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# MULTIPLE BINOMIAL SUMS

ALIN BOSTAN, PIERRE LAIREZ, AND BRUNO SALVY

ABSTRACT. Multiple binomial sums form a large class of multi-indexed sequences, closed under partial summation, which contains most of the sequences obtained by multiple summation of binomial coefficients and also all the sequences with algebraic generating function. We study the representation of the generating functions of binomial sums by integrals of rational functions. The outcome is twofold. Firstly, we show that a univariate sequence is a multiple binomial sum if and only if its generating function is the diagonal of a rational function. Secondly we propose algorithms that decide the equality of multiple binomial sums and that compute recurrence relations for them. In conjunction with geometric simplifications of the integral representations, this approach behaves well in practice. The process avoids the computation of certificates and the problem of accurate summation that afflicts discrete creative telescoping, both in theory and in practice.

## INTRODUCTION

The computation of definite sums in computer algebra is classically handled by the method of *creative telescoping* initiated in the 1990s by Zeilberger [54; 56; 57]. For example, it applies to sums like

$$(1) \quad \sum_{k=0}^n \frac{4^k}{\binom{2k}{k}}, \quad \sum_{k=0}^n \left( \sum_{j=0}^k \binom{n}{j} \right)^3 \quad \text{or} \quad \sum_{i=0}^n \sum_{j=0}^n \binom{i+j}{j}^2 \binom{4n-2i-2j}{2n-2i}.$$

In order to compute a sum  $\sum_k u(n, k)$  of a bivariate sequence  $u$ , this method computes an identity of the form

$$a_0(n)u(n+p, k) + \cdots + a_p(n)u(n, k) = v(n, k+1) - v(n, k).$$

Provided that it is possible to sum both sides over  $k$  and that the sequence  $v$  vanishes at the endpoints of the domain of summation, the left-hand side—called a *telescoper*—gives a recurrence for the sum. The right-hand side is then called the *certificate* of the identity.

In the case of multiple sums, this idea leads to searching for a telescoping identity of the form

$$(2) \quad a_0(n)u(n+p, k_1, \dots, k_m) + \cdots + a_p(n)u(n, k_1, \dots, k_m) = \\ (v_1(n, k_1+1, k_2, \dots, k_m) - v_1(n, k_1, \dots, k_m)) + \cdots \\ + (v_m(n, k_1, \dots, k_m+1) - v_m(n, k_1, \dots, k_m)).$$

Again, under favorable circumstances the sums of the sequences on the right-hand side telescope, leaving a recurrence for the sum on the left-hand side.

This high-level presentation hides practical difficulties. The first one is that it is important to check that the sequences on both sides of the identities above are

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defined over the whole range of summation [1; 3], and more often than not, singularities do appear. To the best of our knowledge, no algorithm based on creative telescoping manages to work around this difficulty; they all let the user handle it. As a consequence, computing the certificate is not merely a useful by-product of the algorithm, but indeed a necessary part of the computation. Unfortunately, the size of the certificate may be much larger than that of the final recurrence and thus costly in terms of computational complexity.

The computation of multiple integrals of rational functions has some similarities with the computation of discrete sums and the method of creative telescoping applies there too. It may also produce extra singularities in the certificate, but in the differential setting this is not an issue anymore: for the integrals we are interested in, the integration path can always be moved to get around any extra singularity. Moreover, we have showed [11; 39] that integration of multivariate rational functions over cycles can be achieved efficiently without computing the corresponding certificate and without introducing spurious singularities. In that case, the algorithm computes a linear differential equation for the parameterized integral. It turns out that numerous multiple sums can be cast into problems of rational integration by passing to generating functions. The algorithmic consequences of this observation form the object of the present work.

**Content.** In §1, we define a class of multivariate sequences, called *(multiple) binomial sums*, that contains the binomial coefficient sequence and that is closed under pointwise addition, pointwise multiplication, linear change of variables and partial summation. Not every sum that creative telescoping handles is a binomial sum: for example, among the three sums in Eq. (1), the second one and the third one are binomial sums but the first one is not, since it contains the inverse of a binomial coefficient; moreover, it cannot be rewritten as a binomial sum (see §1.2). Yet many sums coming from combinatorics and number theory are binomial sums. In §2, we explain how to compute *formal integral representations* of the generating function of a binomial sum in an automated way. The outcome is twofold. Firstly, in §3, we work further on these formal integral representations to show that the generating functions of *univariate* binomial sums are exactly the diagonals of rational power series. This equivalence characterizes binomial sums in an intrinsic way which dismisses the arbitrariness of the definition. All the theory of diagonals transfers to univariate binomial sums and gives many interesting arithmetic properties. Secondly, in §4, we show how to use formal integral representations to actually compute with binomial sums (e.g. find recurrence relations or prove identities automatically) *via* the computation of Picard-Fuchs equations. The direct approach leads to formal integral representations that involve far too many variables to be efficiently handled. In §5, we propose a general method, that we call *geometric reduction*, to reduce tremendously the number of variables in practice. In §6, we describe some variants with the purpose of implementing the algorithms, and finally, in §7, we show how the method applies to some classical identities and more recent ones that were conjectural so far.

All the algorithms that are presented here are implemented in Maple and are available at <https://github.com/lairiez/binomsums>.

**Example.** The following proof of Dixon's identity [24],

$$(3) \quad \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = (-1)^n \frac{(3n)!}{n!^3},$$

illustrates well the main points of the method. The strategy is as follows: find an integral representation of the generating function of the left-hand side; simplify

this integral representation using partial integration; use the simplified integral representation to compute a differential equation of which the generating function is solution; transform this equation into a recurrence relation; solve this recurrence relation.

First of all, the binomial coefficient  $\binom{n}{k}$  is the coefficient of  $x^k$  in  $(1+x)^n$ . Cauchy's integral formula ensures that

$$\binom{n}{k} = \frac{1}{2\pi i} \oint_{|x|=\frac{1}{2}} \frac{(1+x)^n dx}{x^k x}.$$

Therefore, the cube of a binomial coefficient can be represented as a triple integral

$$\binom{2n}{k}^3 = \frac{1}{(2\pi i)^3} \oint_{|x_i|=\frac{1}{2}} \frac{(1+x_1)^{2n}}{x_1^k} \frac{(1+x_2)^{2n}}{x_2^k} \frac{(1+x_3)^{2n}}{x_3^k} \frac{dx_1}{x_1} \frac{dx_2}{x_2} \frac{dx_3}{x_3}.$$

As a result, the generating function of the left-hand side of Equation (3) is

$$\begin{aligned} y(t) &\stackrel{\text{def}}{=} \sum_{n=0}^{\infty} t^n \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 \\ &= \frac{1}{(2i\pi)^3} \oint_{|x_i|=\frac{1}{2}} \sum_{n=0}^{\infty} \sum_{k=0}^{2n} \left( t \prod_{i=1}^3 (1+x_i)^2 \right)^n \left( \frac{-1}{x_1 x_2 x_3} \right)^k \frac{dx_1}{x_1} \frac{dx_2}{x_2} \frac{dx_3}{x_3} \\ &= \frac{1}{(2i\pi)^3} \oint_{|x_i|=\frac{1}{2}} \sum_{n=0}^{\infty} \left( t \prod_{i=1}^3 (1+x_i)^2 \right)^n \frac{1 - \left( \frac{-1}{x_1 x_2 x_3} \right)^{2n+1}}{1 + \frac{1}{x_1 x_2 x_3}} \frac{dx_1}{x_1} \frac{dx_2}{x_2} \frac{dx_3}{x_3} \\ &= \frac{1}{(2i\pi)^3} \oint \frac{\left( x_1 x_2 x_3 - t \prod_{i=1}^3 (1+x_i)^2 \right) dx_1 dx_2 dx_3}{\left( x_1^2 x_2^2 x_3^2 - t \prod_{i=1}^3 (1+x_i)^2 \right) \left( 1 - t \prod_{i=1}^3 (1+x_i)^2 \right)}. \end{aligned}$$

The partial integral with respect to  $x_3$  along the circle  $|x_3| = \frac{1}{2}$  is the sum of the residues of the rational function being integrated at the poles whose modulus is less than  $\frac{1}{2}$ . When  $|t|$  is small and  $|x_1| = |x_2| = \frac{1}{2}$ , the poles coming from the factor  $x_1^2 x_2^2 x_3^2 - t \prod_{i=1}^3 (1+x_i)^2$  all have a modulus that is smaller than  $\frac{1}{2}$ : they behave like  $|t|^{1/2}$ . In contrast, the poles coming from the factor  $1 - t \prod_{i=1}^3 (1+x_i)^2$  behave like  $|t|^{-1/2}$  and have all a modulus that is bigger than  $\frac{1}{2}$ . This criterion *all large or all small* that we checked on each factor of the denominator implies that the partial integral is a rational function of  $t$ ,  $x_1$  and  $x_2$ . Namely

$$y(t) = \frac{1}{(2i\pi)^2} \oint \frac{x_1 x_2 dx_1 dx_2}{x_1^2 x_2^2 - t(1+x_1)^2(1+x_2)^2(1-x_1 x_2)^2}.$$

This formula echoes the original proof of Dixon's [24] in which the left-hand side of (3) is expressed as the coefficient of  $(xy)^{4n}$  in  $((1-y^2)(1-z^2)(1-y^2 z^2))^{2n}$ . Using any algorithm that performs definite integration of rational functions [19; 37; 39] reveals a differential equation satisfied by  $y(t)$ :

$$t(27t+1)y'' + (54t+1)y' + 6y = 0.$$

Looking at the coefficient of  $t^n$  in this equality leads to the recurrence relation

$$3(3n+2)(3n+1)u_n + (n+1)^2 u_{n+1} = 0,$$

where  $u_n = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3$ . Since  $u_0 = 1$ , it leads to a proof of Dixon's identity by induction on  $n$ . The treatment above differs in one important way from what follows: the use of genuine integrals and explicit integration paths rather than *formal residues* that will be introduced in §2.

**Comparison with creative telescoping.** As mentioned above, the computation of multiple binomial sums can be handled by the method of creative telescoping. The amount of work in this direction is considerable and we refer the reader to surveys [20; 38]. In the specific context of multiple sums, the most relevant works are those of Wegschaider [53], Chyzak [19], Apagodu and Zeilberger [7] and Garoufalidis and Sun [31]. We show on the example of Dixon's identity how the method of creative telescoping and the method of generating functions differ fundamentally even on a single sum.

Let  $v_{n,k} = (-1)^k \binom{2n}{k}^3$ . This bivariate sequence satisfies the recurrence relations

$$(4) \quad \begin{aligned} (2n+2-k)^3(2n+1-k)^3v_{n+1,k} - 8(1+n)^3(1+2n)^3v_{n,k} &= 0 \\ \text{and } (k+1)^3v_{n,k+1} + (2n-k)^3v_{n,k} &= 0. \end{aligned}$$

With these relations as input, Zeilberger's algorithm finds the sequence

$$\begin{aligned} w_{n,k} &= \frac{P(n,k)}{2(2n+2-k)^3(2n+1-k)^3} u_{n,k}, \\ \text{where } P(n,k) &= k^3(9k^4n - 90k^3n^2 + 348k^2n^3 - 624kn^4 + 448n^5 \\ &\quad + 6k^4 - 132k^3n + 792k^2n^2 - 1932kn^3 + 1760n^4 - 48k^3 + 594k^2n \\ &\quad - 2214kn^2 + 2728n^3 + 147k^2 - 1113kn + 2084n^2 - 207k + 784n + 116), \end{aligned}$$

that satisfies

$$(5) \quad 3(3n+2)(3n+1)v_{n,k} + (n+1)^2v_{n+1,k} = w_{n,k+1} - w_{n,k}.$$

Whatever the way the sequence  $w_{n,k}$  is found, it is easy to check the telescopic relation (5): using the recurrence relations for  $u_{n,k}$ , each of the four terms in (5) rewrites in the form  $R(n,k)u_{n,k}$ , for some rational function  $R(n,k)$ . However, for some specific values of  $n$  and  $k$ , the sequence  $w_{n,k}$  is not defined, due to the denominator.

To deduce a recurrence relation for  $u_n = \sum_{k=0}^{2n} v_{n,k}$ , it is desirable to sum the telescopic relation (5), over  $k$ , from 0 to  $2n+2$ . Unfortunately, that would hit the forbidden set where  $w_{n,k}$  is not defined. We can only safely sum up to  $k = 2n-1$ . Doing so, we obtain that

$$3(3n+2)(3n+1) \sum_{k=0}^{2n-1} v_{n,k} + (n+1)^2 \sum_{k=0}^{2n-1} v_{n+1,k} = w_{n,2n} - w_{n,0},$$

and then

$$\begin{aligned} 3(3n+2)(3n+1)(u_n - v_{n,2n}) + (n+1)^2(u_{n+1} - v_{n+1,2n+2} - v_{n+1,2n+1} - v_{n+1,2n}) \\ = n^3(8n^5 + 52n^4 + 146n^3 + 223n^2 + 185n + 58). \end{aligned}$$

It turns out that the terms  $3(3n+2)(3n+1)v_{n,2n} + (n+1)^2(v_{n+1,2n+2} + v_{n+1,2n+1} + v_{n+1,2n})$  evaluate exactly to the right-hand side of the above identity, and this leads to Dixon's identity.

In this example, spurious singularities clearly appear in the range of summation. Thus, deriving an identity such as Dixon's from a telescopic identity such as (5) is not straightforward and involves the certificate. This is the problem of *accurate summation*. A few works address this issue for single sums [3; 1], but none for the case of multiple sums: existing algorithms [53; 19; 31] only give the telescopic identity without performing the summation. A recent attempt [21] to check the recurrence satisfied by Apéry's sequence in the proof assistant Coq has shown how difficult it is to formalize the summation step. Note that because of this issue, the *existence* of a linear recurrence for such sums can hardly be inferred from the fact that the algorithm of creative telescoping always terminates with success.

This issue is rooted in the method of creative telescoping by the fact that sequences are *represented* through the linear recurrence relations that they satisfy. Unfortunately, this representation is not very faithful when the leading terms of the relations vanish for some values of the indices. The method of generating functions avoids this issue. For example, the binomial coefficient  $\binom{n}{k}$  is represented unambiguously as the coefficient of  $x^k$  in  $(1+x)^n$  (to be understood as a power series when  $n < 0$ ), rather than as a solution to the recurrence relations  $(n-k)\binom{n}{k} = n\binom{n-1}{k}$  and  $k\binom{n}{k} = n\binom{n-1}{k-1}$ .

**Related works.** The method of generating functions is classical and has been largely studied, in particular by Egorychev [25; 26].

The special case when the generating functions are differentially finite (D-finite) has been studied by Lipshitz [41]. From the effectivity point of view, the starting point is his proof that diagonals of D-finite power series are D-finite [42]. The argument, based on linear algebra, is constructive but does not translate into an efficient algorithm because of the large dimensions involved. This led Wilf and Zeilberger [54, p. 596] to comment that “*This approach, while it is explicit in principle, in fact yields an infeasible algorithm.*” Still, using this construction of diagonals, many closure properties of the sequences under consideration (called P-recursive) can be proved (and, in principle, computed). Then, the representation of a convergent definite sum amounts to evaluating a generating series at 1 and this proves the existence of linear recurrences for the definite sums of all P-recursive sequences. Abramov and Petkovšek showed that in particular, the so-called *proper hypergeometric sequences* are P-recursive in the sense of Lipshitz. The proof is also constructive, relying on Lipshitz’s construction of diagonals to perform products of sequences [4].

While we are close to Lipshitz’s approach, three enhancements make the method of generating functions presented here efficient: we use more efficient algorithms for computing multiple integrals and diagonals that have appeared in the last twenty years [19; 37; 11; 39]; we restrict ourselves to binomial sums, which makes it possible to manipulate the generating functions through rational integral representations (see §2.2 and §2.3); and a third decisive improvement comes with the geometric reduction procedure for simplifying integral representations (see §5).

Creative telescoping is another summation algorithm developed by Zeilberger [58] and proved to work for all proper hypergeometric sequences [54]. This method has the advantage of being applicable and often efficient in practice. However, as already mentioned, it relies on certificates whose size grows fast with the number of variables [11] and, more importantly, whose summation is not straightforward, making the complete automation of the method problematic. For proper hypergeometric sums, a different effective approach developed by Takayama [51] does not suffer from the certificate problem. It consists in expressing the sum as the evaluation of a hypergeometric series and reducing its shifts with respect to a non-commutative Gröbner basis of the contiguity relations of the series, reducing the question to linear algebra in the finite-dimensional quotient.

