{k}$, the upper summation bound n is not really useful when $n \geqslant 0$, we could as well write $\sum_{k=0}^{\infty}\binom{n}{k}$, which defines the same sequence. It is possible to adapt Algorithm 1 to handle infinite sums as long as the underlying summations in the field of iterated Laurent series are convergent. Recall that a

Algorithm 4. Geometric reduction

Input: $F(z_{1\smile d})$, a rational function; an integer $1 \leqslant k \leqslant d$.
Output: FAIL or a rational function $S(z_{1\smile k-1}, z_{k+1\smile d})$.
Specification: If a rational function S is returned and then $\operatorname{res}_{z_k} F = S$.

function GEOMRED(F, k)
  Decompose F as $\sum_{i=1}^{r} a_i/f_i^{n_i} + P(v)$ where the f_i's are irreducible polynomials in $\mathbb{K}[z_{1\smile d}]$ and $a_{1\smile r}, P \in \mathbb{K}(z_{1\smile k-1}, z_{k+1\smile d})[z_k]$.
  $S \leftarrow 0$
  for i from 1 to r **do**
    $\tau \leftarrow$ ALLLARGEORALLSMALL ($\{$exponents of the monomials of $f_i\}, k$)
    if $\tau \subset \{\text{IN}\}$ **then** $S \leftarrow \operatorname{Res}_{z_k=\infty}(a_i/f_i^{n_i})$
    if $\tau = \{\text{IN}, \text{OUT}\}$ **then return** FAIL
  return S

geometric sum $\sum_{n\geqslant 0} f^n$ converges in the field $\mathbb{L}_d$ if and only if $\operatorname{lm}(f) \prec 1$, see Lemma 2.1.

To compute an integral representation of binomial sums involving infinite summations, the principle is to proceed as in §2.2 except when an infinite geometric sum $\sum_{n\geqslant 0} f$ in $\mathbb{L}_d$ shows up, where we check that $\operatorname{lm}(f) \prec 1$ so that Lemma 2.1 applies. If it does, then the summation is performed and the computation continues. If it does not, then the binomial sum is simply rejected.

Example 8. Consider the binomial sum $u_n = \sum_{k=0}^{\infty} \binom{n}{k}$, for $n \geqslant 0$. Note that Algorithm 1 applied to $\sum_{k=0}^{n} \binom{n}{k}$ returns

$$\sum_{k=0}^{n} \binom{n}{k} = [1] \left((1+z)^n \frac{z - z^{-n}}{z-1} \right).$$

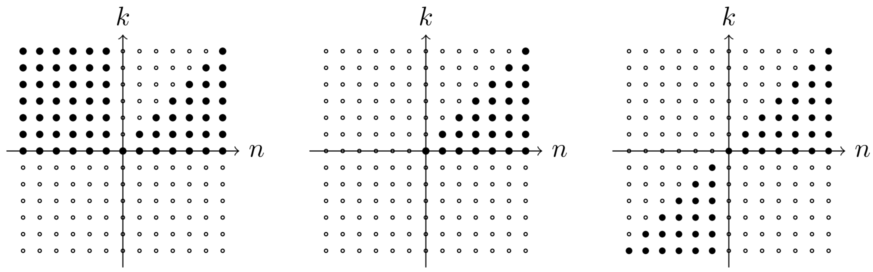
We proceed as in §2.2 except that infinite geometric sums in $\mathbb{L}_d$ are valid when Lemma 2.1 applies. Firstly $\binom{n}{k} = [1](1+z)^n z^{-k}$. Then we consider the infinite sum $\sum_{k=0}^{\infty} (1+z)^n z^{-k}$. Since $1/z \succ 1$, it does not converge, so the binomial sum is rejected. And indeed, when $n < 0$ the sum $\sum_{n=0}^{\infty} \binom{n}{k}$ has infinitely many non zero terms. If we change $\binom{n}{k}$ into $\binom{n}{n-k}$, which is the same when $n \geqslant 0$, we obtain

$$u_n = \sum_{k=0}^{\infty} [1](1+z)^n z^{k-n-1} = [1] \sum_{k=0}^{\infty} \frac{1}{z} \left(1 + \frac{1}{z}\right)^n z^k = [1] \frac{1}{z(1-z)} \left(1 + \frac{1}{z}\right)^n,$$

where this time the sum converges.

6.2. **Building blocks.** Besides the binomial coefficient $\binom{n}{k} = [1](1+z)^n z^{-k}$, we have found useful to have additional building blocks to extend Algorithms 1 and 2 that contruct integral representations. Without enlarging the class of binomial sums, one can add any sequence of the form $\underline{n} \mapsto [1] R_0 R_1^{n_1} \cdots R_d^{n_d}$, where $R_{0\smile d}$ are rational functions, as a new building block. Judicious extra building blocks may give simpler integral representations or speed up the computation. For example, when working with Motzkin numbers

$$M_n \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} (-1)^{n-k} \binom{n}{k} \left(\binom{2k+2}{k+1} - \binom{2k+2}{k+2} \right),$$



(a) $\binom{n}{k}$ *(b)* $\binom{n}{k}'$ *(c)* $\binom{n}{k}''$

Figure 2. Supports of some variants of the binomial coefficient.

