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Kähler-Einstein metrics on group compactifications

Thibaut Delcroix

Abstract

We obtain a necessary and sufficient condition of existence of a Kähler-Einstein metric on a $G \times G$ -equivariant Fano compactification of a complex connected reductive group G in terms of the associated polytope. This condition is not equivalent to the vanishing of the Futaki invariant. The proof relies on the continuity method and its translation into a real Monge-Ampère equation, using the invariance under the action of a maximal compact subgroup $K \times K$.

Introduction

The question of the existence of a Kähler-Einstein metric on a toric Fano manifold, that is, a n -dimensional Fano manifold X acted upon by $(\mathbb{C}^*)^n$ with an open dense orbit isomorphic to $(\mathbb{C}^*)^n$, was fully settled by Wang and Zhu in [WZ04]. They showed that there exists a Kähler-Einstein metric on X if and only if the Futaki invariant of X vanishes. This invariant was known [Mab87] to be linked with the barycenter of the polytope associated to X , so that the condition is equivalent to the fact that this barycenter is the origin.

We consider the existence of Kähler-Einstein metrics on a natural generalization of toric Fano manifolds: the bi-equivariant Fano compactifications of complex connected reductive groups. More precisely, given such a group G , we consider the Fano manifolds X which admit a $G \times G$ -action with an open dense orbit isomorphic to G as a $G \times G$ -homogeneous space under left and right translations. We will call these, for simplicity, *group compactifications*. To such a manifold is, as in the toric case, associated a polytope P^+ , that encodes the information about the boundary $X \setminus G$ and the anticanonical line bundle.

Let G be a complex connected reductive group, choose T a maximal torus in G . Let M denote the group of characters of T , and let $\Phi \subset M$ be the root system of (G, T) . Choose a system of positive roots Φ^+ , denote by 2ρ the sum of positive roots, and by Ξ the relative interior of the cone generated by Φ^+ . The polytope P^+ associated to X is in the positive Weyl chamber in $M \otimes \mathbb{R}$. Let $\text{bar}_{DH}(P^+)$ denote the barycenter of P^+ with respect to the measure $\prod_{\alpha \in \Phi^+} \langle \alpha, p \rangle^2 dp$ where dp is the Lebesgue measure on $M \otimes \mathbb{R}$ normalized by the lattice M . Our main result is the following characterization of Kähler-Einstein Fano group compactifications in terms of the polytope P^+ .

Theorem A. *Let X be a Fano compactification of G , then X admits a Kähler-Einstein metric if and only if $\text{bar}_{DH}(P^+) \in 2\rho + \Xi$.*

Let us remark here that P^+ is related to the moment polytope of X from a symplectic point of view, and that $\text{bar}_{DH}(P^+)$ is the barycenter with respect to the Duistermaat-Heckman measure on this moment polytope.

The condition obtained is computable given an explicit polytope, and we provide in this article both a new example of Kähler-Einstein metric and an example of group compactification without any Kähler-Einstein metric. We refer to the author's PhD thesis [Del15], or new preprint [Dela], for more examples. The example that does not satisfy the condition is especially interesting because its Futaki invariant vanishes. In light of the proof of the Yau-Tian-Donaldson conjecture in the Fano case ([CDS15a, CDS15b, CDS15c] and [Tia15]), and the recent work of Datar and Székelyhidi [DS], it means that there is a non-trivial $G \times G$ -equivariant destabilizing test configuration.

In fact, in [Dela] we obtain through essentially algebraic methods a combinatorial criterion for equivariant K-stability of a Fano spherical variety. Combined with [DS] it provides a combinatorial criterion for the existence of Kähler-Einstein metrics on spherical manifolds. Group compactifications are examples of spherical varieties and the criterion from [Dela] specializes to the criterion from Theorem A in this case. Thus the main result of [Dela] provides a broad generalization of Theorem A. On the other hand, the criterion obtained here served as the guiding intuition for [Dela]. Furthermore, the combination of the two articles provide an independent, and low tech, in the sense that no Gromov-Hausdorff convergence result is needed, proof of the Yau-Tian-Donaldson conjecture for Fano group compactifications.

It should be noted that in [Dela] a criterion is also obtained for the existence of Kähler-Ricci solitons on spherical manifolds, hence on group compactifications. The analytic methods from the present article could easily be extended to the case of Kähler-Ricci solitons to provide another proof. For simplicity, we do not provide details for this case.

It was mentioned to the author by Gábor Székelyhidi that the example with no Kähler-Einstein is also interesting in the study of the Kähler-Ricci flow on Fano manifolds. Indeed, it follows from the work of Chen, Sun and Wang in [CSW] that the Kähler-Ricci flow on a Fano manifold X with no Kähler-Einstein metrics should converge to some Kähler-Ricci soliton on the central fiber of a special test configuration for X . It is reasonable to expect, in the situation where X admits an action of a compact group and the initial metric is invariant, that the corresponding test configuration is equivariant. In our example, the possible central fibers are known (see [AK05, AB04]), and are all singular. No example of such a behavior for the Kähler-Ricci flow is known, so it would be interesting to study this in detail.

Let us mention now a consequence of our result in the variational setting. Darvas and Rubinstein proved in [DR] the equivalence between the existence of Kähler-Einstein metrics and various notions of properness of functionals on spaces of Kähler potentials, allowing to take into account automorphisms of

the manifold. Our result provides a combinatorial criterion for these notions of properness. More precisely, though we refer to [DR] for details, let ω be a reference Kähler form in $2\pi c_1(X)$, and let \mathcal{H} denote the space of Kähler potentials, *i.e.* functions ψ on X such that $\omega + i\partial\bar{\partial}\psi$ is still a Kähler form. Assume X is Kähler-Einstein and $\mathbb{G} = \text{Aut}^0(X)$ is the neutral component of the automorphism group of X . Let \mathbb{K} be a maximal compact subgroup of \mathbb{G} , assume that ω is \mathbb{K} -invariant and let $\mathcal{H}^{\mathbb{K}}$ be the space of \mathbb{K} -invariant Kähler potentials. Then Darvas and Rubinstein prove that the Mabuchi functional is proper on \mathcal{H} modulo the action of \mathbb{G} , and that it is proper on $\mathcal{H}^{\mathbb{K}}$ provided the center of \mathbb{G} is finite. In the case of group compactifications, the automorphism group may very well have a center that is not finite, as it is the case for many toric manifolds. It is interesting to note that the methods from [DR] should allow to prove a statement combining the two statements recalled above, valid with no hypotheses on the automorphism group, namely that the Mabuchi functional is proper on $\mathcal{H}^{\mathbb{K}}$ modulo the action of the normalizer $N_{\mathbb{G}}(\mathbb{K})$ of \mathbb{K} in \mathbb{G} .

Let us now describe our methods. The guiding ideas are to restrict to the open dense orbit G , and consider only $K \times K$ -invariant metrics. The last part is possible because of Matsushima's theorem [Mat57] asserting that the isometry group of a Kähler-Einstein metric is a maximal compact subgroup of the connected group of automorphisms of X . We can then combine the two by using the classical KAK decomposition of reductive groups. This decomposition identifies the quotient of G by $K \times K$ with the closed positive Weyl chamber. Another point of view is that it translates $K \times K$ -invariant geometry on G to W -invariant geometry on $M \otimes \mathbb{R}$, where W is the Weyl group of Φ .

We used this fact in [Delb] to associate to any positively curved hermitian metric on the anticanonical line bundle $-K_X$ a W -invariant convex function, whose asymptotic behavior is controlled by the polytope P^+ . We use this here to translate the Kähler-Einstein equation into a real Monge-Ampère equation on a real vector space. A key point in this process is to compute the complex Hessian of a $K \times K$ -invariant function φ on G in terms of the function it determines on the positive Weyl chamber. This computation then provides an expression of the complex Monge-Ampère $\text{MA}_{\mathbb{C}}(\varphi)$, which is the determinant of the complex Hessian.

Proposition B. *Let φ be a $K \times K$ -invariant smooth function $G \rightarrow \mathbb{R}$, and let u denote the corresponding W -invariant function on $M \otimes \mathbb{R}$. Then for any x in the interior of the positive Weyl chamber,*

$$\text{MA}_{\mathbb{C}}(\varphi)(\exp x) = \text{MA}_{\mathbb{R}}(u)(x) \prod_{\alpha \in \Phi^+} \frac{\langle \alpha, \nabla u(x) \rangle^2}{\sinh^2 \langle \alpha, x \rangle},$$

where $\text{MA}_{\mathbb{R}}(u)$ denotes the determinant of the real Hessian of u .

We use this result to obtain an expression of the potential, with respect to a Haar measure on G , of the restriction to G of a volume form on X determined by a hermitian metric on $-K_X$, in terms of the convex potential of the hermitian

metric. This allows to translate the restriction of the Kähler-Einstein equation to G as a real Monge-Ampère equation in a convex function u reading:

$$\mathrm{MA}_{\mathbb{R}}(u) \prod_{\alpha \in \Phi^+} \langle \alpha, \nabla u \rangle^2 = e^{-u} J,$$

where J is the function defined by $J(x) = \prod_{\alpha \in \Phi^+} \sinh^2 \langle \alpha, x \rangle$.

Another consequence of our computation of the complex Hessian is an analytic derivation of a formula due to Brion [Bri89] and Kazarnovskii [Kaz87] for the degree of an ample line bundle on a group compactification, however up to a multiplicative constant, which does not matter for our barycenters considerations.

Let us digress here to mention that the translation of the Kähler-Einstein equation into a real Monge-Ampère equation was already a key step in the toric case, but also in the case of the horospherical manifolds studied by Podestà and Spiro [PS10]. These manifolds, along with the group compactifications belong to the large family of spherical manifolds, which was highlighted by Donaldson in his survey [Don08] as an interesting family on which to study the existence of Kähler-Einstein metrics. We believe that methods similar to ours can be used for even more spherical manifolds. On another direction, Donaldson suggests not only to study the existence of Kähler-Einstein metrics, but more generally of canonical metrics such as constant scalar curvature. We expect that our computation of the full complex Hessian can be used in relation with this problem. Note that Abreu obtained a formula for the scalar curvature of torus invariant Kähler metrics on toric manifolds [Abr98] which led to important advances in the study of canonical metrics on toric manifolds.

The main difference between the case of toric or horospherical manifolds and the case of group compactifications here is the term J , which is independent on the function u . Indeed, the toric Monge-Ampère equation is just $\mathrm{MA}_{\mathbb{R}}(u) = e^{-u}$ and the horospherical case only adds well controlled gradient terms. We thus have to rework in depth the proof of Wang and Zhu to deal with the term J and the walls of the Weyl chamber. This is how we obtained our condition, to replace the information given by the Futaki invariant in the toric case.

The common part with Wang and Zhu is the use of the classical results on the continuity method, and the study of a well chosen proper and strictly convex function ν_t defined in terms of the solutions of the continuity method equation. In our case this function involves the term J . The key is then to interpret the absence of a priori estimates along the continuity method as the fact that the point at which the minimum of the function ν_t is attained is unbounded as t increases along the continuity method, then to interpret this as a condition on the polytope.

The precise analysis of this behavior further allows to determine the maximal time of existence of solutions along the continuity method. This is called the greatest Ricci lower bound as Székelyhidi [Szé11] proved that it is also the supremum of all $t < 1$ such that there exists a Kähler form in $c_1(X)$ with $\mathrm{Ric}(\omega) \geq t\omega$ where $\mathrm{Ric}(\omega)$ is the Ricci form of ω .

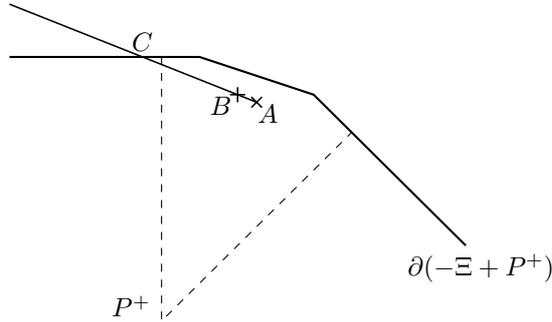


Figure 1: Greatest Ricci lower bound

Theorem C. *Assume there are no Kähler-Einstein metrics on the group compactification X and let $R(X)$ be the greatest Ricci lower bound of X . Then*

$$R(X) = \sup \left\{ t < 1 ; \frac{1}{1-t} (2\rho - t \cdot \text{bar}_{DH}(P^+)) \in -\Xi + P^+ \right\}.$$

The particular case of toric manifolds was obtained by Li [Li11]. Indeed, in the case of a Fano toric manifold X , the polytope P^+ is the usual moment polytope, the root system Φ is trivial, so that $2\rho = 0$, Ξ is restricted to the origin $\{0\}$ and the Duistermaat-Heckman barycenter $\text{bar}_{DH}(P^+)$ is just the barycenter of P^+ with respect to the Lebesgue measure. The condition in Theorem A thus translates as Wang and Zhu's condition, that is, the condition that the barycenter is the origin. To recover Chi Li's result, let us introduce additional notations in the case of groups, as illustrated in Figure 1: let A be $\text{bar}_{DH}(P^+)$, let B be 2ρ and let C denote the intersection of the boundary of $-\Xi + P^+$ with the half line starting from A in the direction of B if it exists. It follows from Theorem C that the intersection exists if and only if $R(X) < 1$ and is then the point $B + s\overrightarrow{AB}$ with $s = R(X)/(1 - R(X))$. We may then check that $|BC|/|AC| = s/(1 + s) = R(X)$. This is precisely the expression given in [Li11] in the toric case, up to a change in notations.

The structure of the article is as follows. Section 1 is devoted to the computation of the Hessian of a $K \times K$ -invariant function on a reductive group G . Proposition B is obtained as a consequence in Corollary 1.3. Section 2 is a short introduction to group compactifications, allowing to introduce the associated polytope and relevant examples. The tool of convex potentials, developed in [Delb], is recalled here and applied to obtain a formula for the degree of an ample linearized line bundle on a group compactification. Section 3 and Section 4 provide the general strategy of the proofs of Theorem A and Theorem B, and preliminary results. Finally, Section 5 contains the proof of the necessity of our condition and of an upper bound on $R(X)$, and Section 6 contains the proof that the condition is sufficient and that the upper bound on $R(X)$ is in fact equal to $R(X)$. Theorem A is a consequence of Proposition 5.3 and Theorem 6.7, and Theorem B is a consequence of Proposition 5.5 and Section 6.4.

The present article along with [Delb] contain the main results of the author's PhD thesis [Del15]. The author would like to thank his advisor Philippe Eyssidieux, and also Michel Brion for helpful discussions and the explanations on the automorphism group of the non Kähler-Einstein example.

Notations

Let us introduce here some notations on groups that will be used throughout the article. We use as reference for Lie groups the books [Kna02, Hel78], and [Bor91] for algebraic groups.

Let G be a connected complex linear reductive group. Let T be a maximal torus in G . We choose a maximal compact subgroup K in G such that $S := K \cap T$ is a maximal compact torus in T . We denote by \mathfrak{g} , \mathfrak{k} , \mathfrak{t} , \mathfrak{s} the respective Lie algebras of G , K , T , S .

Let Φ denote the root system of (G, T) and W the corresponding Weyl group. We choose a system of positive roots Φ^+ . We use the standard notation ρ for the half sum of the positive roots.

Since G is reductive, we have the *Cartan decomposition* $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$, and denote by θ the corresponding Cartan involution on \mathfrak{g} . Define $\mathfrak{a} := \mathfrak{t} \cap i\mathfrak{k}$. We identify \mathfrak{a} with $N \otimes \mathbb{R}$ where N is the group of one parameter subgroups of T . Denote by \mathfrak{a}^+ the positive *open* Weyl chamber defined by Φ^+ .

Recall the *KAK decomposition* for reductive groups [Kna02, Theorem 7.39]: every element $g \in G$ can be written $g = k_1 \exp(a)k_2$ with $a \in \mathfrak{a}^+$ uniquely determined by g , and $k_1, k_2 \in K$.

We can decompose \mathfrak{a} in a toric part and a semisimple part:

$$\mathfrak{a} = \mathfrak{a}_t \oplus \mathfrak{a}_{ss},$$

with $\mathfrak{a}_t = \mathfrak{z}(\mathfrak{g}) \cap i\mathfrak{k}$ and $\mathfrak{a}_{ss} = \mathfrak{a} \cap [\mathfrak{g}, \mathfrak{g}]$. The Killing form of \mathfrak{g} , restricted to \mathfrak{a}_{ss} , gives a scalar product, and we choose a scalar product $\langle \cdot, \cdot \rangle$ on \mathfrak{a} that coincides with the Killing form on \mathfrak{a}_{ss} and leaves \mathfrak{a}_{ss} and \mathfrak{a}_t orthogonal. We use this scalar product to identify \mathfrak{a} and its dual, which is also identified with $M \otimes \mathbb{R}$, where M is the group of characters of T .

1 Complex Monge-Ampère on G

The aim of this section is to compute, for any $K \times K$ -invariant function ψ on G , the function $\text{MA}_{\mathbb{C}}(\psi)$ such that

$$\left(\frac{i}{2\pi} \partial \bar{\partial} \psi \right)^n = \text{MA}_{\mathbb{C}}(\psi) dg,$$

where dg is a Haar volume form on G and n is the complex dimension of G . As the notation suggests, this function is the complex Monge-Ampère of ψ in some local coordinates, *i.e.* the determinant of the complex Hessian of ψ

in these coordinates. We express this Hessian and Monge-Ampère in terms of the restriction of ψ to $\exp(\mathfrak{a})$, using the $K \times K$ -invariance and the KAK -decomposition.

1.1 Complex Hessian matrix on G

We first choose a special basis of $\mathfrak{g} \simeq T_e G$. Recall the root decomposition:

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

where, if ad denotes the adjoint representation of \mathfrak{g} ,

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} ; \text{ad}(h)(x) = \langle \alpha, h \rangle x \quad \forall h \in \mathfrak{t}\}.$$

We can choose a set $\{e_\alpha\}_{\alpha \in \Phi}$ of elements of \mathfrak{g} such that $e_{-\alpha} = -\theta(e_\alpha)$, $[e_\alpha, e_{-\alpha}] = \alpha$ and e_α generates the complex line \mathfrak{g}_α [Hel78, Chapter VI, Lemma 3.1].