The class of sums we consider is a subclass of the sums of proper hypergeometric sequences. We give an algorithm that avoids the computation of certificates in that case, and relies on an efficient method to deal with the integral representations of sums. The same approach has been recently used by Bostan *et alii* [10] on various examples, though in a less systematic manner. Examples in §7 give an idea of the extent of the class we are dealing with. It is a subclass of the *balanced multisums*, shown by Garoufalidis to possess nice asymptotic properties [30]. More recently, a smaller family of *binomial multisums* was studied by Garrabant and Pak: they are further constrained to be diagonals of  $\mathbb{N}$ -rational power series [32].

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## 1. THE ALGEBRA OF BINOMIAL SUMS

**1.1. Basic objects.** For all  $n, k \in \mathbb{Z}$ , the binomial coefficient  $\binom{n}{k}$  is considered in this work as *defined* to be the coefficient of  $x^k$  in the formal power series  $(1+x)^n$ . In other words,

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} \text{ for } k \geq 0 \quad \text{and} \quad \binom{n}{k} = 0 \text{ for } k < 0.$$

For all  $a, b \in \mathbb{Z}$ , we define the *directed sum*  $\sum'_{k=a}^b$  as

$$\sum'_{k=a}^b u_k \stackrel{\text{def}}{=} \begin{cases} \sum_{k=a}^b u_k & \text{if } a \leq b, \\ 0 & \text{if } a = b + 1, \\ -\sum_{k=b+1}^{a-1} u_k & \text{if } a > b + 1, \end{cases}$$

in contrast with the usual convention that  $\sum_{k=a}^b u_k = 0$  when  $a > b$ . This implies the following flexible summation rule for directed sums:

$$\sum'_{k=a}^b u_k + \sum'_{k=b+1}^c u_k = \sum'_{k=a}^c u_k, \quad \text{for all } a, b, c \in \mathbb{Z},$$

and also the geometric summation formula

$$\sum'_{k=a}^b r^k = \frac{r^a - r^{b+1}}{1-r} \quad \text{for any } r \neq 1,$$

valid independently of the relative position of  $a$  and  $b$ .

Let  $\mathbb{K}$  be a field of characteristic zero, and let  $d \geq 1$ . We denote by  $\mathcal{S}_d$  the  $\mathbb{K}$ -algebra of sequences  $\mathbb{Z}^d \rightarrow \mathbb{K}$ , where the addition and the multiplication are performed componentwise. Elements of  $\mathbb{Z}^d$  are denoted using underlined lower case letters, such as  $\underline{n}$ . The algebra  $\mathcal{S}_d$  may be embedded in the algebra of all functions  $\mathbb{Z}^{\mathbb{N}} \rightarrow \mathbb{K}$  by sending a sequence  $u : \mathbb{Z}^d \rightarrow \mathbb{K}$  to the function  $\tilde{u}$  defined by

$$\tilde{u}(n_1, n_2, \dots) = u(n_1, \dots, n_d).$$

Let  $\mathcal{S}$  be the union of the  $\mathcal{S}_d$ 's in the set of all functions  $\mathbb{Z}^{\mathbb{N}} \rightarrow \mathbb{K}$ . For  $u \in \mathcal{S}$  and  $\underline{n} \in \mathbb{Z}^d$ , the notation  $u_{\underline{n}}$  represents  $u_{\underline{n}, 0, 0, \dots}$ .

**Definition 1.1.** The algebra of *binomial sums* over the field  $\mathbb{K}$ , denoted  $\mathcal{B}$ , is the smallest subalgebra of  $\mathcal{S}$  such that:

- (a) The Kronecker delta sequence  $n \in \mathbb{Z} \mapsto \delta_n$ , defined by  $\delta_0 = 1$  and  $\delta_n = 0$  if  $n \neq 0$ , is in  $\mathcal{B}$ .
- (b) The geometric sequences  $n \in \mathbb{Z} \mapsto C^n$ , for all  $C \in \mathbb{K} \setminus \{0\}$ , are in  $\mathcal{B}$ .
- (c) The binomial sequence  $(n, k) \mapsto \binom{n}{k}$  (an element of  $\mathcal{S}_2$ ) is in  $\mathcal{B}$ .
- (d) If  $\lambda : \mathbb{Z}^d \rightarrow \mathbb{Z}^e$  is an affine map and if  $u \in \mathcal{B}$ , then  $\underline{n} \in \mathbb{Z}^d \mapsto u_{\lambda(\underline{n})}$  is in  $\mathcal{B}$ .
- (e) If  $u \in \mathcal{B}$ , then the following *indefinite summation* is in  $\mathcal{B}$ :

$$(\underline{n}, m) \in \mathbb{Z}^d \times \mathbb{Z} \mapsto \sum'_{k=0}^m u_{\underline{n}, k}.$$

Let us give a few useful examples. All polynomial sequences  $\mathbb{Z}^d \rightarrow \mathbb{K}$  are in  $\mathcal{B}$ , because the sequence  $n \mapsto \binom{n}{1} = n$  is in  $\mathcal{B}$ , thanks to points (c) and (d) of the definition. Since  $\mathcal{B}$  is an algebra, it contains all the polynomial sequences.

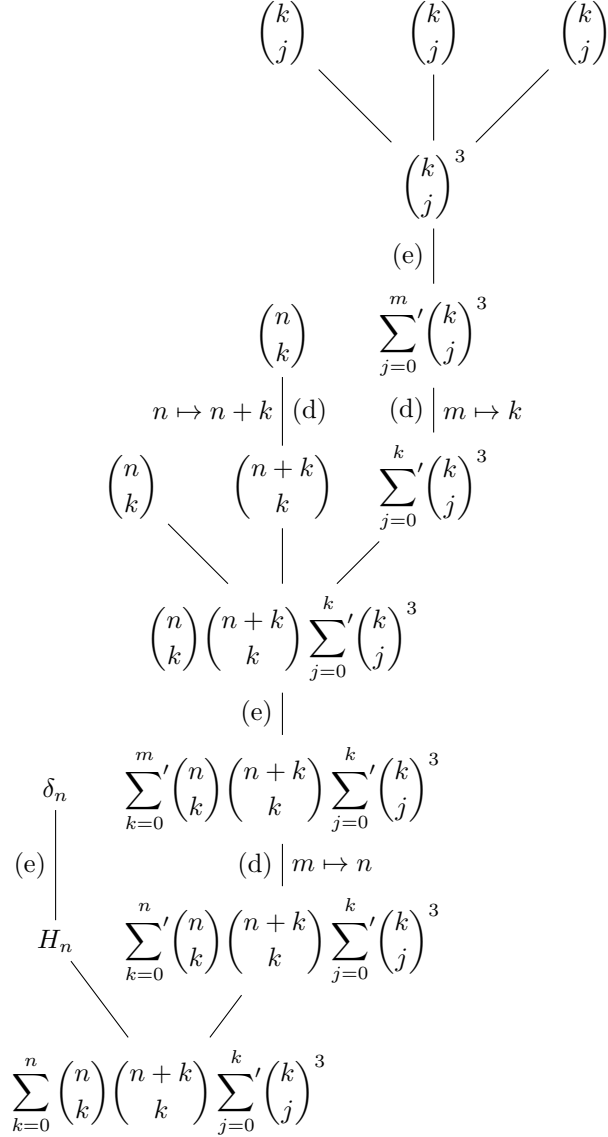


Figure 1. Construction of a binomial sum. The last step replaces the oriented sum by an ordinary sum in order to have nonzero values for  $n \geq 0$  only.

Let  $(H_n)_{n \in \mathbb{Z}}$  be the sequence defined by  $H_n = 1$  if  $n \geq 0$  and  $H_n = 0$  if  $n < 0$ . It is a binomial sum since  $H_n = \sum_{k=0}^n \delta_k$ . As a consequence, we obtain the closure of  $\mathcal{B}$  by usual (indefinite) sums since

$$\sum_{k=0}^m u_{n,k} = H_m \sum_{k=0}^m u_{n,k}.$$

By combining the rules (d) and (e) of the definition, we also obtain the closure of  $\mathcal{B}$  under sums whose bounds depend linearly on the parameter: if  $u \in \mathcal{B}$  and if  $\lambda$



and  $\mu$  are affine maps :  $\mathbb{Z}^d \rightarrow \mathbb{Z}$ , then the sequence

$$(6) \quad n \in \mathbb{Z}^d \mapsto \sum_{k=\lambda(n)}^{\mu(n)} u_{n,k}$$

is a binomial sum.

See also Figure 1 for an example of a classical binomial sum.

**1.2. Characterization of binomial sums.** A few simple criteria make it possible to prove that a given sequence is not a binomial sum. For example:

- The sequence  $(n!)_{n \geq 0}$  is not a binomial sum. Indeed, the set of sequences that grow at most exponentially is closed under the rules that define binomial sums. Since  $\binom{n}{k} \leq 2^{|n|+|k|}$ , every binomial sum grows at most exponentially; but this is not the case for the sequence  $(n!)_{n \geq 0}$ .
- The sequence of all prime integers is not a binomial sum because it does not satisfy any nonzero linear recurrence relation with polynomial coefficients [27], whereas every binomial sum does, see Corollary 3.6.
- The sequence  $(1/n)_{n \geq 1}$  is not a binomial sum. To prove this, we can easily reduce to the case where  $\mathbb{K}$  is a number field and then study the denominators that may appear in the elements of a binomial sum. One may introduce new prime divisors in the denominators only by multiplying with a scalar or with rule (b), so that the denominators of the elements of a given binomial sum contain only finitely many prime divisors. This is clearly not the case for the sequence  $(1/n)_{n \geq 1}$ .

By the same argument, the first sum of Eq. (1) is not a binomial sum. Indeed, by creative telescoping, it can be shown to equal  $(2n+1)4^{n+1}/\binom{2n+2}{n+1} + 1/3$  and thus all prime numbers appear as denominators.

- The sequence  $(u_n)_{n \geq 0}$  defined by  $u_0 = 0, u_1 = 1$  and by the recurrence  $(2n+1)u_{n+2} - (7n+11)u_{n+1} + (2n+1)u_n = 0$  is not a binomial sum. This follows from the asymptotic estimate  $u_n \sim C \cdot (4/(7 - \sqrt{33}))^n \cdot n^{\sqrt{75/44}}$ , with  $C \approx 0.56$ , and the fact that  $\sqrt{75/44}$  is not a rational number [30, Theorem 5].

These criteria are, in substance, the only ones that we know to prove that a given sequence is not a binomial sequence. Conjecturally, they characterize univariate binomial sums. Indeed, we will see that the equivalence between univariate binomial sums and diagonals of rational functions (Theorem 3.5) leads to rephrasing Christol's conjecture as follows: "any sequence  $(u_n)_{n \geq 0}$  of integers that grows at most exponentially and that is solution of a linear recurrence equation with polynomial coefficients is a binomial sum." The equivalence between diagonals and binomial sums has many interesting corollaries; it will be addressed in §3.

## 2. GENERATING FUNCTIONS

To go back and forth between sequences and power series, one can use convergent power series and Cauchy's integrals to extract coefficients, as shown in the introduction with Dixon's identity, or one can use formal power series. This requires the introduction of a *field* of multivariate formal power series that makes it possible, by embedding the rational functions into it, to define what the coefficient of a given monomial is in the power series expansion of an arbitrary multivariate rational function. We choose here to use the field of *iterated Laurent series*. It is an instance of the classical field of Hahn series when the value group is  $\mathbb{Z}^n$  with the lexicographic order. We refer to Xin [55] for a complete treatment of this field and we simply gather here the main definitions and results.

**2.1. Iterated Laurent series.** For a field  $\mathbb{A}$ , let  $\mathbb{A}((t))$  be the field of univariate formal Laurent power series with coefficients in  $\mathbb{A}$ . For  $n \in \mathbb{Z}$  and  $f \in \mathbb{A}((t))$ , let  $[t^n]f$  denote the coefficient of  $t^n$  in  $f$ . For  $d \geq 0$ , let  $\mathbb{L}_d$  be the field of iterated formal Laurent power series  $\mathbb{K}((z_d))((z_{d-1})) \cdots ((z_1))$ . For  $\underline{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$ , let  $z^{\underline{n}}$  denote the monomial  $z_1^{n_1} \cdots z_d^{n_d}$ , and for  $f \in \mathbb{L}_d$ , let  $[z^{\underline{n}}]f$  denote the coefficient of  $z^{\underline{n}}$  in  $f$ , that is  $[z_d^{n_d}] \cdots [z_1^{n_1}]f$ . An element of  $f \in \mathbb{L}_d$  is entirely characterized by its coefficient function  $\underline{n} \in \mathbb{Z}^d \mapsto [z^{\underline{n}}]f$ .

Let  $\prec$  denote the lexicographic ordering on  $\mathbb{Z}^d$ . For  $\underline{n}, \underline{m} \in \mathbb{Z}^d$  we write  $z^{\underline{n}} \prec z^{\underline{m}}$  if  $\underline{m} \prec \underline{n}$  (note the inversion: a monomial is bigger when its exponent is smaller). For  $f \in \mathbb{L}_d$ , the *support* of  $f$ , denoted  $\text{supp}(f)$ , is the set of all  $\underline{n} \in \mathbb{Z}^d$  such that  $[z^{\underline{n}}]f$  is not zero. It is well-known that a function  $\varphi : \mathbb{Z}^d \rightarrow \mathbb{K}$  is the coefficient function of an element of  $\mathbb{L}_d$  if and only if the support of  $\varphi$  is well ordered for the order  $\prec$  (that is to say every subset of the support of  $\varphi$  has a least element).

The *valuation* of  $f \neq 0$ , denoted  $v(f)$ , is the smallest element of  $\text{supp}(f)$  for the ordering  $\prec$ . Since  $\text{supp}(f)$  is well ordered,  $v(f)$  does exist. The *leading monomial* of  $f$ , denoted  $\text{lm}(f)$ , is  $z^{v(f)}$ ; it is the largest monomial that appears in  $f$ .

For  $1 \leq i \leq d$ , the partial derivative  $\partial_i = \partial/\partial z_i$  with respect to the variable  $z_i$ , defined for rational functions, extends to a derivation in  $\mathbb{L}_d$  such that  $[z^{\underline{n}}]\partial_i f = (n_i + 1)[z^{\underline{n}}](f/z_i)$  for any  $f \in \mathbb{L}_d$ .

Let  $\varphi : \mathbb{Z}^d \rightarrow \mathbb{Z}^e$  be a group morphism, strictly increasing with respect to the lexicographic ordering. In particular,  $\varphi$  is injective. For  $f \in \mathbb{L}_d$ , we define  $f^\varphi$  as the unique element of  $\mathbb{L}_e$  such that

$$[z^{\underline{n}}]f^\varphi = \begin{cases} [z^{\varphi^{-1}(\underline{n})}]f & \text{if } \underline{n} \in \varphi(\mathbb{Z}^d), \\ 0 & \text{otherwise.} \end{cases}$$

The map  $f \in \mathbb{L}_d \mapsto f^\varphi \in \mathbb{L}_e$  is a field morphism. For  $1 \leq i \leq d$ , let  $w_i$  be the monomial  $z_i^\varphi$ . The morphism  $\varphi$  is determined by  $w_1, \dots, w_d$ . When  $f$  is a rational function,  $f^\varphi$  coincides with the usual substitution  $f(w_1, \dots, w_d)$ .

Another important construction is the sum of geometric sequences. Let  $f \in \mathbb{L}_d$  be a Laurent power series with  $\text{lm}(f) \prec 1$ . The set of all  $\underline{n} \in \mathbb{Z}^d$  such that there is at least one  $k \in \mathbb{N}$  such that  $[z^{\underline{n}}]f^k \neq 0$  is well ordered [44, Theorem 3.4]. Moreover, for any  $\underline{n} \in \mathbb{Z}^d$ , the coefficient  $[z^{\underline{n}}]f^k$  vanishes for all but finitely many  $k \in \mathbb{N}$  [44, Theorem 3.5]. The following result is easily deduced.

**Lemma 2.1.** *Let  $f \in \mathbb{L}_d$  and let  $z^{\underline{n}}$  be a monomial. If  $\text{lm}(f) \prec 1$ , then  $[z^{\underline{n}}]f^k = 0$  for all but finitely many  $k \in \mathbb{N}$  and moreover*

$$[z^{\underline{n}}]\frac{1}{1-f} = \sum_{k \geq 0} [z^{\underline{n}}]f^k.$$

In what follows, there will be variables  $z_1, \dots, z_d$ , denoted  $z_{1 \setminus d}$  (and sometimes under different names) and we will denote  $z_1 \prec \cdots \prec z_d$  the fact that we consider the field  $\mathbb{L}_d$  with this ordering to define coefficients, residues, etc. The variables are always ordered by increasing index, and  $t_{1 \setminus d} \prec z_{1 \setminus e}$  denotes  $t_1 \prec \cdots \prec t_d \prec z_1 \prec \cdots \prec z_e$ . An element of  $\mathbb{L}_d \cap \mathbb{K}(z_{1 \setminus d})$  is called a *rational Laurent series*, and an element of  $\mathbb{K}[[z_{1 \setminus d}]] \cap \mathbb{K}(z_{1 \setminus d})$  is called a *rational power series*.