one may add the new building block

$$M_n = [1]\frac{(1-x)(1+x)^2}{x}\left(\frac{1+x+x^2}{x}\right)^n.$$

In the applications below, we made use of alternative definitions of the binomial coefficient and their corresponding representations:

$$\binom{n}{k}' \stackrel{\text{def}}{=} H_n H_k \binom{n}{k} = [1]\frac{x^{k-n}y^{-k}}{1-x-y}, \tag{11}$$

$$\binom{n}{k}'' \stackrel{\text{def}}{=} \binom{n}{n-k} = [1]\frac{1}{(1-z)^{k+1}z^{n-k}}. \tag{12}$$

They differ from the binomial coefficient, as defined in §1, only when $n<0$ and agree on their non zero values (see also Figure 2).

6.3. **Summation over a polyhedron.** As in §3.2, let $\Gamma \subset \mathbb{R}^{d+e}_{\geqslant}$ be a rational polyhedron and $u : \mathbb{Z}^{d+e} \to \mathbb{K}$ be a binomial sum. Let us consider the sequence

$$v : \underline{n} \in \mathbb{N}^d \mapsto \sum_{\underline{m}\in\mathbb{Z}^e} u_{\underline{n},\underline{m}} \mathbf{1}_\Gamma(\underline{n},\underline{m}),$$

where $\mathbf{1}_\Gamma(\underline{n},\underline{m}) = 1$ if $(\underline{n},\underline{m}) \in \Gamma$ and 0 otherwise. Proposition 3.12 shows that v is a binomial sum under an additional finiteness hypothesis on Γ, and according to Corollary 2.7, there exists a rational function $R(t_{1\smile d}, z_{1\smile r})$ such that

$$\sum_{\underline{n}\in\mathbb{N}^d} v_{\underline{n}} \underline{t}^{\underline{n}} = \operatorname*{res}_{z_{1\smile r}} R.$$

It is possible to circumvent the construction used in the proof of Proposition 3.12 and compute directly such a rational function R given two ingredients: firstly, the generating function of Γ

$$\varphi_\Gamma(t_{1\smile d}, s_{1\smile e}) \stackrel{\text{def}}{=} \sum_{(\underline{n},\underline{m})\in\mathbb{Z}^{d+e}} \mathbf{1}_\Gamma(\underline{n},\underline{m})\underline{t}^{\underline{n}}\underline{s}^{\underline{m}},$$

which is known to be a rational function (e.g. Brion 1988); and secondly, a representation of the binomial sum u as

$$u_{\underline{n},\underline{m}} = [1]RT_1^{n_1}\cdots T_d^{n_d}S_1^{m_1}\cdots S_e^{m_e}$$

for some rational functions R, $T_{1\smile d}$ and $S_{1\smile e} \in \mathbb{K}(z_{1\smile r})$. Then, with $t_{1\smile e} \prec z_{1\smile d}$,

$$\sum_{\underline{n}\in\mathbb{N}^d} v_{\underline{n}} \underline{t}^{\underline{n}} = [1]R \sum_{(\underline{n},\underline{m})\in\mathbb{Z}^{d+e}} \mathbf{1}_\Gamma(\underline{n},\underline{m})(t_1T_1)^{n_1}\cdots(t_dT_d)^{n_d} S_1^{m_1}\cdots S_e^{m_e}$$
$$= \operatorname*{res}_{z_{1\smile r}} \frac{R\cdot\varphi_\Gamma(t_1T_1,\dots,t_dT_d,S_1,\dots,S_d)}{z_1\cdots z_d},$$

provided that the sums converge in $\mathbb{L}_{d+e}$. This method is interesting because it is known how to compute efficiently compact representations of the rational function φ_Γ (Barvinok 2008).

7. Applications

7.1. **Andrews-Paule identity.**

We detail the proof of the following identity:

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{i+j}{j}^2\binom{4n-2i-2j}{2n-2i} = (2n+1)\binom{2n}{n}^2. \tag{13}$$

This identity appeared first as a problem in the *American Mathematical Monthly* (Blodgett 1990) and was subsequently proved by Andrews and Paule (1993) and Wegschaider (1997) using various tools from the method of creative telescoping. It attracted attention because the theory of creative telescoping was unable to give a complete automated proof at that time.

Let u_n denote the left-hand side. It can be written as an infinite sum

$$u_n = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{i+j}{j}'^2\binom{(2n-2i)+(2n-2j)}{2n-2i}',$$

where $\binom{n}{k}'$ is the *natural* binomial defined by $\binom{n}{k}' = H_nH_k\binom{n}{k}$, see §6.2. Of course, we could have stuck to the former definition and used finite sums, but while the natural binomials introduce two variables instead of one in the integral representation—see Equation (11)—the integral representation obtained after the geometric reduction step is often simpler when using the natural binomial.