We would like to choose a complex basis of \mathfrak{g} which is also a real basis of \mathfrak{k} . Define, for $\alpha \in \Phi^+$,

$$\mathfrak{k}_\alpha := \left\{ x \in \mathfrak{k} ; \text{ad}(h)^2(x) = \langle \alpha, h \rangle^2 x \quad \forall h \in \mathfrak{s} \right\}.$$

Then each \mathfrak{k}_α is of real dimension two and we have $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} = \mathfrak{k}_\alpha \oplus i\mathfrak{k}_\alpha$. We obtain a real basis of \mathfrak{k}_α by considering $e_\alpha + \theta(e_\alpha) = e_\alpha - e_{-\alpha}$ and $ie_\alpha + \theta(ie_\alpha) = ie_\alpha + ie_{-\alpha}$. We complete this with a real basis of \mathfrak{s} to obtain a real basis of $\mathfrak{k} = \mathfrak{s} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{k}_\alpha$, and thus a complex basis of \mathfrak{g} . We denote this complex basis of \mathfrak{g} by $(l_j)_{j=1}^n$.

Since the exponential map is a biholomorphism from a neighborhood of $0 \in \mathfrak{g}$ to a neighborhood of the neutral element e in G , we get holomorphic coordinates on G near e . Then, using a translation on the right by $g \in G$, this defines holomorphic coordinates on a neighborhood of g . More precisely, the map corresponding to the local coordinates is the map $\mathbb{C}^n \rightarrow G$ defined by

$$(z_1, \dots, z_n) \mapsto \exp(z_1 l_1 + \dots + z_n l_n)g.$$

We will compute the complex Hessian with respect to these coordinates. If ψ is a function on G we denote by $\text{Hess}_{\mathbb{C}}(\psi)(g)$ the complex Hessian and by $\text{MA}_{\mathbb{C}}(\psi)(g)$ the determinant of the complex Hessian of ψ at g , called the complex Monge-Ampère, in both cases with respect to the coordinates given above.

If f is a function on \mathfrak{a} , then we denote by $\text{MA}_{\mathbb{R}}(f)(x)$ the determinant of its real Hessian at x , and by ∇f the gradient of f with respect to the scalar product $\langle \cdot, \cdot \rangle$ on \mathfrak{a} .

By the KAK decomposition, a $K \times K$ -invariant function ψ on G is completely determined by the function f defined on \mathfrak{a} by $f(x) = \psi(\exp x)$.

Let us begin by a remark that shows that even though we use a specific set of coordinates to compute the complex Hessian, the complex Monge-Ampère thus obtained is the potential of $\frac{i}{2\pi} \partial \bar{\partial} \psi$ with respect to a Haar measure, up to a multiplicative constant.

Remark 1.1. Let dg be a Haar volume form on G (thus a $G \times G$ -invariant volume form, because a reductive group is unimodular [Kna02, Section VIII.2]). Then there exists a constant $C > 0$ such that for any $K \times K$ -invariant function ψ on G , we have,

$$\left(\frac{i}{2\pi} \partial \bar{\partial} \psi \right)^n = C \text{MA}_{\mathbb{C}}(\psi) dg.$$

Indeed, in local coordinates z_j , we can write

$$\frac{i}{2\pi} \partial \bar{\partial} \psi = \sum_{j,k} \frac{i}{2\pi} \frac{\partial^2 \psi}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k,$$

so in the coordinates above, we have locally

$$\left(\frac{i}{2\pi} \partial \bar{\partial} \psi \right)^n = \frac{1}{(2\pi)^n} \text{MA}_{\mathbb{C}}(\psi) (i^n dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge dz_n \wedge d\bar{z}_n).$$

But by the construction of the coordinates, the global form defined locally as $i^n dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge dz_n \wedge d\bar{z}_n$ is a Haar volume form. To conclude, remark that all Haar volume forms are positive scalar multiples of one another.

Theorem 1.2. *Let ψ be a $K \times K$ invariant function on G , and f the associated function on \mathfrak{a} . Then in the coordinates above and for $a \in \mathfrak{a}^+$, the complex Hessian matrix of ψ is diagonal by blocks, equal to:*

$$\text{Hess}_{\mathbb{C}}(\psi)(\exp(a)) = \begin{pmatrix} \frac{1}{4} \text{Hess}_{\mathbb{R}}(f)(a) & 0 & & 0 \\ 0 & M_{\alpha_1}(a) & & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & & 0 \\ 0 & 0 & & M_{\alpha_p}(a) \end{pmatrix}$$

where the $(\alpha_i)_{i \in \{1, \dots, p\}}$ describe the positive roots of Φ and M_{α} is defined by:

$$M_{\alpha}(a) = \frac{1}{2} \langle \alpha, \nabla f(a) \rangle \begin{pmatrix} \coth \langle \alpha, a \rangle & i \\ -i & \coth \langle \alpha, a \rangle \end{pmatrix}.$$

As a corollary, we obtain Proposition B:

Corollary 1.3. *Let ψ be a $K \times K$ invariant function on G , and f the associated function on \mathfrak{a} . Then in the coordinates above and at $a \in \mathfrak{a}^+$, if r denotes the rank of G and p the number of positive roots, we have*

$$\text{MA}_{\mathbb{C}}(\psi)(\exp(a)) = \frac{1}{4^{r+p}} \text{MA}_{\mathbb{R}}(f)(a) \frac{1}{J(a)} \prod_{\alpha \in \Phi^+} \langle \alpha, \nabla f(a) \rangle^2$$

where we denote by J the function defined on \mathfrak{a} by

$$J(a) := \prod_{\alpha \in \Phi^+} \sinh^2 \langle \alpha, a \rangle.$$

Proof. Since $\text{MA}_{\mathbb{R}}(f)(a) = \det(\text{Hess}_{\mathbb{R}}(f)(a))$, we just have to compute the determinant of M_{α} . This is

$$\begin{aligned} \det(M_{\alpha}) &= \left(\frac{1}{2} \langle \alpha, \nabla f(a) \rangle \right)^2 (\coth^2 \langle \alpha, a \rangle - 1) \\ &= \frac{1}{4} \langle \alpha, \nabla f(a) \rangle^2 \frac{\cosh^2 \langle \alpha, a \rangle - \sinh^2 \langle \alpha, a \rangle}{\sinh^2 \langle \alpha, a \rangle} \\ &= \frac{1}{4} \langle \alpha, \nabla f(a) \rangle^2 \frac{1}{\sinh^2 \langle \alpha, a \rangle} \quad \square \end{aligned}$$

Example 1.4. Consider the case $G = \text{PSL}_2(\mathbb{C})$. Then $\mathfrak{a}^+ \simeq \mathbb{R}_+^*$, and there is only one positive root that we can identify with the identity on \mathbb{R} . Then

$$\text{Hess}_{\mathbb{C}}(\psi)(\exp(a)) = \frac{1}{2} \begin{pmatrix} f''(a)/2 & 0 & 0 \\ 0 & f'(a) \coth(a) & i f'(a) \\ 0 & -i f'(a) & f'(a) \coth(a) \end{pmatrix}$$

and the complex Monge-Ampère reads:

$$\text{MA}_{\mathbb{C}}(\psi)(\exp(a)) = \frac{1}{16} f''(a) (f'(a))^2 \frac{1}{\sinh^2(a)}.$$

The rest of the section is devoted to the proof of the theorem. The technique of the proof is based on the work of Bielawski [Bie04]. In particular, the idea to use the decomposition in Lemma 1.6 and the Baker-Campbell-Hausdorff formula appears in this article. We begin by introducing these two tools.

1.2 The Baker-Campbell-Hausdorff formula

As a formal series in the variables x and y , the logarithm of $\exp(x) \exp(y)$ is well defined. We denote this by $\text{BCH}(x, y)$. The Baker-Campbell-Hausdorff formula is the following.

Proposition 1.5. [Hoc65, Theorem X.3.1] *There exists a neighborhood U of 0 in \mathfrak{g} such that for all x and y in U , $\text{BCH}(x, y)$ is convergent and defines an element of \mathfrak{g} , and we have*

$$\exp(x) \exp(y) = \exp(\text{BCH}(x, y)).$$

Furthermore we know explicitly the terms of $\text{BCH}(x, y)$. We will only use the following:

$$\text{BCH}(x, y) = x + y + \frac{1}{2}[x, y] + O$$

where O denotes terms of order higher than 2 in x and y .

1.3 A decomposition in \mathfrak{g}

Let $a \in \mathfrak{a}^+$. Let $\text{Exp}(\text{ad}(a))$ be the linear application $\mathfrak{g} \rightarrow \mathfrak{g}$ defined by

$$\text{Exp}(\text{ad}(a))(x) = \sum_{n=0}^{\infty} \frac{\text{ad}(a)^n(x)}{n!}.$$

Recall that G acts on \mathfrak{g} through the adjoint action Ad , and that we have the general relation

$$\text{Exp}(\text{ad}(a))(x) = \text{Ad}(\exp a)(x)$$

for $x \in \mathfrak{g}$.

Lemma 1.6. *Let $l \in \mathfrak{g}$ and $a \in \mathfrak{a}^+$. Then*

1. *there exists $A \in \mathfrak{k}$, $B \in \mathfrak{a}$ and $C \in \text{Ad}(\exp a) \left(\bigoplus_{\alpha \in \Phi^+} \mathfrak{k}_\alpha \right)$ such that*

$$l = A + B + C;$$

2. *if $l \in \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$ then $B = 0$;*

3. *if $l \in \mathfrak{k}_\alpha$, and l' denotes $\text{ad}(a)(il) / \langle \alpha, a \rangle$, then $l' \in \mathfrak{k}_\alpha$ and the decomposition above for the element $il \in \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ reads*

$$il = -(\cosh \langle \alpha, a \rangle)l' + \frac{1}{\sinh \langle \alpha, a \rangle} \text{Ad}(\exp a)(l');$$

4. *if $l = e_\alpha + \theta(e_\alpha)$ then $l' = ie_\alpha - i\theta(e_\alpha)$;*

5. *if $l = ie_\alpha - i\theta(e_\alpha)$ then $l' = -e_\alpha - \theta(e_\alpha)$.*

In the statement, the result is more and more precise as we know more precisely the element considered. In the proof we will begin by the very precise case of the elements of the basis and work our way up by linearity.

Proof. Let $a \in \mathfrak{a}^+$. We begin by the two last points. By definition of l' , we have, if $l = e_\alpha + \theta(e_\alpha)$,

$$\begin{aligned} l' &= \text{ad}(a)(il) / \langle \alpha, a \rangle \\ &= \text{ad}(a)(ie_\alpha) / \langle \alpha, a \rangle + \text{ad}(a)(i\theta(e_\alpha)) / \langle \alpha, a \rangle \\ &= ie_\alpha - i\theta(e_\alpha) \end{aligned}$$

and if $l = ie_\alpha - i\theta(e_\alpha)$,

$$\begin{aligned} l' &= \text{ad}(a)(il) / \langle \alpha, a \rangle \\ &= \text{ad}(a)(-e_\alpha) / \langle \alpha, a \rangle + \text{ad}(a)(\theta(e_\alpha)) / \langle \alpha, a \rangle \\ &= -e_\alpha - \theta(e_\alpha). \end{aligned}$$

In particular, in both cases, l' is in \mathfrak{k}_α . By linearity this is also true of l' for any $l \in \mathfrak{k}_\alpha$.

To prove the decomposition in the third point, it suffices to compute that, using the definition of \mathfrak{k}_α ,

$$\begin{aligned} \text{Exp}(\text{ad}(a))(l') &= (\cosh \langle \alpha, a \rangle)l' + (\sinh \langle \alpha, a \rangle)il \\ &= \text{Ad}(\exp a)(x). \end{aligned}$$

Then by linearity the first point holds true for any $l \in i \bigoplus_{\alpha \in \Phi^+} \mathfrak{k}_\alpha$, with $B = 0$. But we have

$$\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha = \bigoplus_{\alpha \in \Phi^+} \mathfrak{k}_\alpha \oplus i \bigoplus_{\alpha \in \Phi^+} \mathfrak{k}_\alpha,$$

so we have the decomposition for any $l \in \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$, with $B = 0$.

Finally for $l \in \mathfrak{t}$, it suffices to decompose l along $\mathfrak{t} = \mathfrak{s} \oplus \mathfrak{a}$. By linearity and the root decomposition, we obtain the proposition for any $l \in \mathfrak{g}$. \square

1.4 Using the Baker-Campbell-Hausdorff formula

We want to compute the complex Hessian of ψ in the chosen system of coordinates, at a point $\exp(a)$ for a in the open Weyl chamber \mathfrak{a}^+ . If l_1 and l_2 are two vectors in the chosen basis of \mathfrak{k} , we thus want to compute:

$$H_{l_1, l_2}(a) := \left. \frac{\partial^2}{\partial z_1 \partial \bar{z}_2} \right|_{z_1, z_2=0} \psi(\exp(z_1 l_1 + z_2 l_2) \exp(a)).$$

There are different cases, according to the subspaces where l_1 and l_2 lie. We will first describe the part of the argument that is used in all cases, which relies on the Baker-Campbell-Hausdorff formula, and then deal with each case separately.

Using the decomposition from Lemma 1.6 on $z_1 l_1 + z_2 l_2$ we can write

$$z_1 l_1 + z_2 l_2 = A_1 + B_1 + C_1$$

with $A_1 \in \mathfrak{k}$, $B_1 \in \mathfrak{a}$ and $C_1 \in \text{Ad}(\exp a)(\mathfrak{k})$, and all are of homogeneous degree one in z_1 and z_2 . Let

$$D_1 = \frac{1}{2}([B_1, A_1] + [C_1, A_1] + [C_1, B_1]),$$

it is of order two in z_1 and z_2 .

Let us now use again Lemma 1.6 to get

$$D_1 = A_2 + B_2 + C_2.$$

with $A_2 \in \mathfrak{k}$, $B_2 \in \mathfrak{a}$ and $C_2 \in \text{Ad}(\exp a)(\mathfrak{k})$, and all are of homogeneous degree two in z_1 and z_2 .

The Baker-Campbell-Hausdorff formula allows to prove the following lemma, which can be seen as an explicit infinitesimal KAK decomposition. Note that to lighten the notations we do not write the dependence on z_1, z_2 , but all the terms defined above A_j, B_j, C_j and D_1 are in fact functions of these two complex variables.

Lemma 1.7. *We can write*

$$\exp(z_1 l_1 + z_2 l_2) \exp(a) = k_1 \exp(B_1 + B_2 + a + O) k_2$$

where $k_1, k_2 \in K$, and O denotes terms of order greater than two in z_1 and z_2 .

Proof. We begin by applying Proposition 1.5 to $\exp(-A_1) \exp(A_1 + B_1 + C_1)$, and get that this is equal to

$$\exp(B_1 + C_1 + [-A_1, B_1 + C_1]/2 + O_1),$$

where O_1 denotes terms of order greater than 2 in z_1 and z_2 .

Then we multiply on the right by $\exp(-C_1)$ and get, with Proposition 1.5 again,

$$\exp(B_1 + [-A_1, B_1 + C_1]/2 + [B_1, -C_1]/2 + O_2),$$

where O_2 denotes terms of order greater than 2 in z_1 and z_2 . By definition of D_1 , we have proved

$$\exp(-A_1) \exp(z_1 l_1 + z_2 l_2) \exp(-C_1) = \exp(B_1 + D_1 + O_2).$$

Recall that $D_1 = A_2 + B_2 + C_2$, and that all of these are of degree two in z_1 and z_2 . We apply another time the Proposition 1.5, to $\exp(-A_2) \exp(B_1 + D_1 + O_2)$, but here we only need to use the first term in the development of BCH. We might say that A_2 commutes up to order two with elements of degree greater or equal to one in z_1, z_2 . We get

$$\exp(-A_2) \exp(B_1 + D_1 + O_2) = \exp(B_1 + B_2 + C_2 + O_3),$$

where O_3 denotes terms of order greater than 2 in z_1 and z_2 .

One further use of the Baker-Campbell-Hausdorff formula yields

$$\exp(-A_2) \exp(B_1 + D_1 + O_2) \exp(-C_2) = \exp(B_1 + B_2 + O_4),$$

where O_4 denotes terms of order greater than 2 in z_1 and z_2 .

Consider now $\exp(C_2) \exp(C_1)$. Since $C_1, C_2 \in \text{Ad}(\exp a)(\mathfrak{k})$, we have

$$\exp(C_2) \exp(C_1) = \exp(a) k_2 \exp(-a)$$

for some $k_2 \in K$. On the other hand, we have $k_1 := \exp(A_1) \exp(A_2) \in K$.

Summing up we have proved that

$$\exp(z_1 l_1 + z_2 l_2) = k_1 \exp(B_1 + B_2 + O_4) \exp(a) k_2 \exp(-a).$$

But then

$$\exp(z_1 l_1 + z_2 l_2) \exp(a) = k_1 \exp(B_1 + B_2 + O_4) \exp(a) k_2,$$

and one last application of Proposition 1.5 gives the lemma, because B_1, B_2 and a commute:

$$\exp(z_1 l_1 + z_2 l_2) \exp(a) = k_1 \exp(B_1 + B_2 + a + O) k_2$$

where O denotes terms of order greater than 2 in z_1 and z_2 . □

Lemma 1.8. *We have*

$$H_{l_1, l_2}(a) = \frac{\partial^2}{\partial z_1 \partial \bar{z}_2} \Big|_0 f(a + B_2 + B_1).$$

Proof. We first use $K \times K$ -invariance of ψ and Lemma 1.7 to write

$$\psi(\exp(z_1 l_1 + z_2 l_2) \exp(a)) = \psi(\exp(a + B_1 + B_2 + O)).$$

Then

$$\begin{aligned} H_{l_1, l_2}(a) &= \frac{\partial^2}{\partial z_1 \partial \bar{z}_2} \Big|_{z_1, z_2=0} \psi(\exp(z_1 l_1 + z_2 l_2) \exp(a)) \\ &= \frac{\partial^2}{\partial z_1 \partial \bar{z}_2} \Big|_0 \psi(\exp(a + B_1 + B_2 + O)) \end{aligned}$$

because O is of order greater than two, this becomes

$$= \frac{\partial^2}{\partial z_1 \partial \bar{z}_2} \Big|_0 \psi(\exp(a + B_1 + B_2))$$

since $a + B_1 + B_2 \in \mathfrak{a}$, this is

$$= \frac{\partial^2}{\partial z_1 \partial \bar{z}_2} \Big|_0 f(a + B_1 + B_2) \quad \square$$

It remains to determine $B_1 + B_2$ for all coefficients of the Hessian, and then to compute the coefficient. For that, since we want to reduce to real coordinates, we recall that if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then

$$\frac{\partial^2}{\partial z_1 \partial \bar{z}_2} = \frac{1}{4} \left(\frac{\partial^2}{\partial x_1 \partial x_2} + \frac{\partial^2}{\partial y_1 \partial y_2} \right) + \frac{i}{4} \left(\frac{\partial^2}{\partial x_1 \partial y_2} - \frac{\partial^2}{\partial y_1 \partial x_2} \right).$$

1.5 Determining $H_{l_1, l_2}(a)$

Lemma 1.9. *Assume $l_1, l_2 \in \mathfrak{s}$. Then $H_{l_1, l_2}(a)$ is the corresponding coefficient of $\text{Hess}_{\mathbb{R}}(f)(a)/4$:*

$$H_{l_1, l_2}(a) = \frac{1}{4} \frac{\partial^2}{\partial y_1 \partial y_2} \Big|_0 f(a + y_1 i l_1 + y_2 i l_2).$$

Proof. In this case we have $z_1 l_1 + z_2 l_2 = A_1 + B_1 + 0$ with $A_1 = x_1 l_1 + x_2 l_2 \in \mathfrak{s}$ and $B_1 = y_1 l_1 + y_2 l_2 \in \mathfrak{a}$, and A_1 and B_1 commute, so $D_1 = 0$ and $B_2 = 0$.