**Example 1.** Since  $\mathbb{L}_d$  is a field containing all the rational functions in the variables  $z_1, \dots, z_d$ , one may define the *coefficient of a monomial in a rational function*. However, it strongly depends on the ordering of the variables. For example, in  $\mathbb{L}_2 = \mathbb{K}((z_2))((z_1))$ , the coefficient of 1 in  $z_2/(z_1 + z_2)$  is 1 because

$$\frac{z_2}{z_1 + z_2} = 1 - \frac{1}{z_2}z_1 + \mathcal{O}(z_1^2),$$

whereas the coefficient of 1 in  $z_1/(z_1 + z_2)$  is 0 because

$$\frac{z_1}{z_1 + z_2} = \frac{1}{z_2} z_1 + \mathcal{O}(z_1^2).$$

**2.2. Binomial sums and coefficients of rational functions.** Binomial sums can be obtained as certain extractions of coefficients of rational functions. The main step is the following:

**Theorem 2.2.** *Every binomial sum is a linear combination of finitely many sequences of the form  $\underline{n} \in \mathbb{Z}^d \mapsto [1](R_0 R_1^{n_1} \cdots R_d^{n_d})$ , where  $R_{0 \sim d}$  are rational functions of ordered variables  $z_{1 \sim r}$ .*

*Proof.* It is clear that  $\delta_n = [1]z^n$ , that  $C^n = [1]C^n$  and that  $\binom{n}{k} = [1](1+z)^n z^{-k}$ , for all  $n, k \in \mathbb{Z}$ . Thus, it is enough to prove that the vector space generated by the sequences of the form  $[1](R_0 R_1^{n_1} \cdots R_d^{n_d})$  is a subalgebra of  $\mathcal{S}$  which is closed by the rules defining binomial sums, see §1.1.

First, it is closed under product: if  $R_{0 \sim d}$  and  $R'_{0 \sim d}$  are rational functions in the variables  $z_{1 \sim r}$  and  $z'_{1 \sim r'}$  respectively, then

$$[1] \left( R_0 \prod_{i=1}^d R_i^{n_i} \right) [1] \left( R'_0 \prod_{i=1}^d R'_i{}^{n'_i} \right) = [1] \left( R_0 R'_0 \prod_{i=1}^d (R_i R'_i)^{n_i} \right),$$

with the ordering  $z_{1 \sim r} \prec z'_{1 \sim r'}$ , for example. Then, to prove the closure under change of variables, it is enough to reorder the factors:

$$[1] \left( R_0 \prod_{i=1}^d R_i^{\sum_{j=1}^e a_{ij} n_j + b_i} \right) = [1] \left( \left( R_0 \prod_{i=1}^d R_i^{b_i} \right) \prod_{j=1}^e \left( \prod_{i=1}^d R_i^{a_{ij}} \right)^{n_j} \right).$$

Only the closure under partial sum remains. Let  $u$  be a sequence in the form

$$(\underline{n}, k) \in \mathbb{Z}^d \times \mathbb{Z} \mapsto [1] \left( T^k R_0 \prod_{i=1}^d R_i^{n_i} \right),$$

where  $T$  and  $R_{0 \sim d}$  are rational functions. If  $T = 1$ , then

$$\sum'_{k=0}^m u_{n,k} = [1] \left( (m+1) R_0 \prod_{i=1}^d R_i^{n_i} \right) = [1] \left( \frac{(1+v)R_0}{v} (1+v)^m \prod_{i=1}^d R_i^{n_i} \right),$$

where  $v$  is a new variable, because  $[v](1+v)^{m+1} = m+1$ . If  $T \neq 1$ , then

$$\sum'_{k=0}^m u_{n,k} = [1] \left( \frac{R_0}{1-T} \prod_{i=1}^d R_i^{n_i} \right) - [1] \left( \frac{R_0 T}{1-T} T^m \prod_{i=1}^d R_i^{n_i} \right),$$

which concludes the proof.  $\square$

The previous proof is algorithmic and Algorithm 1 sums up the procedure with a slight modification: when dealing with a sum of geometric ratio 1, we introduce a polynomial factor in the variables  $n_{1 \sim d}$  rather than introducing a new variable. This is possible because for any polynomial  $p(n)$  and any constant  $a \neq 0$ , there exists a polynomial  $q(n)$  such that  $q(n+1)a^{n+1} - q(n)a^n = p(n)a^n$  for  $n \in \mathbb{N}$ .

Sequences of the form  $u : \underline{n} \in \mathbb{Z}^d \mapsto [1](R_0 R_1^{n_1} \cdots R_d^{n_d})$  generalize several notions. For example if  $R_0 \in \mathbb{K}(z_1, \dots, z_d)$  and  $R_i = 1/z_i$ , then  $u$  is simply the coefficient function of  $R_0$ ; and if  $d = 1$ ,  $R_0 \in \mathbb{K}(z_1, \dots, z_r)$  and  $R_1 = 1/(z_1 \cdots z_r)$ , then  $u$  is the sequence of the diagonal coefficients of  $R_0$ .

**Example 2.** Algorithm 1 gives that

$$\sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} = [1] \left( \frac{(1+z_1)^n (1+z_2)^n z_1 z_2}{1+z_2 - z_1 z_2} \left( \left( \frac{1+z_2}{z_2 z_1} \right)^{n+1} - 1 \right) \right).$$

**Corollary 2.3.** *For any binomial sum  $u : \mathbb{Z}^d \rightarrow \mathbb{K}$  with  $\text{supp } u \subset \mathbb{N}^d$ , there exists a rational function  $R(t_{1 \setminus d}, z_{1 \setminus r})$ , with  $t_{1 \setminus d} \prec z_{1 \setminus r}$ , such that  $u_n = [t^n z^0]R$  for all  $n \in \mathbb{Z}^d$ .*

(See Proposition 3.10 below for a converse.)

*Proof.* It is enough to prove the claim for a generating set of the vector space of all binomial sums with support in  $\mathbb{N}^d$ . In accordance with Theorem 2.2, let  $u : \mathbb{Z}^d \rightarrow \mathbb{K}$  be a binomial sum with  $\text{supp } u \subset \mathbb{N}^d$  of the form

$$n \in \mathbb{Z}^d \mapsto [1] (S_0 S_1^{n_1} \cdots S_d^{n_d}),$$

where  $S_{0 \setminus d}$  are rational functions of ordered variables  $z_{1 \setminus r}$ . Let  $R(t_{1 \setminus d}, z_{1 \setminus r})$  be the rational function

$$R = S_0 \prod_{i=1}^d \frac{1}{1 - t_i S_i}.$$

Lemma 2.1 implies that for any  $n \in \mathbb{Z}^d$

$$[t^n z^0]R = \sum_{k_1, \dots, k_d \geq 0} [t^n z^0] (S_0 \cdot (t_1 S_1)^{k_1} \cdots (t_d S_d)^{k_d}).$$

Since the variables  $t_i$  do not appear in the  $S_i$ 's, the coefficient under the  $\Sigma$  is not zero only if  $n = k$ . Thus

$$[t^n z^0]R = [t^n z^0] (S_0 \cdot (t_1 S_1)^{n_1} \cdots (t_d S_d)^{n_d}) = [1] (S_0 S_1^{n_1} \cdots S_d^{n_d}) = u_n,$$

which concludes the proof.  $\square$

**Example 3.** Continuing Example 2 and following the steps of the proof of Corollary 2.3, we compute that

$$\sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} = [t^n z_1^0 z_2^0] \left( \frac{z_1 z_2}{(z_1 z_2 - (1+z_2)^2 (1+z_1)t)(1 - (1+z_2)(1+z_1)t)} \right).$$

**2.3. Residues.** The notion of residue makes it possible to represent the full generating function of a binomial sum. It is a key step toward their computation. For  $f \in \mathbb{L}_d$  and  $1 \leq i \leq d$ , the *formal residue* of  $f$  with respect to  $z_i$ , denoted  $\text{res}_{z_i} f$ , is the unique element of  $\mathbb{L}_d$  such that

$$[z^n]_{z_i} (\text{res}_{z_i} f) = \begin{cases} [z^n](z_i f) & \text{if } n_i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

This is somehow the coefficient of  $1/z_i$  in  $f$ , considered as a power series in  $z_i$  (though  $f$  is not a Laurent series in  $z_i$  since it may contain infinitely many negative powers of  $z_i$ ). When  $f$  is a rational function, care should be taken not to confuse the formal residue with the *rational residue* at  $z_i = 0$ , which is the coefficient of  $1/z_i$  in the partial fraction decomposition of  $f$  with respect to  $z_i$ . However, the former can be expressed in terms of rational residues, see §5, and like the rational residues, it vanishes on derivatives:

**Lemma 2.4.**  $\text{res}_{z_i} \partial_i f = 0$  for all  $f \in \mathbb{L}_d$ .

If  $\alpha$  is a set of variables  $\{z_{i_1}, \dots, z_{i_r}\}$ , let  $\text{res}_\alpha f$  denote the iterated residue  $\text{res}_{z_{i_1}} \cdots \text{res}_{z_{i_r}} f$ . It is easily checked that this definition does not depend on the order in which the variables appear. This implies, together with Lemma 2.4 that

**Lemma 2.5.**  $\text{res}_\alpha (\sum_{v \in \alpha} \partial_v f_v) = 0$ , for any family  $(f_v)_{v \in \alpha}$  of elements of  $\mathbb{L}_d$ .

The following lemma will also be useful:

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*Algorithm 1.* Computation of a constant term representation of a binomial sum

**Input:** A binomial sum  $u : \mathbb{Z}^d \rightarrow \mathbb{K}$  given as an abstract syntax tree (as in Figure 1)

**Output:**  $P(n_{1 \setminus d}; z_{1 \setminus r})$  a linear combination of expressions of the form  $n_j^\alpha R_0 R_1^{n_1} \cdots R_d^{n_d}$  where  $1 \leq j \leq d$ ,  $\alpha$  is a nonnegative integer and  $R_{0 \setminus d}$  are rational functions of  $z_{1 \setminus r}$ .

**Specification:**  $u_{\underline{n}} = [1]P(n_{1 \setminus d}; z_{1 \setminus r})$  for any  $\underline{n} \in \mathbb{Z}^d$ .

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function SUMTOCT( $u$ )
  if  $u_{\underline{n}} = \delta_{n_1}$  then return  $z_1^{n_1}$ 
  else if  $u_{\underline{n}} = a^{n_1}$  for some  $a \in \mathbb{K}$  then return  $a^{n_1}$ 
  else if  $u_{\underline{n}} = \binom{n_1}{n_2}$  then return  $(1 + z_1)^{n_1} / z_1^{n_2}$ 
  else if  $u_{\underline{n}} = v_{\underline{n}} + w_{\underline{n}}$  then return SUMTOCT( $v$ ) + SUMTOCT( $w$ )
  else if  $u_{\underline{n}} = v_{\underline{n}} w_{\underline{n}}$  then
     $P(n_{1 \setminus d}; z_{1 \setminus r}) \leftarrow$  SUMTOCT( $v$ )
     $Q(n_{1 \setminus e}; z_{1 \setminus s}) \leftarrow$  SUMTOCT( $w$ )
    return  $P(n_{1 \setminus d}; z_{1 \setminus r}) Q(n_{1 \setminus e}; z_{r+1 \setminus r+s})$ 
  else if  $u_{\underline{n}} = v_{\lambda(\underline{n})}$  for some affine map  $\lambda : \mathbb{Z}^d \rightarrow \mathbb{Z}^e$  then
     $P(n_{1 \setminus d}; z_{1 \setminus r}) \leftarrow$  SUMTOCT( $v$ )
    return  $P(\lambda(n_{1 \setminus d}); z_{1 \setminus r})$ 
  else if  $u_{n_1, n_2, \dots} = \sum_{k=0}^{n_1} v_{k, n_2, \dots}$  then
     $P(n_{1 \setminus d}; z_{1 \setminus r}) \leftarrow$  SUMTOCT( $v$ )
    Compute  $Q(n_{1 \setminus d}; z_{1 \setminus r})$  s.t.  $Q(n_1 + 1, n_{2 \setminus d}; z_{1 \setminus r}) - Q(n_{1 \setminus d}; z_{1 \setminus r}) = P$ 
    return  $Q(n_1 + 1, n_{2 \setminus d}; z_{1 \setminus r}) - Q(0, n_{2 \setminus d}; z_{1 \setminus r})$ 

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**Lemma 2.6.** *Let  $\alpha, \beta \subset \{z_{1 \setminus d}\}$  be disjoint sets of variables. If  $f \in \mathbb{L}_d$  does not depend on the variables in  $\beta$  and if  $g \in \mathbb{L}_d$  does not depend on the variables in  $\alpha$ , then*

$$\left( \operatorname{res}_{\alpha} f \right) \left( \operatorname{res}_{\beta} g \right) = \operatorname{res}_{\alpha \cup \beta} (fg).$$

*Proof.*  $\operatorname{res}_{\alpha \cup \beta} fg = \operatorname{res}_{\alpha} \operatorname{res}_{\beta} fg = \operatorname{res}_{\alpha} (f \operatorname{res}_{\beta} g) = (\operatorname{res}_{\alpha} f) (\operatorname{res}_{\beta} g)$ .  $\square$

The order on the variables matters. For example, let  $F(x, y)$  be a rational function, with  $x \prec y$ . The residue  $\operatorname{res}_x F$  is a rational function of  $y$ . Indeed, if  $-n$  is the exponent of  $x$  in  $\operatorname{lm}(F)$ , which we assume to be negative, then

$$\operatorname{res}_x F = \frac{1}{(n-1)!} \partial_x^{n-1} (x^n F)|_{x=0}.$$

In contrast, the residue with respect to  $y$  (or any other variable which is not the smallest) is not, in general, a rational function. It is an important point because we will represent generating series of binomial sums—which need be neither rational nor algebraic—as residues of rational functions.

**Example 4.** Let  $F = \frac{1}{xy(y^2+y-x)}$ . If  $y \prec x$  (that is to say, in the field  $\mathbb{K}((x))((y))$ ), then  $\operatorname{res}_y F = -\frac{1}{x^2}$  because

$$F = -\frac{1}{x^2 y} + \frac{(y+1)}{x^2 (y^2 + y - x)}$$

and the second term does not contain any negative power of  $y$ . On the other hand, if  $x \prec y$ , then

$$\operatorname{res}_y F = -\frac{1}{x^2} + \frac{2}{x + 4x^2 - x(1+4x)^{1/2}} = -\frac{1}{x} + 3 - 10x + 35x^2 + \mathcal{O}(x^3).$$

We will return to this example in §5.2.

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*Algorithm 2.* Computation of a formal integral representation of a binomial sum

**Input:** A binomial sum  $u : \mathbb{Z}^d \rightarrow \mathbb{K}$  given as an abstract syntax tree (as in Figure 1)

**Output:** A rational function  $R(t_{1 \setminus d}, z_{1 \setminus r})$

**Specification:**  $\sum_{n \in \mathbb{N}^d} u_n t^n = \text{res}_{z_{1 \setminus r}} R$ , where  $t_{1 \setminus d} \prec z_{1 \setminus r}$

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**function** SUMTORES( $u$ )

$$\sum_{k=1}^m n_{j_k}^{\alpha_k} R_{k,0} R_{k,1}^{n_{k,1}} \cdots R_{k,d}^{n_{k,d}} \leftarrow \text{SUMTOCT}(u)$$

$$\text{return } \frac{1}{z_1 \cdots z_r} \sum_{k=1}^m R_{k,0} F_{j_k, \alpha_k}(t_{1 \setminus d}, R_{k,1}, \dots, R_{k,d})$$

$$\text{where } F_{j, \alpha}(t_{1 \setminus d}, u_{1 \setminus d}) = \left( u_j \frac{\partial}{\partial u_j} \right)^\alpha \cdot \frac{1}{(1 - t_1 u_1) \cdots (1 - t_d u_d)}$$


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Residues of rational functions can be used to represent any binomial sum; it is the main point of the method.