So we obtain the following integral representation:

$$\sum_{n\geqslant 0} u_n t^n = \operatorname*{res}_{z_{1\smile 6}} \frac{z_1z_2}{(z_1^2z_2^2-t)(z_4z_6-z_1^2)(z_3z_5-z_2^2)(1-z_3-z_4)(1-z_5-z_6)(1-z_1-z_2)},$$

with the ordering $t \prec z_1 \prec z_2 \prec z_3 \prec z_4 \prec z_5 \prec z_6$. As expected, each binomial coefficient brings two extra variables so we end up with six variables in addition to the parameter t.

Geometric reduction applies successively with respect to the variables z_1, z_3, z_4 and z_5. For example, the poles w.r.t. the variable z_1, are $\{\pm tz_2^{-1/2}\}$, $\{\pm z_4^{1/2}z_6^{1/2}\}$ and $\{1-z_2\}$, gathered by conjugacy classes. The first pair of conjugate poles is $\prec z_1$ whereas the second one is $\succ z_1$. The rational root $1-z_2$ is $\succ z_1$. Thus, by application of Proposition 5.2,

$$\sum_{n\geqslant 0} u_n t^n = \operatorname*{res}_{z_{2\smile 6}} \frac{(1-z_2)z_2^3}{(t-(1-z_2)^2z_2^2)\,(1-z_3-z_4)\,(z_2^2-z_3z_5)\,(1-z_5-z_6)\,(t-z_2^2z_4z_6)}.$$

In the end, repeated application of Algorithm 4 leads to

$$\sum_{n\geqslant 0} u_n t^n = \operatorname*{res}_{z_4,z_6} \frac{(z_6-1)z_6^3}{(t-z_6^2(z_6-1)^2)((z_4-1)t-z_4z_6^2(z_6^2+z_4-1))}.$$

Using the algorithm of Lairez (2016), we obtain (in about one second) a differential operator annihilating $\sum_{n\geqslant 0} u_n t^n$:

$$\begin{aligned}16t^4(256t^2+736t+81)(16t-1)^2\partial_t^6 \\ +16t^3(16t-1)(86016t^3+256256t^2+20976t-1053)\partial_t^5 \\ +4t^2(36601856t^4+113760256t^3+6103168t^2-908088t+14823)\partial_t^4 \\ +16t(22691840t^4+75716608t^3+6677824t^2-459552t+3645)\partial_t^3 \\ +(305827840t^4+1109626112t^3+139138736t^2-4247073t+9720)\partial_t^2 \\ +(60272640t^3+244005120t^2+42117840t-374625)\partial_t+691200t^2+3369600t+996300\end{aligned}$$

The roots of the indicial equation are 0, 1, $-\frac{1}{2}$ et $\frac{1}{2}$. This differential operator corresponds to the Picard-Fuchs equation associated to the integral representation, but it is not the minimal-order operator annihilating $\sum_{n\geqslant 0} u_n t^n$. This may happen because the differential operator that we compute annihilates more that simply the residue in which we ar interested: it annihilates every period of the integral of the rational function inside the residue. Of course, this is not a issue as long as we do obtain a differential equation.

Concerning the right-hand side, we find the integral representation

$$\sum_{n\geqslant 0}(2n+1)\binom{2n}{n}^2 t^n = \operatorname*{res}_{z_1,z_2} \frac{t+u_2(u_2-1)u_1(u_1-1)}{(t-u_2(u_2-1)u_1(u_1-1))^2},$$

which cannot be simplified further with the geometric reduction. We compute (in about 0.1s) the annihilating operator: $t(16t-1)\partial_t^2+(48t-1)\partial_t+12$. The Andrews-Paule identity follows with the equality test described in §4.2.

In this case the right-hand side is a hypergeometric sequence, so it can be *discovered* automatically: the differential equation of order 6 leads to a recurrence relation of order 4 for u_n from which the algorithm of Petkovšek (1992) finds the hypergeometric solutions and the initial conditions are enough to identify the right-hand side.

7.2. **Several known identities.** This section shows the integral representations and the Picard-Fuchs equations appearing in the proofs of known identities. The integral representations have been obtained with the method presented in this article and the variants presented in §6. Note that the computation of the annihilating operator never takes longer than 4 seconds.

7.2.1. *Strehl (1994).* $\displaystyle\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2 = \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\sum_{j=0}^{k}\binom{k}{j}^3.$

This identity relates the Apéry numbers (left) and Franel numbers (the inner sum in the right-hand side).

$$\text{g.f.l.h.s}^2 = \operatorname*{res}_{z_{1\smile 3}} \frac{1}{(1-z_1)(1-z_2)(1-z_3)z_1z_2z_3 - t(z_1+z_2z_3-z_1z_2z_3)}$$

$$\text{g.f.r.h.s}^3 = \operatorname*{res}_{z_{1\smile 3}} \frac{1}{(1-z_1)(1-z_2)(1-z_3)z_1z_2z_3 - t(1-z_3-z_2(1-(2+z_1(1-z_2)(1-z_3))z_3))}$$

$$\text{ann. op.}^4 = t^2(t^2-34t+1)\partial_t^3+3t(2t^2-51t+1)\partial_t^2+(7t^2-112t+1)\partial_t+t-5$$

[2]Generating function of the left-hand side
[3]Generating function of the right-hand side
[4]Annihilating operator of both right and left-hand sides