Then by Lemma 1.8,

$$\begin{aligned} H_{l_1, l_2}(a) &= \frac{\partial^2}{\partial z_1 \partial \bar{z}_2} \Big|_0 f(a + y_1 i l_1 + y_2 i l_2) \\ &= \frac{1}{4} \frac{\partial^2}{\partial y_1 \partial y_2} \Big|_0 f(a + y_1 i l_1 + y_2 i l_2) \quad \square \end{aligned}$$

Lemma 1.10. *Assume $l_1 \in \mathfrak{k}_\alpha$ and $l_2 \in \mathfrak{s}$, then $H_{l_1, l_2}(a) = 0$.*

Proof. Let us first determine the A_1, B_1, C_1 such that $z_1 l_1 + z_2 l_2 = A_1 + B_1 + C_1$. Using Lemma 1.6, write

$$il_1 = -(\coth \langle \alpha, a \rangle) l'_1 + \frac{1}{\sinh \langle \alpha, a \rangle} (\text{Ad}(\exp a)(l'_1))$$

with $l'_1 = \text{ad}(a)(il) / \langle \alpha, a \rangle$.

Then we have

$$\begin{aligned} A_1 &= x_1 l_1 - y_1 (\coth \langle \alpha, a \rangle) l'_1 + x_2 l_2 \\ B_1 &= y_2 il_2 \\ C_1 &= \frac{y_1}{\sinh \langle \alpha, a \rangle} (\text{Ad}(\exp a)(l'_1)) = y_1 il_1 + y_1 (\cosh \langle \alpha, a \rangle) l'_1 \end{aligned}$$

We must now compute $D_1 = ([B_1, A_1] + [C_1, A_1] + [C_1, B_1]) / 2$. In fact we must only determine B_2 which is the part of D_1 that lies in \mathfrak{a} .

We have

$$\begin{aligned} [B_1, A_1] &= [y_2 il_2, x_1 l_1 - y_1 (\coth \langle \alpha, a \rangle) l'_1 + x_2 l_2] \\ &= -y_1 y_2 (\coth \langle \alpha, a \rangle) [il_2, l'_1] + x_1 y_2 [il_2, l_1] \end{aligned}$$

Now $il_2 \in \mathfrak{a}$ and $l_1, l'_1 \in \mathfrak{k}_\alpha \subset \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ so $[il_2, l'_1], [il_2, l_1] \in \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$, and the third point of Lemma 1.6 applies to show that the \mathfrak{a} component of $[B_1, A_1]$ is zero.

For the second part, write

$$\begin{aligned} [C_1, A_1] &= x_1 y_1 (\cosh \langle \alpha, a \rangle) [l'_1, l_1] - y_1^2 (\coth \langle \alpha, a \rangle) [il_1, l'_1] \\ &\quad + x_2 y_1 [il_1, l_2] + x_2 y_1 (\cosh \langle \alpha, a \rangle) [l'_1, l_2] \end{aligned}$$

We have here $[l'_1, l_1], [l'_1, l_2] \in \mathfrak{k}$, and $[il_1, l_2] \in \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ as above, so only $[il_1, l'_1]$ matters. By the properties of the root decomposition,

$$[il_1, l'_1] \in (\mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{2\alpha}) \cap i\mathfrak{k}$$

and $\mathfrak{g}_{-2\alpha} = \mathfrak{g}_{2\alpha} = \{0\}$. So $[il_1, l'_1] \in \mathfrak{a}$. But in fact we don't need to determine it more explicitly because it appears as a term in y_1^2 and these are ignored in the computation of $\partial \bar{\partial}$.

For the third part,

$$[C_1, B_1] = y_1 y_2 [il_1, il_2] + y_1 y_2 (\cosh \langle \alpha, a \rangle) [l'_1, il_2]$$

with $[il_1, il_2] \in \mathfrak{k}$ and $[l'_1, il_2] \in \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ so there is no contribution to B_2 .

We have thus proved that

$$B_2 = -\frac{1}{2} y_1^2 (\coth \langle \alpha, a \rangle) [il_1, l'_1].$$

Lemma 1.8 now gives

$$\begin{aligned} H_{l_1, l_2}(a) &= \frac{\partial^2}{\partial z_1 \partial \bar{z}_2} \Big|_0 f(a + B_2 + B_1) \\ &= \frac{\partial^2}{\partial z_1 \partial \bar{z}_2} \Big|_0 f(a + y_2 il_2 - y_1^2 (\coth \langle \alpha, a \rangle) [il_1, l'_1]/2) \\ &= 0 \end{aligned} \quad \square$$

Lemma 1.11. *Assume $l_1 \in \mathfrak{k}_{\alpha_1}$ and $l_2 \in \mathfrak{k}_{\alpha_2}$, with $\alpha_1 \neq \alpha_2$ positive roots. Then $H_{l_1, l_2}(a) = 0$.*

Proof. Using Lemma 1.6, we write

$$\begin{aligned} il_1 &= -(\coth \langle \alpha_1, a \rangle) l'_1 + \frac{1}{\sinh \langle \alpha_1, a \rangle} \text{Ad}(\exp a)(l'_1) \\ il_2 &= -(\coth \langle \alpha_2, a \rangle) l'_2 + \frac{1}{\sinh \langle \alpha_2, a \rangle} \text{Ad}(\exp a)(l'_2) \end{aligned}$$

with $l'_1 = \text{ad}(a)(il_1)/\langle \alpha_1, a \rangle$ and $l'_2 = \text{ad}(a)(il_2)/\langle \alpha_2, a \rangle$.

Then we have

$$\begin{aligned} A_1 &= x_1 l_1 + x_2 l_2 - y_1 (\coth \langle \alpha_1, a \rangle) l'_1 - y_2 (\coth \langle \alpha_2, a \rangle) l'_2 \\ B_1 &= 0 \\ C_1 &= y_1 \frac{1}{\sinh \langle \alpha_1, a \rangle} (\text{Ad}(\exp a)(l'_1) + y_2 \frac{1}{\sinh \langle \alpha_2, a \rangle} (\text{Ad}(\exp a)(l'_2)) \\ D_1 &= [C_1, A_1]/2 \\ &= [y_1 il_1 + y_2 il_2 + y_1 (\cosh \langle \alpha_1, a \rangle) l'_1 + y_2 (\cosh \langle \alpha_2, a \rangle) l'_2, A_1]/2 \end{aligned}$$

We have $y_1 (\cosh \langle \alpha_1, a \rangle) l'_1 + y_2 (\cosh \langle \alpha_2, a \rangle) l'_2$ and A_1 in \mathfrak{k} , so their bracket remains in \mathfrak{k} and doesn't appear in B_2 . We compute $[y_1 il_1 + y_2 il_2, A_1]$ which is equal to

$$\begin{aligned} &x_2 y_1 [il_1, l_2] - y_1^2 (\coth \langle \alpha_1, a \rangle) [il_1, l'_1] - y_1 y_2 (\coth \langle \alpha_1, a \rangle) [il_2, l'_1] \\ &+ x_1 y_2 [il_2, l_1] - y_2^2 (\coth \langle \alpha_2, a \rangle) [il_2, l'_2] - y_1 y_2 (\coth \langle \alpha_2, a \rangle) [il_1, l'_2]. \end{aligned}$$

Again the properties of the root decomposition tell us that $[il_1, l_2]$, $[il_1, l'_2]$, $[il_2, l_1]$, and $[il_2, l'_1]$ are in $\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$, so the corresponding terms do not contribute to B_2 . As before, $[il_1, l'_1]$ and $[il_2, l'_2]$ are in \mathfrak{a} , so we get

$$B_2 = \frac{1}{2} (-y_1^2 (\coth \langle \alpha_1, a \rangle) [il_1, l'_1] - y_2^2 (\coth \langle \alpha_2, a \rangle) [il_2, l'_2]).$$

Applying Lemma 1.8, we get

$$\begin{aligned} H_{l_1, l_2}(a) &= \frac{\partial^2}{\partial z_1 \partial \bar{z}_2} \Big|_0 f(a + B_2 + B_1) \\ &= \frac{\partial^2}{\partial z_1 \partial \bar{z}_2} \Big|_0 f(a - y_1^2 \coth(\langle \alpha_1, a \rangle) [il_1, l'_1]/2 - y_2^2 \coth(\langle \alpha_2, a \rangle) [il_2, l'_2]/2) \\ &= 0 \end{aligned} \quad \square$$

Suppose now that $\alpha_1 = \alpha_2 = \alpha$. The subspace \mathfrak{k}_α is two dimensional, and we have chosen a basis formed by the vectors $e_\alpha + \theta(e_\alpha)$ and $ie_\alpha - i\theta(e_\alpha)$.

First we deal with the case when $l_1 \neq l_2$.

Lemma 1.12. *Suppose $l_1 = e_\alpha + \theta(e_\alpha)$ and $l_2 = ie_\alpha - i\theta(e_\alpha)$. Then*

$$H_{l_1, l_2}(a) = \frac{i}{2} \langle \alpha, \nabla f(a) \rangle = -H_{l_2, l_1}(a).$$

Proof. Using Lemma 1.6, we write, just as in the previous proof

$$\begin{aligned} il_1 &= -(\coth \langle \alpha, a \rangle)l'_1 + \frac{1}{\sinh \langle \alpha, a \rangle} \text{Ad}(\exp a)(l'_1) \\ il_2 &= -(\coth \langle \alpha, a \rangle)l'_2 + \frac{1}{\sinh \langle \alpha, a \rangle} \text{Ad}(\exp a)(l'_2) \end{aligned}$$

with $l'_1 = l_2$ and $l'_2 = -l_1$.

Then we have

$$\begin{aligned} A_1 &= (x_1 + y_2 \coth \langle \alpha, a \rangle)l_1 + (x_2 - y_1 \coth \langle \alpha, a \rangle)l_2 \\ B_1 &= 0 \\ C_1 &= y_1 \frac{1}{\sinh \langle \alpha, a \rangle} \text{Ad}(\exp a)(l'_1) + y_2 \frac{1}{\sinh \langle \alpha, a \rangle} \text{Ad}(\exp a)(l'_2) \\ D_1 &= [C_1, A_1]/2 \\ &= [y_1 il_1 + y_2 il_2 + y_1 \cosh(\langle \alpha, a \rangle)l_2 - y_2 \cosh(\langle \alpha, a \rangle)l_1, A_1]/2 \end{aligned}$$

Once again the bracket of $y_1(\cosh \langle \alpha, a \rangle)l_2 - y_2(\cosh \langle \alpha, a \rangle)l_1$ with A_1 yields only terms in \mathfrak{k} so we compute $[y_1 il_1 + y_2 il_2, A_1]$, which is equal to

$$y_2(x_1 + y_2 \coth \langle \alpha, a \rangle)[il_2, l_1] + y_1(x_2 - y_1 \coth \langle \alpha, a \rangle)[il_1, l_2].$$

Using the explicit choices of l_1 and l_2 we have

$$\begin{aligned} -[il_2, l_1] &= [il_1, l_2] = [i(e_\alpha + \theta(e_\alpha)), ie_\alpha - i\theta(e_\alpha)] \\ &= 2[e_\alpha, \theta(e_\alpha)] \\ &= -2[e_\alpha, e_{-\alpha}] \\ &= -2\alpha. \end{aligned}$$

Finally we have

$$B_2 = (y_2 x_1 + y_2^2 \coth \langle \alpha, a \rangle - y_1 x_2 + y_1^2 \coth \langle \alpha, a \rangle) \alpha.$$

Applying Lemma 1.8, we get

$$\begin{aligned} H_{l_1, l_2}(a) &= \frac{\partial^2}{\partial z_1 \partial \bar{z}_2} \Big|_0 f(a + B_2 + B_1) \\ &= \frac{i}{4} \left(\frac{\partial^2}{\partial x_1 \partial y_2} - \frac{\partial^2}{\partial y_1 \partial x_2} \right) \Big|_0 f(a + (y_2 x_1 - y_1 x_2) \alpha) \\ &= \frac{i}{2} (Df)_a(\alpha) \end{aligned}$$

where $(Df)_a$ denotes the differential of f at a , so

$$H_{l_1, l_2}(a) = \frac{i}{2} \langle \alpha, \nabla f(a) \rangle. \quad \square$$

Lemma 1.13. *Suppose now that $l_1 = l_2 = e_\alpha + \theta(e_\alpha)$, then*

$$H_{l_1, l_2}(a) = \frac{1}{2} \langle \alpha, \nabla f(a) \rangle \coth \langle \alpha, a \rangle.$$

Proof. Using Lemma 1.6, we write

$$il_2 = il_1 = -(\coth \langle \alpha, a \rangle)l'_1 + \frac{1}{\sinh \langle \alpha, a \rangle} \text{Ad}(\exp a)(l'_1)$$

with

$$l'_2 = l'_1 = ie_\alpha - i\theta(e_\alpha).$$

Then we have

$$\begin{aligned} A_1 &= (x_1 + x_2)l_1 - (y_1 \coth \langle \alpha, a \rangle + y_2 \coth \langle \alpha, a \rangle)l'_1 \\ B_1 &= 0 \\ C_1 &= y_1 \frac{1}{\sinh \langle \alpha, a \rangle} \text{Ad}(\exp a)(l'_1) + y_2 \frac{1}{\sinh \langle \alpha, a \rangle} \text{Ad}(\exp a)(l'_2) \\ D_1 &= [C_1, A_1]/2 \\ &= [(y_1 + y_2)il_1 + (y_1 \cosh \langle \alpha, a \rangle + y_2 \cosh \langle \alpha, a \rangle)l'_1, A_1]/2 \end{aligned}$$

Once again the bracket of $(y_1 \cosh \langle \alpha, a \rangle + y_2 \cosh \langle \alpha, a \rangle)l'_1$ with A_1 yields only terms in \mathfrak{k} , so we just compute

$$[(y_1 + y_2)il_1, A_1] = -(y_1 + y_2)^2 (\coth \langle \alpha, a \rangle) [il_1, l'_1].$$

Using the explicit choices of l_1 we have

$$\begin{aligned} [il_1, l'_1] &= [i(e_\alpha + \theta(e_\alpha)), ie_\alpha - i\theta(e_\alpha)] \\ &= -2\alpha. \end{aligned}$$

Finally we have

$$B_2 = (y_1 + y_2)^2 (\coth \langle \alpha, a \rangle) \alpha.$$

Applying Lemma 1.8, we get

$$\begin{aligned} H_{l_1, l_2}(a) &= \frac{\partial^2}{\partial z_1 \partial \bar{z}_2} \Big|_0 f(a + B_2 + B_1) \\ &= \frac{1}{4} \frac{\partial^2}{\partial y_1 \partial y_2} \Big|_0 f(a + (y_1^2 + 2y_1 y_2 + y_2^2) (\coth \langle \alpha, a \rangle) \alpha) \\ &= \frac{\coth \langle \alpha, a \rangle}{2} (Df)_a(\alpha) \\ &= \frac{\coth \langle \alpha, a \rangle}{2} \langle \alpha, \nabla f(a) \rangle. \quad \square \end{aligned}$$

The last step is to compute the coefficient of the Hessian with $l_1 = l_2 = ie_\alpha - i\theta(e_\alpha)$, and the result is exactly the same as in the previous case:

Lemma 1.14. *Assume that $l_1 = l_2 = ie_\alpha - i\theta(e_\alpha)$, then*

$$H_{l_1, l_2}(a) = \frac{1}{2} \langle \alpha, \nabla f(a) \rangle \coth \langle \alpha, a \rangle.$$

Proof. Using Lemma 1.6, we write

$$il_2 = il_1 = -(\coth \langle \alpha, a \rangle)l'_1 + \frac{1}{\sinh \langle \alpha, a \rangle} \text{Ad}(\exp a)(l'_1)$$

with $l'_2 = l'_1 = -e_\alpha - \theta(e_\alpha)$.