**Corollary 2.7** (Formal integral representations). *For any binomial sum  $u : \mathbb{N}^d \rightarrow \mathbb{K}$ , there exists a rational function  $R(t_{1 \setminus d}, z_{1 \setminus r})$ , with  $t_{1 \setminus d} \prec z_{1 \setminus r}$ , such that*

$$\sum_{n \in \mathbb{N}^d} u_n t^n = \text{res}_{z_{1 \setminus r}} R.$$

*Proof.* Let  $R(t_{1 \setminus d}, z_{1 \setminus r})$  be such that  $u_n = [t^n z^0] R$  for all  $n \in \mathbb{Z}^d$ . Then

$$\sum_{n \in \mathbb{N}^d} u_n t^n = \sum_{n \in \mathbb{N}^d} ([t^n z^0] R) t^n = \text{res}_{z_{1 \setminus r}} \left( \frac{R}{z_1 \cdots z_r} \right). \quad \square$$

Algorithm 2 summarizes the procedure to compute the rational function  $R$ .

In §3, we prove an equivalence between univariate binomial sums and *diagonals of rational functions* that are a special kind of residues.

**2.4. Analytic integral representations.** When  $\mathbb{K}$  is a subfield of  $\mathbb{C}$ , then formal residues can be written as integrals.

**Proposition 2.8.** *Let  $R(t_{1 \setminus d}, z_{1 \setminus e})$  be a rational function whose denominator does not vanish when  $t_1 = \cdots = t_d = 0$ . There exist positive real numbers  $s_{1 \setminus d}$  and  $r_{1 \setminus e}$  such that on the set  $\{(t_{1 \setminus d}) \in \mathbb{C}^d \mid |t_i| \leq s_i\}$ , the power series  $\text{res}_{z_{1 \setminus e}} R \in \mathbb{C}[[t_{1 \setminus d}]]$  converges and*

$$\text{res}_{z_{1 \setminus e}} R = \frac{1}{(2\pi i)^e} \oint_{\gamma} R(t_{1 \setminus d}, z_{1 \setminus e}) dz_{1 \setminus e},$$

where  $\gamma = \{z \in \mathbb{C}^e \mid \forall 1 \leq i \leq e, |z_i| = r_i\}$ .

*Proof.* When  $R$  is a Laurent monomial, the equality follows from Cauchy's integral formula. By linearity, it still holds when  $R$  is a Laurent polynomial.

In the general case, let  $R$  be written as  $a/f$ , where  $a$  and  $f$  are polynomials. We may assume that the leading coefficient of  $f$  is 1 and so  $f$  decomposes as  $\text{lm}(f)(1-g)$  where  $g$  is a Laurent polynomial with monomials  $\prec 1$ . The hypothesis that  $f$  does not vanish when  $t_1 = \cdots = t_d = 0$  implies that  $\text{lm}(f)$  depends only on the variables  $z_{1 \setminus e}$  and that  $g$  contains no negative power of the variables  $t_{1 \setminus d}$ . This and the fact that all monomials of  $g$  are  $\prec 1$  imply that there exist positive real numbers  $s_{1 \setminus d}$  and  $r_{1 \setminus e}$  such that  $|g(t_{1 \setminus d}, z_{1 \setminus e})| \leq \frac{1}{2}$  if  $|t_i| \leq s_i$  and  $|z_i| = r_i$ . For example, we can take  $s_i = \exp(-\exp(N/i))$  and  $t_i = \exp(-\exp(N/(d+i)))$ , for some large enough  $N$ , because

$$\exp(-\exp(N/i))^p = o(\exp(-\exp(N/i))^q), \quad N \rightarrow \infty,$$

for any  $p, q > 0$  and  $i < j$ . On the one hand

$$\operatorname{res}_{z_1 \sim e} R = \sum_{k \geq 0} \operatorname{res}_{z_1 \sim e} \left( \frac{ag^k}{\operatorname{Im}(f)} \right),$$

because of Lemma 2.1, and on the other hand, if  $|t_i| \leq s_i$ , for  $1 \leq i \leq d$  then

$$\oint_{\gamma} R(t_{1 \sim d}, z_{1 \sim e}) dz_{1 \sim e} = \sum_{k \geq 0} \oint_{\gamma} \frac{ag^k}{\operatorname{Im}(f)} dz_{1 \sim e}.$$

where  $\gamma = \{z \in \mathbb{C}^e \mid \forall 1 \leq i \leq e, |z_i| = r_i\}$ , because the sum  $\sum_{k \geq 0} g^k$  converges uniformly on  $\gamma$ , since  $|g| \leq \frac{1}{2}$ . And the lemma follows from the case where  $R$  is a Laurent polynomial.  $\square$

### 3. DIAGONALS

Let  $R(z_{1 \sim d}) = \sum_{n \in \mathbb{N}^d} a_n \underline{z}^n$  be a rational power series in  $\mathbb{K}(z_1, \dots, z_d)$ . The *diagonal* of  $R$  is the univariate power series

$$\operatorname{diag} R \stackrel{\text{def}}{=} \sum_{n \geq 0} a_{n, \dots, n} t^n.$$

Diagonals have been introduced to study properties of the Hadamard product of power series [14; 29]. They also appear in the theory of G-functions [15]. They can be written as residues: with  $t \prec z_{2 \sim d}$ , it is easy to check that

$$(7) \quad \operatorname{diag} R = \operatorname{res}_{z_2, \dots, z_d} \frac{1}{z_2 \cdots z_d} R \left( \frac{t}{z_2 \cdots z_d}, z_2, \dots, z_d \right).$$

Despite their simplistic appearance, diagonals have very strong properties, the first of which is *differential finiteness*:

**Theorem 3.1** (Christol [16], Lipshitz [42]). *Let  $R(z_{1 \sim d})$  be a rational power series in  $\mathbb{K}(z_1, \dots, z_d)$ . There exist polynomials  $p_{0 \sim r} \in \mathbb{K}[t]$ , not all zero, such that  $p_r f^{(r)} + \dots + p_1 f' + p_0 f = 0$ , where  $f = \operatorname{diag} R$ .*

We recall that a power series  $f \in k[[t]]$  over a field  $k$  is called *algebraic* if there exists a nonzero polynomial  $P \in k[x, y]$  such that  $P(t, f) = 0$ .

**Theorem 3.2** (Furstenberg [29]). *A power series  $f \in \mathbb{K}[[t]]$  is algebraic if and only if  $f$  is the diagonal of a bivariate rational power series.*

Diagonals of rational power series in more than two variables need not be algebraic, as shown by

$$\sum_{n \geq 0} \frac{(3n)!}{n!^3} t^n = \operatorname{diag} \left( \frac{1}{1 - z_1 - z_2 - z_3} \right).$$

However, the situation is simpler modulo any prime number  $p$ .

**Theorem 3.3** (Furstenberg [29]). *Let  $f \in \mathbb{Q}[[t]]$  be the diagonal of a rational power series with rational coefficients. Finitely many primes may divide the denominator of a coefficient of  $f$ . For all prime  $p$  except those, the power series  $f \pmod{p} \in \mathbb{F}_p[[t]]$  is algebraic.*

**Example 5.** It is easy to check that the series  $f = \sum_{n \geq 0} \frac{(3n)!}{n!^3} t^n$  satisfies

$$f \equiv (1+t)^{-\frac{1}{4}} \pmod{5},$$

$$f \equiv (1+6t+6t^2)^{-\frac{1}{6}} \pmod{7},$$

$$f \equiv (1+6t+2t^2+8t^3)^{-\frac{1}{10}} \pmod{11}, \text{ etc.}$$

The characterization of diagonals of rational power series remains largely an open problem, despite very recent attempts [17].

**Conjecture 3.4** (Christol [18]). *Let  $f \in \mathbb{Z}[[t]]$  be a power series with integer coefficients, positive radius of convergence and such that  $p_r f^{(r)} + \dots + p_1 f' + p_0 f = 0$  for some polynomials  $p_{0 \dots r} \in \mathbb{Q}[t]$ , not all zero. Then  $f$  is the diagonal of a rational power series.*

For  $f \in \mathbb{Q}[[t]]$ , let  $N(f)$  be the least integer  $n$ , if any, such that  $f$  is the diagonal of a rational power series in  $n$  variables, or else  $N(f) = \infty$ <sup>1</sup> If is clear that  $N(f) = 1$  if and only if  $f$  is rational. According to Theorem 3.2,  $N(f) = 2$  if and only if  $f$  is algebraic and not rational. It is easy to find power series  $f$  with  $N(f) = 3$ , for example  $N(\sum_n \frac{(3n)!}{n!^3} t^n) = 3$  because it is the diagonal of a trivariate rational power series but it is not algebraic. However, we do not know of any explicit series  $f$  such that  $N(f) > 3$ .

In this section, we prove the following:

**Theorem 3.5.** *A sequence  $u : \mathbb{N} \rightarrow \mathbb{K}$  is a binomial sum if and only if the generating function  $\sum_{n \geq 0} u_n t^n$  is the diagonal of a rational power series.*

Note that that Garrabrant and Pak proved a similar, although essentially different, result: the subclass of binomial multisums considered in [32, Theorem 1.3] corresponds to the subclass of diagonals of  $\mathbb{N}$ -rational power series.

Theorem 3.5 and the statements above imply right away interesting corollaries.

**Corollary 3.6.** *If  $u : \mathbb{Z} \rightarrow \mathbb{K}$  is a binomial sum, then there exist polynomials  $p_{0 \dots r}$  in  $\mathbb{K}[t]$ , not all zero, such that  $p_r(n)u_{n+r} + \dots + p_1(n)u_{n+1} + p_0(n)u_n = 0$  for all  $n \in \mathbb{Z}$ .*

As mentioned in the introduction, this is a special case of a more general result for proper hypergeometric sums that can alternately be obtained as a consequence of the results of Lipshitz, Abramov and Petkovšek [41; 4].

**Corollary 3.7** (Flajolet, Soria [28]). *If the generating function  $\sum_n u_n t^n$  of a sequence  $u : \mathbb{N} \rightarrow \mathbb{K}$  is algebraic then  $u$  is a binomial sum.*

**Corollary 3.8.** *Let  $u : \mathbb{N} \rightarrow \mathbb{Q}$  be a binomial sum. Finitely many primes may divide the denominators of values of  $u$ . For all primes  $p$  except those, the generating function of a binomial sum is algebraic modulo  $p$ .*

Moreover, Christol's conjecture is equivalent to the following:

**Conjecture 3.9.** *If an integer sequence  $u : \mathbb{N} \rightarrow \mathbb{Z}$  grows at most exponentially and satisfies a recurrence  $p_r(n)u_{n+r} + \dots + p_0(n)u_n = 0$ , for some polynomials  $p_{0 \dots r}$  with integer coefficients, not all zero, then  $u$  is a binomial sum.*

The proof of Theorem 3.5 also gives information on binomial sums depending on several indices, in the form of a converse of Corollary 2.3.

**Proposition 3.10.** *A sequence  $\mathbb{Z}^d \rightarrow \mathbb{K}$  with  $\text{supp } u \subset \mathbb{N}^d$  is a binomial sum if and only if there exists a rational function  $R(t_{1 \dots d}, z_{1 \dots r})$ , with  $t_{1 \dots d} \prec z_{1 \dots r}$ , such that  $u_{\underline{n}} = [t^{\underline{n}} z^{\underline{0}}]R$  for all  $\underline{n} \in \mathbb{Z}^d$ .*

In other words, binomial sums are exactly the constant terms of rational power series, where the largest variables are eliminated.

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<sup>1</sup>This notion is closely related to the *grade* of a power series [5].



**3.1. Binomial sums as diagonals.** Corollary 2.7 and Equation (7) provide an expression of the generating function of a binomial sum (of one free variable) as the diagonal of a rational *Laurent* series, but more is needed to obtain the diagonal of a rational *power* series and thus obtain the first part of Theorem 3.5.

Let  $u : \mathbb{N} \rightarrow \mathbb{K}$  be a binomial sum. In this section, we aim at constructing a rational power series  $S$  such that  $\sum_{n \geq 0} u_n t^n = \text{diag } S$ . By Corollary 2.3, there exists a rational function  $S_0(z_{1 \setminus d}) = \frac{a}{f}$  such that  $u_n = [z_1^n] S_0$  for all  $n \geq 0$ .

Recall the notation  $f^\varphi$ , for  $f \in \mathbb{L}_d$  and  $\varphi$  an increasing endomorphism of  $\mathbb{Z}^d$ , introduced in §2.1. For example, if  $f = z_1 + z_2 \in \mathbb{L}_2$  and  $\varphi(n_1, n_2) = (n_1, n_1 + n_2)$  then  $f^\varphi = z_1 z_2 + z_2 = z_2(1 + z_1)$ .

**Lemma 3.11.** *Let  $f$  be a polynomial in  $K[z_{1 \setminus d}]$ . Then there exists an increasing endomorphism  $\varphi$  of  $\mathbb{Z}^d$  such that  $f^\varphi = C z^{\underline{m}}(1 + g)$ , for some  $\underline{m} \in \mathbb{N}^d$ ,  $C \in \mathbb{K} \setminus \{0\}$  and  $g \in \mathbb{K}[z_{1 \setminus d}]$  with  $g(0, \dots, 0) = 0$ .*

*Proof.* Let  $z^a$  and  $z^b$  be monomials such that  $z^a \prec z^b$ . Let  $i$  be the smallest integer such that  $a_i \neq b_i$ . By definition  $a_i > b_i$ . For  $k \in \mathbb{N}$ , let  $\varphi_k : \mathbb{Z}^d \rightarrow \mathbb{Z}^d$  be defined by

$$\varphi_k : (n_1, \dots, n_d) \in \mathbb{Z}^d \mapsto (n_1, \dots, n_i, n_{i+1} + kn_i, \dots, n_d + kn_i) \in \mathbb{Z}^d.$$

It is strictly increasing and if  $k$  is large enough then  $\varphi_k(a) \geq \varphi_k(b)$  componentwise, that is  $(z^b)^{\varphi_k}$  divides  $(z^a)^{\varphi_k}$ . We may apply repeatedly this argument to construct an increasing endomorphism  $\varphi$  of  $\mathbb{Z}^d$  such that the leading monomial of  $f^\varphi$  divides all the monomials of  $f^\varphi$ , which proves the Lemma.  $\square$

To conclude the proof of Theorem 3.5, let  $\varphi$  be the endomorphism given by the lemma above. For  $1 \leq i \leq d$ , let  $w_i = z_i^\varphi$ . Then

$$S_0(w_{1 \setminus d}) = S_0^\varphi = \frac{a^\varphi}{C z^{\underline{m}}(1 + g)},$$

as explained in §2.1. And by definition of  $S_0^\varphi$ , we have  $u_n = [w_1^n] S_0(w_{1 \setminus d})$  for all  $n \in \mathbb{N}$ . Let  $R$  be the rational power series  $\frac{a^\varphi}{C(1+g)}$ , so that  $u_n = [w_1^n](R/z^{\underline{m}})$ . We now prove that we can reduce to the case where  $\underline{m} = 0$ . If  $\underline{m} \neq 0$ , let  $i$  be the smallest integer such that  $m_i \neq 0$ . The specialization  $R|_{z_i=0}$  is a rational power series and  $R - R|_{z_i=0} = z_i T$  for some rational power series  $T$ . For all  $n \geq 0$ , the coefficient of  $w_1^n$  in  $R|_{z_i=0}/z^{\underline{m}}$  vanishes because the exponent of  $z_i$  in  $w_1^n$  is nonnegative while the exponent of  $z_i$  in every monomial in  $R|_{z_i=0}/z^{\underline{m}}$  is  $-m_i < 0$ . Thus  $u_n = [w_1^n] \frac{T}{z^{\underline{m}/z_i}}$ , and we can replace  $R$  by  $T$  and subtract 1 to the first nonzero coordinate of  $\underline{m}$ , which makes  $\underline{m}$  decrease for the lexicographic ordering. Iterating this procedure leads to  $\underline{m} = 0$  and thus  $u_n = [w_1^n] T$  for some rational power series  $T$ .

Let us write  $w_1$  as  $z_1^{a_1} \cdots z_d^{a_d}$ , with  $a_{1 \setminus d} \in \mathbb{N}$ . If all the  $a_i$ 's were equal to 1, then  $\sum_n u_n t^n$  would be the diagonal of  $T$ . To reduce to this case, let us consider the following rational power series:

$$U = \frac{1}{a_1 \cdots a_d} \sum_{\varepsilon_1^{a_1}=1} \cdots \sum_{\varepsilon_d^{a_d}=1} T(\varepsilon_1 z_1, \dots, \varepsilon_d z_d),$$

where the  $\varepsilon_i$  ranges over the  $a_i^{\text{th}}$  roots of unity. By construction, if  $m$  is a monomial in the  $z_i^{a_i}$ , then  $[m]U = [m]T$ . In particular  $u_n = [w_1^n]U$ .