7.2.2. *Graham et al. (1989, p. 33) and Wegschaider (1997, §5.7.6).*

$$\sum_{r\geqslant 0}\sum_{s\geqslant 0}(-1)^{n+r+s}\binom{n}{r}\binom{n}{s}\binom{n+s}{s}\binom{n+r}{r}\binom{2n-r-s}{n}=\sum_{k\geqslant 0}\binom{n}{k}^4$$

$$\text{g.f.l.h.s} = \operatorname*{res}_{z_1 \smile 3} \frac{1}{(1-z_1)(1-z_2)(1-z_3)z_1z_2z_3+(z_2-z_3)(z_1-z_3)t}$$

$$\text{g.f.r.h.s} = \operatorname*{res}_{z_1 \smile 3} \frac{1}{(1-z_1)(1-z_2)(1-z_3)z_1z_2z_3+(1-z_1-z_2)z_3t-(1-z_1)(1-z_2)t}$$

$$\text{ann. op.} = t^2(4t+1)(16t-1)\partial_t^3+3t(128t^2+18t-1)\partial_t^2 t+(444t^2+40t-11)\partial_t+60t+2$$

7.2.3. *Dent.* The following identity is due to Dent and used as an example by Wegschaider (1997, p. 90):

$$\sum_{k=0}^{n_1+2n_2}\sum_{j\geqslant 0}(-1)^j\binom{k}{j}\binom{2n_2+n_1-k}{2n_2-j}\binom{n_1}{k-j}=2^{n_1}\binom{n_1+n_2}{n_1}, \text{with } n_1,n_2\geqslant 0.$$

$$\text{g.f.l.h.s} = \text{g.f.r.h.s} = \frac{1}{1-2t_1-t_2}$$

Here, the generating series is rational and geometric reduction performs the entire computation, there is no need to compute a Picard-Fuchs equation.

7.2.4. *Dixon (1891).* $\displaystyle\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^3=(-1)^n\frac{(3n)!}{n!^3}$

$$\text{g.f.l.h.s} = \operatorname*{res}_{z_1 \smile 2} \frac{(1-z_2)(1-z_1)z_1z_2}{z_1^2z_2^2(1-z_2)^2(1-z_1)^2-(1-z_1-z_2)^2t}$$

$$\text{g.f.r.h.s} = \operatorname*{res}_{z_1 \smile 3} \frac{1}{t+z_1z_2(1-z_1-z_2)}$$

$$\text{ann. op.} = t(27t+1)\partial_t^2+(54t+1)\partial_t+6$$

7.2.5. *Moriarty (see Egorychev 1984, p. 11).*

$$\sum_{k=n}^{m}(-4)^k\binom{k}{m}\frac{n}{n+k}\binom{n+k}{2k}=(-1)^n 4^m\frac{n}{n+m}\binom{n+m}{2m}$$

Because of the division by $n+m$, the right-hand side is not obviously a binomial sum. However, it becomes obvious after observing that

$$\frac{n}{n+k}\binom{n+k}{2k}=\binom{n+k-1}{2k}+\frac{1}{2}\binom{n+k-1}{2k-1}.$$

$$\text{g.f.l.h.s} = \text{g.f.r.h.s} = \frac{1}{2}\frac{(1-t_1)(1+t_1)}{t_1^2+4t_1t_2+2t_1+1}$$

Here again it is a rational power series and the geometric reduction finds it.

7.2.6. *Davletshin et al. (2015, Theorem 1.2).*

$$1+\sum_{q=1}^{\infty}2^{q-1}\binom{n_2}{q}\left(\sum_{m=1}^{n_1/2}\binom{m-1}{q-1}+\sum_{m=1}^{n_1}\binom{m-1}{q-1}\right)$$

$$=\sum_{q=1}^{\infty}2^{q-1}\binom{n_2}{q}\left(\binom{n_1}{q}+\binom{\lfloor n_1/2\rfloor}{q}\right)$$

$$\text{g.f.l.h.s} = \text{g.f.r.h.s} = \frac{t_1t_2(1+t_1-t_2-t_1t_2-2t_1^2-2t_1^2t_2)}{(1-t_2)(1-t_1)(1-t_2-t_1^2t_2-t_1^2)(1-t_1-t_2-t_1t_2)}$$

The summation bound $n_1/2$ and the integer part $\lfloor n_1/2 \rfloor$ may look problematic, until we observe that

$$\sum_{m=1}^{n/2}\binom{m-1}{q-1} = \sum_{m=1}^{\infty}\binom{m-1}{q-1}H_{n-2m}$$
$$\text{and}\quad \binom{\lfloor n/2\rfloor}{q} = \sum_{k=0}^{\infty}\binom{k}{q}\left(\delta_{2k-n}+\delta_{2k+1-n}\right),$$

using the binomial sums δ and H defined in §1.

