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Ratikanta Behera[1], Sylvain Meignen[2], and Thomas Oberlin[3]
[1,2]Jean Kuntzmann laboratory,
Joseph Fourier University,
51 rue des mathematiques,
38041 Grenoble cedex 09,France.
[3] INP - ENSEEIHT Toulouse,
2 rue Charles Camichel, B.P. 7122,
31071 Toulouse Cedex 7, FRANCE.

Abstract

We consider in this article the analysis of multicomponent signals, defined as superpositions of modulated waves also called modes. More precisely, we focus on the analysis of a variant of the second-order synchrosqueezing transform, which was introduced recently in [13], to deal with modes containing strong frequency modulation. Before going into this analysis, we revisit the case where the modes are assumed to be with weak frequency modulation as in the seminal paper of Daubechies et *al.* [7] to show that the constraint on the compactness of the analysis window in the Fourier domain can be alleviated. We also explain why the hypotheses made on the modes making up the multicomponent signal must be different when one considers either wavelet or short time Fourier transform -based synchrosqueezing. The rest of the paper is devoted to the theoretical analysis of the variant of the second order synchrosqueezing transform [13] and numerical simulations illustrate the performance of the latter.

Keywords: time-frequency analysis; synchrosqueezing transform; multicomponent signals.

[1]Email:ratikanta.behera@imag.fr
[2]E-mail:sylvain.meignen@imag.fr
[3]thomas.oberlin@enseeiht.fr

1. Introduction

Multicomponent signals (MCs) are encountered in a number of fields of practical interest such as meteorology, structural stability analysis, and medical studies - see, e.g. [5, 6, 9, 10]. Linear time-frequency (TF) analysis techniques have been extensively used to analyze and process these signals [11], the method to be used depending on the nature of the modes making up the signal. Standard linear TF methods such as the short-time Fourier transform (STFT) and the continuous wavelet transform (CWT) are commonly used to analyze such signals. In that context, the reassignment method (RM) was developed [1] to improve the readability of these linear TF representations. Unfortunately, since RM applies to the magnitude on the studied TF transform, the method results in a loss of information and the reassigned representation is not sufficient to recover the original signal.

Recently, Daubechies et *al.* [7] showed interesting results on the so-called synchrosqueezing transform (SST) to represent MCs, a method introduced in the mid-1990s for audio signal analysis [8]. SST combines the localization and sparsity properties of RM with the invertibility of a traditional TF representation. Originally proposed as a post-processing method applied to the CWT, SST can alternatively be applied to STFT with minor changes [12], to obtain the so-called FSST. Since the seminal paper of Daubechies et al. [7], many new developments have been carried out in various directions. First, an extension was proposed to the bidimensional case in [4], while a generalization of the wavelet approach by means of wavelet packets decomposition for both the one dimensional and bidimensional cases is available in [17, 16]. Finally, it is also worth noting that a study of synchrosqueezing applied to a more general class of multicomponent signals was done in [15].

In the present paper, we focus on what essentially limits the applicability of SST which is the hypothesis of weak frequency modulation for the modes making up the signal. To better take into account this frequency modulation, FSST has recently been extended to the case of MCs containing modes with strong frequency modulation [13, 2]. However, in these papers, no theoretical analysis of the proposed new synchrosqueezing transform was provided. To deal with this issue is one of the aim of this paper. For that purpose, we first revisit the case of weak modulation for the modes, putting the emphasis

on the fact that the analysis window need not be compactly supported in the Fourier domain, which consists in a new result. We then prove a new approximation theorem on the novel synchrosqueezing transform we propose, which we call second order synchrosqueezing.

The outline of the paper is as follows: in Section 2, we recall some notation and definitions. In Section 3, we introduce FSST and wavelet-based SST (WSST) along with the corresponding approximation results and explain why the hypotheses on the phase of the modes have to be different. Then, after having introduced some necessary ingredients, we derive, in Section 4, the approximation theorem for the second order synchrosqueezing transform, showing that this new transform is fully adapted for analyzing modes with strong frequency modulations. Finally, numerical illustrations conclude the paper.

2. Definitions

2.1. Short-time Fourier transform

We denote by $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ the space of integrable, and square integrable functions. Consider a signal $f \in L^1(\mathbb{R})$, and take a window g in the Schwartz class, $\mathcal{S}(\mathbb{R})$, the space of smooth functions with fast decaying derivatives of any order; its Fourier transform is defined by:

$$\widehat{f}(\eta) = \int_{\mathbb{R}} f(\tau) e^{-2i\pi\eta\tau} d\tau. \tag{1}$$

The need for time-localized frequency descriptors leads to short-time Fourier transform (STFT) which is obtained through the use of a sliding window $g \in L^2(\mathbb{R})$ defined by:

$$V_f^g(\eta, t) \quad = \int_{\mathbb{R}} f(\tau) g(\tau - t) e^{-2i\pi\eta(\tau - t)} d\tau = \quad \mathbb{A}(\eta, t) e^{2i\pi\Phi(\eta, t)}. \tag{2}$$

The representation of $|V_f^g(\eta, t)|^2$ in the TF plane is called the spectrogram of f. The STFT admits the following synthesis formula (provided the window g does not vanish and is continuous at 0):

$$f(t) = \frac{1}{g(0)} \int_{\mathbb{R}} V_f^g(\eta, t) d\eta. \tag{3}$$

Further, if the signal is analytic (i.e. $\eta \leq 0 \Rightarrow \hat{f}(\eta) = 0$) then the integral domain for η is restricted to $\mathbb{R}_+$.