We may consider  $T$  and  $U$  as elements of the extension of the field  $\mathbb{K}(z_1^{a_1}, \dots, z_d^{a_d})$  by the roots of the polynomials  $X^{a_i} - z_i$ , for  $1 \leq i \leq d$ . By construction, the rational function  $U$  is left invariant by the automorphisms of this extension. Thus  $U \in \mathbb{K}(z_1^{a_1}, \dots, z_d^{a_d})$ . Let  $S(z_{1 \setminus d})$  be the unique rational function such that  $U = S(z_1^{a_1}, \dots, z_d^{a_d})$ . It is a rational power series and  $u_n = [z_1^n \cdots z_d^n] S$ . Thus  $\sum_{n \geq 0} u_n t^n = \text{diag } S$ , which concludes the proof that binomial sums are diagonals of rational functions (first part of Theorem 3.5).

**3.2. Summation over a polyhedron.** To prove that diagonals of rational power series are generating functions of binomial sums, we prove a property of closure of binomial sums under certain summations that generalize the indefinite summation of rule (e). Let  $u : \mathbb{Z}^{d+e} \rightarrow \mathbb{K}$  be a binomial sum and let  $\Gamma \subset \mathbb{R}^{d+e}$  be a rational polyhedron, that is to say

$$\Gamma = \bigcap_{\lambda \in \Lambda} \{x \in \mathbb{R}^{d+e} \mid \lambda(x) \geq 0\},$$

for a finite set  $\Lambda$  of linear maps  $\mathbb{R}^{d+e} \rightarrow \mathbb{R}$  with integer coefficients in the canonical bases. This section is dedicated to the proof of the following:

**Proposition 3.12.** *If for all  $\underline{n} \in \mathbb{Z}^d$  the set  $\{\underline{m} \in \mathbb{Z}^e \mid (\underline{n}, \underline{m}) \in \Gamma\}$  is finite, then the sequence*

$$v : \underline{n} \in \mathbb{Z}^d \mapsto \sum_{\underline{m} \in \mathbb{Z}^e} u_{\underline{n}, \underline{m}} \mathbf{1}_\Gamma(\underline{n}, \underline{m})$$

is a binomial sum, where  $\mathbf{1}_\Gamma(\underline{n}, \underline{m}) = 1$  if  $(\underline{n}, \underline{m}) \in \Gamma$  and 0 otherwise.

Recall that  $H : \mathbb{Z} \rightarrow \mathbb{K}$  is the sequence defined by  $H(n) = 1$  if  $n \geq 0$  and 0 otherwise, it is a binomial sum, see §1.1. Thus  $\mathbf{1}_\Gamma$  is a binomial sum, because  $\mathbf{1}_\Gamma(\underline{n}) = \prod_{\lambda \in \Lambda} H(\lambda(\underline{n}))$  and each factor of the product is a binomial sum, thanks to rule (d) of the definition of binomial sums. However, the summation defining  $v$  ranges over an infinite set, rule (e) is not enough to conclude directly, neither is the generalized summation of Equation (6).

For  $\underline{m} \in \mathbb{R}^e$ , let  $|\underline{m}|_\infty$  denote  $\max_{1 \leq i \leq e} |m_i|$ . For  $\underline{n} \in \mathbb{Z}^d$ , let  $B(\underline{n})$  be

$$B(\underline{n}) \stackrel{\text{def}}{=} \max \{|\underline{m}|_\infty \mid \underline{m} \in \mathbb{R}^e \text{ and } (\underline{n}, \underline{m}) \in \Gamma\} \cup \{-\infty\}.$$

A rational polyhedron is bounded if and only if it contains a finite number of integer points. So by hypothesis on  $\Gamma$ ,  $B(\underline{n}) < \infty$  for all  $\underline{n} \in \mathbb{Z}^d$ .

**Lemma 3.13.** *There exists  $C > 0$  such that  $B(\underline{n}) \leq C(1 + |\underline{n}|_\infty)$ , for all  $\underline{n} \in \mathbb{Z}^d$ .*

*Proof.* By contradiction, let us assume that such a  $C$  does not exist, that is there exists a sequence  $\underline{p}_k = (\underline{n}_k, \underline{m}_k)$  of elements of  $\Gamma$  such that  $|\underline{m}_k|_\infty / |\underline{n}_k|_\infty$  and  $|\underline{n}_k|_\infty$  tend to  $\infty$ . Up to considering a subsequence, we may assume that  $\underline{m}_k / |\underline{m}_k|_\infty$  has a limit  $\ell \in \mathbb{R}^e$ , which is nonzero. For all  $\alpha \in [0, 1]$  and  $k \geq 0$ , the point  $\underline{p}_0 + \alpha(\underline{p}_k - \underline{p}_0)$  is in  $\Gamma$ , because  $\Gamma$  is convex. Let  $u > 0$  and let  $\alpha_k = u / |\underline{m}_k|_\infty$ . If  $k$  is large enough, then  $0 \leq \alpha_k \leq 1$ . Moreover

$$\underline{p}_0 + \alpha_k(\underline{p}_k - \underline{p}_0) \xrightarrow[k \rightarrow \infty]{} (\underline{n}_0, \underline{m}_0 + u\ell).$$

Yet  $\Gamma$  is closed so  $(\underline{n}_0, \underline{m}_0 + u\ell) \in \Gamma$ . By definition  $|\underline{m}_0 + u\ell|_\infty \leq B(\underline{n}_0) < \infty$ . This is a contradiction because  $\ell \neq 0$  and  $u$  is arbitrarily large.  $\square$

Thus, for all  $\underline{n} \in \mathbb{Z}^d$ ,

$$(8) \quad v_{\underline{n}} = \sum_{\substack{\underline{m} \in \mathbb{Z}^e \\ |\underline{m}| \leq C(1 + |\underline{n}|_\infty)}} u_{\underline{n}, \underline{m}} \mathbf{1}_\Gamma(\underline{n}, \underline{m}).$$

For  $1 \leq i \leq d$ , let  $w^{i,+}$  be the sequence

$$w_{\underline{n}}^{i,+} \stackrel{\text{def}}{=} H_{n_i} \prod_{j=1}^{i-1} (H_{n_i - n_j - 1} H_{n_i + n_j - 1}) \prod_{j=i+1}^d (H_{n_i - n_j} H_{n_i + n_j}),$$

so that  $w_{\underline{n}}^{i,+} = 1$  if  $n_i = |n|_\infty$  and  $|n_j| < |n_i|$  for all  $j < i$ ; and 0 otherwise. Likewise, let  $w^{i,-}$  be the sequence

$$w_{\underline{n}}^{i,-} \stackrel{\text{def}}{=} H_{-n_i-1} \prod_{j=1}^{i-1} (H_{-n_i-n_j-1} H_{-n_i+n_j-1}) \prod_{j=i+1}^d (H_{-n_i-n_j} H_{-n_i+n_j}),$$

so that  $w_{\underline{n}}^{i,-} = 1$  if  $n_i = -|n|_\infty < 0$  and  $|n_j| < |n_i|$  for all  $j < i$ ; and 0 otherwise. The sequences  $w^{i,+}$  and  $w^{i,-}$  are binomial sums that partition 1: for all  $\underline{n} \in \mathbb{Z}^d$

$$1 = \sum_{i=1}^d w_{\underline{n}}^{i,+} + \sum_{i=1}^d w_{\underline{n}}^{i,-}.$$

By design, the sum in Equation (8) rewrites as

$$\begin{aligned} v_{\underline{n}} = & \sum_{i=1}^d w_{\underline{n}}^{i,+} \sum_{m_1=-C(1+n_i)}^{C(1+n_i)} \cdots \sum_{m_e=-C(1+n_i)}^{C(1+n_i)} u_{\underline{n}, \underline{m}} \mathbf{1}_\Gamma(\underline{n}, \underline{m}) \\ & + \sum_{i=1}^d w_{\underline{n}}^{i,-} \sum_{m_1=-C(1-n_i)}^{C(1-n_i)} \cdots \sum_{m_e=-C(1-n_i)}^{C(1-n_i)} u_{\underline{n}, \underline{m}} \mathbf{1}_\Gamma(\underline{n}, \underline{m}), \end{aligned}$$

which concludes the proof of Proposition 3.12, because now the summation bounds are affine functions with integer coefficients of  $\underline{n}$ .

From a practical point of view, summations over polyhedra can be handled in a very different way, see §6.3.

**3.3. Diagonals as binomial sums.** We now prove the second part of Theorem 3.5: the diagonal of a rational function is the generating function of a binomial sum. Let  $R(z_{1 \cup d})$  be a rational power series and let us prove that the sequence defined by  $u_n = [z_1^n \cdots z_d^n] R$  is a binomial sum. Since the binomial sums are closed under linear combinations, it is enough to consider the case where the numerator of  $R$  is a monomial and where the constant term of its denominator is 1. (It cannot be zero since  $R$  is a power series.) Thus  $R$  has the form

$$R = \frac{z^{m_0}}{1 + \sum_{k=1}^e a_k z^{m_k}},$$

where the  $m_k$ 's have nonnegative coordinates and  $m_k \neq 0$  if  $k \neq 0$ . Let  $y_{0 \cup e}$  be new variables and let  $S$  be the rational power series

$$S(y_{0 \cup e}) \stackrel{\text{def}}{=} \frac{y_0}{1 + \sum_{k=1}^e a_k y_k} = y_0 \sum_{\underline{k} \in \mathbb{N}^e} \underbrace{\binom{k_1 + \cdots + k_e}{k_1, \dots, k_e}}_{C_{\underline{k}}} a_1^{k_1} \cdots a_e^{k_e} y_1^{k_1} \cdots y_e^{k_e}.$$

The coefficient sequence  $C_{\underline{k}}$  of this power series is a binomial sum because the multinomial coefficient is a product of binomial coefficients:

$$\binom{k_1 + \cdots + k_e}{k_1, \dots, k_e} = \binom{k_1 + \cdots + k_e}{k_1} \binom{k_2 + \cdots + k_e}{k_2} \cdots \binom{k_{e-1} + k_e}{k_{e-1}}.$$

Let  $\Gamma \subset \mathbb{R} \times \mathbb{R}^e$  be the rational polyhedron defined by

$$\Gamma \stackrel{\text{def}}{=} \left\{ (n, \underline{k}) \in \mathbb{R} \times \mathbb{R}^e \mid k_1 \geq 0, \dots, k_e \geq 0 \text{ and } m_0 + \sum_{i=1}^e k_i m_i = (n, \dots, n) \right\}.$$

By construction  $R(z_{1 \cup d}) = S(z^{m_0}, \dots, z^{m_e})$ , and the image of a monomial  $y_0 y_1^{k_1} \cdots y_e^{k_e}$  is a diagonal monomial  $z_1^n \cdots z_d^n$  if and only if  $(n, \underline{k}) \in \Gamma$ . Thus

$$[z_1^n \cdots z_d^n] R = \sum_{\underline{k} \in \mathbb{Z}^e} C_{\underline{k}} \mathbf{1}_\Gamma(n, \underline{k}).$$

Thanks to the positivity conditions on the  $m_k$ 's, it is obvious that  $\Gamma$  satisfies the finiteness hypothesis of Proposition 3.12. Thus the sequence  $u_n$  is a binomial sum.

#### 4. COMPUTING BINOMIAL SUMS

*Computing* may have different meanings. When manipulating sequences like binomial sums, one may want, for example, to compute recurrence relations that they satisfy, to decide their equality, to compute their asymptotic behavior or to find *simple* closed form formulas for them. The representation of the generating series of binomial sums as residues or diagonals of rational functions gives an interesting tool to tackle these goals for binomial sums, though some questions remain open.

While the customary tool to compute recurrence relations satisfied by binomial sums is *creative telescoping*, integral representations of binomial sums give another approach: given a binomial sum we first compute a formal integral representation (this is fast and easy, see Algorithm 1), and next we compute a differential equation satisfied by the integral, which translates immediately into a recurrence relation for the binomial sum. The decision problem  $A = B$  can be solved by computing recurrence relations for  $A - B$  and checking sufficiently many initial conditions, see §4.4 for more details.

A recurrence relation is also a good starting point for deriving the asymptotic behavior of a univariate binomial sum [e.g. 43]. However, some constants that determine the asymptotic behavior can be computed numerically but it is not known how to decide their nullity, which makes it difficult to catch subdominant behavior. A more direct method is possible once the generating function has been written as the diagonal of a rational power series [46].

The problem of simplifying binomial sums is still largely open: for example, given the left-hand side of Strehl's identity (see §7.2.1), it is not known how to *discover* automatically the right-hand side. The only known automatic simplification procedures consist in computing a recurrence relation and applying Petkovšek's algorithm [47] to find hypergeometric solutions or Abramov and Petkovšek's algorithm [2] to find D'Alembertian solutions. See §7.2 and §7.3 for numerous examples.

The computation of a recurrence relation for a binomial sum is done in two steps. Firstly, an integral representation is computed.

**4.1. Picard-Fuchs equations.** Let  $\mathbb{L}$  be a field of characteristic zero and let  $R(z_{1 \cup r})$  be a rational function with coefficients in  $\mathbb{L}$ , written as  $R = \frac{P}{F}$  where  $P$  and  $F$  are polynomials. Let  $A_F$  be the localized ring  $\mathbb{L}[z_{1 \cup r}, F^{-1}]$ . It is known that the  $\mathbb{L}$ -linear quotient space

$$H_F \stackrel{\text{def}}{=} A_F / \left( \frac{\partial}{\partial z_1} A_F + \cdots + \frac{\partial}{\partial z_r} A_F \right)$$

is finite dimensional [34]. Let us assume that there is a derivation  $\partial$  defined on  $\mathbb{L}$ . It extends naturally to  $A_F$  with  $\partial z_i = 0$  and it commutes with the derivations  $\frac{\partial}{\partial z_i}$ , so that  $\partial$  defines a derivation on the  $\mathbb{L}$ -linear space  $H_F$ . Since  $H_F$  is finite dimensional, there exist  $c_{0 \cup m} \in \mathbb{L}$ , not all zero, such that

$$c_m \partial^m R + \cdots + c_1 \partial R + c_0 R \in \frac{\partial}{\partial z_1} A_F + \cdots + \frac{\partial}{\partial z_r} A_F.$$

Now let us assume that  $\mathbb{L}$  is the field of rational function  $\mathbb{K}(t_{1 \cup d})$  and that  $\partial$  is the derivation  $\frac{\partial}{\partial t_i}$  for some  $i$ . Then the operator  $\partial$  commutes with the operator  $\text{res}_{z_{1 \cup r}}$ , as do the multiplications by elements of  $\mathbb{L}$ , and Lemma 2.5 implies that

$$c_m \frac{\partial^m}{\partial t_i^m} f + \cdots + c_1 \frac{\partial}{\partial t_i} f + c_0 f = 0,$$

where  $f = \text{res}_{z_{1 \cup r}} R$ , with  $t_{1 \cup d} \prec z_{1 \cup r}$ . Differential equations that arise in this way are called *Picard-Fuchs equations*.

A Laurent series  $f(t_{1\cup d}) \in \mathbb{L}_d$  is called *differentially finite* if the  $\mathbb{K}(t_{1\cup d})$ -linear subspace of  $\mathbb{L}_d$  generated by the derivatives  $\partial^{n_1+\dots+n_d} f / \partial t_1^{n_1} \dots \partial t_d^{n_d}$ , for  $n_{1\cup d} \geq 0$ , is finite dimensional. In particular, a univariate Laurent series  $f \in \mathbb{L}_1$  is differentially finite if and only if there exist  $m \geq 0$  and polynomials  $p_{0\cup m} \in \mathbb{K}[t]$ , not all zero, such that  $p_m f^{(m)} + \dots + p_1 f' + p_0 f = 0$ . In that case, we say that  $f$  is solution of a differential equation of *order*  $m$  and *degree*  $\max_k \deg p_k$ . The above argument implies the following classical theorem of which Lipshitz [42] gave an elementary proof.

**Theorem 4.1.** *For any rational function  $R(t_{1\cup d}, z_{1\cup r})$  with  $t_{1\cup d} \prec z_{1\cup r}$ , the residue  $\text{res}_{z_{1\cup r}} R$  is differentially finite.*

In previous work [11, Theorem 12; 40, §I.35.3], we proved the following quantitative result about the size of Picard-Fuchs equations and the complexity of their computation. For an efficient algorithm to compute Picard-Fuchs equations, see [39].