7.2.7. *Davletshin et al. (2015, Theorem 1.1).*

$$2+\sum_{q=1}^{\infty}\binom{n_2}{q}\left(\sum_{m=1}^{n_1/2}\binom{m-1}{q-1}+\sum_{k_1=1}^{\infty}\cdots\sum_{k_q=1}^{\infty}\delta_{k_1+\cdots+k_q-m}\prod_{i=1}^{q}(k_i+1)\right)$$
$$= \binom{n_1+2n_2}{2n_2}+\binom{\lfloor n_1/2\rfloor+n_2}{n_2}$$

$$\text{g.f.l.h.s} = \text{g.f.r.h.s} = \frac{t_1t_2(1-t_2-2t_1^2+t_1^3-2t_1^2)}{(1-t_2)(1-t_1)(1-t_2-t_1^2)(1-t_2-2t_1+t_1^2)}$$

Again, the left-hand side does not have the appearance of a binomial sum until we remark that

$$\sum_{k_1=1}^{\infty}\cdots\sum_{k_q=1}^{\infty}\delta_{k_1+\cdots+k_q-m}\prod_{i=1}^{q}(k_i+1) = [1]\left(\left(\frac{t(2-t)}{(1-t)^2}\right)^q\frac{1}{t^m}\right).$$

7.2.8. *Chapman and Tauraso (2016).*

$$\sum_{k=1}^{n}(-4)^{-k}\binom{n-k}{k-1}\sum_{j=1}^{3m}(-2)^{-j}\binom{n+1-2k}{j-1}\binom{m-k}{3m-j} = 0, \qquad n, m > 0,$$

where we consider the binomial coefficient as defined in §1.1 and not one of the variants of §6.2. This is an important clarification because negative values may appear in the upper arguments of the binomial coefficients. The geometric reduction is enough to prove this equality, no integration step is required.

7.3. **Proof of some conjectures.**

7.3.1. *Le Borgne's identity.* The following identity for Baxter's numbers arises as a conjecture in an unpublished work by Yvan Le Borgne. With the methods presented here we can prove it automatically.

$$1+F_n^{-1,-1}+2F_n^{0,0}-F_n^{0,1}+F_n^{1,0}-3F_n^{1,1}+F_n^{1,2}-F_n^{3,1}+3F_n^{3,2}$$
$$-F_n^{3,3}-2F_n^{4,2}+F_n^{4,3}-F_n^{5,2} = \sum_{m=0}^{n}\frac{\binom{n+2}{m}\binom{n+2}{m+1}\binom{n+2}{m+2}}{\binom{n+2}{1}\binom{n+2}{2}},$$

$$\text{where } F_n^{a,b} = \sum_{d=0}^{n-1}\sum_{c=0}^{d-a}\binom{d-a-c}{c}\binom{n}{d-a-c}\left(\binom{n+d+1-2a-2c+2b}{n-a-c+b}-\binom{n+d+1-2a-2c+2b}{n+1-a-c+b}\right).$$

The sum in the right-hand side does not have the appearance of a binomial sum (even if it is such), but the identity becomes clearly an identity between binomial sums after multiplication by $\binom{n+2}{1}\binom{n+2}{2}$. Alternatively, one can use the following equality to write the right-hand side as a binomial sum:

$$\frac{\binom{n+2}{m}\binom{n+2}{m+1}\binom{n+2}{m+2}}{\binom{n+2}{1}\binom{n+2}{2}} = \det \begin{pmatrix} \binom{n}{k} & \binom{n}{k+1} & \binom{n}{k+2} \\ \binom{n}{k-1} & \binom{n}{k} & \binom{n}{k+1} \\ \binom{n}{k-2} & \binom{n}{k-1} & \binom{n}{k} \end{pmatrix},$$

7.3.2. *Identities from Brent et al. (2014).* We have been able to prove the following identities conjectured by Brent et al. (2014). The left-hand sides involve absolute values of nonlinear polynomials. They are nevertheless binomial sums for two reasons. The first one is that all the nonlinear polynomials under consideration split into linear or positive factors. For example $|i^3 - j^3| = |i-j|(i^2+ij+j^2)$ and $|i-j| = (i-j)(H_{i-j} - H_{j-i})$ is a binomial sum.

The second reason, which we used for the computation, is that we can eliminate the absolute values by using the symmetries of the sums. For example

$$\sum_{i,j} \binom{2n}{n+i}\binom{2n}{n+j}|i^3 - j^3| = 4\sum_{i=0}^{n}\sum_{j=-i}^{i-1} \binom{2n}{n+i}\binom{2n}{n+j} i^3.$$

The integral representations obtained are too lengthy to be presented here, they can be found online[5]. Since the right-hand sides are hypergeometric sequences, they can be computed from the recurrence relations satisfied by the left-hand sides with the algorithm of Petkovšek (1992).