2.2. Continuous wavelet transform

Let us consider an admissible wavelet $\psi \in L^2(\mathbb{R})$, satisfying $0 < C_\psi = \int_0^\infty |\widehat{\psi}(\xi)|^2 \frac{d\xi}{\xi} < \infty$ and then define for any time t and scale $a > 0$, the *continuous wavelet transform* (CWT) of f by:

$$W_f(a,t) = \frac{1}{a}\int_{\mathbb{R}} f(\tau)\psi\left(\frac{\tau - t}{a}\right)^* d\tau, \tag{4}$$

where z^* denotes the complex conjugate of z. Assuming that ψ is analytic, i.e. $\mathrm{Supp}(\widehat{\psi}) \subset [0,\infty[$, and f real-valued, the CWT admits the following synthesis formula (Morlet formula):

$$f(t) = 2\mathcal{R}\left\{\frac{1}{C'_\psi}\int_0^\infty W_f(a,t)\frac{da}{a}\right\}, \tag{5}$$

where $\mathcal{R}$ denotes the real part of a complex number and $C'_\psi = \displaystyle\int_0^\infty \widehat{\psi}^*(\xi)\frac{d\xi}{\xi}$.

2.3. Multicomponent signal

In the present paper, we analyze so-called *multicomponent signals* of the form,

$$f(t) = \sum_{k=1}^{K} f_k(t), \quad \text{with } f_k(t) = A_k(t)e^{2i\pi\phi_k(t)}, \tag{6}$$

for some K, where $A_k(t)$ and $\phi_k(t)$ are time-varying amplitude and phase functions respectively such that $A_k(t) > 0$, $\phi'_k(t) > 0$ and $\phi'_{k+1}(t) > \phi'_k(t)$ for all t. In the following, $\phi'_k(t)$ is often called instantaneous frequency (IF) of mode k and $A_k(t)$ its instantaneous amplitude (IA). One of the goal of TF analysis is to recover the instantaneous frequencies $\{\phi'_k(t)\}_{1\le k\le K}$ and amplitudes $\{A_k(t)\}_{1\le k\le K}$, from a given TF representation of f. Note that, if we are given real-valued functions, i.e. $f_k(t) = A_k(t)\cos(2\pi\phi_k(t))$, we can still derive the same analysis by considering analytic windows or wavelets, as soon as $\phi'_1(t) > \Delta\ \forall t$.

3. Synchrosqueezing transform

The synchrosqueezing transform (SST) is an approach originally introduced in the context of audio signals [8], whose theoretical analysis was recently carried out in [7]. It belongs to the family of TF reassignment methods

and corresponds to a nonlinear operator that sharpens the TF representation of a signal while enabling mode reconstruction. Moreover, SST combines the localization and sparsity properties of TF reassignment with the invertibility property of traditional TF transforms, and is robust to a variety of signal perturbations [14]. The key ingredient to SST is an IF estimate computed from the TF representation, which we introduce now.

3.1. Computation of IF Estimate in the TF plane

In the STFT framework, the IF of signal f at time t and frequency η can be estimated by [1]:

$$\begin{aligned} \widehat{\omega}_f(\eta,t) &:= \frac{1}{2\pi}\partial_t arg V_f^g(\eta,t), \\ &= \partial_t \Phi(\eta,t) \text{ defined in (2)}, \\ &= \mathcal{R}\left(\frac{1}{2i\pi}\frac{\partial_t V_f^g(\eta,t)}{V_f^g(\eta,t)}\right). \end{aligned} \tag{7}$$

Note that the quantity of which we take the real part:

$$\widetilde{\omega}_f(\eta,t) = \frac{\partial_t V_f^g(\eta,t)}{2i\pi V_f^g(\eta,t)}, \tag{8}$$

is useful to more theoretical investigations but should not be used to estimate real quantities such as IF. Similar quantities can be derived in the CWT case as follows: $\widehat{\omega}_f(a,t) = \mathcal{R}\left(\widetilde{\omega}_f(a,t)\right)$, with $\widetilde{\omega}_f(a,t) = \frac{1}{2i\pi}\frac{\partial_t W_f(a,t)}{W_f(a,t)}$.