The beginning of the computation doesn't change: we have

$$\begin{aligned} A_1 &= (x_1 + x_2)l_1 - (y_1 \coth \langle \alpha, a \rangle + y_2 \coth \langle \alpha, a \rangle)l'_1 \\ B_1 &= 0 \\ C_1 &= y_1 \frac{1}{\sinh \langle \alpha, a \rangle} \text{Ad}(\exp a)(l'_1) + y_2 \frac{1}{\sinh \langle \alpha, a \rangle} \text{Ad}(\exp a)(l'_2) \\ D_1 &= [(y_1 + y_2)il_1 + (y_1 \cosh \langle \alpha, a \rangle + y_2 \cosh \langle \alpha, a \rangle)l'_1, A_1]/2 \end{aligned}$$

Once again the bracket of $(y_1 \cosh \langle \alpha, a \rangle + y_2 \cosh \langle \alpha, a \rangle)l'_1$ with A_1 yields only terms in \mathfrak{k} , so we just compute

$$[(y_1 + y_2)il_1, A_1] = -(y_1 + y_2)^2 (\coth \langle \alpha, a \rangle) [il_1, l'_1].$$

Now the expression of l_1 has changed, but we have

$$\begin{aligned} [il_1, l'_1] &= [i(ie_\alpha - i\theta(e_\alpha)), -e_\alpha - \theta(e_\alpha)] \\ &= 2[e_\alpha, \theta(e_\alpha)] \\ &= -2\alpha. \end{aligned}$$

In other words we have again

$$B_2 = (y_1 + y_2)^2 (\coth \langle \alpha, a \rangle) \alpha,$$

and applying Lemma 1.8, we get again

$$\begin{aligned} H_{l_1, l_2}(a) &= \frac{\partial^2}{\partial z_1 \partial \bar{z}_2} \Big|_0 f(a + B_2 + B_1) \\ &= \frac{\coth \langle \alpha, a \rangle}{2} \langle \alpha, \nabla f(a) \rangle. \end{aligned} \quad \square$$

2 Fano group compactifications

We present here in a concise way group compactifications and the associated polytopes, then recall results from [Delb] about the convex potential of a hermitian metric, giving more precise statements for smooth metrics. We then combine this with the computation of the complex Monge-Ampère (Corollary 1.3)

to partially recover the formula of Brion and Kazarnovskii for the degree of an ample line bundle on a group compactification. The reader willing to learn more details about group compactifications can consult, for example, [AB04, AK05] or [BK05, Chapter 6]. For more general references about spherical varieties, see [Tim11, Per14].

2.1 Definition and examples

Definition 2.1. We say that a projective manifold X is a *compactification of G* (or a *group compactification*, when G is not specified) if X admits a holomorphic $G \times G$ action with an open and dense orbit isomorphic to G as a $G \times G$ -homogeneous space. We say that (X, L) is a *polarized compactification of G* if X is a compactification of G and L is a $G \times G$ -linearized ample line bundle on X .

Recall that a *G -linearized line bundle* over a G -manifold X is a line bundle L on X equipped with an action of G lifting the action on X , and such that the morphisms between the fibers induced by this action are linear.

Example 2.2. If $G = T \simeq (\mathbb{C}^*)^n$ is a torus, then the compactifications of T are the projective toric manifolds. One goes from the T -action to the $T \times T$ action through the morphism $T \times T \rightarrow T, (t_1, t_2) \mapsto t_1 t_2^{-1}$.

Example 2.3. Assume that G is an adjoint semisimple group, that is, a reductive group with trivial center. Then De Concini and Procesi [DCP83] proved the existence of a special compactification of G , called the wonderful compactification of G . It is the only compactification of G satisfying the following property: its boundary $X \setminus G$ (where we identify the open and dense orbit in X with G) is a union of simple normal crossing prime divisors $D_i, i \in \{1, \dots, r\}$, such that for any subset $I \subset \{1, \dots, r\}$, the intersection $X \cap \bigcap_{i \in I} D_i$ is the closure of a unique $G \times G$ -orbit, and all $G \times G$ -orbits appear this way. The integer r is equal to the rank of G , which is the dimension of a maximal torus in G .

2.2 Polytopes associated to a polarized group compactification

Theorem 2.4. [AB04, Section 2] *Let (X, L) be a polarized group compactification of G . Denote by Z the closure of T in X . Then Z is a toric manifold, equipped with a W -action, and $L|_Z$ is a W -linearized ample toric line bundle on Z .*

We denote by $P(X, L)$, or P for simplicity, the polytope associated to the ample toric line bundle $L|_Z$ by the theory of toric varieties [Ful93, Oda88]. The polytope P is a lattice polytope in $M \otimes \mathbb{R}$, that we identified with \mathfrak{a} , and it is W -invariant. Define $P^+(X, L) := P(X, L) \cap \overline{\mathfrak{a}^+}$. It is a polytope in \mathfrak{a} , and $P(X, L)$ is the union of the images of $P^+(X, L)$ by W .

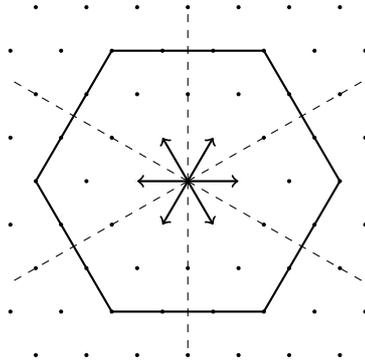


Figure 2: Polytope $P(X_1, -K_{X_1})$

The polytope $P^+(X, L)$ encodes the structure of $G \times G$ -representation of the space of holomorphic sections of L , generalizing the same property for toric line bundles.

Proposition 2.5. [AB04, Section 2.2] *Let (X, L) be a polarized group compactification, then*

$$H^0(X, L) \simeq \bigoplus \{\text{End}(V_\alpha) ; \alpha \in M \cap P^+(X, L)\}$$

where V_α is an irreducible representation of G with highest weight α .

Example 2.6. [BK05, Proposition 6.1.11] The wonderful compactification X of an adjoint semisimple group is Fano. The corresponding polytope $P(X, -K_X)$ is the convex hull of the images by the Weyl group W of the weight $2\rho + \sum_{i=1}^r \alpha_i$, where the α_i are the simple roots of Φ^+ , and 2ρ is the sum of all positive roots.

We will use the three following examples to illustrate our results.

Example 2.7. The wonderful compactification X_0 of $\text{PGL}_2(\mathbb{C})$ is especially simple: it is \mathbb{P}^3 considered as $\mathbb{P}(\text{Mat}_{2,2}(\mathbb{C}))$ equipped with the action of $\text{PGL}_2(\mathbb{C}) \times \text{PGL}_2(\mathbb{C})$ induced by the multiplication of matrices on the left and on the right.

If we identify \mathfrak{a} with \mathbb{R} by sending the only positive root to 1, then the lattice of characters is $(1/2)\mathbb{Z}$, the polytope P associated to the anticanonical line bundle is $[-2, 2]$ and the polytope P^+ is $[0, 2]$. Remark that X_0 is in fact homogeneous under a bigger group. This is in general not the case for wonderful compactifications, as the next example shows.

Example 2.8. Consider the adjoint semisimple group $\text{PGL}_3(\mathbb{C})$. Then its wonderful compactification, that we denote by X_1 , is Fano. Figure 2 represents the polytope $P(X_1, -K_{X_1})$, along with the Weyl chambers decomposition, and the weight lattice M .

Brion computed the automorphism group $\text{Aut}(X)$ of all wonderful compactifications X of adjoint semisimple groups [Bri07, Example 2.4.5]. In particular,

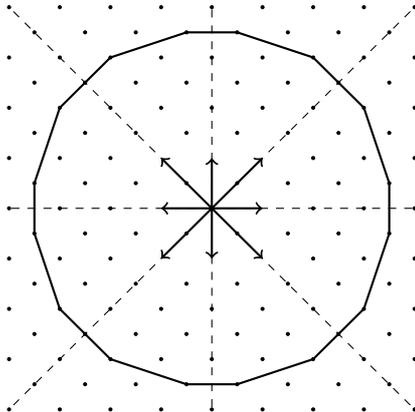


Figure 3: Polytope $P(X_2, -K_{X_2})$

the connected component of the identity $\text{Aut}^0(X_1) \subset \text{Aut}(X_1)$ is the image of $\text{PGL}_3(\mathbb{C}) \times \text{PGL}_3(\mathbb{C})$. This shows that the manifold X_1 is not homogeneous under a bigger group, and not toric. The same statement is true for any wonderful compactification of an adjoint semisimple group with no factor of rank one.

Example 2.9. Consider now the simply connected semisimple group $\text{Sp}_4(\mathbb{C})$. It is the subgroup of $\text{GL}_4(\mathbb{C})$ whose elements are the matrices M such that $M^t \Omega M = \Omega$, where M^t denotes the transpose of M and

$$\Omega = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}.$$

It is a semisimple group of type B_2 , which means that its associated root system is the root system B_2 . While it is not an adjoint group, it also admits a wonderful compactification (see for example [GR13]). Let X_2 be the blow up of this wonderful compactification along the closed orbit. Then X_2 is a Fano compactification of $\text{Sp}_4(\mathbb{C})$ [Ruz12].

The moment polytope associated to the anticanonical line bundle is obtained easily in the following way: Ruzzi classifies in [Ruz12] all Fano compactifications of reductive groups of rank two, by giving for each the usual data identifying a spherical variety. Then we can use the description by Gagliardi and Hofscheier [GH15] of the moment polytope of the anticanonical line bundle in these terms. Figure 3 represents the polytope $P(X_2, -K_{X_2})$, along with the Weyl chambers decomposition, and the weight lattice M .

For this manifold also, $\text{Aut}^0(X_2)$ is the image of $\text{Sp}_4(\mathbb{C}) \times \text{Sp}_4(\mathbb{C})$. This is proved using Blanchard's lemma [Bla56, Proposition I.1] applied to the blow up, and the fact that the connected component of the identity in the automorphism group of the wonderful compactification of $\text{Sp}_4(\mathbb{C})$ is also the image of $\text{Sp}_4(\mathbb{C}) \times \text{Sp}_4(\mathbb{C})$ [Pez09].

2.3 Convex potentials

Let (X, L) be a polarized compactification of G . We recall in this section how to associate to a $K \times K$ -invariant hermitian metric with positive curvature on L a convex function, called its convex potential (see [Delb, Section 2.2]). We let $P := P(X, L)$, $P^+ := P^+(X, L)$, and let v denote the support function of the dilated polytope $2P$, defined by $v(x) = \sup\{\langle p, x \rangle ; p \in 2P\}$.

Let s be a $G \times \{e\}$ -equivariant trivialization of $L|_G$, where we identify G with the open $G \times G$ -orbit in X . Let h be a smooth, $K \times K$ -invariant, positively curved hermitian metric on L . Let φ be the local potential of h with respect to the trivialization s , defined for $g \in G$ by:

$$\varphi(g) := -\ln |s(g)|_h^2.$$

Then the function φ is a smooth strictly plurisubharmonic function on G , $K \times K$ -invariant by [Delb, Proposition 2.2].

The function u defined on \mathfrak{a} by $u(x) = \varphi(\exp x)$ is then called the *convex potential* of h and satisfies the following properties.

Proposition 2.10. *Let h be a smooth $K \times K$ -invariant hermitian metric with positive curvature on L , and let u be its convex potential. Then u is a W -invariant, smooth and strictly convex function such that:*

1. $u(x) \geq v(x) + C_1$ for all $x \in \mathfrak{a}$, for some constant C_1 depending on u ;
2. given any $x_0 \in \mathfrak{a}$, $u(x) \leq v(x - x_0) + u(x_0)$ for all $x \in \mathfrak{a}$;
3. the gradient ∇u of u defines a diffeomorphism from \mathfrak{a} to the interior of $2P$;
4. in particular, there exists a constant d depending only on L such that $|\nabla u(x)| \leq d$ for all $x \in \mathfrak{a}$;
5. the restriction of ∇u to \mathfrak{a}^+ is a diffeomorphism to the interior of $2P^+$.

Proof. The fact that u is a W -invariant smooth and strictly convex function follows from its definition and [AL92].

A smooth metric has locally bounded potentials, so [Delb, Theorem 2.4] implies that there exists constants C_1 and C_2 depending on u such that

$$v(x) + C_1 \leq u(x) \leq v(x) + C_2.$$

In particular, the first point is proved. Let us prove the second point.

Let $x_0 \in \mathfrak{a}$. For any $0 \neq y \in \mathfrak{a}$, consider the slope

$$\frac{u(x_0 + ty) - u(x_0)}{t},$$

with $t > 0$. By convexity and the two previous inequalities, we see that this slope increases and converges to $v(y)$ as t tends to infinity. This shows that for any $x = x_0 + y \in \mathfrak{a} \setminus \{x_0\}$, we have

$$u(x) \leq v(x - x_0) + u(x_0).$$

This inequality is obviously also satisfied at x_0 , so the second point is proved.

Since u is smooth and strictly convex, ∇u is a diffeomorphism. It remains to determine its image. It is clear that this image is open and included in $2P$ (for example because the domain of the convex conjugate of u is $2P$, see [Delb, Proposition 3.12]). Assume now that one point from the interior of $2P$ is not attained by ∇u . Then by convexity, we can find a $0 \neq y \in \mathfrak{a}$ and an $\epsilon > 0$ such that $\forall x \in \mathfrak{a}$, $v(y) - \langle \nabla u(x), y \rangle > \epsilon$. But considering the ray starting from any x_0 and in the direction y as above leads to a contradiction. This implies that $\nabla u(\mathfrak{a}) = \text{Int}(2P)$. By W -invariance, we also have $\nabla u(\mathfrak{a}^+) = \text{Int}(2P^+)$.

Finally, the fourth point is a direct consequence of the third because the polytope $2P$ is bounded. \square

Remark 2.11. We fixed a choice of a $G \times \{e\}$ -equivariant trivialization of L on G to define the convex potential. All such trivializations are non zero scalar multiples of one another, so the convex potentials defined with respect to two different such trivializations differ by a constant scalar independent of the metric. In particular, it does not change the asymptotic behavior.

2.4 Degree of an ample linearized line bundle

Proposition 2.12. *Let (X, L) be a polarized compactification of G , and let $P^+ = P^+(X, L)$. Then*

$$\deg(L) = C \int_{2P^+} \prod_{\alpha \in \Phi^+} \langle \alpha, p \rangle^2 dp$$

for some constant C depending only on the group G . Furthermore, if u is the convex potential of a smooth positively curved $K \times K$ -invariant metric on L , then

$$\deg(L) = C \int_{\mathfrak{a}^+} \prod_{\alpha \in \Phi^+} \langle \alpha, \nabla u(a) \rangle^2 \text{MA}_{\mathbb{R}}(u)(a) da.$$

Proof. Let h be a smooth positively curved $K \times K$ -invariant hermitian metric on L , with curvature the Kähler form $\omega \in 2\pi c_1(L)$. Then we have

$$\deg(L) = \int_X \left(\frac{\omega}{2\pi} \right)^n.$$

Let s be a $G \times \{e\}$ -equivariant section of L , and φ the potential of h with respect to s . We have $\omega = i\partial\bar{\partial}\varphi$ on G . Using Remark 1.1, Choose a Haar volume form dg on G so that, independently of φ ,

$$\left(\frac{i}{2\pi} \partial\bar{\partial}\varphi \right)^n = \text{MA}_{\mathbb{C}}(\varphi) dg,$$

where $\text{MA}_{\mathbb{C}}(\varphi)$ is the Monge-Ampère of φ in the coordinates chosen in Section 1.1. Then

$$\begin{aligned} \deg(L) &= \int_X \left(\frac{\omega}{2\pi}\right)^n = \int_G \left(\frac{\omega}{2\pi}\right)^n \\ &= \int_G \left(\frac{i}{2\pi} \partial\bar{\partial}\varphi\right)^n \\ &= \int_G \text{MA}_{\mathbb{C}}(\varphi) dg. \end{aligned}$$

By the *KAK* integration formula, this is, for a constant C depending only on the choice of Haar measure,

$$\deg(L) = C \int_{\mathfrak{a}^+} \text{MA}_{\mathbb{C}}(\varphi)(\exp a) J(a) da$$

where da is the Lebesgue measure on \mathfrak{a} normalized by N . Let u denote the convex potential of h . From Corollary 1.3 we obtain that this is

$$\deg(L) = C' \int_{\mathfrak{a}^+} \prod_{\alpha \in \Phi^+} \langle \alpha, \nabla u(a) \rangle^2 \text{MA}_{\mathbb{R}}(u)(a) da$$

for a constant C' still depending only on G . We then use the variable change $p = \nabla u(a)$, and obtain, by Proposition 2.10,

$$\deg(L) = C' \int_{2P^+} \prod_{\alpha \in \Phi^+} \langle \alpha, p \rangle^2 dp. \quad \square$$

3 Strategy of proof

We describe in this section the global strategy of the proof, starting by recalling classical results on the continuity method. We then determine the expression of the continuity method equation in restriction to G , in terms of convex potentials. We finally introduce the function ν_t and gather some estimates on this function or on $-\ln J$ that will be used later.

3.1 The continuity method

Let X be a Fano manifold. Fix a reference Kähler form ω_{ref} in the class $2\pi c_1(X)$. The Kähler forms in $2\pi c_1(X)$ can all be written as $\omega_{\text{ref}} + i\partial\bar{\partial}\psi$ with ψ a smooth and ω_{ref} -strictly psh function on X (i.e. such that $\omega_{\text{ref}} + i\partial\bar{\partial}\psi > 0$).

The Kähler-Einstein equation $\text{Ric}(\omega) = \omega$ on X translates, in terms of ω_{ref} -psh functions, as the Monge-Ampère equation

$$(\omega_{\text{ref}} + i\partial\bar{\partial}\psi)^n = e^{f_{\text{ref}} - \psi} \omega_{\text{ref}}^n, \quad (1)$$

where f_{ref} is the *normalized Ricci potential* of ω_{ref} defined as the ω_{ref} -psh function that satisfies $\omega_{\text{ref}} + i\partial\bar{\partial}f_{\text{ref}} = \text{Ric}(\omega_{\text{ref}})$ and $\int_X e^{f_{\text{ref}}} \omega_{\text{ref}}^n = \int_X \omega_{\text{ref}}^n$.

Let h_{ref} be a smooth hermitian metric on $-K_X$ with curvature form ω_{ref} . Then it determines a volume form dV on X defined in a local trivialization s of $-K_X$ by $dV = |s|_{h_{\text{ref}}}^2 s^{-1} \wedge \bar{s}^{-1}$. Then up to a constant, the Ricci potential f_{ref} is equal to the logarithm of the potential of dV with respect to ω_{ref}^n . We choose h_{ref} (by multiplying by a scalar) such that f_{ref} is indeed equal to that.

The following family of equations is the one used in the usual continuity method for the Kähler-Einstein equation:

$$(\omega_{\text{ref}} + i\partial\bar{\partial}\psi_t)^n = e^{f_{\text{ref}} - t\psi_t} \omega_{\text{ref}}^n. \quad (2)$$

To show the existence of a Kähler-Einstein metric on X , it is enough to show that the set I of $0 \leq t \leq 1$ such that this equation admits a solution is exactly $[0, 1]$.