**Theorem 4.2.** *Let  $R \in \mathbb{K}(t, z_{1\cup r})$  be a rational function, written as  $R = \frac{P}{F}$ , with  $P$  and  $F$  polynomials. Let*

$$N = \max(\deg_{z_{1\cup r}} P + r + 1, \deg_{z_{1\cup r}} F) \text{ and } d_t = \max(\deg_t P, \deg_t F).$$

*Then  $\text{res}_{z_{1\cup r}} R$ , with  $t \prec z_{1\cup r}$ , is solution of a linear differential equation of order at most  $N^r$  and degree at most  $(\frac{5}{8}N^{3r} + N^r) \exp(r)d_t$ . Moreover, this differential equation can be computed with  $\mathcal{O}(\exp(5r)N^{8r}d_t)$  arithmetic operations in  $\mathbb{K}$ , uniformly in all the parameters.*

**4.2. Power series solutions of differential equations.** When a power series satisfies a given linear differential equation with polynomial coefficients, one only needs a few *initial conditions* to determine entirely the power series.

Let  $\mathcal{L} \in \mathbb{K}[t]\langle \partial_t \rangle$  be a linear differential operator in  $\partial_t$  with polynomial coefficients in  $t$ . There exists a unique  $n \in \mathbb{Z}$  and a unique  $b_{\mathcal{L}} \in \mathbb{K}[a]$  such that  $\mathcal{L}(t^a) = b_{\mathcal{L}}(a)t^{a+n} + o(t^{a+n})$  for all  $a \in \mathbb{Z}$  and  $t \rightarrow 0$ . The polynomial  $b_{\mathcal{L}}$  is the *indicial polynomial* of  $\mathcal{L}$  at  $t = 0$ . For more details about the indicial polynomial, see [35]. It is easy to check that if  $f \in \mathbb{K}[[t]]$  is annihilated by  $\mathcal{L}$ , then its leading monomial  $t^n$  satisfies  $b_{\mathcal{L}}(n) = 0$ .

**Proposition 4.3.** *Let  $f \in \mathbb{K}[[t]]$ . If  $\mathcal{L}(f) = 0$  and if  $[t^n]f = 0$  for all  $n \in \mathbb{N}$  such that  $b_{\mathcal{L}}(n) = 0$ , then  $f = 0$ .*

It is worth noting that the indicial equation of a Picard-Fuchs equation is very special:

**Theorem 4.4** ([36]). *If  $\mathcal{L}$  is a Picard-Fuchs equation, then the degree of  $b_{\mathcal{L}}$  equals the order of  $\mathcal{L}$  and all the roots of  $b_{\mathcal{L}}$  are rational.*

The data of a differential operator  $\mathcal{L}$  and elements of  $\mathbb{K}$  for each nonnegative integer root of  $b_{\mathcal{L}}$  (that we will call here *sufficient initial conditions*) determines entirely an element of  $\mathbb{K}[[t]]$ . It is an excellent data structure for manipulating power series [49]. For example, it lets one compute efficiently the coefficients of the power series  $\sum_n u_n t^n$ : the differential equation translates into a recurrence relation

$$p_r(n)u_{n+r} + \dots + p_1(n)u_{n+1} + p_0(n)u_n = 0$$

for some polynomials  $p_{0\cup r} \in \mathbb{K}[n]$  and the sufficient initial conditions translate into initial conditions for the recurrence, exactly where we need them.

**4.3. Equality test for univariate differentially finite power series.** Let  $f \in \mathbb{K}[[t]]$  be a power series given by a differential operator  $\mathcal{L} \in \mathbb{K}[t]\langle\partial_t\rangle$  such that  $\mathcal{L}(f) = 0$ , and by sufficient initial conditions. Let  $\mathcal{M}$  be another differential operator. We may decide if  $\mathcal{M}(f) = 0$  in the following way. Firstly, we compute the right g.c.d. of  $\mathcal{M}$  and  $\mathcal{L}$ : this is the operator  $\mathcal{D}$  of the largest order such that  $\mathcal{M} = \mathcal{M}'\mathcal{D}$  and  $\mathcal{L} = \mathcal{L}'\mathcal{D}$  for some operators  $\mathcal{M}'$  and  $\mathcal{L}'$  in  $\mathbb{K}(t)\langle\partial_t\rangle$ . Then, it is enough to compute the indicial equation  $b_{\mathcal{L}'}$  and to compute  $[t^n]\mathcal{D}(f)$  for each nonnegative integer root  $n$  of  $b_{\mathcal{L}'}$ . We will find only zeros if and only if  $\mathcal{M}(f) = 0$ . Indeed,  $\mathcal{M}(f) = 0$  if and only if  $\mathcal{D}(f) = 0$  and since  $\mathcal{L}'(\mathcal{D}(f)) = 0$ , we may apply Proposition 4.3 to check whether  $\mathcal{D}(f) = 0$  or not.

Now, let  $g \in \mathbb{K}[[t]]$  be another power series given by a differential operator  $\mathcal{M} \in \mathbb{K}[t]\langle\partial_t\rangle$  and sufficient initial conditions. To decide if  $f = g$ , it is enough to check that  $\mathcal{M}(f) = 0$ , with the above method, and to check that the coefficients of  $f$  and  $g$  corresponding to the nonnegative integer roots of  $b_{\mathcal{M}}$  are the same.

**4.4. Equality test for binomial sums.** Let  $u : \mathbb{Z} \rightarrow \mathbb{K}$  be a binomial sum. Up to considering separately the binomial sums  $H_n u_n$  and  $H_n u_{-n}$ , it is enough to look at the case where  $u_n = 0$  for  $n < 0$ . In this case  $u$  is entirely determined by its generating function  $f = \sum_{n \geq 0} u_n t^n$ . Using Algorithm 1 we obtain an integral representation of  $f$ , and then, as explained in §4.1, we obtain a differential operator  $\mathcal{L}$  that annihilates  $f$ . Since  $u$  is a binomial sum given explicitly, we can compute sufficient initial conditions. Given another binomial sum  $v : \mathbb{N} \rightarrow \mathbb{K}$ , we can check that  $u = v$  by computing a differential operator annihilating the generating function  $g$  of  $v$  together with sufficient initial conditions and by checking that  $f = g$  with the method explained in §4.2.

Let us now consider the multivariate case. As above, we can reduce the equality test for binomial sums  $\mathbb{Z}^d \rightarrow \mathbb{K}$  to the equality test for binomial sums  $\mathbb{N}^d \rightarrow \mathbb{K}$ . And this equality test can be reduced to the equality test for binomial sums  $\mathbb{N}^{d-1} \rightarrow \mathbb{K}$  as follows. Let  $u : \mathbb{N}^d \rightarrow \mathbb{K}$  be a binomial sum. It is determined by its generating function  $f(t_{1 \setminus d}) = \sum_{n \in \mathbb{N}^d} u_n t^n \in \mathbb{L}_d$ . With Algorithm 1, we compute a rational function  $R(t_{1 \setminus d}, z_{1 \setminus r})$  such that  $f = \text{res}_{z_{1 \setminus r}} R$ . Let  $\mathbb{K}'$  be the field  $\mathbb{K}(t_{1 \setminus d-1})$ . Following §4.1, we can compute an operator  $\mathcal{L} \in \mathbb{K}'[t_d]\langle\partial_{t_d}\rangle$  such that  $\mathcal{L}(f) = 0$ . This gives a differential equation for  $f$  considered as a power series in  $t_d$  with coefficients in  $\mathbb{L}_{d-1}$ . The sufficient initial conditions to determine  $f$  are given by the power series  $\sum_{n \in \mathbb{N}^{d-1}} u_{n,k} t^n$ , where  $k$  ranges over the nonnegative integer roots of  $b_{\mathcal{L}}$ . These power series are the generating functions of binomial sums in  $d-1$  variables. This reduces the equality test for binomial sums in  $d$  variables to the equality test for binomial sums in  $d-1$  variables.

## 5. GEOMETRIC REDUCTION

Putting into practice the computation of binomial sums through integral representations shows immediately that the number of integration variables is high and makes the computation of the Picard-Fuchs equations slow. However, the rational functions obtained this way are not ordinary. In particular, their denominator often factors into small pieces. This section presents a sufficient condition under which a residue  $\text{res}_v F$  of a rational function  $F$  is rational. This leads to rewriting an iterated residue of a rational function, like the ones given by Corollary 2.7, into another one with one or several variables less. This simplification procedure is very efficient on the residues we are interested in and reduces the number of variables significantly.

Conceptually the simplification procedure is simple: in terms of integrals it boils down to computing partial integrals in specific cases where we know that they are

rational. The rational nature of a partial integral depends on the integration cycle and integration algorithms usually forget about this cycle. Instead, they compute the Picard-Fuchs equations—see §4.1—that annihilate a given integral whatever its integration cycle. In our setting, the integration cycle underlies the notion of residue—see §2.4. This provides a symbolic treatment that we call *geometric reduction* since it takes into account the geometry of the cycle and decreases the number of variables for which the general integration algorithm above is actually needed. The time required by the computation is dramatically reduced by this symbolic treatment.

**5.1. Rational poles.** Let us consider variables  $v$  and  $z_{1 \setminus d}$ , where  $v$  can appear anywhere in the variable ordering. Let  $F(v, z_{1 \setminus d}) = a/f$  be a rational function. In general,  $\text{res}_v F$  is not a rational function—except if  $v$  is the smallest variable, see §2.3.

Let  $\rho \in \mathbb{L}_d$  be a power series in the variables  $z_{1 \setminus d}$ . The *rational* residue of  $F$  at  $v = \rho$ , denoted  $\text{Res}_{v=\rho} F$ , is the coefficient of  $1/(v - \rho)$  in the partial fraction decomposition of  $F$  as an element of  $\mathbb{L}_d(v)$ . Contrary to the *formal* residue  $\text{res}_v F$ , the rational residue is always in the field generated by  $\rho$  and the coefficients of  $F$ . Similarly to the formal residues, the rational residues of a derivative  $\partial G/\partial v$  are all zero.

Rational residues can be computed in a simple way: if  $\rho$  is not a pole of  $F$ , then  $\text{Res}_{v=\rho} F = 0$ ; if  $\rho$  is a pole of order 1, then  $\text{Res}_{v=\rho} F = ((v - \rho)F)|_{v=\rho}$ ; and if  $\rho$  is a pole of order  $r > 1$ , then

$$\text{Res}_{v=\rho} F = \frac{1}{(r-1)!} \left( \frac{\partial^{r-1}(v - \rho)^r F}{\partial^{r-1} v} \right) \Big|_{v=\rho}.$$

In its simplest form, the geometric reduction applies when  $f$  factors as a product of factors of degree 1:

$$f = C(z_{1 \setminus d}) \prod_{\rho \in U} (v - \rho)^{n_\rho},$$

where  $U$  is a finite subset of  $\mathbb{K}(z_{1 \setminus d})$ . Then the partial fraction decomposition of  $F$  writes as

$$F = \sum_{\rho \in U} \left( \frac{a_\rho}{v - \rho} + \sum_{k > 1} \frac{b_{\rho,k}}{(v - \rho)^k} \right) + P(v),$$

where  $a_\rho \in \mathbb{K}(z_{1 \setminus d})$  is  $\text{Res}_{v=\rho} F$ , where  $b_{\rho,k} \in \mathbb{K}(z_{1 \setminus d})$  and where  $P \in \mathbb{K}(z_{1 \setminus d})[v]$ . The terms with multiple poles and the polynomial term are derivatives and thus their formal residue is zero. Hence

$$\text{res}_v F = \sum_{\rho \in U} \text{res}_v \left( \frac{a_\rho}{v - \rho} \right).$$

Let  $\rho \in U$ . Depending on the leading monomial  $\text{lm}(\rho)$  of  $\rho$ , as an element of  $\mathbb{L}_d$ , two situations may occur. Either  $\text{lm}(\rho) \prec v$ , in which case

$$\frac{a_\rho}{v - \rho} = \frac{a_\rho}{v} \sum_{n=0}^{\infty} \left( \frac{\rho}{v} \right)^n,$$

and  $\text{res}_v \frac{a_\rho}{v - \rho} = a_\rho$ ; or  $\text{lm}(\rho) \succ v$ , in which case

$$\frac{a_\rho}{v - \rho} = -\frac{a_\rho}{\rho} \sum_{n=0}^{\infty} \left( \frac{v}{\rho} \right)^n,$$

hence  $\text{res}_v \frac{a_\rho}{v-\rho} = 0$ . In the end, we obtain that

$$(9) \quad \text{res}_v F = \sum_{\rho \in U} [\text{lm}(\rho) \prec v] \text{Res}_{v=\rho} F,$$

where the bracket is 1 if the inequality inside is true and 0 otherwise. In particular, the right-hand side is a rational function which we can compute.

**Example 6.** Let  $d > 0$  be an integer and let us consider the binomial sum

$$u_n = \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{dk}{n}.$$

We will show that  $u_n = (-d)^n$ . This example is interesting because the geometric reduction procedure alone is able to compute entirely the double integral representing the generating function of  $u$ , whereas Zeilberger's algorithm finds a recurrence relation of order  $d-1$  [45], far from the minimal recurrence relation  $u_{n+1} + du_n = 0$ .

Algorithm 1 computes that

$$\sum_{n \geq 0} u_n t^n = \text{res}_{z_1, z_2} \frac{z_2}{(z_2 - t(1+z_1))(z_1 z_2 + t(1+z_1)(1+z_2)^d)},$$

with  $t \prec z_1 \prec z_2$ . Let  $F$  denote the rational function on the right-hand side. Each factor of the denominator of  $F$  has degree 1 with respect to  $z_1$ . Thus Equation (9) applies. The roots of the denominator are

$$\rho_1 = \frac{z_2}{t} - 1 \quad \text{and} \quad \rho_2 = \frac{-1}{1 + \frac{z_2}{t(1+z_2)^d}},$$

moreover  $\text{lm}(\rho_1) = z_2/t \succ z_1$  and  $\text{lm}(\rho_2) = t/z_2 \prec z_1$ . Thus  $\text{res}_{z_1} F = \text{Res}_{z_1=\rho_2} F$  and we obtain

$$\sum_{n \geq 0} u_n t^n = \text{res}_{z_2} \text{res}_{z_1} F = \text{res}_{z_2} \frac{1}{t(1+z_2)^d + z_2 - t}.$$

If  $d > 2$ , the denominator of the latter rational function does not split into factors of degree 1 and Equation (9) does not apply. However, the study of nonrational poles can lead to a further reduction.

**5.2. Arbitrary poles.** Equation (9) extends to the general case. To describe this generalization, we need an algebraic closure of  $\mathbb{L}_d$ . Let  $\mathbb{L}_{d,N}$  be the field

$$\mathbb{L}_{d,N} \stackrel{\text{def}}{=} \mathbb{K}((z_d^{1/N}))((z_{d-2}^{1/N})) \dots ((z_2^{1/N}))((z_1^{1/N})).$$

It is the algebraic extension of  $\mathbb{L}_d$  generated by the  $z_i^{1/N}$ . Let  $\mathbb{L}_{d,\infty}$  be the union of all  $\mathbb{L}_{d,N}$ ,  $N > 0$ , and let  $\overline{\mathbb{K}} \otimes_{\mathbb{K}} \mathbb{L}_{d,\infty}$  be the compositum of  $\mathbb{L}_{d,\infty}$  and  $\overline{\mathbb{K}}$ , where  $\overline{\mathbb{K}}$  is an algebraic closure of  $\mathbb{K}$ . The following is classical [e.g. 48].

**Lemma 5.1** (Iterated Puiseux theorem). *The field  $\overline{\mathbb{K}} \otimes_{\mathbb{K}} \mathbb{L}_{d,\infty}$  is an algebraic closure of  $\mathbb{L}_d$ .*

The field  $\overline{\mathbb{K}} \otimes_{\mathbb{K}} \mathbb{L}_{d,\infty}$  is thus simply denoted  $\overline{\mathbb{L}_d}$ . The valuation defined on  $\mathbb{L}_d$  is extended to a valuation defined on  $\overline{\mathbb{L}_d}$  with values in the group  $\mathbb{Q}^d$ , ordered lexicographically. The leading monomial  $\text{lm}(\rho)$  of a  $f \in \overline{\mathbb{L}_d}$  is also defined as  $z^{v(f)}$ . The argument of §5.1 applies and shows that

$$(10) \quad \text{res}_v F = \sum_{\rho \text{ pole of } F} [\text{lm}(\rho) \prec v] \text{Res}_{v=\rho} F,$$

where this time, the poles are in  $\overline{\mathbb{L}_d}$ . A root  $\rho$  is called *small* if  $\rho = 0$  or  $\text{lm}(\rho) \prec v$  and *large* otherwise.