Remark 1. In [7], the estimate $\widetilde{\omega}_f(a,t)$ is used instead of $\widehat{\omega}_f(a,t)$. However, most theorems are valid only for a real estimate as will be shown later.

3.2. STFT-based synchrosqueezing transform

Definition 1. *Let $\varepsilon > 0$ and $\Delta > 0$. The set $\mathcal{B}_{\Delta,\varepsilon}$ of multicomponent signals with modulation ε and separation Δ is the set of all multicomponent signals defined in* (6) *satisfying:*

$$\begin{aligned} &A_k \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \phi_k \in C^2(\mathbb{R}), \\ &\sup_{t\in\mathbb{R}} \phi_k'(t) < \infty, \phi_k'(t) > 0, A_k(t) > 0, \quad \forall t \\ &|A_k'(t)| \leq \varepsilon, |\phi_k''(t)| \leq \varepsilon \quad \forall t \in \mathbb{R}. \end{aligned} \tag{9}$$

Further, the f_ks are separated with resolution Δ, i.e., for all $k \in \{1, ..., K-1\}$ and all t,

$$\phi'_{k+1}(t) - \phi'_k(t) \geq 2\Delta. \tag{10}$$

Definition 2. *Let h be a L^1 normalized positive function belonging to $C_c^\infty(\mathbb{R})$, the space of compactly supported smooth function, and pick $\gamma, \delta > 0$. The STFT-based synchrosqueezing (FSST) of f with threshold γ and accuracy δ is defined by:*

$$T_f^{\delta,\gamma}(\omega, t) = \frac{1}{g(0)} \int_{|V_f^g(\eta,t)|>\gamma} V_f^g(\eta, t) \frac{1}{\delta} h\left(\frac{\omega - \widehat{\omega}_f(\eta, t)}{\delta}\right) d\eta. \tag{11}$$

If we make δ and γ tend to zero, then $T_f^{\delta,\gamma}(\omega, t)$ tends to some value which we formally write as:

$$T_f(\omega, t) = \frac{1}{g(0)} \int_{\mathbb{R}} V_f^g(\eta, t) \delta(\omega - \widehat{\omega}_f(\eta, t)) d\eta, \tag{12}$$

called FSST in the sequel.

Theorem 1. *Consider $f \in \mathcal{B}_{\Delta,\varepsilon}$ and put $\tilde{\varepsilon} = \varepsilon^{1/3}$. Let $g \in \mathcal{S}(\mathbb{R})$, the Schwartz class, be such that supp$(\widehat{g}) \subset [-\Delta, \Delta]$. Then, if ε is small enough, the following holds:*

(a) $|V_f^g(\eta, t)| > \tilde{\varepsilon}$ only when there exists $k \in \{1, ..., K\}$ such that $(\eta, t) \in Z_k := \{(\eta, t),\ s.t.\ |\eta - \phi'_k(t)| < \Delta\}$.

(b) For all $k \in \{1, ..., K\}$ and all $(\eta, t) \in Z_k$ such that $|V_f^g(\eta, t)| > \tilde{\varepsilon}$, we have

$$|\widehat{\omega}_f(\eta, t) - \phi'_k(t)| \leq \tilde{\varepsilon}. \tag{13}$$

(c) For all $k \in \{1, ..., K\}$, there exists a constant C such that for all $t \in \mathbb{R}$,

$$\left| \lim_{\delta \to 0} \left(\int_{|\omega - \phi'_k(t)|<\tilde{\varepsilon}} T_f^{\delta,\tilde{\varepsilon}}(\omega, t) dw \right) - f_k(t) \right| \leq C\tilde{\varepsilon}. \tag{14}$$

Proof. This theorem gives a strong approximation result for the class $\mathcal{B}_{\Delta,\varepsilon}$, since it ensures that the non-zero coefficients of the FSST are localized around the curves $(t, \phi'_k(t))$, and that the reconstruction of the modes is easily obtained from the concentrated representation. The main steps of the proofs are detailed hereafter.

(a) For any $(\eta, t) \in \mathbb{R}^2$, a zeroth order Taylor expansion of the amplitude and first order expansion of the phase of the modes lead to:

$$|V_f^g(\eta, t) - \sum_{k=1}^{K} f_k(t)\hat{g}(\eta - \phi_k'(t))| \leq \varepsilon\Gamma_1(t), \tag{15}$$

where $\Gamma_1(t) = KI_1 + \pi I_2 \sum_{k=1}^{K} A_k(t)$, and $I_n = \int_{\mathbb{R}} |x|^n |g(x)| dx$. If

$$\tilde{\varepsilon} \leq \|\Gamma_1\|_\infty^{-\frac{1}{2}}, \tag{16}$$

and since $\hat{g}$ is compactly supported in $[-\Delta, \Delta]$, and the IFs of the modes are separated by more than 2Δ, we have, for any $(\eta, t) \notin \bigcup_{k=1}^{K} Z_k$:

$$|V_f^g(\eta, t)| \leq \varepsilon\Gamma_1(t) \leq \tilde{\varepsilon}. \tag{17}$$