By the work of Aubin [Aub76] and Yau [Yau78], $0 \in I$, and I is open. Furthermore, it is enough to know uniform a priori estimates on the C^0 norm of ψ_t , to ensure the closure of I , and thus the existence of a solution at $t = 1$, i.e. a Kähler-Einstein metric. We recall that by C^0 estimates, we mean, as in most of the literature, a uniform control on $\sup_{x \in X} |\psi_t(x)|$. In fact, we can even concentrate only on a uniform upper bound on ψ_t (see [Siu88, Proposition 2.1] or [Tia87, pages 235 and 236]).

Notation 3.1. Let us fix some $0 < t_0 \in I$, which exists since $0 \in I$ and I is open.

Let us summarize the consequence of what we have recalled in this section.

Proposition 3.2. *Assume that $[t_0, t_1[\subset I$, that ψ_t denotes the solution at $t \in [t_0, t_1[$, and that there exists a constant C such that $\psi_t \leq C \forall t \in [t_0, t_1[$. Then $t_1 \in I$.*

3.2 Reduction to the open orbit

Suppose now that X is a $G \times G$ -equivariant smooth and Fano compactification of G . Let P be the polytope associated to the anticanonical bundle $-K_X$.

We choose ω_{ref} and h_{ref} $K \times K$ -invariant. Then a solution of equation (2) at $t < 1$ is $K \times K$ -invariant. This follows from the uniqueness result for twisted (or generalized) Kähler-Einstein metrics (see [ZZ14, Corollary 1.4] or [DS]).

By continuity of the solutions ψ_t , it is enough to prove a uniform upper bound on the restrictions of ψ_t to the open and dense orbit $G \subset X$. Let φ_t denote the function on \mathfrak{a} induced by ψ_t . It is enough to give an upper bound for φ_t . We also denote by h_t the hermitian metric on $-K_X$ whose potential with respect to h_{ref} is ψ_t and whose curvature form is $\omega_{\text{ref}} + i\partial\bar{\partial}\psi_t$.

Any $G \times \{e\}$ -equivariant trivialization s of $-K_G$ defines a left G -invariant volume form $s^{-1} \wedge \bar{s}^{-1}$ on G , and so a Haar volume form. All Haar volume forms on G are obtained this way. By Remark 1.1, we can thus choose s and

the corresponding Haar volume form $dg := s^{-1} \wedge \overline{s^{-1}}$ such that for all smooth $K \times K$ -invariant function ψ on G ,

$$(i\partial\bar{\partial}\psi)^n = \text{MA}_{\mathbb{C}}(\psi)dg,$$

where $\text{MA}_{\mathbb{C}}(\psi)$ is the complex Monge-Ampère of ψ in the local coordinates given in Section 1.1.

Let u_{ref} be the convex potential of h_{ref} with respect to the trivialization s , defined on \mathfrak{a} , and denote by u_t the convex function $u_{\text{ref}} + \varphi_t$ which is the convex potential of the metric h_t . Finally, we denote by w_t the function $tu_t + (1-t)u_{\text{ref}}$.

Proposition 3.3. *Suppose ψ_t is a $K \times K$ -invariant solution of equation (2). Then for $x \in \mathfrak{a}$,*

$$\text{MA}_{\mathbb{R}}(u_t)(x) \prod_{\alpha \in \Phi^+} \langle \alpha, \nabla u_t(x) \rangle^2 = e^{-w_t(x)} J(x). \quad (3)$$

Proof. Let ψ_{ref} be the potential of the reference metric h_{ref} with respect to the section s . By the choice of s , we have, on G ,

$$\begin{aligned} (\omega_{\text{ref}} + i\partial\bar{\partial}\psi_t)|_G^n &= (i\partial\bar{\partial}(\psi_{\text{ref}} + \psi_t))^n \\ &= \text{MA}_{\mathbb{C}}(\psi_{\text{ref}} + \psi_t)dg \end{aligned}$$

Now the computation of the complex Monge-Ampère in local coordinates from Section 1.1 gives

$$(\omega_{\text{ref}} + i\partial\bar{\partial}\psi_t)^n(\exp x) = \text{MA}_{\mathbb{R}}(u_t)(x) \prod_{\alpha \in \Phi^+} \langle \alpha, \nabla u_t(x) \rangle^2 \frac{1}{J(x)} dg$$

for $x \in \mathfrak{a}^+$.

On the other hand, the definition of the normalized Ricci potential f_{ref} implies that

$$e^{f_{\text{ref}}}\omega_{\text{ref}}^n = e^{-\psi_{\text{ref}}} dg,$$

which allows to write the right hand side of equation (2) as

$$e^{f_{\text{ref}} - t\psi_t}\omega_{\text{ref}}^n = e^{-t\psi_t - \psi_{\text{ref}}} dg.$$

For $x \in \mathfrak{a}$, we have

$$\begin{aligned} (-t\psi_t - \psi_{\text{ref}})(\exp x) &= -t\varphi_t(x) - u_{\text{ref}}(x) \\ &= -tu_t(x) - (1-t)u_{\text{ref}}(x) \\ &= -w_t(x). \end{aligned}$$

In conclusion, at a point $\exp(x)$ for $x \in \mathfrak{a}^+$, equation (2) reads

$$\text{MA}_{\mathbb{R}}(u_t)(x) \prod_{\alpha \in \Phi^+} \langle \alpha, \nabla u_t(x) \rangle^2 \frac{1}{J(x)} dg = e^{-w_t(x)} dg.$$

It is equivalent to the equality of the potentials with respect to dg . Furthermore, by multiplying both sides by $J(x)$, we obtain the equation of the statement, that is well defined on the whole of \mathfrak{a} , and it is satisfied by W -invariance and smoothness. \square

3.3 Strategy

To find a uniform upper bound for φ_t we will introduce another function ν_t , and study this function, following the strategy used by Wang and Zhu in the toric case. More precisely, let j be the function on the open Weyl chamber \mathfrak{a}^+ defined by $j(x) = -\ln J(x)$. We consider the function $\nu_t = w_t + j$ defined on \mathfrak{a}^+ . We will show that it is a strictly convex function on \mathfrak{a}^+ . It is proper in the following sense: As x goes to infinity, or x goes to a wall of \mathfrak{a}^+ , $\nu_t(x)$ goes to infinity.

These two properties of ν_t imply that it admits a unique minimum. Let m_t be the minimum of ν_t and x_t be the point of \mathfrak{a}^+ where this minimum is attained. We will obtain estimates on both the value m_t of the minimum and on the distance from the origin $|x_t|$ of the point where it is attained. Then we need to relate these estimates with the function that we want to control. Namely we will go from ν_t to w_t then u_t and finally φ_t .

To summarize, the strategy to prove estimates on φ_t is in three steps:

- reduce to estimates on $|m_t|$ and $|x_t|$,
- find uniform estimates $|m_t| \leq C$,
- get a uniform control $|x_t| \leq C$ of x_t .

We will also have to prove the necessity of the condition in Theorem A and the upper bound on $R(X)$. Before that, we gather some preliminary results.

3.4 The function $j = -\ln J$

The half sum of positive roots ρ is in \mathfrak{a}^+ , so $\langle \alpha, \rho \rangle > 0$ for all $\alpha \in \Phi^+$. We will use this as a reference to control the distance to the walls of \mathfrak{a}^+ . We also choose a basis $\{e_i\}$ of \mathfrak{a} , and corresponding coordinates $\{x_i\}$ when necessary.

Lemma 3.4. *There exists a constant c such that for any $x \in \mathfrak{a}^+$, we have*

$$j(x) \geq \langle -4\rho, x \rangle + c.$$

Furthermore, for any $b > 0$, there exists constants $C = C(b)$, $C' = C'(b)$ such that for all $x \in b\rho + \mathfrak{a}^+$,

$$j(x) \leq \langle -4\rho, x \rangle + C \leq C'.$$

Proof. Write

$$\sinh \langle \alpha, x \rangle = e^{\langle \alpha, x \rangle} \left(\frac{1 - e^{-2\langle \alpha, x \rangle}}{2} \right) \leq \frac{e^{\langle \alpha, x \rangle}}{2}$$

for $x \in \mathfrak{a}^+$. Then

$$j(x) = -2 \sum_{\alpha \in \Phi^+} \ln \sinh \langle \alpha, x \rangle \geq -2 \sum_{\alpha \in \Phi^+} \langle \alpha, x \rangle + c,$$

where $c = 2 \ln(2) \text{Card}(\Phi^+)$.

If we assume $x \in b\rho + \mathfrak{a}^+$, then using the same expression of $\sinh \langle \alpha, x \rangle$, we get $j(x) \leq \langle -4\rho, x \rangle + C$ with

$$C = -2 \sum_{\alpha \in \Phi^+} \ln \left(\frac{1 - e^{-2b\langle \alpha, \rho \rangle}}{2} \right).$$

and $j(x) \leq C'$ with

$$C' = C - 2b \sum_{\alpha \in \Phi^+} \langle \alpha, \rho \rangle. \quad \square$$

Let us now turn to the derivatives of j . Recall that the directional derivative of j in the direction ξ is defined by:

$$\frac{\partial j}{\partial \xi}(x) := \langle \nabla j(x), \xi \rangle.$$

Lemma 3.5. *For any $\xi \in \mathfrak{a}^+$, we have, for all $x \in \mathfrak{a}^+$,*

$$\frac{\partial j}{\partial \xi}(x) < \langle -4\rho, \xi \rangle.$$

If $\xi \in \mathfrak{a}_t$, then for all $x \in \mathfrak{a}^+$,

$$\frac{\partial j}{\partial \xi}(x) = 0.$$

Proof. The second statement is clear because for any $\alpha \in \Phi$, $\langle \alpha, x \rangle$ depends only on the projection of x on \mathfrak{a}_{ss} .

For the first statement, we compute

$$\frac{\partial j}{\partial \xi}(x) = -2 \sum_{\alpha \in \Phi^+} \langle \alpha, \xi \rangle \coth \langle \alpha, x \rangle.$$

Since $\xi \in \mathfrak{a}^+$, we have $\langle \alpha, \xi \rangle > 0$ for all $\alpha \in \Phi^+$. Since x is also in \mathfrak{a}^+ , we have $\coth \langle \alpha, x \rangle > 1$. We deduce from this that

$$\frac{\partial j}{\partial \xi}(x) < -2 \sum_{\alpha \in \Phi^+} \langle \alpha, \xi \rangle = \langle -4\rho, \xi \rangle. \quad \square$$

We will also need to control the derivatives of j away from the walls. This is achieved by the following lemma.

Lemma 3.6. *For any $b > 0$, there exists a constant C such that for any $x \in b\rho + \mathfrak{a}^+$,*

$$|\nabla(j)(x)| \leq C.$$

Proof. Recall that

$$\frac{\partial j}{\partial x_i}(x) = -2 \sum_{\alpha \in \Phi^+} \langle \alpha, e_i \rangle \coth \langle \alpha, x \rangle$$

For $x \in b\rho + \mathfrak{a}^+$, we have $1 < \coth \langle \alpha, x \rangle < \coth(b \langle \alpha, \rho \rangle)$, so for any i ,

$$\left| \frac{\partial j}{\partial x_i}(x) \right| \leq 2 \sum_{\alpha \in \Phi^+} |\alpha| \coth(b \langle \alpha, \rho \rangle). \quad \square$$

Lemma 3.7. *The function j is strictly convex on \mathfrak{a}^+ .*

Proof. An easy computation shows that

$$\frac{\partial^2}{\partial x_j \partial x_i} (-\ln \sinh \langle \alpha, x \rangle) = \frac{\langle \alpha, e_i \rangle \langle \alpha, e_j \rangle}{\sinh^2 \langle \alpha, x \rangle}.$$

So the Hessian of j is a sum of semipositive matrices, and it is easy to check that the whole sum is definite, so the Hessian of j is positive definite. \square

3.5 The function ν_t

Lemma 3.8. *The function ν_t is strictly convex and admits a unique minimum.*

Proof. From Lemma 3.7, and the fact that w_t is strictly convex as a convex combination of strictly convex functions, we obtain that $\nu_t = w_t + j$ is strictly convex. The function w_t is bounded below, and j tends to $+\infty$ when x approaches a Weyl wall, so $\nu_t(x)$ also tends to $+\infty$ when x approaches a Weyl wall. To prove the existence of a minimum x_t it remains to explain why ν_t goes to infinity at infinity.

Proposition 2.10 implies that $w_t(x) \geq v(x) + C_1$ for some constant C_1 , where v is the support function of the polytope $2P$, so $\nu_t(x) \geq v(x) + j(x) + C_1$. Then

$$\nu_t(x) \geq v(x) - \langle 4\rho, x \rangle + c + C_1$$

by Lemma 3.4. Finally, the fact that X is Fano implies, by [Delb, Remark 4.10] that $4\rho \in \text{Int}(2P)$, so ν_t is indeed proper. \square

Let m_t denote the minimum value of ν_t and x_t be such that $\nu_t(x_t) = m_t$.

Lemma 3.9. *There exists a constant $b_1 > 0$ independent of t such that*

$$x_t \in b_1\rho + \mathfrak{a}^+$$

Proof. By definition of x_t , the derivative of ν_t at x_t vanishes. In particular, we have

$$\frac{\partial \nu_t}{\partial \rho}(x_t) = 0.$$

Recall that $\nu_t = w_t + j$, and that the derivatives of w_t are bounded by Proposition 2.10. In particular we get a bound

$$\left| \frac{\partial w_t}{\partial \rho}(x_t) \right| \leq C.$$

On the other hand, recall that:

$$\frac{\partial j}{\partial \rho}(x_t) = -2 \sum_{\alpha \in \Phi^+} \langle \alpha, \rho \rangle \coth \langle \alpha, x_t \rangle.$$

So we have

$$\left| 2 \sum_{\alpha \in \Phi^+} \langle \alpha, \rho \rangle \coth \langle \alpha, x_t \rangle \right| \leq C$$

but since all the terms of the sum are positive and all the $\langle \alpha, \rho \rangle$ are strictly positive, this implies that for all $\alpha \in \Phi^+$, $\coth \langle \alpha, x_t \rangle \leq C$. Observe that the function \coth tends to $+\infty$ at 0, so we obtain $\langle \alpha, x_t \rangle \geq C'$ for all α for a constant $C' > 0$ independent of t .

To conclude, observe that the intersection of the half spaces defined by $\langle \alpha, x \rangle \geq C'$ is contained in a translate $b_1 \rho + \mathfrak{a}^+$ for some $b_1 > 0$ sufficiently small, independent of t . \square

We will also need to control from below the value of ν_t near the walls. This will be achieved by the following technical proposition. For now we cannot control ν_t uniformly close to the walls, but we will as soon as we control m_t . We will use twice the proposition, first to obtain a lower bound on m_t , then to ensure $e^{-\nu_t}$ is uniformly sufficiently small near the walls. Remark also that this proposition can be seen as a precise statement of what we called the properness of ν_t near the walls.

Proposition 3.10. *For any $M > 0$, there exists a constant $b > 0$ independent of t such that for any $x \in \mathfrak{a}^+$ satisfying $\langle \alpha, x \rangle < b \langle \alpha, \rho \rangle$ for some root $\alpha \in \Phi^+$ defining a wall of \mathfrak{a}^+ , we have*

$$\nu_t(x) \geq m_t + M.$$

Recall that the roots defining the walls are also the simple roots of Φ^+ .

Proof. Let $x \in \mathfrak{a}^+$ be such that $\langle \alpha, x \rangle < b_1 \langle \alpha, \rho \rangle$ for some simple root $\alpha \in \Phi^+$. Consider the ray $\{x + s\rho, s \geq 0\}$ starting from x . It meets the boundary $\partial(b_1 \rho + \mathfrak{a}^+)$ of $b_1 \rho + \mathfrak{a}^+$ at a unique point $y = x + s_0 \rho$. Furthermore y is in $b_1 \rho + \alpha^\perp$ for a simple root α . We can then write $x = y - s_0 \rho$, and s_0 satisfies $0 < b_1 - s_0 < b$.

Consider $\alpha \in \Phi^+$ a simple root, and $y \in (b_1 \rho + \alpha^\perp) \cap \partial(b_1 \rho + \mathfrak{a}^+)$. We will show that there exists a constant $b > 0$ independent of t such that $\nu_t(y - s\rho) \geq m_t + M$ for all s such that $0 < b_1 - s < b$, and that this b can be chosen independent of y and α .

This is enough to prove the proposition because any x as in the statement is of the form above for some α , y and s as shown at the beginning.

Consider the function $g(s) = \nu_t(y - s\rho)$ on $[0, b_1[$. We have $g(0) = \nu_t(y) \geq m_t$ by definition of m_t .

We consider now the derivative of g . Remember that the derivatives of w_t are uniformly bounded, by d , in absolute value by Proposition 2.10. Then

$$g'(s) \geq -d + 2 \sum_{\beta \in \Phi^+} \langle \beta, \rho \rangle \coth \langle \beta, y - s\rho \rangle.$$

Since all the terms in the sum are positive, we have in particular

$$g'(s) \geq -d + 2 \langle \alpha, \rho \rangle \coth \langle \alpha, y - s\rho \rangle.$$

From the assumptions, we compute

$$\langle \alpha, y - s\rho \rangle = b_1 \langle \alpha, \rho \rangle - s \langle \alpha, \rho \rangle = (b_1 - s) \langle \alpha, \rho \rangle.$$

Observe that the positive function \coth is not integrable near 0^+ , so

$$\int_0^{b_1} \coth((b_1 - s) \langle \alpha, \rho \rangle) ds = +\infty.$$

Together with the fact that g is greater than m_t at 0, it means that for any M , we can find a $b_\alpha > 0$ such that $g(s) \geq M + m_t$ for $b_1 - s \leq b_\alpha$.

Remark that none of what we have done depends on the choice of y . Furthermore, since there are only a finite number of roots α , we can choose a $b > 0$ such that $b < b_\alpha$ for all α , and it concludes the proof. \square

3.6 Reduction to estimates on m_t and x_t

Lemma 3.11. *Suppose we have uniform estimates $|m_t| < C_m$ and $|x_t| < C_x$ for t in some interval $[t_0, t_1[\subset [0, 1]$. Then there is an uniform upper bound for φ_t on $[t_0, t_1[$.*

Proof. Recall that it is enough to obtain a uniform upper bound on $u_t - u_{\text{ref}}$ which is a function defined on \mathfrak{a} . We have, by Proposition 2.10 with $x_0 = x_t$, that

$$u_t(x) \leq v(x - x_t) + u_t(x_t)$$

where v is the support function of $2P$. Using the other inequality for u_{ref} we have

$$u_{\text{ref}}(x) \geq v(x) + C_1 \geq v(x - x_t) + C_1 - d|x_t|.$$

Combining these two gives

$$\begin{aligned} (u_t - u_{\text{ref}})(x) &\leq v(x - x_t) + u_t(x_t) - v(x - x_t) - C_1 + d|x_t| \\ &\leq u_t(x_t) - C_1 + d|x_t| \\ &\leq u_t(x_t) - C_1 + dC_x \end{aligned}$$

so we just have to control $u_t(x_t)$.