**Example 7.** Let  $F = \frac{1}{xy(y^2+y-x)}$ , with  $y \prec x$ . With respect to  $y$ , the poles of  $F$  are 0, and

$$\rho_1 = -\frac{1}{2} + \frac{\sqrt{1+4x}}{2} \quad \text{and} \quad \rho_2 = -\frac{1}{2} - \frac{\sqrt{1+4x}}{2}.$$

Only 0 and  $\rho_1$  are *small*. Thus

$$\operatorname{res}_y F = \operatorname{Res}_{y=0} F + \operatorname{Res}_{y=\rho_1} F = -\frac{1}{x^2} + \frac{2}{x+4x^2-x\sqrt{1+4x}}.$$

Equation (10) does not look as interesting as Equation (9) because the right-hand side is algebraic and need not be a rational function. However, if all roots are large, then the residue is zero, which is interesting. On the contrary, if they are all small, then  $\operatorname{res}_v F$  is the sum of all the rational residues of  $F$ , which equals the residue at infinity:

$$\operatorname{res}_v F = \operatorname{Res}_{v=\infty} F \stackrel{\text{def}}{=} \operatorname{Res}_{v=0} \left( \frac{1}{v^2} F|_{v \leftarrow 1/v} \right),$$

which is a rational function. Thus, in the case where the poles are either all small or all large, Equation (10) shows that the residue  $\operatorname{res}_v F$  is rational and how to compute it.

Actually it is enough to check that any two conjugate poles (two poles are conjugate if they annihilate the same irreducible factor of the denominator of  $F$ ) are simultaneously large or small. Indeed, we can write  $f = \prod_{k=1}^r f_k^{n_k}$  where  $f_{1 \prec r}$  are irreducible polynomials in  $\mathbb{K}(z_{1 \prec d})[v]$ , and the partial fraction decomposition leads to

$$F = \sum_{k=1}^r \frac{a_k}{f_k^{n_k}}$$

for some polynomials  $a_{1 \prec r}$ . Equation (10) applies to each term of this sum. If  $U_k$  denotes the set of all the roots of  $f_k$ , then

$$\operatorname{res}_v F = \sum_{k=1}^r \sum_{\rho \in U_k} [\operatorname{lm}(\rho) \prec v] \operatorname{Res}_{v=\rho} \frac{a_k}{f_k^{n_k}},$$

and we can apply the *all large or all small* criterion to each subsum separately.

**Proposition 5.2.** *With the notations above, if for all  $k$ , there exists  $\varepsilon_k \in \{0, 1\}$  such that  $[\operatorname{lm}(\rho) \prec v] = \varepsilon_k$  for all  $\rho \in U_k$ , then*

$$\operatorname{res}_v F = \sum_{k=1}^r \varepsilon_k \operatorname{Res}_{v=\infty} \frac{a_k}{f_k^{n_k}}.$$

It remains to explain how to compute the leading monomials of the roots of a polynomial  $f \in \mathbb{K}(z_{1 \prec d})[v]$ . If  $d = 1$ , then it is the classical method of Newton's polygon for the resolution of bivariate polynomial equations using Puiseux series [e.g. 52; 22]. When  $d \geq 2$ , we may apply this method recursively by considering  $\mathbb{L}_d$  as a subfield of the field of Puiseux series with coefficients in  $\mathbb{L}_{d-1}$ . Based on this idea, Algorithm 3 proposes an implementation which avoids all technicalities. Algorithm 4 sums up the geometric reduction procedure with respect to the variable  $v$ . Of course one should apply the procedure iteratively with every variable until no further reduction is possible.

**Example 8.** Continuing Example 6, let us consider  $\operatorname{res}_z F_1$ , where  $F_1 = 1/(t(1+z)^d + z - t)$ , where  $t \prec z$ . The denominator factors as  $zg$ , where  $g = 1 + t \sum_{k=0}^{d-1} \binom{d}{k+1} z^k$ . Thus, the leading monomial of any root of  $f$  is  $t^{-\frac{1}{d-1}}$ , which

---

*Algorithm 3.* Computation of the *all large or all small* criterion

**Input:**  $S \subset \mathbb{N}^{d+1}$  finite; an integer  $0 \leq k < d$ .

**Output:** A subset of  $\{\text{IN}, \text{OUT}\}$ .

**Specification:** Let  $P = \sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{N}^d} a_{n,m} v^n z^m \in \mathbb{K}[z_1 \smile \dots \smile z_d][v]$  be a polynomial. The variables satisfy  $z_1 \prec \dots \prec z_d$  and  $z_k \prec v \prec z_{k+1}$ . If  $S = \{(n, m) \mid a_{n,m} \neq 0\}$  then  $\text{ALLLARGEORALLSMALL}(S, k)$  contains  $\text{OUT}$  (resp.  $\text{IN}$ ) if and only if there exists a nonzero root  $\rho \in \overline{\mathbb{L}_d}$  of  $P(v)$  such that  $\text{lm}(\rho) \succ v$  (resp.  $\text{lm}(\rho) \prec v$ ).

---

**function**  $\text{ALLLARGEORALLSMALL}(S, k)$   
**if**  $\max_{m \in S} m_1 = \min_{m \in S} m_1$  **then return**  $\emptyset$   
**if**  $k = 0$  or  $S \subset \mathbb{N}^1$  **then return**  $\{\text{OUT}\}$   
 $\mu \leftarrow \min_{m \in S} m_2$   
 $M \leftarrow \{(m_1, m_{3 \smile \dots \smile n}) \in \mathbb{N}^d \mid m \in S \text{ and } m_2 = \mu\}$   
 $r \leftarrow \text{ALLLARGEORALLSMALL}(M, k - 1)$   
**if**  $\max_{m \in S} m_1 > \max_{m \in M} m_1$  **then**  $r \leftarrow r \cup \{\text{OUT}\}$   
**if**  $\min_{m \in S} m_1 < \min_{m \in M} m_1$  **then**  $r \leftarrow r \cup \{\text{IN}\}$   
**return**  $r$

---

it not  $\prec z$ . All the roots of  $g$  are *large*, so only the pole  $z = 0$  remains and  $\text{res}_z F_1 = \text{Res}_{z=0} F_1$ , by Proposition 5.2. Thus  $\text{res}_z F_1 = \frac{1}{1+d}$  and it follows that

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{dk}{n} = (-d)^n.$$

If we compute the Picard-Fuchs equation of  $F_1$ , we would find a differential equation of order  $d$  because the Picard-Fuchs operator annihilates *all* the periods, and the periods associated to the roots of  $g$  are algebraic of degree  $d - 1$ .

## 6. OPTIMIZATIONS

**6.1. Infinite sums.** So far, we have only considered binomial sums in which the bounds of the  $\sum$  symbols are finite and explicit. It is possible and desirable to consider infinite sums, or more exactly *syntactically* infinite sums which in fact reduce to finite sums. This often leads to simpler integral representations.

For example, in the sum  $\sum_{k=0}^n \binom{n}{k}$ , the upper summation bound  $n$  is not really useful when  $n \geq 0$ , we could as well write  $\sum_{k=0}^{\infty} \binom{n}{k}$ , which defines the same sequence. It is possible to adapt Algorithm 1 to handle infinite sums as long as the underlying summations in the field of iterated Laurent series are convergent. Recall that a geometric sum  $\sum_{n \geq 0} f^n$  converges in the field  $\mathbb{L}_d$  as soon as  $\text{lm}(f) \prec 1$ , see Lemma 2.1.

To compute an integral representation of binomial sums involving infinite summations, the principle is to proceed as in §2.2 except when an infinite geometric sum  $\sum_{n \geq 0}$  in  $\mathbb{L}_d$  shows up, where we check that  $\text{lm}(f) \prec 1$  so that Lemma 2.1 applies. If it does, then the summation is performed and the computation continues. If it does not, then the binomial sum is simply rejected.

**Example 9.** Let us devise an integral representation for the binomial sum  $u_n = \sum_{k=0}^{\infty} \binom{n}{k}$ . Note that Algorithm 1 applied to  $\sum_{k=0}^n \binom{n}{k}$  returns

$$\sum_{k=0}^n \binom{n}{k} = [1] \left( (1+z)^n \frac{z - z^{-n}}{z - 1} \right).$$

---

*Algorithm 4.* Geometric reduction

**Input:**  $F(z_{1 \setminus d}, v)$ , a rational function; an integer  $0 \leq k \leq d$ .

**Output:** FAIL or  $S(z_{1 \setminus d})$ .

**Specification:** If  $S(z_{1 \setminus d})$  is returned and if  $z_k \prec v \prec z_{k+1}$  then  $\text{res}_v F = S$ .

---

**function** GEOMRED( $F, k$ )

Decompose  $F$  as  $\sum_{i=1}^r a_i / f_i^{n_i} + P(v)$  where the  $f_k$ 's are irreducible polynomials in  $\mathbb{K}[z_{1 \setminus d}, v]$ ,  $a_k \in \mathbb{K}(z_{1 \setminus d})[v]$  and  $P \in \mathbb{K}(z_{1 \setminus d})[v]$

$S \leftarrow 0$

**for**  $i$  from 1 to  $r$  **do**

$\tau \leftarrow \text{ALLLARGEORALLSMALL}(\{\text{exponents of the monomials of } f_i\}, k)$

**if**  $\tau \subset \{\text{IN}\}$  **then**  $S \leftarrow \text{Res}_{v=\infty}(a_i / f_i^{n_i})$

**if**  $\tau = \{\text{IN}, \text{OUT}\}$  **then return** FAIL

**return**  $S$

---

We proceed as in §2.2 except that infinite geometric sums in  $\mathbb{L}_d$  are valid when Lemma 2.1 applies. Firstly  $\binom{n}{k} = [1](1+z)^n z^{-k}$ . Then

$$u_n \stackrel{?}{=} [1] \sum_{k=0}^{\infty} (1+z)^n z^{-k}.$$

Unfortunately, since  $1/z \succ 1$ , the sum does not converge, so the binomial sum is rejected. This is disappointing but the problem can be fixed in two different ways. The first way is to exploit the fact that  $\binom{n}{k} = \binom{n}{n-k}$ :

$$u_n = \sum_{k=0}^{\infty} [1](1+z)^n z^{k-n-1} = [1] \sum_{k=0}^{\infty} \frac{1}{z} \left(1 + \frac{1}{z}\right)^n z^k = [1] \frac{1}{z(1-z)} \left(1 + \frac{1}{z}\right)^n,$$

where this time the sum converges. The second way is explained in the next paragraph.

**6.2. Building blocks.** The definition  $\binom{n}{k} = [1](1+z)^n z^{-k}$  is arbitrary and it has the drawback that the infinite sum  $\sum_{k=0}^{\infty} \binom{n}{k}$  has to be rejected. In our current implementation<sup>2</sup>, we use the following:

$$\binom{n}{k} = [1] \frac{1}{(1-z)^{k+1} z^{n-k}}.$$

Besides, we may add new building blocks. For example, the expression  $\binom{n+k}{k}$  often appears in binomial sums. It is often a good idea not to write it as  $[1] \frac{1}{(1-z)^{k+1} z^n}$  but rather as  $[1] \frac{x^{-k} y^{-n}}{1-x-y}$ , which is an equivalent expression when  $n, k \geq 0$ . Even though this formula uses two variables instead of one it often leads to simpler integral representations after the geometric reduction step. See §7.1 for an example.

Another example, if one is working with the Motzkin numbers

$$M_n \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} (-1)^{n-k} \frac{1}{k+2} \binom{2k+2}{k+1} \binom{n}{k},$$

which does not form obviously a binomial sum because of the division by  $k+2$ , then one might want to add the new building block

$$M_n = [1] \frac{(1-z)(1+z)^2}{z t^n ((z^2 + z + 1)t - z)}, \text{ with } t \prec u_1.$$

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<sup>2</sup><https://github.com/lairer/binomsums>

More generally, one can add any sequence in the form  $\underline{n} \mapsto [1]R_0R_1^{n_1} \cdots R_d^{n_d}$ , where  $R_{0 \cup d}$  are rational functions, as a new building block.

**6.3. Summation over a polyhedron.** As in §3.2, let  $\Gamma \subset \mathbb{R}_{\geq}^{d+e}$  be a rational polyhedron and  $u : \mathbb{Z}^{d+e} \rightarrow \mathbb{K}$  be a binomial sum. Let us consider the sequence

$$v : \underline{n} \in \mathbb{N}^d \mapsto \sum_{\underline{m} \in \mathbb{Z}^e} u_{\underline{n}, \underline{m}} \mathbf{1}_{\Gamma}(\underline{n}, \underline{m}),$$

where  $\mathbf{1}_{\Gamma}(\underline{n}, \underline{m}) = 1$  if  $(\underline{n}, \underline{m}) \in \Gamma$  and else 0. Under an additional finiteness hypothesis on  $\Gamma$ , Lemma 3.12 shows that  $v$  is a binomial sum, and according to Corollary 2.7, there exists a rational function  $R(t_{1 \cup d}, z_{1 \cup r})$  such that

$$\sum_{\underline{n} \in \mathbb{N}^d} v_{\underline{n}} t^{\underline{n}} = \operatorname{res}_{z_{1 \cup r}} R.$$

It is possible to circumvent the construction used in the proof of Lemma 3.12 and compute directly such a rational function  $R$  given two things: firstly, the generating function of  $\Gamma$

$$\varphi_{\Gamma}(t_{1 \cup d}, s_{1 \cup e}) \stackrel{\text{def}}{=} \sum_{(\underline{n}, \underline{m}) \in \mathbb{Z}^{d+e}} \mathbf{1}_{\Gamma}(\underline{n}, \underline{m}) t^{\underline{n}} s^{\underline{m}},$$

which is known to be a rational function [e.g. 13]; and secondly, a representation of the binomial sum  $u$  as

$$u_{\underline{n}, \underline{m}} = [1]RT_1^{n_1} \cdots T_d^{n_d} S_1^{m_1} \cdots S_e^{m_e}$$

for some rational functions  $R, T_{1 \cup d}$  and  $S_{1 \cup e} \in \mathbb{K}(z_{1 \cup r})$ . Then, with  $t_{1 \cup e} \prec z_{1 \cup d}$ ,

$$\begin{aligned} \sum_{\underline{n} \in \mathbb{N}^d} v_{\underline{n}} t^{\underline{n}} &= [1]R \sum_{(\underline{n}, \underline{m}) \in \mathbb{Z}^{d+e}} \mathbf{1}_{\Gamma}(\underline{n}, \underline{m}) (t_1 T_1)^{n_1} \cdots (t_d T_d)^{n_d} S_1^{m_1} \cdots S_e^{m_e} \\ &= \operatorname{res}_{z_{1 \cup r}} \frac{R \cdot \varphi_{\Gamma}(t_1 T_1, \dots, t_d T_d, S_1, \dots, S_d)}{z_1 \cdots z_d}, \end{aligned}$$

provided that the sums converge in  $\mathbb{L}_{d+e}$ . This method is interesting because it is known how to compute efficiently compact representations of the rational function  $\varphi_{\Gamma}$  [8].

## 7. APPLICATIONS

**7.1. Andrews-Paule identity.** We show in detail the proof of the following identity:

$$(11) \quad \sum_{i=0}^n \sum_{j=0}^n \binom{i+j}{j}^2 \binom{4n-2i-2j}{2n-2i} = (2n+1) \binom{2n}{n}^2.$$

This identity appeared first as a problem in the *American Mathematical Monthly* [9] and was subsequently proved by Andrews and Paule [6] and Wegschaider [53] using various tools from the method of creative telescoping. It attracted attention because the theory of creative telescoping was unable to give a complete automated proof.

Let  $u_n$  denote the left-hand side. It can be written as an infinite sum

$$u_n = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \binom{i+j}{j}'^2 \binom{(2n-2i)+(2n-2j)}{2n-2i}' ,$$

where  $\binom{n}{k}'$  is the *natural* binomial defined by  $\binom{n}{k}' = [1](1-x-y)^{-1}/x^k/y^{n-k}$ . It differs from the binomial coefficient defined in §1.1 when  $n < 0$ : in that case  $\binom{n}{k}'$  vanishes whereas  $\binom{n}{k}$  need not vanish. Of course, we could have stuck to the former definition and used finite sums, but while the natural binomials introduce two variables instead of one in the integral representation, the integral representation

obtained after the geometric reduction step is simpler when using the natural binomial.

So we obtain the following integral representation:

$$\sum_{n \geq 0} u_n t^n = \operatorname{res}_{z_1 \succ 6} \frac{z_1 z_2}{(z_1^2 z_2^2 - t)(z_4 z_6 - z_1^2)(z_3 z_5 - z_2^2)(1 - z_3 - z_4)(1 - z_5 - z_6)(1 - z_1 - z_2)},$$

with the ordering  $t \prec z_1 \prec z_2 \prec z_3 \prec z_4 \prec z_5 \prec z_6$ . As expected, each binomial coefficient brings two extra variables so we end up with six variables in addition to the parameter  $t$ .