(Brent et al. 2014, Eq. 5.7) $\displaystyle\sum_{i,j} \binom{2n}{n+i}\binom{2n}{n+j}|i^3 - j^3| = \frac{2n^2(5n-2)}{4n-1}\binom{4n}{2n}$

(Brent et al. 2014, Eq. 5.8)

$$\sum_{i,j} \binom{2n}{n+i}\binom{2n}{n+j}|i^5 - j^5| = \frac{2n^2(43n^3 - 70n^2 + 36n - 6)}{(4n-1)(4n-3)}\binom{4n}{2n}$$

(Brent et al. 2014, Eq. 5.9)

$$\sum_{i,j} \binom{2n}{n+i}\binom{2n}{n+j}|i^7 - j^7| = \frac{2n^2(531n^5 - 1960n^4 + 2800n^3 - 1952n^2 + 668n - 90)}{(4n-1)(4n-3)(4n-5)}\binom{4n}{2n}$$

(Brent et al. 2014, Eq. 5.12)

$$\sum_{i,j} \binom{2n}{n+i}\binom{2n}{n+j}|ij(i^2 - j^2)| = \frac{2n^3(n-1)}{2n-1}\binom{2n}{n}^2$$

(Brent et al. 2014, Eq. 5.14)

$$\sum_{i,j} \binom{2n}{n+i}\binom{2n}{n+j}|i^3j^3(i^2 - j^2)| = \frac{2n^4(n-1)(3n^2-6n+2)}{(2n-3)(2n-1)}\binom{2n}{n}^2$$

7.4. **Computational limitations.** The method is mainly limited by the integration step. When the integral representation has more than four variables (in addition to the parameter and after the geometric reduction), then computation of the Picard-Fuchs equation becomes challenging. For example, in Strehl's second

[5] https://github.com/lairez/binomsums

identity

$$\sum_{k=0}^{n} \binom{n}{k}^3 \binom{n+k}{k}^3 = \sum_{k=0}^{\infty} \binom{n}{k} \binom{n+k}{k} \sum_{i=0}^{\infty} \binom{k}{i}^2 \binom{2i}{i}^2 \binom{2i}{k-i},$$

the integral representation of the generating function of each side has five variables and a parameter. For the left-hand side, we obtain

$$\operatorname*{res}_{z_{1\smile 5}} \frac{1}{(z_1z_3z_4z_5+z_2z_3z_4z_5-z_1z_2z_3z_4z_5-z_3z_4z_5+z_1z_2)t+z_1z_2z_3z_4z_5(1-z_1)(1-z_2)(1-z_3)(1-z_4)(1-z_5)}.$$

The integral representation of the right-hand side is more complicated ant still has five variables, in addition to the parameter. The computation requires several hours with the current algorithms. Without the geometric reduction, it involves nine variables and a parameter.

References

Abramov, S. A. (2006). "On the summation of P-recursive sequences". In: *Proceedings of the international symposium on symbolic and algebraic computation*. ISSAC '06. ACM Press, pp. 17–22. DOI: 10.1145/1145768.1145779.

Abramov, S. A. and M. Petkovšek (1994). "D'Alembertian solutions of linear differential and difference equations". In: *Proceedings of the international symposium on symbolic and algebraic computation*. ISSAC '94. ACM Press, pp. 169–174. DOI: 10.1145/190347.190412.

– (2002). "On the structure of multivariate hypergeometric terms". In: *Advances in Applied Mathematics* 29.3, pp. 386–411. DOI: 10.1016/S0196-8858(02)00022-2.

Abramov, S. A. and M. Petkovšek (2005). "Gosper's algorithm, accurate summation, and the discrete Newton-Leibniz formula". In: *Proceedings of the international symposium on symbolic and algebraic computation*. ISSAC '05. ACM Press, pp. 5–12. DOI: 10.1145/1073884.1073888.

Allouche, J.-P. and M. Mendès France (2011). "Hadamard grade of power series". In: *Journal of Number Theory* 131.11, pp. 2013–2022. DOI: 10.1016/j.jnt.2011.04.011.

Andrews, G. E. and P. Paule (1993). "Some questions concerning computer-generated proofs of a binomial double-sum identity". In: *Journal of Symbolic Computation* 16.2, pp. 147–153. DOI: 10.1006/jsco.1993.1038.

Apagodu, M. and D. Zeilberger (2006). "Multi-variable Zeilberger and Almkvist-Zeilberger algorithms and the sharpening of Wilf-Zeilberger theory". In: *Advances in Applied Mathematics* 37.2, pp. 139–152. DOI: 10.1016/j.aam.2005.09.003.

Barvinok, A. (2008). *Integer points in polyhedra*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich. DOI: 10.4171/052.

Blodgett, R. J. (1990). "Problem E3376". In: *American Mathematical Monthly* 97.3, p. 240.

Bostan, A., P. Lairez, and B. Salvy (2013a). "Creative telescoping for rational functions using the Griffiths-Dwork method". In: *Proceedings of the international symposium on symbolic and algebraic computation*. ISSAC '13. ACM Press, pp. 93–100. DOI: 10.1145/2465506.2465935.

Bostan, A., S. Boukraa, G. Christol, S. Hassani, and J.-M. Maillard (2013b). “Ising n-fold integrals as diagonals of rational functions and integrality of series expansions”. In: *Journal of Physics. A. Mathematical and Theoretical* 46.18, pp. 185202, 44. DOI: 10.1088/1751-8113/46/18/185202.

Brent, R. P., H. Ohtsuka, J.-A. H. Osborn, and H. Prodinger (2014). *Some binomial sums involving absolute values.* arXiv: 1411.1477.