(b) Now, we derive the same kind of estimate for $\partial_t V_f^g$. Let us first remark that $V_f^g(\eta, t)$ is differentiable with respect to both variables, and that for any $(\eta, t) \in \mathbb{R}^2$:

$$\partial_t V_f^g(\eta, t) = 2i\pi\eta V_f^g(\eta, t) - V_f^{g'}(\eta, t). \tag{18}$$

The counterpart of equation (15) is then:

$$|\partial_t V_f^g(\eta, t) - 2i\pi \sum_{k=1}^{K} f_k(t)\phi_k'(t)\hat{g}(\eta - \phi_k'(t))| \leq \varepsilon \left(\Gamma_2(t) + 2\pi|\eta|\Gamma_1(t)\right), \tag{19}$$

where $\Gamma_2(t) = KI_1' + \pi I_2' \sum_{k=1}^{K} A_k(t)$, with $I_n' = \int_{\mathbb{R}} |t|^n |g'(t)| dt$.

Now, we can remark that, since the sets Z_k are disjoint, there is at most one non-zero term in the sums involved in equations (15) and (19). We can thus write for any $(\eta, t) \in \bigcup_{k=1}^{K} Z_k$:

$$\begin{aligned}
&|\hat{\omega}_f(\eta, f) - \phi_k'(t)| \\
&\leq \left| \frac{\partial_t V_f^g(\eta, t) - 2i\pi\phi_k'(t) f_k(t)\hat{g}(\eta - \phi_k'(t))}{2i\pi V_f^g(\eta, t)} \right| + \left| \frac{\phi_k'(t)(f_k(t)\hat{g}(\eta - \phi_k'(t)) - V_f^g(\eta, t))}{V_f^g(\eta, t)} \right| \\
&\leq \frac{\varepsilon\left(\Gamma_2(t) + 2\pi|\eta|\Gamma_1(t)\right)}{2\pi\tilde{\varepsilon}} + \phi_k'(t)\frac{\varepsilon\Gamma_1(t)}{\tilde{\varepsilon}} \\
&\leq \tilde{\varepsilon}^2 \left(\frac{\Gamma_2(t)}{2\pi} + (|\eta| + \phi_k'(t))\Gamma_1(t) \right).
\end{aligned}$$

Hence, if ε is sufficiently small, i.e. if it satisfies for all t

$$\tilde{\varepsilon} \leq \left(\frac{\Gamma_2(t)}{2\pi} + (2\phi_k'(t) + \Delta)\Gamma_1(t) \right)^{-1}, \tag{20}$$

one obtains

$$|\hat{\omega}(\eta, f) - \phi_k'(t)| \leq \tilde{\varepsilon}. \tag{21}$$

(c) Let $t \in \mathbb{R}$, we have:

$$\begin{aligned}
&\lim_{\delta \to 0} \int_{|\omega - \phi_k'(t)| < \tilde{\varepsilon}} T_f^{\delta, \tilde{\varepsilon}}(\omega, t)\, d\omega \\
&= \lim_{\delta \to 0} \int_{|\omega - \phi_k'(t)| < \tilde{\varepsilon}} \frac{1}{g(0)} \int_{|V_f^g(\eta,t)| > \tilde{\varepsilon}} V_f^g(\eta, t) \frac{1}{\delta} h\left(\frac{\omega - \hat{\omega}(\eta, t)}{\delta} \right) d\eta d\omega \\
&= \lim_{\delta \to 0} \int_{|V_f^g(\eta,t)| > \tilde{\varepsilon}} \frac{1}{g(0)} V_f^g(\eta, t) \int_{|\omega - \phi_k'(t)| < \tilde{\varepsilon}} \frac{1}{\delta} h\left(\frac{\omega - \hat{\omega}(\eta, t)}{\delta} \right) d\omega d\eta \\
&= \frac{1}{g(0)} \int_{|V_f^g(\eta,t)| > \tilde{\varepsilon}} V_f^g(\eta, t) \lim_{\delta \to 0} \int_{|\omega - \phi_k'(t)| < \tilde{\varepsilon}} \frac{1}{\delta} h\left(\frac{\omega - \hat{\omega}(\eta, t)}{\delta} \right) d\omega d\eta \\
&= \frac{1}{g(0)} \int_{\{|V_f^g(\eta,t)| > \tilde{\varepsilon}\} \bigcap \{|\hat{\omega}(\eta,t) - \phi_k'(t)| < \tilde{\varepsilon}\}} V_f^g(\eta, t)\, d\eta,
\end{aligned}$$

where we use the Fubini and dominated convergence theorems, all the functions being bounded on bounded integration sets, hence integrable. We now prove that the integration interval of the last equation, denoted by $X = \{|V_f(\eta, t)| > \tilde{\varepsilon}\} \bigcap \{|\hat{\omega}(\eta, t) - \phi_k'(t)| < \tilde{\varepsilon}\}$, equals the following set:

$$Y = \{|V_f^g(\eta, t)| > \tilde{\varepsilon}\} \bigcap \{|\eta - \phi_k'(t)| < \Delta\}.$$