We have $|m_t| = |\nu_t(x_t)| \leq C_m$, i.e

$$|tu_t(x_t) + (1-t)u_{\text{ref}}(x_t) + j(x_t)| \leq C_m.$$

Now we have:

- $t \geq t_0 > 0$,
- $|j(x_t)| \leq C_2$ for some constant C_2 because $x_t \in b_1\rho + \mathfrak{a}^+$,
- and $u_{\text{ref}}(x_t) \leq \sup\{u_{\text{ref}}(y) ; y \in B(0, C_x)\} =: C_3$.

So

$$u_t(x_t) \leq \frac{C_m + C_2 + C_3}{t_0}.$$

Finally we have proved the uniform upper bound

$$(u_t - u_{\text{ref}})(x) \leq C_4 := \frac{C_m + C_2 + C_3}{t_0} - C_1 + dC_x. \quad \square$$

4 Estimates on $|m_t|$

In this section, we essentially follow the work of Wang and Zhu to obtain estimates on $|m_t|$ and a uniform estimates on the (at least linear) growth of ν_t . A key step is proposition 4.3, which is a direct consequence of Proposition 2.12 and will be used in the next sections also.

We consider the set

$$A_t := \{x \in \mathfrak{a}^+ ; m_t \leq \nu_t(x) \leq m_t + 1\} \subset \mathfrak{a}^+.$$

We will obtain upper and lower bound for the volume of A_t . The upper bound will depend on m_t . The key is to obtain an upper bound that is small enough to give information.

Note that the set A_t is a bounded and convex set. Indeed, since m_t is the minimum of ν_t , A_t is a sublevel set of ν_t which is convex, so A_t is convex. Furthermore, by the properness of ν_t , A_t is a bounded set.

Lemma 4.1. *There is an upper bound on the volume of A_t :*

$$\text{Vol}(A_t) \leq Ce^{m_t/2}$$

where the constant $C > 0$ does not depend on $t \geq t_0$.

Proof. Fritz John proved in [Joh48, Theorem III] that for any convex and bounded subset A of \mathbb{R}^r , there exists an ellipsoid E such that

$$\frac{1}{r}E \subset A \subset E$$

where $(1/r)E$ is the dilation of E of factor $1/r$ centered at the center of the ellipsoid E .

We can then find such an ellipsoid E_t for A_t . Let T be a linear transformation, of determinant one, such that $T(E) = B(y, \delta)$ is a ball. We will obtain an upper bound on δ , thus getting an upper bound for the volume of $\text{Vol}(A_t)$ because

$$\text{Vol}(A_t) \leq \text{Vol}(E) = \text{Vol}(T(E)) = C\delta^r.$$

Let ν'_t be the function defined by $\nu'_t(x) = \nu_t(T^{-1}(x))$. We want to use a comparison principle on $B(y, \delta/r)$. For that we first show that $\text{MA}_{\mathbb{R}}(\nu'_t)(x) \geq Ce^{-m_t}$ on $T(A_t)$. This is equivalent to showing that $\text{MA}_{\mathbb{R}}(\nu_t)(x) \geq Ce^{-m_t}$ on A_t . First remark that since the Hessian $\text{Hess}_{\mathbb{R}}\nu_t$ of ν_t satisfies:

$$\text{Hess}_{\mathbb{R}}\nu_t = t\text{Hess}_{\mathbb{R}}u_t + (1-t)\text{Hess}_{\mathbb{R}}u_{\text{ref}} + \text{Hess}_{\mathbb{R}}j,$$

we have

$$\det(\text{Hess}_{\mathbb{R}}\nu_t) \geq \det(t\text{Hess}_{\mathbb{R}}u_t),$$

i.e.

$$\text{MA}_{\mathbb{R}}(\nu_t)(x) \geq t^r \text{MA}_{\mathbb{R}}(u_t)(x).$$

Using Proposition 3.3 we deduce that

$$\begin{aligned} \text{MA}_{\mathbb{R}}(\nu_t)(x) &\geq t^r J(x) e^{-w_t(x)} \prod_{\alpha \in \Phi^+} \frac{1}{\langle \alpha, \nabla u_t(x) \rangle^2} \\ &\geq t^r e^{-\nu_t(x)} \prod_{\alpha \in \Phi^+} \frac{1}{\langle \alpha, \nabla u_t(x) \rangle^2}. \end{aligned}$$

We treat the factors separately:

- We have $t \geq t_0 > 0$ for t_0 defined in Notation 3.1.
- By definition of A_t , we have $e^{-\nu_t(x)} \geq e^{-m_t-1}$ on A_t .
- For any $x \in \mathfrak{a}$, we have $\nabla u_t(x) \in 2P$, so for any $\alpha \in \Phi^+$, $\langle \alpha, \nabla u_t(x) \rangle$ is bounded above independently of t . This implies that

$$\prod_{\alpha \in \Phi^+} \frac{1}{\langle \alpha, \nabla u_t(x) \rangle^2} \geq c$$

for some positive constant c .

In conclusion, we indeed have an inequality $\text{MA}_{\mathbb{R}}(\nu_t)(x) \geq Ce^{-m_t}$ on A_t , with C a positive constant independent of $t \geq t_0$.

Now we use the comparison principle on $B(y, \delta/r)$ for real Monge-Ampère equations: let g be the auxiliary function defined by

$$g(x) = C^{1/r} e^{-m_t/r} \left(|x - y|^2 - \frac{\delta^2}{r^2} \right) + m_t + 1.$$

Then we have

- $g(x) = m_t + 1 \geq \nu'_t(x)$ for $x \in \partial B(y, \delta/r)$, and
- $\text{MA}_{\mathbb{R}}(g)(x) = Ce^{-m_t} \leq \text{MA}_{\mathbb{R}}(\nu'_t)(x)$ on $B(y, \delta/r)$.

So the comparison principle gives that $\nu'_t(x) \leq g(x)$ on $B(y, \delta/r)$. In particular, we have

$$\begin{aligned}
m_t &\leq \nu_t(T^{-1}(y)) \\
&\leq \nu'_t(y) \\
&\leq g(y) \\
&\leq C^{1/r} e^{-m_t/r} \left(-\frac{\delta^2}{r^2} \right) + m_t + 1.
\end{aligned}$$

We deduce from that the following upper bound for δ :

$$\delta \leq \sqrt{\frac{1}{C^{1/r}} r e^{m_t/2r}}.$$

Putting everything together, we obtain

$$\text{Vol}(A_t) \leq \text{Vol}(B(y, \delta)) \leq C' e^{m_t/2}. \quad \square$$

We turn now to a lower bound on $\text{Vol}(A_t)$.

Lemma 4.2. *There exists a constant $c > 0$ independent of t such that*

$$\text{Vol}(A_t) \geq c.$$

Proof. There exists a constant b_2 independent of t such that $0 < b_2 < b_1$ and

$$A_t \subset b_2 \rho + \mathfrak{a}^+.$$

This is a corollary of Proposition 3.10, taking b_2 corresponding to $M = 1$.

Then by Lemma 3.6 and Proposition 2.10, on $b_2 \rho + \mathfrak{a}^+$, $|\nabla \nu_t|$ is bounded independently of t , say by C . Then it is clear that the ball $B(x_t, 1/C)$ is contained in A_t . So $\text{Vol}(A_t) \geq c$ for some $c > 0$ independent of t . \square

Proposition 4.3. *The following integral is independent of t :*

$$\int_{\mathfrak{a}^+} e^{-\nu_t(x)} dx = \int_{2P^+} \prod_{\alpha \in \Phi^+} \langle \alpha, p \rangle^2 dp =: V.$$

Proof. Applying Proposition 2.12 with the ample line bundle $-K_X$, we have, for some constant C depending only on G , and for any convex potential u of a smooth $K \times K$ -invariant positively curved hermitian metric on $-K_X$,

$$\begin{aligned}
\deg(-K_X) &= C \int_{\mathfrak{a}^+} \prod_{\alpha \in \Phi^+} \langle \alpha, \nabla u(x) \rangle^2 \text{MA}_{\mathbb{R}}(u)(x) dx \\
&= C \int_{2P^+} \prod_{\alpha \in \Phi^+} \langle \alpha, p \rangle^2 dp
\end{aligned}$$

We apply this to the convex potential u_t , which by Proposition 3.3 satisfies

$$e^{-\nu_t(x)} = \prod_{\alpha \in \Phi^+} \langle \alpha, \nabla u_t(x) \rangle^2 \text{MA}_{\mathbb{R}}(u_t)(x)$$

and obtain the statement, with $V = \text{deg}(-K_X)/C$. \square

We can now prove the main result of this subsection.

Proposition 4.4.

- *There exists a constant C independent of t , such that $|m_t| \leq C$.*
- *There exist a constant $\kappa > 0$ and a constant C , both independent of t , such that for $x \in \mathfrak{a}^+$,*

$$\nu_t(x) \geq \kappa|x - x_t| - C.$$

Proof. Following here Donaldson [Don08] rather than Wang and Zhu, we write

$$\begin{aligned} \int_{\mathfrak{a}^+} e^{-\nu_t(x)} dx &= \int_{\mathfrak{a}^+} \int_{\nu_t(x)}^{+\infty} e^{-s} ds dx \\ &= \int_{-\infty}^{+\infty} e^{-s} \int_{\mathfrak{a}^+} 1_{\{\nu_t(x) \leq s\}} dx ds \\ &= \int_{m_t}^{+\infty} e^{-s} \text{Vol}(\{\nu_t \leq s\}) ds \\ &= e^{-m_t} \int_0^{+\infty} e^{-s} \text{Vol}(\{\nu_t \leq m_t + s\}) ds \end{aligned}$$

Now remark that $\{\nu_t \leq m_t + s\} \subset s \cdot A_t$ by convexity of ν_t , where $s \cdot A_t$ is the s -dilation of A_t with center x_t . We deduce from that

$$\text{Vol}(\{\nu_t \leq m_t + s\}) \leq s^r \text{Vol}(A_t) \leq C s^r e^{m_t/2}.$$

Applying this to the formula above we obtain

$$\begin{aligned} \int_{\mathfrak{a}^+} e^{-\nu_t(x)} dx &\leq e^{-m_t} C e^{m_t/2} \int_0^{+\infty} e^{-s} s^r ds \\ &\leq C' e^{-m_t/2}. \end{aligned}$$

The left hand side being constant, this inequality gives an upper bound on m_t .

For the lower bound, remark that

$$\begin{aligned} V &= \int_{\mathfrak{a}^+} e^{-\nu_t(x)} dx = e^{-m_t} \int_0^{+\infty} e^{-s} \text{Vol}(\{\nu_t \leq m_t + s\}) ds \\ &\geq e^{-m_t} \int_1^{+\infty} e^{-s} \text{Vol}(\{\nu_t \leq m_t + s\}) ds \\ &\geq e^{-m_t} \text{Vol}(A_t) \int_1^{+\infty} e^{-s} ds. \end{aligned}$$

By Lemma 4.2, $\text{Vol}(A_t)$ admits a lower bound c independent of t , so

$$V \geq e^{-m_t} c \int_1^{+\infty} e^{-s} ds,$$

and we deduce that

$$-m_t \leq \ln(V) - \ln\left(c \int_1^{+\infty} e^{-s} ds\right)$$

so $-m_t$ is bounded above by a constant independent of t . Thus we have showed estimates on $|m_t|$.

Now for linear growth, the estimate on $|m_t|$ implies that we know both an upper bound C_1 and a lower bound C_2 independent of t for the volume of A_t . Since this set is convex, and contains a ball $B(x_t, \delta_0)$ of fixed radius δ_0 independent of t by the proof of Lemma 4.2, this implies that A_t is included in a ball $A_t \subset B(x_t, \delta)$ where δ only depends on C_1 and δ_0 .

By convexity of ν_t , this implies that $\nu_t(x) \geq |x - x_t|/\delta + m_t$ outside of the ball, and we can extend this inequality to the whole of \mathfrak{a}^+ simply by subtracting 1:

$$\nu_t(x) \geq |x - x_t|/\delta + m_t - 1$$

everywhere. Using again the fact that m_t is uniformly bounded we get the result with $\kappa = 1/\delta$. \square

5 Obstruction, and upper bound on $R(X)$

We will here explain how our condition appears as a necessary condition, then obtain an upper bound on the greatest Ricci lower bound. Everything relies on the following vanishing statement, which will also be of major importance in the next section.

5.1 A vanishing integral

Proposition 5.1. *Let u be the convex potential of a smooth positive metric on $-K_X$. Define $\nu := u + j$. Let ξ be any vector in $\overline{\mathfrak{a}^+}$. Then*

$$\int_{\mathfrak{a}^+} \frac{\partial \nu}{\partial \xi} e^{-\nu} dx = 0.$$

Before we get to the proof, let us remark that the function considered is integrable. More generally, we can remark first that for any potential u_0 , and any vector ξ , the function $\frac{\partial u_0}{\partial \xi} e^{-\nu}$ is integrable on \mathfrak{a}^+ . This is the case because $\nabla u_0 \in 2P$, and $e^{-\nu} \leq C e^{-(\nu-4\rho)+C}$ (by Proposition 2.10 and Lemma 3.4) is obviously integrable.

Secondly, we have to show that the function $\frac{\partial j}{\partial \xi} e^{-\nu} = \frac{\partial j}{\partial \xi} J e^{-u}$ is integrable. Write

$$\begin{aligned} \frac{\partial j}{\partial \xi}(x)J(x) &= -2 \sum_{\alpha \in \Phi^+} \langle \alpha, \xi \rangle \coth \langle \alpha, x \rangle \prod_{\beta \in \Phi^+} \sinh^2 \langle \beta, x \rangle \\ &= -2 \sum_{\alpha \in \Phi^+} \langle \alpha, \xi \rangle \cosh \langle \alpha, x \rangle \sinh \langle \alpha, x \rangle \prod_{\beta \neq \alpha} \sinh^2 \langle \beta, x \rangle. \end{aligned}$$

Then by a computation similar to Lemma 3.4, we have

$$\left| e^{4\rho} \frac{\partial j}{\partial \xi} J \right| \leq C e^{4\rho},$$

so again $\frac{\partial j}{\partial \xi} e^{-\nu}$ is integrable.

Proof. Choose a basis $(e_i)_{i=1..s}$ of the semisimple part \mathfrak{a}_{ss} which generate $\mathfrak{a}^+ \cap \mathfrak{a}_{ss}$ as a cone, and a basis $(f_j)_{j=1..r-s}$ of the toric part \mathfrak{a}_t . Consider the following sets, for $0 \leq \epsilon < M$:

$$Q(\epsilon, M) := \left\{ \sum_i x_i e_i + \sum_j y_j f_j ; \forall i \epsilon \leq x_i \leq M, \forall j -M \leq y_j \leq M \right\}.$$

Note that $\partial Q(\epsilon, M) = S_1(\epsilon, M) \cup S_2(\epsilon, M)$, where:

$$S_1(\epsilon, M) := \left\{ \sum_i x_i e_i + \sum_j y_j f_j \in Q(\epsilon, M) ; \exists i x_i = \epsilon \right\}$$

$$S_2(\epsilon, M) := \left\{ \sum_i x_i e_i + \sum_j y_j f_j \in Q(\epsilon, M) ; \exists i x_i = M \text{ or } \exists j |y_j| = M \right\}.$$

Remark that $\frac{\partial \nu}{\partial \xi} e^{-\nu} = -\frac{\partial e^{-\nu}}{\partial \xi}$. Then by the divergence formula applied to $e^{-\nu}$ we have for $\epsilon > 0$,

$$\int_{Q(\epsilon, M)} \frac{\partial \nu}{\partial \xi} e^{-\nu} dx = \int_{S_1(\epsilon, M) \cup S_2(\epsilon, M)} e^{-\nu} \langle \xi, \mu \rangle d\sigma$$

where μ is the exterior normal and $d\sigma$ is the surface area.

Write now $e^{-\nu} = e^{-u} J$. This is a continuous function on \mathfrak{a} , and it vanishes on the Weyl walls. Fixing M , we can thus let ϵ tend to 0, and we have that $e^{-\nu}$ tends uniformly to 0 on $S_1(\epsilon, M)$, so $\int_{S_1(\epsilon, M)} e^{-\nu} \langle \xi, \mu \rangle d\sigma$ tends to 0.

We thus have

$$\int_{Q(0, M)} \frac{\partial \nu}{\partial \xi} e^{-\nu} dx = \int_{S_2(0, M)} e^{-\nu} \langle \xi, \mu \rangle d\sigma.$$

Now as we have seen before, we have $e^{-\nu} \leq C e^{-(v-4\rho)+C}$, so $e^{-\nu}(x)$ decreases exponentially as $|x|$ tends to infinity. Since the area of $S_2(0, M)$ grows polynomially, this ensures that $\int_{S_2(0, M)} e^{-\nu} \langle \xi, \mu \rangle d\sigma$ tends to zero as M tends to ∞ . This ends the proof. \square

5.2 Obstruction to the existence of a Kähler-Einstein metric

Let us now explain the relation between convex potentials and the barycenter $\text{bar}_{DH}(P^+)$. Recall that V denotes the volume of the polytope P^+ with respect to the Duistermaat-Heckman measure, and is equal, up to a multiplicative constant, to the volume of the anticanonical line bundle.