Geometric reduction applies successively with respect to the variables  $z_1, z_3, z_4$  and  $z_5$ . For example, the poles w.r.t. the variable  $z_1$ , are  $\{\pm t z_2^{-1/2}\}$ ,  $\{\pm z_4^{1/2} z_6^{1/2}\}$  and  $\{1 - z_2\}$ , gathered by conjugacy classes. The first pair of conjugate poles is  $\prec z_1$  whereas the second one is  $\succ z_1$ . The rational root  $1 - z_2$  is  $\succ z_1$ . Thus, by application of Proposition 5.2,

$$\sum_{n \geq 0} u_n t^n = \operatorname{res}_{z_2 \succ 6} \frac{(1 - z_2) z_2^3}{(t - (1 - z_2)^2 z_2^2)(1 - z_3 - z_4)(z_2^2 - z_3 z_5)(1 - z_5 - z_6)(t - z_2^2 z_4 z_6)}.$$

In the end, repeated application of Algorithm 4 leads to

$$\sum_{n \geq 0} u_n t^n = \operatorname{res}_{z_4, z_6} \frac{(z_6 - 1) z_6^3}{(t - z_6^2 (z_6 - 1)^2)((z_4 - 1)t - z_4 z_6^2 (z_6^2 + z_4 - 1))}.$$

Using Lairez's algorithm [39], we obtain (in about one second) a differential operator annihilating  $\sum_{n \geq 0} u_n t^n$ :

$$\begin{aligned} & 16t^4(256t^2 + 736t + 81)(16t - 1)^2 \partial_t^6 \\ & + 16t^3(16t - 1)(86016t^3 + 256256t^2 + 20976t - 1053) \partial_t^5 \\ & + 4t^2(36601856t^4 + 113760256t^3 + 6103168t^2 - 908088t + 14823) \partial_t^4 \\ & + 16t(22691840t^4 + 75716608t^3 + 6677824t^2 - 459552t + 3645) \partial_t^3 \\ & + (305827840t^4 + 1109626112t^3 + 139138736t^2 - 4247073t + 9720) \partial_t^2 \\ & + (60272640t^3 + 244005120t^2 + 42117840t - 374625) \partial_t + 691200t^2 + 3369600t + 996300 \end{aligned}$$

The roots of the indicial equation are 0, 1,  $-\frac{1}{2}$  et  $\frac{1}{2}$ . This differential operator corresponds to the Picard-Fuchs equation associated to the integral representation, but it is not the minimal-order operator annihilating  $\sum_{n \geq 0} u_n t^n$ . This happens sometimes. Of course, this is not a issue as long as we do obtain a differential equation.

Concerning the right-hand side, we find the integral representation

$$\sum_{n \geq 0} (2n + 1) \binom{2n}{n} t^n = \operatorname{res}_{z_1, z_2} \frac{t + u_2(u_2 - 1)u_1(u_1 - 1)}{(t - u_2(u_2 - 1)u_1(u_1 - 1))^2},$$

which cannot be simplified further with the geometric reduction. We compute (in about 0.1s) the annihilating operator:  $t(16t - 1)\partial_t^2 + (48t - 1)\partial_t + 12$ . The Andrews-Paule identity follows with the equality test described in §4.2.

In this case the right-hand side is a hypergeometric sequence, so it can be *discovered* automatically: the differential equation of order 6 leads to a recurrence relation of order 4 for  $u_n$  from which Petkovšek's algorithm [47] finds the hypergeometric solutions and the initial conditions are enough to identify the right-hand side.

**7.2. Several known identities.** This section shows the integral representations and the Picard-Fuchs equations appearing in the proofs of known identities. The integral representations have been obtained with the method presented in this article and the variants presented in §6. Note that the computation of the annihilating operator never takes longer than 4 seconds.

$$7.2.1. \text{ Strehl [50]. } \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \sum_{j=0}^k \binom{k}{j}^3.$$

This identity relates the Apéry numbers (left) and Franel numbers (the inner sum in the right-hand side).

$$\text{g.f.l.h.s}^3 = \text{res}_{z_1 \sim z_3} \frac{1}{(1-z_1)(1-z_2)(1-z_3)z_1z_2z_3 - t(z_1 + z_2z_3 - z_1z_2z_3)}$$

$$\text{g.f.r.h.s}^4 = \text{res}_{z_1 \sim z_3} \frac{1}{(1-z_1)(1-z_2)(1-z_3)z_1z_2z_3 - t(1-z_3 - z_2(1-(2+z_1(1-z_2))(1-z_3))z_3)}$$

$$\text{ann. op.}^5 = t^2(t^2 - 34t + 1)\partial_t^3 + 3t(2t^2 - 51t + 1)\partial_t^2 + (7t^2 - 112t + 1)\partial_t + t - 5$$

7.2.2. [33, p. 33; 53, §5.7.6].

$$\sum_{r \geq 0} \sum_{s \geq 0} (-1)^{n+r+s} \binom{n}{r} \binom{n}{s} \binom{n+s}{s} \binom{n+r}{r} \binom{2n-r-s}{n} = \sum_{k \geq 0} \binom{n}{k}^4$$

$$\text{g.f.l.h.s} = \text{res}_{z_1 \sim z_3} \frac{1}{(1-z_1)(1-z_2)(1-z_3)z_1z_2z_3 + (z_2 - z_3)(z_1 - z_3)t}$$

$$\text{g.f.r.h.s} = \text{res}_{z_1 \sim z_3} \frac{1}{(1-z_1)(1-z_2)(1-z_3)z_1z_2z_3 + (1-z_1 - z_2)z_3t - (1-z_1)(1-z_2)t}$$

$$\text{ann. op.} = t^2(t^2 - 34t + 1)\partial_t^3 + 3(2t^2 - 51t + 1)\partial_t^2 + (7t^2 - 112t + 1)\partial_t + t - 5$$

7.2.3. *Dent.* The following identity is due to Dent and used as an example by Wegschaider [53, p. 90]:

$$\sum_{k=0}^{n_1+2n_2} \sum_{j \geq 0} (-1)^j \binom{k}{j} \binom{2n_2 + n_1 - k}{2n_2 - j} \binom{n_1}{k-j} = 2^{n_1} \binom{n_1 + n_2}{n_1}, \text{ with } n_1, n_2 \geq 0.$$

$$\text{g.f.l.h.s} = \text{g.f.r.h.s} = \frac{1}{1 - 2t_1 - t_2}$$

Here, the generating series is rational and geometric reduction performs the entire computation, there is no need to compute a Picard-Fuchs equation.

$$7.2.4. \text{ Dixon [24]. } \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = (-1)^n \frac{(3n)!}{n!^3}$$

$$\text{g.f.l.h.s} = \text{res}_{z_1 \sim z_2} \frac{(1-z_2)(1-z_1)z_1z_2}{z_1^2z_2^2(1-z_2)^2(1-z_1)^2 - (1-z_1-z_2)^2t}$$

$$\text{g.f.r.h.s} = \text{res}_{z_1 \sim z_3} \frac{1}{t + z_1z_2(1-z_1-z_2)}$$

$$\text{ann. op.} = t(27t + 1)\partial_t^2 + (54t + 1)\partial_t + 6$$

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<sup>3</sup>Generating function of the left-hand side

<sup>4</sup>Generating function of the right-hand side

<sup>5</sup>Annihilating operator

$$7.2.5. \text{ Moriarty [25, p. 11]. } \sum_{k=n_1}^{n_2} (-4)^k \binom{k}{m} \frac{n}{n+k} \binom{n+k}{2k} = (-1)^{n_1} 4^{n_1} \frac{n}{n+m} \binom{n+m}{2m}$$

Because of the division by  $n+m$ , the right-hand side is not obviously a binomial sum. However, it becomes obvious after observing that

$$\frac{n}{n+k} \binom{n+k}{2k} = \binom{n+k-1}{2k} + \frac{1}{2} \binom{n+k-1}{2k-1}.$$

$$\text{g.f.l.h.s} = \text{g.f.r.h.s} = \frac{1}{2} \frac{(1-t_1)(1+t_1)}{t_1^2 + 4t_1t_2 + 2t_1 + 1}$$

Here again it is a rational power series and the geometric reduction finds it.

7.2.6. *Davletshin, Egorychev, and Krivokolesko [23, Theorem 1.2].*

$$1 + \sum_{q=1}^{\infty} 2^{q-1} \binom{n_2}{q} \left( \sum_{m=1}^{\lfloor n_1/2 \rfloor} \binom{m-1}{q-1} + \sum_{m=1}^{n_1} \binom{m-1}{q-1} \right) = \sum_{q=1}^{\infty} 2^{q-1} \binom{n_2}{q} \left( \binom{n_1}{q} + \binom{\lfloor n_1/2 \rfloor}{q} \right)$$

$$\text{g.f.l.h.s} = \text{g.f.r.h.s} = \frac{t_1t_2(1+t_1-t_2-t_1t_2-2t_1^2-2t_1^2t_2)}{(1-t_2)(1-t_1)(1-t_2-t_1^2t_2-t_1^2)(1-t_1-t_2-t_1t_2)}$$

The summation bound  $n_1/2$  and the integer part  $\lfloor n_1/2 \rfloor$  may look problematic, but

$$\sum_{m=1}^{\lfloor n_1/2 \rfloor} \binom{m-1}{q-1} = \sum_{m=1}^{\infty} \binom{m-1}{q-1} H_{n-2m} \quad \text{and} \quad \binom{\lfloor n_1/2 \rfloor}{q} = \sum_{k=0}^{\infty} \binom{k}{q} (\delta_{2k-n} + \delta_{2k+1-n}),$$

using the binomial sums  $\delta$  and  $H$  defined in §1.

7.2.7. *Davletshin, Egorychev, and Krivokolesko [23, Theorem 1.1].*

$$2 + \sum_{q=1}^{\infty} \binom{n_2}{q} \left( \sum_{m=1}^{\lfloor n_1/2 \rfloor} \binom{m-1}{q-1} + \sum_{k_1=1}^{\infty} \cdots \sum_{k_q=1}^{\infty} \delta_{k_1+\dots+k_q-m} \prod_{i=1}^q (k_i + 1) \right) = \binom{n_1 + 2n_2}{2n_2} + \binom{\lfloor n_1/2 \rfloor + n_2}{n_2}$$

$$\text{g.f.l.h.s} = \text{g.f.r.h.s} = \frac{t_1t_2(1-t_2-2t_1^2+t_1^3-2t_1^2t_2)}{(1-t_2)(1-t_1)(1-t_2-t_1^2t_2)(1-t_2-2t_1+t_1^2)}$$

Again, the left-hand side does not have the appearance of a binomial sum until we remark that

$$\sum_{k_1=1}^{\infty} \cdots \sum_{k_q=1}^{\infty} \delta_{k_1+\dots+k_q-m} \prod_{i=1}^q (k_i + 1) = [1] \left( \left( \frac{t(2-t)}{(1-t)^2} \right)^q \frac{1}{t^m} \right).$$

### 7.3. Proof of some conjectures.

7.3.1. *Le Borgne's identity.* The following identity for Baxter's numbers arises as a conjecture in an unpublished work by Yvan Le Borgne. With the methods presented here we can prove it automatically.

$$1 + F_n^{-1,-1} + 2F_n^{0,0} - F_n^{0,1} + F_n^{1,0} - 3F_n^{1,1} + F_n^{1,2} - F_n^{3,1} + 3F_n^{3,2} - F_n^{3,3} - 2F_n^{4,2} + F_n^{4,3} - F_n^{5,2} = \sum_{m=0}^n \frac{\binom{n+2}{m} \binom{n+2}{m+1} \binom{n+2}{m+2}}{\binom{n+2}{1} \binom{n+2}{2}},$$

$$\text{where } F_n^{a,b} = \sum_{d=0}^{n-1} \sum_{c=0}^{d-a} \binom{d-a-c}{c} \binom{n}{d-a-c} \left( \binom{n+d+1-2a-2c+2b}{n-a-c+b} - \binom{n+d+1-2a-2c+2b}{n+1-a-c+b} \right).$$

The sum in the right-hand side do not have the appearance of a binomial sum. To check the identity, one can either use the known recurrence relation for these numbers or use the equality

$$\frac{\binom{n+2}{m}\binom{n+2}{m+1}\binom{n+2}{m+2}}{\binom{n+2}{1}\binom{n+2}{2}} = \det \begin{pmatrix} \binom{n}{k} & \binom{n}{k+1} & \binom{n}{k+2} \\ \binom{n}{k-1} & \binom{n}{k} & \binom{n}{k+1} \\ \binom{n}{k-2} & \binom{n}{k-1} & \binom{n}{k} \end{pmatrix},$$

which can be checked automatically after multiplication by  $\binom{n+2}{1}\binom{n+2}{2}$ .

**7.3.2. Identities from Brent, Ohtsuka, Osborn, and Prodinger.** We have been able to prove the following identities conjectured in [12]. The left-hand sides involve absolute values of nonlinear polynomials. They are nevertheless binomial sums for two reasons. The first one is that all the nonlinear polynomials under consideration split into linear or positive factors. For example  $|i^3 - j^3| = |i - j|(i^2 + ij + j^2)$  and  $|i - j| = (i - j)(H_{i-j} - H_{j-i})$  is a binomial sum.

The second reason, which we used for the computation, is that we can eliminate the absolute values by using the symmetries of the sums. For example

$$\sum_{i,j} \binom{2n}{n+i} \binom{2n}{n+j} |i^3 - j^3| = 4 \sum_{i=0}^n \sum_{j=-i}^{i-1} \binom{2n}{n+i} \binom{2n}{n+j} i^3.$$

The formal integral representations obtained are too lengthy to be presented here, they can be found online<sup>6</sup>. Since the right-hand sides are hypergeometric sequences, they can be computed from the recurrence relations satisfied by the left-hand sides with Petkovšek's algorithm [47].

$$[12, \text{Eq. 5.7}] \quad \sum_{i,j} \binom{2n}{n+i} \binom{2n}{n+j} |i^3 - j^3| = \frac{2n^2(5n-2)}{4n-1} \binom{4n}{2n}$$

$$[12, \text{Eq. 5.8}] \quad \sum_{i,j} \binom{2n}{n+i} \binom{2n}{n+j} |i^5 - j^5| = \frac{2n^2(43n^3 - 70n^2 + 36n - 6)}{(4n-1)(4n-3)} \binom{4n}{2n}$$

$$[12, \text{Eq. 5.9}] \quad \sum_{i,j} \binom{2n}{n+i} \binom{2n}{n+j} |i^7 - j^7| = \frac{2n^2(531n^5 - 1960n^4 + 2800n^3 - 1952n^2 + 668n - 90)}{(4n-1)(4n-3)(4n-5)} \binom{4n}{2n}$$

$$[12, \text{Eq. 5.12}] \quad \sum_{i,j} \binom{2n}{n+i} \binom{2n}{n+j} |ij(i^2 - j^2)| = \frac{2n^3(n-1)}{2n-1} \binom{2n}{n}^2$$

$$[12, \text{Eq. 5.14}] \quad \sum_{i,j} \binom{2n}{n+i} \binom{2n}{n+j} |i^3 j^3 (i^2 - j^2)| = \frac{2n^4(n-1)(3n^2 - 6n + 2)}{(2n-3)(2n-1)} \binom{2n}{n}^2$$

**7.4. Computational limitations.** The method is mainly limited by the integration step. When the integral representation has more than four variables (in addition to the parameter and after the geometric reduction), then computation of the Picard-Fuchs equation becomes challenging. For example, in Strehl's second identity

$$\sum_{k=0}^n \binom{n}{k}^3 \binom{n+k}{k}^3 = \sum_{k=0}^{\infty} \binom{n}{k} \binom{n+k}{k} \sum_{i=0}^{\infty} \binom{k}{i}^2 \binom{2i}{i}^2 \binom{2i}{k-i},$$

<sup>6</sup><https://github.com/lairer/binomsums>



the integral representation of the generating function of each side has five variables and a parameter. For the left-hand side, we obtain

$$\operatorname{res}_{z_1 \sim z_5} \frac{1}{(z_1 z_3 z_4 z_5 + z_2 z_3 z_4 z_5 - z_1 z_2 z_3 z_4 z_5 - z_3 z_4 z_5 + z_1 z_2) t + z_1 z_2 z_3 z_4 z_5 (1-z_1)(1-z_2)(1-z_3)(1-z_4)(1-z_5)}.$$

The formal integral representation of the right-hand side is more complicated but still has only five variables, in addition to the parameter. The computation would require several hours with the current algorithms. Without the geometric reduction, it involves nine variables and a parameter.

Another limitation concerns the computation of binomial sums with several parameters. The method still applies but the integration step is considerably harder.

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