If $\eta \in X$, it is such that $|V_f^g(\eta, t)| > \tilde{\varepsilon}$ and $|\hat{\omega}(\eta, t) - \phi_k'(t)| < \tilde{\varepsilon}$. Also, from (a) there exists a unique l such that $(\eta, t) \in Z_l$. If $l \neq k$, we would have:

$$\begin{aligned}
|\hat{\omega}_f(\eta, t) - \phi_k'(t)| &\geq |\phi_l'(t) - \phi_k'(t)| - |\hat{\omega}_f(\eta, t) - \phi_l'(t)| \\
&\geq 2\Delta - \tilde{\varepsilon},
\end{aligned}$$

because of (b). If $\tilde{\varepsilon}$ satisfies

$$\tilde{\varepsilon} \leq \Delta, \tag{22}$$

we would get $|\hat{\omega}_f(\eta,t)-\phi_k'(t)|\geq\tilde{\varepsilon}$, which contradicts $\eta\in X$. Hence, $l=k$ and $X\subset Y$. Conversely, if $\eta\in Y$, equation (25) immediately shows $\eta\in Y$, hence $X=Y$.

Finally,

$$\begin{aligned}
&\left|\lim_{\delta\to 0}\left(\int_{|\omega-\phi_k'(t)|<\tilde{\varepsilon}} T_f^{\delta,\tilde{\varepsilon}}(\omega,t)\,d\omega\right)-f_k(t)\right| \\
&=\left|\frac{1}{g(0)}\int_{\{|V_f^g(\eta,t)|>\tilde{\varepsilon}\}\bigcap\{|\eta-\phi_k'(t)|<\Delta\}} V_f^g(\eta,t)\,d\eta-f_k(t)\right| \\
&=\left|\frac{1}{g(0)}\int_{\{|\eta-\phi_k'(t)|<\Delta\}} V_f^g(\eta,t)\,d\eta-f_k(t)-\frac{1}{g(0)}\int_{\{|\eta-\phi_k'(t)|<\Delta\}\bigcap\{|V_f^g(\eta,t)|\leq\tilde{\varepsilon}\}} V_f^g(\eta,t)\,d\eta\right| \\
&\leq\left|\frac{1}{g(0)}\int_{\{|\eta-\phi_k'(t)|<\Delta\}} V_f(\eta,t)\,d\eta-f_k(t)\right|+\frac{2\Delta}{|g(0)|}\tilde{\varepsilon} \\
&\leq\left|\frac{1}{g(0)}\int_{\{|\eta-\phi_k'(t)|<\Delta\}} f_k(t)\hat{g}(\eta-\phi_k'(t))\,d\eta-f_k(t)\right| \\
&\qquad+\frac{1}{|g(0)|}\int_{\{|\eta-\phi_k'(t)|<\Delta\}}\left|V_f^g(\eta,t)-f_k(t)\hat{g}(\eta-\phi_k'(t))\right|\,d\eta+\frac{2\Delta}{|g(0)|}\tilde{\varepsilon} \\
&\leq 0+\frac{2\Delta}{|g(0)|}\varepsilon\Gamma_1(t)+\frac{2\Delta}{|g(0)|}\tilde{\varepsilon} \\
&\leq\frac{4\Delta}{|g(0)|}\tilde{\varepsilon}.
\end{aligned}$$

□

3.3. Generalization to non compactly supported $\hat{g}$

In the above section, we assumed that $\hat{g}$ was compactly supported. We here explain that we can state the same approximation result while relaxing this strong condition.

Theorem 2. *Assume that there exist constants K_1, $K_2>0$ such that $g\in\mathcal{S}(\mathbb{R})$ satisfies:*

$$\begin{aligned}
&\textit{For all } |\eta|>\Delta,\ |\hat{g}(\eta)|\leq K_1\varepsilon, \\
&\textit{and } \int_{\{|\eta|>\Delta\}}|\hat{g}(\eta)|\,d\eta\leq K_2\tilde{\varepsilon}.
\end{aligned}$$

Then all conclusions of Theorem 1 remain valid.