Proposition 5.2. *Assume that u is the convex potential of a smooth, positively curved hermitian metric on $-K_X$, then*

$$\int_{\mathfrak{a}^+} \frac{\partial u}{\partial \xi} \prod_{\alpha \in \Phi^+} \langle \alpha, \nabla u \rangle^2 \text{MA}_{\mathbb{R}}(u) dx = \langle \xi, \text{bar}_{DH}(2P^+) \rangle V.$$

Proof. We simply use the change of variable $p = \nabla u(x)$, just as in the proof of Proposition 2.12. It gives:

$$\begin{aligned} \int_{\mathfrak{a}^+} \frac{\partial u}{\partial \xi} \prod_{\alpha \in \Phi^+} \langle \alpha, \nabla u \rangle^2 \text{MA}_{\mathbb{R}}(u) dx &= \int_{2P^+} \langle p, \xi \rangle \prod_{\alpha \in \Phi^+} \langle \alpha, p \rangle^2 dp \\ &= \langle \xi, \text{bar}_{DH}(2P^+) \rangle V. \quad \square \end{aligned}$$

We now prove an obstruction to the existence of Kähler-Einstein metrics on X . Recall that Ξ denotes the relative interior of the cone generated by the positive roots Φ^+ .

Proposition 5.3. *Assume there exists a Kähler-Einstein metric on X . Then*

$$\text{bar}_{DH}(2P^+) \in 4\rho + \Xi.$$

Proof. By Proposition 3.3, the Kähler-Einstein equation restricted to the open orbit reads:

$$\text{MA}_{\mathbb{R}}(u) \prod_{\alpha \in \Phi^+} \langle \alpha, \nabla u \rangle^2 = e^{-u} J.$$

Suppose that there exists a solution u . Applying Proposition 5.1 to u gives

$$\int_{\mathfrak{a}^+} \frac{\partial \nu}{\partial \xi} e^{-\nu} dx = 0,$$

so by definition of $\nu = u + j$,

$$\int_{\mathfrak{a}^+} \frac{\partial u}{\partial \xi} e^{-\nu} dx + \int_{\mathfrak{a}^+} \frac{\partial j}{\partial \xi} e^{-\nu} dx = 0.$$

Since u is solution to the Kähler-Einstein equation, we have

$$e^{-\nu} = e^{-u} J = \text{MA}_{\mathbb{R}}(u) \prod_{\alpha \in \Phi^+} \langle \alpha, \rho \rangle^2.$$

In particular, by Proposition 4.3,

$$\int_{\mathfrak{a}^+} e^{-\nu} dx = V$$

is constant, and by Proposition 5.2,

$$\int_{\mathfrak{a}^+} \frac{\partial u}{\partial \xi} e^{-\nu} dx = \langle \xi, \text{bar}_{DH}(2P^+) \rangle V.$$

By Lemma 3.5, we have for $\xi \in \mathfrak{a}^+$,

$$\int_{\mathfrak{a}^+} \frac{\partial j}{\partial \xi} e^{-\nu} dx < -4 \langle \rho, \xi \rangle V.$$

Combining these facts, we have:

$$\begin{aligned} \langle \xi, \text{bar}_{DH}(2P^+) \rangle V - 4 \langle \rho, \xi \rangle V &> \int_{\mathfrak{a}^+} \frac{\partial u}{\partial \xi} e^{-\nu} dx + \int_{\mathfrak{a}^+} \frac{\partial j}{\partial \xi} e^{-\nu} dx \\ &> 0. \end{aligned}$$

Dividing by V we obtain that for any $\xi \in \mathfrak{a}^+$,

$$\langle \xi, \text{bar}_{DH}(2P^+) - 4\rho \rangle > 0.$$

The large inequality would mean $\text{bar}_{DH}(2P^+) - 4\rho \in (\mathfrak{a}^+)^\vee$, and here the strict inequality means that $\text{bar}_{DH}(2P^+) - 4\rho$ is in the relative interior of $(\mathfrak{a}^+)^\vee$, which is precisely Ξ . \square

Example 5.4. This obstruction allows to show that the manifold X_2 does not admit any Kähler-Einstein metric. Recall that this is a compactification of $\text{Sp}_4(\mathbb{C})$ so the corresponding root system is B_2 . Denote by α_1 and α_2 the simple roots, with α_2 the long root. Then the other positive roots are $\alpha_1 + \alpha_2$ and $2\alpha_1 + \alpha_2$, so $2\rho = 4\alpha_1 + 3\alpha_2$.

Choosing a realization of B_2 in the euclidean plane with $\alpha_1 = (1, 0)$ and $\alpha_2 = (-1, 1)$, we compute that, for $p = x\alpha_1 + y\alpha_2$,

$$\prod_{\alpha \in \Phi^+} \langle \alpha, p \rangle^2 = x^2 y^2 (x - y)^2 (-x + 2y)^2.$$

We can then compute

$$\text{bar}_{DH}(P^+) = \frac{278037566905}{66955221696} \alpha_1 + \frac{3043253830}{1046175339} \alpha_2.$$

In particular the coordinate in α_2 is strictly smaller than 3 so there exists no Kähler-Einstein metric on X_2 . Figure 4 gives a representation of the polytope P^+ , the barycenter is represented by a cross and the cone delimited by the dashed lines is $2\rho + \Xi$.

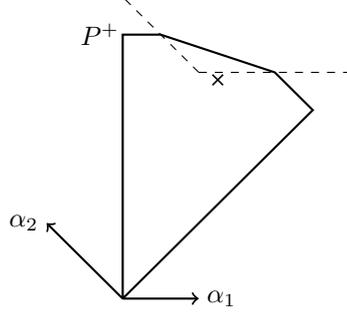


Figure 4: Barycenter for X_2

5.3 Upper bound on $R(X)$

Proposition 5.5. *Assume that X admits no Kähler-Einstein metrics, then the greatest Ricci lower bound of X is lower than or equal to the supremum of all $t < 1$ such that*

$$4\rho + \frac{t}{t-1}(\text{bar}_{DH}(2P^+) - 4\rho) \in -\Xi + 2P^+.$$

Proof. Consider the equation at time t :

$$\text{MA}_{\mathbb{R}}(u_t) \prod_{\alpha \in \Phi^+} \langle \alpha, \nabla u_t \rangle^2 = e^{-\nu_t}.$$

Apply Proposition 5.1 to w_t . This gives for any $\xi \in \overline{\mathfrak{a}^+}$,

$$\int_{\mathfrak{a}^+} \frac{\partial \nu_t}{\partial \xi} e^{-\nu_t} dx = 0.$$

This is equivalent to

$$t \int_{\mathfrak{a}^+} \frac{\partial u_t}{\partial \xi} e^{-\nu_t} + (1-t) \int_{\mathfrak{a}^+} \frac{\partial u_{\text{ref}}}{\partial \xi} e^{-\nu_t} + \int_{\mathfrak{a}^+} \frac{\partial j}{\partial \xi} e^{-\nu_t} = 0.$$

Without loss of generality we can assume $t < 1$ and divide by $(t-1)V$ to get

$$\frac{t}{t-1} \int_{\mathfrak{a}^+} \frac{\partial u_t}{\partial \xi} \frac{e^{-\nu_t}}{V} + \frac{1}{t-1} \int_{\mathfrak{a}^+} \frac{\partial j}{\partial \xi} \frac{e^{-\nu_t}}{V} = \int_{\mathfrak{a}^+} \frac{\partial u_{\text{ref}}}{\partial \xi} \frac{e^{-\nu_t}}{V}.$$

If v is the support function of $2P$, we have for any $x \in \mathfrak{a}^+$, $\frac{\partial u_{\text{ref}}}{\partial \xi}(x) \leq v(\xi)$, so

$$\int_{\mathfrak{a}^+} \frac{\partial u_{\text{ref}}}{\partial \xi} \frac{e^{-\nu_t}}{V} \leq v(\xi).$$

On the other hand, we can use here also Lemma 3.5 to get, for $\xi \in \mathfrak{a}^+$,

$$\frac{1}{t-1} \int_{\mathfrak{a}^+} \frac{\partial j}{\partial \xi} \frac{e^{-\nu_t}}{V} > \frac{1}{t-1}(-4 \langle \rho, \xi \rangle).$$

We thus have, using Proposition 5.2,

$$\begin{aligned}
v(\xi) &> \frac{t}{t-1} \int_{\mathfrak{a}^+} \frac{\partial u_t}{\partial \xi} \frac{e^{-\nu_t}}{V} + \frac{1}{t-1} \int_{\mathfrak{a}^+} \frac{\partial j}{\partial \xi} \frac{e^{-\nu_t}}{V} \\
&> \frac{t}{t-1} \langle \xi, \text{bar}_{DH}(2P^+) \rangle - \frac{1}{t-1} 4 \langle \rho, \xi \rangle \\
&> \left\langle \xi, 4\rho + \frac{t}{t-1} (\text{bar}_{DH}(2P^+) - 4\rho) \right\rangle
\end{aligned}$$

The fact that this is true for all $\xi \in \mathfrak{a}^+$ means, since v is the support function of $2P$, that

$$4\rho + \frac{t}{t-1} (\text{bar}_{DH}(2P^+) - 4\rho) \in -\Xi + 2P^+. \quad \square$$

6 Absence of estimates on $|x_t|$

6.1 Consequence

We will assume now that there are no Kähler-Einstein metrics on X . We will denote by $t_\infty := R(X)$ the greatest Ricci lower bound.

Our assumption implies that $|x_t|$ is unbounded as t tends to t_∞ . Indeed if it was not the case, then we would have estimates on $|x_t|$ and so by Proposition 3.11 and Lemma 4.4, there would be a solution at time t_∞ and by openness for times greater than t_∞ if t_∞ . This is a contradiction.

We can find a sequence t_i such that $t_i \rightarrow t_\infty$ and $|x_{t_i}| \rightarrow \infty$. Define $\xi_t \in \mathfrak{a}^+$ as the direction given by the minimum $\xi_t := x_t/|x_t|$. Up to taking a subsequence, we can also assume that ξ_t admits a limit $\xi_\infty \in \mathfrak{a}^+$ as $t_i \rightarrow t_\infty$.

We will consider an integral equality involving ν_t and consider the limit as $t_i \rightarrow t_\infty$. The integral equality follows from Proposition 5.1 applied to w_t :

$$\int_{\mathfrak{a}^+} \frac{\partial \nu_t}{\partial \xi} e^{-\nu_t} dx = 0.$$

Recall that $w_t = tu_t + (1-t)u_{\text{ref}}$ by definition, so

$$\nu_t = tu_t + (1-t)u_{\text{ref}} + j = t(u_t + j) + (1-t)(u_{\text{ref}} + j).$$

The vanishing integral thus gives

$$t \int_{\mathfrak{a}^+} \frac{\partial u_t}{\partial \xi_t} e^{-\nu_t} dx + (1-t) \int_{\mathfrak{a}^+} \frac{\partial u_{\text{ref}} + j}{\partial \xi_t} e^{-\nu_t} dx + \int_{\mathfrak{a}^+} \frac{\partial j}{\partial \xi_t} e^{-\nu_t} dx = 0,$$

which can also be written

$$t \int_{\mathfrak{a}^+} \frac{\partial u_t + j}{\partial \xi_t} e^{-\nu_t} dx = (t-1) \int_{\mathfrak{a}^+} \frac{\partial u_{\text{ref}} + j}{\partial \xi_t} e^{-\nu_t} dx.$$

We will compute the limit of each of these terms as $t_i \rightarrow t_\infty$, and obtain the following result.

Proposition 6.1. *We have*

$$t_\infty \langle \text{bar}_{DH}(2P^+) - 4\rho, \xi_\infty \rangle = (t_\infty - 1)(v(\xi_\infty) - \langle 4\rho, \xi_\infty \rangle).$$

6.2 Proof of Proposition 6.1

Let us first consider

$$\int_{\mathfrak{a}^+} \frac{\partial u_t}{\partial \xi_t} e^{-\nu_t} dx.$$

Since

$$e^{-\nu_t} = \prod_{\alpha \in \Phi^+} \langle \alpha, \nabla u_t \rangle^2 \text{MA}_{\mathbb{R}}(u_t),$$

Proposition 5.2 implies

$$\int_{\mathfrak{a}^+} \frac{\partial u_t}{\partial \xi_t} e^{-\nu_t} dx = \langle \xi_t, \text{bar}_{DH}(2P^+) \rangle V.$$

In particular, the limit as $t_i \rightarrow t_\infty$ is

$$\langle \xi_\infty, \text{bar}_{DH}(2P^+) \rangle V.$$

For the other terms we need more work to compute the limits. For simplicity, we will often omit the indices i in t_i . We will prove the two following propositions.

Proposition 6.2. *We have*

$$\lim_{t_i \rightarrow t_\infty} \int_{\mathfrak{a}^+} \frac{\partial j}{\partial \xi_t} e^{-\nu_t} = -\langle 4\rho, \xi_\infty \rangle V.$$

Proposition 6.3. *We have*

$$\lim_{t_i \rightarrow t_\infty} \int_{\mathfrak{a}^+} \frac{\partial u_{\text{ref}}}{\partial \xi_t} e^{-\nu_t} = v(\xi_\infty) V.$$

Let us first find a domain $D(\epsilon)$ of the form $B(x_t, \delta) \cap (b\rho + \mathfrak{a}^+)$ where $e^{-\nu_t} dx$ puts all the mass up to $2\epsilon > 0$. When we write $B(x_t, \delta)$, we in general mean $B(x_t, \delta) \cap \mathfrak{a}^+$.

Lemma 6.4. *For any $\epsilon > 0$, there exists a constant $\delta = \delta(\epsilon)$ independent of t such that*

$$\int_{\mathfrak{a}^+ \setminus B(x_t, \delta)} e^{-\nu_t} dx < \epsilon$$

and

$$e^{-\kappa\delta + C} \sigma_r \delta^{r-1} < \epsilon,$$

where σ_r is the area of a sphere of radius 1 in \mathbb{R}^r .

Proof. Recall from Proposition 4.4 that $\nu_t(x) \geq \kappa|x - x_t| - C$, for some $\kappa > 0$, C independent of t . Observe that the function $e^{-\kappa|x - x_t| + C}$ is well defined on \mathfrak{a} , positive and integrable. So there exists a $\delta > 0$ such that

$$\int_{\mathfrak{a} \setminus B(x_t, \delta)} e^{-\kappa|x - x_t| + C} dx < \epsilon.$$

But then we also have

$$\int_{\mathfrak{a}^+ \setminus B(x_t, \delta)} e^{-\nu_t} dx \leq \int_{\mathfrak{a}^+ \setminus B(x_t, \delta)} e^{-\kappa|x-x_t|+C} < \epsilon.$$

Of course, since $e^{-\kappa y+C}$ decreases exponentially with respect to y , we can increase δ so as to have the second condition:

$$e^{-\kappa\delta+C} \sigma_r \delta^{r-1} < \epsilon. \quad \square$$

Lemma 6.5. *For any $\epsilon > 0$, let $\delta = \delta(\epsilon)$ be given by Lemma 6.4. There exists a constant $b = b(\epsilon) > 0$ such that if we denote by $D = D(\epsilon)$ the domain $B(x_t, \delta) \cap (b\rho + \mathfrak{a}^+)$, we have*

$$\int_{B(x_t, \delta) \setminus D} e^{-\nu_t} dx < \epsilon,$$

and

$$\int_{\partial D} e^{-\nu_t} d\sigma < \epsilon,$$

where $d\sigma$ is the area measure of ∂D , which is piecewise smooth.

Proof. Here we want to use Proposition 3.10. Now that we know that m_t is uniformly bounded, we can choose M and the corresponding b so that:

$$\forall x \in \mathfrak{a}^+ \setminus (b\rho + \mathfrak{a}^+), \quad e^{-\nu_t(x)} < \max\left(\frac{\epsilon}{\sigma_r \delta^{r-1}}, \frac{\epsilon}{\delta^r \omega_r}\right)$$

where ω_r is the volume of the ball of radius 1 in \mathbb{R}^r .

Let us prove that $e^{-\nu_t} < \epsilon/(\sigma_r \delta^{r-1})$ on ∂D . A point $x \in \partial D$ is either on the sphere of radius δ centered at x_t , or on $\partial(b\rho + \mathfrak{a}^+)$. In the first case, we have, by Proposition 4.4,

$$\begin{aligned} e^{-\nu_t(x)} &\leq e^{-\kappa|x-x_t|+C} \\ &\leq e^{-\kappa\delta+C} \\ &< \frac{\epsilon}{\sigma_r \delta^{r-1}} \end{aligned}$$

by the second consequence of Lemma 6.4. In the second case, $x \in \partial(b\rho + \mathfrak{a}^+)$, so by the choice of b above, using the first term in the maximum, we have

$$e^{-\nu_t(x)} < \frac{\epsilon}{\sigma_r \delta^{r-1}}.$$

Obviously the volume of ∂D is $\leq \sigma_r \delta^{r-1}$, so

$$\begin{aligned} \int_{\partial D} e^{-\nu_t} d\sigma &< \int_{\partial D} \frac{\epsilon}{\sigma_r \delta^{r-1}} d\sigma \\ &< \epsilon. \end{aligned}$$

For the other conclusion, we use the fact that $e^{-\nu_t(x)} < \epsilon/(\delta^r \omega_r)$ on $B(x_t, \delta) \setminus D \subset \mathbf{a}^+ \setminus (b\rho + \mathbf{a}^+)$, which implies that

$$\int_{B(x_t, \delta) \setminus D} e^{-\nu_t} dx < \int_{B(x_t, \delta) \setminus D} \frac{\epsilon}{\delta^r \omega_r} dx < \epsilon$$

using the fact that the volume of $B(x_t, \delta) \setminus D$ is $\leq \delta^r \omega_r$. \square

Lemma 6.6. *Let $\epsilon > 0$ and $D = D(\epsilon)$ be the domain given by Lemma 6.5. We have*

$$\left| \int_{\mathbf{a}^+ \setminus D} \frac{\partial j}{\partial \xi_t} e^{-\nu_t} \right| < (2d + 1)\epsilon.$$

Proof. Let us write:

$$\begin{aligned} \int_{\mathbf{a}^+ \setminus D} \frac{\partial j}{\partial \xi_t} e^{-\nu_t} dx &= \int_{\mathbf{a}^+ \setminus D} \frac{\partial(\nu_t - w_t)}{\partial \xi_t} e^{-\nu_t} dx \\ &= \int_{\mathbf{a}^+} \frac{\partial \nu_t}{\partial \xi_t} e^{-\nu_t} dx - \int_D \frac{\partial \nu_t}{\partial \xi_t} e^{-\nu_t} dx - \int_{\mathbf{a}^+ \setminus D} \frac{\partial w_t}{\partial \xi_t} e^{-\nu_t} dx \end{aligned}$$

The first of these three integrals is zero by Proposition 5.1.