Brion, M. (1988). “Points entiers dans les polyèdres convexes”. In: *Annales Scientifiques de l’École Normale Supérieure. Quatrième Série* 21.4, pp. 653–663.

Cameron, R. H. and W. T. Martin (1938). “Analytic continuation of diagonals and Hadamard compositions of multiple power series”. In: *Transactions of the American Mathematical Society* 44.1, pp. 1–7. DOI: 10.2307/1990100.

Chapman, R. and R. Tauraso (2016). “Problem 11914”. In: *American Mathematical Monthly* 123.5, p. 613.

Christol, G. (1985). “Diagonales de fractions rationnelles et équations de Picard-Fuchs”. In: *Study group on ultrametric analysis, 12th year, 1984/85, No. 1.* Paris: Secrétariat Math., Exp. No. 13, 12.

– (1988). “Diagonales de fractions rationnelles”. In: *Séminaire de Théorie des Nombres, Paris 1986–87.* Vol. 75. Progr. Math. Boston, MA: Birkhäuser Boston, pp. 65–90.

– (1990). “Globally bounded solutions of differential equations”. In: *Analytic number theory (Tokyo, 1988).* Vol. 1434. Lecture Notes in Math. Berlin: Springer, pp. 45–64. DOI: 10.1007/BFb0097124.

– (2015). “Diagonals of rational fractions”. In: *Eur. Math. Soc. Newsl.* 97, pp. 37–43.

Chyzak, F. (2000). “An extension of Zeilberger’s fast algorithm to general holonomic functions”. In: *Discrete Mathematics* 217.1-3. Formal power series and algebraic combinatorics (Vienna, 1997), pp. 115–134. DOI: 10.1016/S0012-365X(99)00259-9.

– (2014). *The ABC of Creative Telescoping: Algorithms, Bounds, Complexity.* Mémoire d’habilitation à diriger les recherches.

Chyzak, F., A. Mahboubi, T. Sibut-Pinote, and E. Tassi (2014). “A Computer-Algebra-Based Formal Proof of the Irrationality of $\zeta(3)$”. In: *5th International Conference on Interactive Theorem Proving.* Vienna, Austria. DOI: 10.1007/978-3-319-08970-6_11.

Cutkosky, S. D. (2004). *Resolution of singularities.* Vol. 63. Graduate Studies in Mathematics. American Mathematical Society. DOI: 10.1090/gsm/063.

Davletshin, M., G. Egorychev, and V. Krivokolesko (2015). *New applications of the Egorychev method of coefficients of integral representation and calculation of combinatorial sums.* arXiv: 1506.03596.

Dixon, A. C. (1891). “On the sum of the cubes of the coefficients in a certain expansion by the binomial theorem”. In: *Messenger of mathematics* 20, pp. 79–80.

Egorychev, G. P. (1984). *Integral representation and the computation of combinatorial sums.* Vol. 59. Translations of Mathematical Monographs. Translated from the Russian by H. H. McFadden, Translation edited by Lev J. Leifman. Providence, RI: American Mathematical Society.

Egorychev, G. P. and E. V. Zima (2008). “Integral representation and algorithms for closed form summation”. In: *Handbook of algebra. Vol. 5.* Vol. 5. Handb.

Algebr. Elsevier/North-Holland, Amsterdam, pp. 459–529. DOI: 10.1016/S1570-7954(07)05009-7.

Flajolet, P. and M. Soria (1998). *Coefficients of algebraic series*. Tech. rep. 3504. Algorithms seminar 1997-1998. Inria, pp. 27–30.

Flajolet, P., S. Gerhold, and B. Salvy (2005). "On the Non-Holonomic Character of Logarithms, Powers, and the nth Prime Function". In: *Electr. J. Comb.* 11.2.

Furstenberg, H. (1967). "Algebraic Functions over Finite Fields". In: *Journal of Algebra* 7.2, pp. 271–277. DOI: 10.1016/0021-8693(67)90061-0.

Garoufalidis, S. (2009). "G-functions and multisum versus holonomic sequences". In: *Advances in Mathematics* 220.6, pp. 1945–1955. DOI: 10.1016/j.aim.2008.11.012.

Garoufalidis, S. and X. Sun (2010). "A new algorithm for the recursion of hypergeometric multisums with improved universal denominator". In: *Gems in experimental mathematics*. Vol. 517. Contemp. Math. Providence, RI: Amer. Math. Soc., pp. 143–156. DOI: 10.1090/conm/517/10138.

Garrabrant, S. and I. Pak (2014). *Counting with irrational tiles*. arXiv: 1407.8222.

Graham, R. L., D. E. Knuth, and O. Patashnik (1989). *Concrete mathematics. A foundation for computer science*. Addison-Wesley Publishing Company.

Grothendieck, A. (1966). "On the de Rham cohomology of algebraic varieties". In: *Institut des Hautes Études Scientifiques. Publications Mathématiques* 29, pp. 95–103.

Ince, E. L. (1944). *Ordinary Differential Equations*. Dover Publications, New York.

Katz, N. M. (1970). "Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin". In: *Institut des Hautes Études Scientifiques. Publications Mathématiques* 39, pp. 175–232.