Proof. (a) We first remark that equation (15) remains valid for any (η, t):

$$|V_f^g(\eta,t) - \sum_{k=1}^{K} f_k(t)\widehat{g}(\eta - \phi_k'(t))| \leq \varepsilon\Gamma_1(t).$$

If we assume that for any t,

$$\tilde{\varepsilon} \leq \min\left[\frac{1}{2}\left(K_1\sum_{k=1}^{K}A_k(t)\right)^{-1}, \frac{1}{2}\Gamma_1(t)^{-\frac{1}{2}}\right], \tag{23}$$

then we get, for any $(\eta,t) \notin \bigcup_{k=1}^{K} Z_k$:

$$|V_f^g(\eta,t)| \;\leq\; \varepsilon\left(K_1\sum_{k=1}^{K}A_k(t) + \Gamma_1(t)\right) \leq \tilde{\varepsilon}.$$

If $(\eta,t) \in Z_k$, we have $|V_f^g(\eta,t) - f_k(t)\widehat{g}(\eta-\phi_k'(t))| \leq \varepsilon\left(K_1\sum_{k=1}^{K}A_k(t) + \Gamma_1(t)\right)$.

(b) Similarly, we still have estimate (19) for any (η,t). If we assume that for all $k \in \{1,\cdots,K\}$ and for all t,

$$\tilde{\varepsilon} \leq \min\left[\frac{1}{2}\left(K_1\sum_{l=1}^{K}A_l(t)\right)^{-1}, \frac{1}{2}\left(\frac{\Gamma_2(t)}{2\pi} + (2\phi_k'(t)+\Delta)\Gamma_1(t)\right)^{-1}\right], \tag{24}$$

then we get for any $(\eta,t) \in Z_k$ such that $|V_f^g(\eta,t)| \geq \tilde{\varepsilon}$:

$$|\hat{\omega}_f(\eta,f) - \phi_k'(t)| \leq \tilde{\varepsilon}. \tag{25}$$

(c) Regarding point (c) of Theorem 1, the only difference when one considers a non compactly supported window appears at the very end of the proof. More precisely, the following term is not zero anymore, but can be bounded in the following way:

$$\begin{aligned}\left|\frac{1}{g(0)}\int_{\{|\eta-\phi_k'(t)|<\Delta\}} f_k(t)\hat{g}(\eta-\phi_k'(t))\,d\eta - f_k(t)\right| &\leq \frac{A_k(t)}{|g(0)|}\int_{|\eta|>\Delta}|\hat{g}(\eta)|\,d\eta \\ &\leq \frac{A_k(t)K_2}{|g(0)|}\tilde{\varepsilon}.\end{aligned}$$

Thus, point (c) of the theorem still holds, but with constant

$$C = \frac{4\Delta + \max_k \|A_k\|_\infty K_2}{|g(0)|}.$$

□

Remark 2. Let us remark that this extended result requires stronger assumptions on $\tilde{\varepsilon}$: conditions (16) and (20) are replaced by (23) and (24), respectively. Note also that here, the separation parameter Δ depends on $\tilde{\varepsilon}$.

Remark 3. It is interesting to remark also that $f_k(t)\hat{g}(\eta - \phi'_k(t)) = V^g_{f^1_k}(\eta, t)$ with $f^1_k(\tau) = A_k(t)e^{2i\pi(\phi_k(t)+(\tau-t)\phi'_k(t))}$. Then, the right hand side of equation (15) and (19) can be rewritten respectively as $|V^g_f(\eta,t) - V^g_{f^1_k}(\eta,t)|$ and $|\partial_t V^g_f(\eta,t) - \partial_t V^g_{f^1_k}(\eta,t)|$ so that Theorem 1 is based on the STFT of a constant amplitude and first order phase approximation of the kth mode.

3.4. Wavelet based synchrosqueezing transform [7]

We now introduce SST in the wavelet framework.

Definition 3. *Let $\varepsilon > 0$ and $\Delta \in (0,1)$. The set $\mathcal{A}_{\Delta,\varepsilon}$ of multicomponent signals with modulation ε and separation Δ corresponds to signals defined in* (6) *with f satisfying:*

$$\begin{aligned} &A_k \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \phi \in C^2(\mathbb{R}), \\ &\inf_{t\in\mathbb{R}} \phi'_k(t) > 0, \sup_{t\in\mathbb{R}} \phi'_k(t) < \infty, A_k(t) > 0, \\ &|A'_k(t)| \le \varepsilon|\phi'_k(t)|, |\phi''_k(t)| \le \varepsilon|\phi'_k(t)| \quad \forall t \in \mathbb{R}. \end{aligned} \tag{26}$$

Further, the f_ks are separated with resolution Δ, i.e., for all $k \in \{1, \cdots, K-1\}$ and all t

$$|\phi'_{k+1}(t) - \phi'_k(t)| \ge \Delta(\phi'_{k+1}(t) + \phi'_k(t)). \tag{27}$$

Definition 4. *Let h be a positive L^1 normalized window belonging to $C^\infty_c(\mathbb{R})$, and consider $\gamma, \delta > 0$, the wavelet-based synchrosqueezing (WSST) of f with threshold γ and accuracy δ is defined by:*

$$S^{\delta,\gamma}_f(\omega,t) := C'^{-1}_\psi \int_{|W_f(a,t)|>\gamma} W_f(a,t)\frac{1}{\delta}h\left(\frac{\omega - \hat{\omega}_f(a,t)}{\delta}\right)\frac{da}{a}. \tag{28}$$