For the third term, we have

$$\begin{aligned} \int_{\mathbf{a}^+ \setminus D} \left| \frac{\partial w_t}{\partial \xi_t} \right| e^{-\nu_t} dx &\leq d \int_{\mathbf{a}^+ \setminus D} e^{-\nu_t} dx \\ &< 2d\epsilon \end{aligned}$$

by Lemma 6.5.

It remains to deal with the second integral. We apply the divergence theorem on D to the function $e^{-\nu_t}$:

$$\int_D \frac{\partial \nu_t}{\partial \xi_t} e^{-\nu_t} dx = - \int_D \frac{\partial e^{-\nu_t}}{\partial \xi_t} dx = \int_{\partial D} e^{-\nu_t} \langle \xi_t, n \rangle d\sigma$$

where n is the exterior normal to D and $d\sigma$ the area measure.

We conclude using Lemma 6.5 that

$$\left| \int_D \frac{\partial \nu_t}{\partial \xi_t} e^{-\nu_t} dx \right| \leq \int_{\partial D} e^{-\nu_t} d\sigma < \epsilon.$$

Putting everything together, we see that

$$\left| \int_{\mathbf{a}^+ \setminus D} \frac{\partial j}{\partial \xi_t} e^{-\nu_t} dx \right| < (2d + 1)\epsilon \quad \square$$

Proof of Proposition 6.2. Let $\epsilon > 0$. Set $\theta = \epsilon/(3(2d + 1 + 8|\rho|))$ and let $D := D(\theta)$. Write

$$\begin{aligned} \left| \int_{\mathfrak{a}^+} \frac{\partial j}{\partial \xi_t} e^{-\nu_t} + \langle 4\rho, \xi_\infty \rangle V \right| &\leq \left| \int_{\mathfrak{a}^+ \setminus D} \frac{\partial j}{\partial \xi_t} e^{-\nu_t} \right| + \left| \int_D \frac{\partial j}{\partial \xi_t} e^{-\nu_t} + \langle 4\rho, \xi_\infty \rangle V \right| \\ &\leq (2d + 1)\theta + \left| \int_D \frac{\partial j}{\partial \xi_t} e^{-\nu_t} + \langle 4\rho, \xi_\infty \rangle V \right| \end{aligned}$$

by Lemma 6.6.

Then we have

$$\left| \int_D \frac{\partial j}{\partial \xi_t} e^{-\nu_t} + \langle 4\rho, \xi_\infty \rangle V \right| \leq \left| \int_D \frac{\partial j}{\partial \xi_t} e^{-\nu_t} + \langle 4\rho, \xi_t \rangle V \right| + |\langle 4\rho, \xi_\infty - \xi_t \rangle V|$$

The second term tends to zero so there exists an i_0 such that for all $i \geq i_0$,

$$|\langle 4\rho, \xi_\infty - \xi_t \rangle V| \leq \frac{\epsilon}{3}.$$

We now deal with the second term:

$$\begin{aligned} \left| \int_D \frac{\partial j}{\partial \xi_t} e^{-\nu_t} + \langle 4\rho, \xi_t \rangle V \right| &\leq \left| \int_D \left(\frac{\partial j}{\partial \xi_t} + \langle 4\rho, \xi_t \rangle \right) e^{-\nu_t} \right| + \left| \int_{\mathfrak{a}^+ \setminus D} \langle 4\rho, \xi_t \rangle e^{-\nu_t} \right| \\ &\leq \left| \int_D \left(\frac{\partial j}{\partial \xi_t} + \langle 4\rho, \xi_t \rangle \right) e^{-\nu_t} \right| + \left| \langle 4\rho, \xi_t \rangle \int_{\mathfrak{a}^+ \setminus D} e^{-\nu_t} \right| \\ &\leq \left| \int_D \left(\frac{\partial j}{\partial \xi_t} + \langle 4\rho, \xi_t \rangle \right) e^{-\nu_t} \right| + |4\rho| \left| \int_{\mathfrak{a}^+ \setminus D} e^{-\nu_t} \right| \end{aligned}$$

and by construction of D we deduce:

$$\leq \left| \int_D \left(\frac{\partial j}{\partial \xi_t} + \langle 4\rho, \xi_t \rangle \right) e^{-\nu_t} \right| + |4\rho| \cdot 2\theta$$

We consider now the quantity

$$\frac{\partial j}{\partial \xi_t}(x) + \langle 4\rho, \xi_t \rangle$$

for $x \in D$. It is negative by Lemma 3.5 since $\xi_t \in \overline{\mathfrak{a}^+}$.

Recall that $D(\theta) \subset b(\theta)\rho + \mathfrak{a}^+$ for some $b(\theta) > 0$, and more precisely that $D(\theta) = B(x_t, \delta(\theta)) \cap (b(\theta)\rho + \mathfrak{a}^+)$. Choose $b_0 > 0$ such that $B(b_0\rho, \delta(\theta)) \subset b(\theta)\rho + \mathfrak{a}^+$.

We can write

$$\begin{aligned} \xi_t &= \frac{x_t}{|x_t|} \\ &= \frac{x_t - b_0\rho}{|x_t|} + \frac{b_0\rho}{|x_t|} \\ &= \frac{|x_t - b_0\rho|}{|x_t|} \xi_1 + \frac{|b_0\rho|}{|x_t|} \xi_2 \end{aligned}$$

where $\xi_1 = (x_t - b_0\rho)/|x_t - b_0\rho|$ and $\xi_2 = b_0\rho/|b_0\rho|$.

It gives

$$\frac{\partial}{\partial \xi_t} = \frac{|x_t - b_0\rho|}{|x_t|} \frac{\partial}{\partial \xi_1} + \frac{|b_0\rho|}{|x_t|} \frac{\partial}{\partial \xi_2}.$$

Let $x = x_t + y \in D$, we consider the restriction of j to the line starting from $b_0\rho + y$ and of direction ξ_1 , which contains x . By convexity, we have

$$\frac{\partial j}{\partial \xi_1}(x) \geq \frac{j(x) - j(y + b_0\rho)}{|x_t - b_0\rho|}.$$

Recall from Lemma 3.4 that

$$j(x) \geq -2 \sum_{\alpha \in \Phi^+} \langle \alpha, x \rangle + C = -\langle 4\rho, x \rangle + C$$

on \mathfrak{a}^+ , for some constant C .

Applying this gives

$$\frac{\partial j}{\partial \xi_1}(x) \geq \frac{-\langle 4\rho, x \rangle + C - j(y + b_0\rho)}{|x_t - b_0\rho|}.$$

Now going back to $\frac{\partial j}{\partial \xi_t}(x)$, we have

$$\begin{aligned} \frac{\partial j}{\partial \xi_t}(x) &= \frac{|x_t - b_0\rho|}{|x_t|} \frac{\partial j}{\partial \xi_1}(x) + \frac{|b_0\rho|}{|x_t|} \frac{\partial j}{\partial \xi_2}(x) \\ &\geq \frac{|x_t - b_0\rho|}{|x_t|} \frac{-\langle 4\rho, x \rangle + C - j(y + b_0\rho)}{|x_t - b_0\rho|} + \frac{|b_0\rho|}{|x_t|} \frac{\partial j}{\partial \xi_2}(x) \\ &\geq \frac{-\langle 4\rho, x_t + y \rangle + C - j(y + b_0\rho)}{|x_t|} + \frac{|b_0\rho|}{|x_t|} \frac{\partial j}{\partial \xi_2}(x) \end{aligned}$$

so

$$0 \geq \frac{\partial j}{\partial \xi_t}(x) + \left\langle 4\rho, \frac{x_t}{|x_t|} \right\rangle \geq \frac{-\langle 4\rho, y \rangle + C - j(y + b_0\rho) + |b_0\rho| \frac{\partial j}{\partial \xi_2}(x)}{|x_t|}$$

Now $y \in B(b_0\rho, \delta(\theta))$ is bounded, j is bounded on $b(\theta)\rho + \mathfrak{a}^+$ by Lemma 3.4, and the derivatives of j are bounded on $b(\theta)\rho + \mathfrak{a}^+$ by Lemma 3.6, so there is a (negative) constant C' such that

$$0 \geq \frac{\partial j}{\partial \xi_t}(x) + \left\langle 4\rho, \frac{x_t}{|x_t|} \right\rangle \geq \frac{C'}{|x_t|}.$$

Applying this to the sequence t_i , we find that there exists a i_1 such that for $i > i_1$, and uniformly for $x \in D$,

$$\left| \frac{\partial j}{\partial \xi_t}(x) + \langle 4\rho, \xi_t \rangle \right| \leq \frac{\epsilon}{3V}.$$

Then for $i > i_1$,

$$\begin{aligned} \left| \int_D \left(\frac{\partial j}{\partial \xi_t} + \langle 4\rho, \xi_t \rangle \right) e^{-\nu_t} \right| &\leq \int_D \left| \frac{\partial j}{\partial \xi_t} + \langle 4\rho, \xi_t \rangle \right| e^{-\nu_t} \\ &\leq \frac{\epsilon}{3V} \int_D e^{-\nu_t} \\ &\leq \frac{\epsilon}{3} \end{aligned}$$

Gathering everything gives, for $i > \max(i_0, i_1)$,

$$\left| \int_{\mathfrak{a}^+} \frac{\partial j}{\partial \xi_t} e^{-\nu_t} + \langle 4\rho, \xi_\infty \rangle V \right| \leq (2d+1)\theta + 8|\rho|\theta + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \quad \square$$

Proof of Proposition 6.3. Let $\epsilon > 0$. Set $\theta := \epsilon/6d$ and let $\delta = \delta(\theta)$. First, by Lemma 6.4, we have

$$\left| \int_{\mathfrak{a}^+ \setminus B(x_t, \delta)} \frac{\partial u_{\text{ref}}}{\partial \xi_t} e^{-\nu_t} \right| < d\theta.$$

On $B(x_t, \delta)$, we always have $\frac{\partial u_{\text{ref}}}{\partial \xi_t} \leq v(\xi_t)$. Now consider the ray starting from $x - x_t$ and going to x . By convexity, we have

$$\begin{aligned} \frac{\partial u_{\text{ref}}}{\partial \xi_t}(x) &\geq \frac{u_{\text{ref}}(x) - u_{\text{ref}}(x - x_t)}{|x_t|} \\ &\geq \frac{v(x) + C}{|x_t|} \end{aligned}$$

for some constant C independent of x in $B(x_t, \delta)$, by Proposition 2.10 and because u_{ref} is bounded on $B(0, \delta)$. Then we can write

$$\begin{aligned} \frac{\partial u_{\text{ref}}}{\partial \xi_t}(x) &\geq v(\xi_t + \frac{x - x_t}{|x_t|}) + \frac{C}{|x_t|} \\ &\geq v(\xi_t) + \frac{C'}{|x_t|} \end{aligned}$$

The last step holds because v is Lipschitz.

For $i > i_0$ for some i_0 , we thus have, for $x \in B(x_t, \delta)$,

$$\left| \frac{\partial u_{\text{ref}}}{\partial \xi_t}(x) - v(\xi_t) \right| < \frac{\epsilon}{3} V.$$

Integrating on the ball gives

$$\left| \int_{B(x_t, \delta)} \frac{\partial u_{\text{ref}}}{\partial \xi_t} e^{-\nu_t} - \int_{B(x_t, \delta)} v(\xi_t) e^{-\nu_t} \right| < \frac{\epsilon}{3}.$$

Applying Lemma 6.4 again gives

$$\left| \int_{B(x_t, \delta)} v(\xi_t) e^{-\nu_t} - \int_{\mathfrak{a}^+} v(\xi_t) e^{-\nu_t} \right| < d\theta,$$

with

$$\int_{\mathfrak{a}^+} v(\xi_t) e^{-\nu_t} = v(\xi_t) V.$$

Finally, since ξ_t converges to ξ_∞ , there exists i_1 such that for $i > i_1$,

$$|v(\xi_t) V - v(\xi_\infty) V| < \frac{\epsilon}{3}.$$

We have proved that for $i > i_0, i_1$, we have

$$\left| \int_{\mathfrak{a}^+} \frac{\partial u_{\text{ref}}}{\partial \xi_t} e^{-\nu_t} - v(\xi_\infty) V \right| < 2\frac{\epsilon}{3} + 2d\theta = \epsilon \quad \square$$

This concludes the proof of Proposition 6.1, by considering the limit at t_∞ and dividing by V .

6.3 Sufficient condition

We can now prove that our condition is sufficient for the existence of a Kähler-Einstein metric and thus conclude the proof of Theorem A.

Theorem 6.7. *If $\text{bar}_{DH}(2P^+) \in 4\rho + \Xi$, then X admits a Kähler-Einstein metric.*

Proof. Assume that X admits no Kähler-Einstein metric. Then Proposition 6.1 gives

$$t_\infty \langle \text{bar}_{DH}(2P^+) - 4\rho, \xi_\infty \rangle = (t_\infty - 1)(v(\xi_\infty) - \langle 4\rho, \xi_\infty \rangle).$$

In particular, since v is the support function of $2P$ and $2\rho \in \text{Int}(P)$, and $0 < t_\infty \leq 1$, we have

$$\langle \text{bar}_{DH}(2P^+) - 4\rho, \xi_\infty \rangle \leq 0.$$

Assume that $\text{bar}_{DH}(2P^+) \in 4\rho + \Xi$. Then by the definition of Ξ , the only possibility is that $\xi_\infty \in \mathfrak{a}_t$ and $\langle \text{bar}_{DH}(2P^+) - 4\rho, \xi_\infty \rangle = 0$.

To prove that this is impossible, consider the vanishing

$$\int_{\mathfrak{a}^+} \frac{\partial \nu_t}{\partial \xi_\infty} e^{-\nu_t} dx = 0.$$

The difference with what we have done before is that we fix ξ_∞ instead of considering ξ_t .

Since $\xi_\infty \in \mathfrak{a}_t$, we have $\frac{\partial j}{\partial \xi_\infty} = 0$ and so we deduce from the vanishing of the integral the following equality, valid for $t < t_\infty$.

$$t \int_{\mathfrak{a}^+} \frac{\partial u_t}{\partial \xi_\infty} e^{-\nu_t} dx = (t-1) \int_{\mathfrak{a}^+} \frac{\partial u_{\text{ref}}}{\partial \xi_\infty} e^{-\nu_t} dx$$

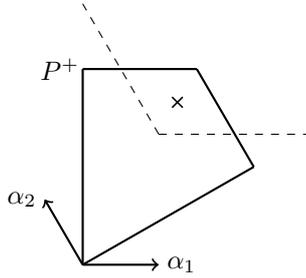


Figure 5: Barycenter for X_1

The left hand side term is zero because we assumed

$$0 = \langle \text{bar}_{DH}(2P^+) - 4\rho, \xi_\infty \rangle = \langle \text{bar}_{DH}(2P^+), \xi_\infty \rangle.$$

We thus have, for all $t < t_\infty$,

$$\int_{\mathfrak{a}^+} \frac{\partial u_{\text{ref}}}{\partial \xi_\infty} e^{-\nu t} dx = 0.$$

This is a contradiction: let $m := \min\{v(\xi) ; \xi \in \mathfrak{a}, |\xi| = 1\} > 0$. For any $\delta > 0$ fixed, there exists an $\epsilon > 0$ such that if $t_\infty - \epsilon < t < t_\infty$, $\frac{\partial u_{\text{ref}}}{\partial \xi_\infty} \geq m/2$ on $B(x_t, \delta)$. This is because $|x_t|$ goes to ∞ and u_{ref} is asymptotic to v . Choose now $\delta = \delta(m/4)$ given by Lemma 6.4, then for t close to t_∞ , we obtain

$$\int_{\mathfrak{a}^+} \frac{\partial u_{\text{ref}}}{\partial \xi_\infty} e^{-\nu t} dx \geq m/4 > 0. \quad \square$$

Combined with the obstruction proved earlier, it ends the proof of Theorem A.

Example 6.8. Consider the example X_1 , which we recall is the wonderful compactification of $\text{PGL}_3(\mathbb{C})$. The corresponding root system is A_2 . We denote by α_1 and α_2 the simple roots. The third positive root is then $\alpha_1 + \alpha_2$, and $2\rho = 2(\alpha_1 + \alpha_2)$. For $p = x\alpha_1 + y\alpha_2$,

$$\prod_{\alpha \in \Phi^+} \langle \alpha, p \rangle^2 = (x - y/2)^2 (-x/2 + y)^2 (x/2 + y/2)^2.$$

We computed the coordinates x and y of the barycenter $\text{bar}_{DH}(P^+)$ and obtained

$$\text{bar}_{DH}(P^+) = \frac{24641}{9888}(\alpha_1 + \alpha_2).$$

As a consequence, X_1 admits a Kähler-Einstein metric. Figure 5 gives a representation of P^+ , where the cross is the barycenter, and the convex cone delimited by the dashed lines is $2\rho + \Xi$.

6.4 Lower bound on $R(X)$

Assume first that $t_\infty < 1$, then we can write

$$\frac{t_\infty}{t_\infty - 1} \langle \text{bar}_{DH}(2P^+) - 4\rho, \xi_\infty \rangle = v(\xi_\infty) - \langle 4\rho, \xi_\infty \rangle,$$

or

$$\left\langle 4\rho + \frac{t_\infty}{1 - t_\infty} (-\text{bar}_{DH}(2P^+) + 4\rho), \xi_\infty \right\rangle = v(\xi_\infty).$$

For $t = 0$, we have $4\rho \in \text{Int}(2P)$. The function $t \mapsto t/(1 - t)$ is strictly increasing and its image is $[0, \infty[$. Besides, since v is the support function of $2P$, the value $v(\xi_\infty)$ is attained by $\langle m, \xi_\infty \rangle$ if and only if m is in the supporting hyperplane of $2P$ defined by ξ_∞ . We deduce that necessarily t_∞ is the unique value of t for which

$$4\rho + \frac{t}{1 - t} (-\text{bar}_{DH}(2P^+) + 4\rho) \in \partial(-\Xi + 2P^+),$$

if it exists. If it doesn't exist, then $t_\infty = 1$.

Combining this with the upper bound on $R(X)$, we have proved Theorem C.

Example 6.9. This allows to compute exactly the greatest Ricci lower bound for X_2 , which is

$$R(X_2) = \frac{1046175339}{1236719713}.$$

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