Koutschan, C. (2010). "A fast approach to creative telescoping". In: *Mathematics in Computer Science* 4.2-3, pp. 259–266. DOI: 10.1007/s11786-010-0055-0.

– (2013). "Creative telescoping for holonomic functions". In: *Computer Algebra in Quantum Field Theory*. Ed. by C. Schneider and J. Blümlein. Texts & Monographs in Symbolic Computation. Springer, pp. 171–194.

Lairez, P. (2014). "Périodes d'intégrales rationnelles : algorithmes et applications". PhD thesis. École polytechnique. TEL: 01089130.

– (2016). "Computing periods of rational integrals". In: *Mathematics of Computation* 85.300, pp. 1719–1752. DOI: 10.1090/mcom/3054.

Lipshitz, L. (1988). "The diagonal of a D-finite power series is D-finite". In: *Journal of Algebra* 113.2, pp. 373–378. DOI: 10.1016/0021-8693(88)90166-4.

– (1989). "D-Finite Power Series". In: *Journal of Algebra* 122.2, pp. 353–373.

Mezzarobba, M. and B. Salvy (2010). "Effective bounds for P-recursive sequences". In: *Journal of Symbolic Computation* 45.10, pp. 1075–1096. DOI: 10.1016/j.jsc.2010.06.024.

Neumann, B. H. (1949). "On ordered division rings". In: *Transactions of the American Mathematical Society* 66, pp. 202–252. DOI: 10.1090/S0002-9947-1949-0032593-5.

Paule, P. and M. Schorn (1995). "A Mathematica Version of Zeilberger's Algorithm for Proving Binomial Coefficient Identities". In: *Journal of Symbolic Computation* 20.5-6, pp. 673–698. DOI: 10.1006/jsco.1995.1071.

Pemantle, R. and M. C. Wilson (2013). *Analytic Combinatorics in Several Variables*. Vol. 140. Cambridge University Press. DOI: 10.1017/CBO9781139381864.

Petkovšek, M. (1992). "Hypergeometric solutions of linear recurrences with polynomial coefficients". In: *Journal of Symbolic Computation* 14.2-3, pp. 243–264. DOI: 10.1016/0747-7171(92)90038-6.

Rayner, F. J. (1974). "Algebraically closed fields analogous to fields of Puiseux series". In: *Journal of the London Mathematical Society.* 2nd ser. 8, pp. 504–506. DOI: 10.1112/jlms/s2-8.3.504.

Salvy, B. and S. Melczer (2016). "Symbolic-numeric tools for analytic combinatorics in several variables". In: *Proceedings of the international symposium on symbolic and algebraic computation.* ISSAC '16. ACM Press. DOI: 10.1145/2930889.2930913.

Salvy, B. and P. Zimmermann (1994). "Gfun: a Maple package for the manipulation of generating and holonomic functions in one variable". In: *ACM Transactions on Mathematical Software* 20.2, pp. 163–177. DOI: 10.1145/178365.178368.

Strehl, V. (1994). "Binomial identities. Combinatorial and algorithmic aspects". In: *Discrete Mathematics* 136.1-3. Trends in discrete mathematics, pp. 309–346. DOI: 10.1016/0012-365X(94)00118-3.

Takayama, N. (1995). "An algorithm for finding recurrence relations of binomial sums and its complexity". In: *Journal of Symbolic Computation* 20.5-6. Symbolic computation in combinatorics Δ_1 (Ithaca, NY, 1993), pp. 637–651. DOI: 10.1006/jsco.1995.1069.

Walker, R. J. (1950). *Algebraic Curves.* Princeton University Press.

Wegschaider, K. (1997). "Computer Generated Proofs of Binomial Multi-Sum Identities". MA thesis. RISC, J. Kepler University.

Wilf, H. S. and D. Zeilberger (1992). "An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities". In: *Inventiones Mathematicae* 108.3, pp. 575–633. DOI: 10.1007/BF02100618.

Xin, G. (2004). "The ring of Malcev-Neumann series and the residue theorem". PhD thesis. Brandeis University. arXiv: 0405133.

Zeilberger, D. (1990). "A holonomic systems approach to special functions identities". In: *Journal of Computational and Applied Mathematics* 32.3, pp. 321–368. DOI: 10.1016/0377-0427(90)90042-X.

– (1991a). "A Maple program for proving hypergeometric identities". In: *SIGSAM Bulletin* 25.3, pp. 4–13. DOI: 10.1145/122514.122515.

– (1991b). "The method of creative telescoping". In: *Journal of Symbolic Computation* 11, pp. 195–204. DOI: 10.1016/S0747-7171(08)80044-2.

(A. Bostan) Inria Saclay Île-de-France, Bâtiment Alan Turing, 1 rue Honoré d'Estienne d'Orves, 91120 Palaiseau, France
E-mail address: alin.bostan@inria.fr

(P. Lairez) Technische Universität Berlin, Fakultät II, Sekretariat 3-2, Strasse des 17. Juni 136, 10623 Berlin, Germany
E-mail address: pierre@lairez.fr

(B. Salvy) Inria, LIP (U. Lyon, CNRS, ENS Lyon, UCBL), France
E-mail address: bruno.salvy@inria.fr