Theorem 3. *Let $f \in \mathcal{A}_{\Delta,\varepsilon}$, and set $\tilde{\varepsilon} = \varepsilon^{\frac{1}{3}}$. Let ψ be a wavelet belonging to $\mathcal{S}(\mathbb{R})$ such that $\widehat{\psi}$ is supported in $[1-\triangle, 1+\triangle]$. Then, provided ε is sufficiently small, the following hold:*

(a) $|W_f(a,t)| \geq \tilde{\varepsilon}$ only when, there exists $k \in \{1, ..., K\}$, such that for each pair $(a,t) \in Z_k := \{(a,t),\ s.t.\ |a\phi_k'(t) - 1| < \Delta\}$.

(b) For each $k \in \{1, ..., K\}$ and then for all $(a,t) \in Z_k$ for which holds $|W_f(a,t)| > \tilde{\varepsilon}$, we have

$$|\widehat{\omega}_f(a,t) - \phi_k'(t)| \leq \tilde{\varepsilon}. \tag{29}$$

(c) Moreover, for each $k \in \{1, ..., K\}$, there exists a constant C such that for any $t \in \mathbb{R}$

$$\left| \lim_{\delta \to 0} \int_{|\omega - \phi_k'(t)| < \tilde{\varepsilon}} S_f^{\delta,\tilde{\varepsilon}}(\omega, t) d\omega - f_k(t) \right| \leq C\tilde{\varepsilon}. \tag{30}$$

Proof. This theorem is proved in [7] and gives a strong approximation result for the class $\mathcal{A}_{\Delta,\varepsilon}$, since it ensures that the non-zero coefficients of the WSST are localized around the curves $(t, \frac{1}{\phi_k'(t)})$ in the time-scale space, and that the reconstruction of the modes is easily obtained from the concentrated representation. □

Remark 4. Pure waves obviously satisfy the assumptions of Theorems 1 and 3 for any ε. However, in these theorems, we assumed different hypotheses for $\phi''(t)$, when considering FSST, from the ones originally made with WSST. In the case on FSST, we can motivate the hypothesis put on $\phi''(t)$, by studying the IF estimate on a linear chirp, i.e., a wave with linear instantaneous frequency. To carry out this study, we consider the STFT computed with a Gaussian window $g(t) = e^{-\pi t^2}$. To choose such a window is interesting for our purpose, because one obtains a closed form expression for the STFT of a linear chirp, as shown hereafter.

To carry out the study of the IF estimate $\widehat{\omega}_f(\eta, t)$ for a linear chirp, we need the following lemma:

Lemma 1. *Consider $u(t) = e^{-\pi z t^2}$, where $z = re^{i\theta}$ with $\cos\theta > 0$, so that u is integrable. Then its Fourier transform reads*

$$\hat{u}(\xi) = r^{\frac{-1}{2}} e^{-i\frac{\theta}{2}} e^{-\frac{-\pi}{re^{i\theta}}\xi^2}. \tag{31}$$

Proof. This result is straightforward as one can proceed as in the case z real. □

Then, consider the linear chirp $h_c(\tau) = Ae^{2i\pi\phi(\tau)}$, where ϕ is a quadratic polynomial. To start with, we remark that $h_c(\tau)$ can be written in the following form:

$$h_c(\tau) = h_c(t)e^{2i\pi[\phi'(t)(\tau-t)+\frac{1}{2}\phi''(t)(\tau-t)^2]}. \tag{32}$$

Then, we have the following result.

Proposition 1. *The STFT of h_c, computed using the Gaussian window g admits the following closed-form expression:*

$$V_{h_c}^g(\eta, t) = h_c(t) r^{-\frac{1}{2}} e^{-i\frac{\theta}{2}} e^{-\frac{\pi^2(1+i\phi''(t)^2)(\eta-\phi'(t))^2}{1+\phi''(t)^2}}. \tag{33}$$

Proof. We remark that

$$V_{h_c}^g(\eta, t) \;=\; h_c(t)\widehat{e^{-\pi(1-i\phi''(t))\tau^2}}(\eta-\phi'(t)) = h_c(t)\widehat{e^{-\pi z\tau^2}}(\eta-\phi'(t)),$$

where $z = 1 - i\phi''(t) = re^{i\theta}$. Lemma 1 then gives

$$V_{h_c}^g(\eta, t) = h_c(t) z^{-\frac{1}{2}} e^{-\pi\frac{(\eta-\phi'(t))^2}{z}} = h_c(t) z^{-\frac{1}{2}} e^{-\pi\frac{(\eta-\phi'(t))^2}{r}} e^{-i\theta}. \tag{34}$$

□

From this, we immediately get that:

$$\hat{\omega}_{h_c}(\eta, t) - \phi'(t) = \frac{\phi''(t)^2}{1+\phi''(t)^2}(\eta - \phi'(t)).$$

So now, if $|\eta - \phi'(t)| < \varepsilon$, we obtain:

$$|\hat{\omega}_{h_c}(\eta, t) - \phi'(t)| < \varepsilon|1 - \frac{1}{1+\phi''(t)^2}|.$$

Here we see that the quality of the estimate $\widehat{\omega}_{h_c}(\eta, t)$ close to the curve $(t, \phi'(t))$ only depends on the magnitude of $\phi''(t)$: if the latter goes to zero, the estimate tends to $\phi'(t)$. This justifies why, in Theorem 1, we assume the modes satisfy $\phi''(t) \leq \varepsilon$: what matters is how small the modulation is, $\phi'(t)$ playing no role.

4. New FSST based on second order approximation of the phase

The original FSST assumes $\phi''(t)$ is negligible. When it is not the case, Theorems 1 and 3 are no longer valid, because they apply to the wrong type of modes. Recently, an extension of FSST based on a second order approximation of the phase was proposed [13], but, to our knowledge, no theoretical analysis of the proposed estimate is available. We propose to bridge that gap in this section.

4.1. Computation of the new IF estimate

Remember $\tilde{\omega}_f(\eta,t) = \frac{\partial_t V_f^g(\eta,t)}{2i\pi V_f^g(\eta,t)}$ and then introduce:

$$\tilde{t}_f(\eta,t) \;=\; t - \frac{\partial_\eta V_f^g(\eta,t)}{2i\pi V_f^g(\eta,t)}. \tag{35}$$

Similarly to $\widehat{\omega}_f(\eta,t) = \mathcal{R}(\tilde{\omega}_f(\eta,t))$, we define $\widehat{t}_f(\eta,t) = \mathcal{R}(\tilde{t}_f(\eta,t))$. First, we recall an estimate of the frequency modulation introduced in [13].

Definition 5. *Let $f \in L^2(\mathbb{R})$ and consider when $V_f^g(\eta,t) \neq 0$ and $\partial_t\left(\frac{\partial_\eta V_f^g(\eta,t)}{V_f^g(\eta,t)}\right) \neq 2i\pi$ the quantity*

$$\tilde{q}_f(\eta,t) \;=\; \frac{\partial_t \tilde{\omega}_f(\eta,t)}{\partial_t \tilde{t}_f(\eta,t)} = \frac{\partial_t\left(\frac{\partial_t V_f^g(\eta,t)}{V_f^g(\eta,t)}\right)}{2i\pi - \partial_t\left(\frac{\partial_\eta V_f^g(\eta,t)}{V_f^g(\eta,t)}\right)}. \tag{36}$$

An estimate of the frequency modulation is then defined by

$$\hat{q}_f(\eta,t) = \mathcal{R}\left(\tilde{q}_f(\eta,t)\right). \tag{37}$$

The estimate (37) was used in [13] where it was proved that $\hat{q}_f(\eta,t) = \phi''(t)$, when f is a Gaussian modulated linear chirp, i.e. a chirp where both ϕ and $\log(A)$ are quadratic. Further, a new estimate of $\phi'(t)$ was also derived there, namely:

$$\phi'(t) \approx \widehat{\omega}_f(\eta,t) + \hat{q}_f(\eta,t)(t - \hat{t}_f(\eta,t)), \tag{38}$$

which is exact for constant amplitude linear chirp. Here we propose a slightly different IF estimate than the one introduced in [13], which allows for mathematical study.

Definition 6. *Let $f \in L^2(\mathbb{R})$, we define the second order IF complex estimate of f as:*

$$\tilde{\omega}_f^{(2)}(\eta,t) = \begin{cases} \tilde{\omega}_f(\eta,t) + \tilde{q}_f(\eta,t)(t-\tilde{t}_f(\eta,t)) & \text{if } \partial_t \tilde{t}_f(\eta,t) \neq 0 \\ \tilde{\omega}_f(\eta,t) & \text{otherwise,} \end{cases} \tag{39}$$

and then its real part $\hat{\omega}_f^{(2)}(\eta,t) = \mathcal{R}(\tilde{\omega}_f^{(2)}(\eta,t))$.

Proposition 2. *Let $f \in L^2(\mathbb{R})$, then the IF estimate $\hat{\omega}_f^{(2)}(\eta,t)$ can be expressed by means of five different STFTs, since we have:*

$$\begin{aligned} \tilde{\omega}_f(\eta,t) &= \eta - \frac{1}{2i\pi}\frac{V_f^{g'}(\eta,t)}{V_f^g(\eta,t)}, \\ \tilde{q}_f(\eta,t) &= \frac{1}{2i\pi}\frac{V_f^{g''}(\eta,t)V_f^g(\eta,t) - (V_f^{g'}(\eta,t))^2}{(V_f^g(\eta,t))^2 + V_f^{tg}(\eta,t)V_f^{g'}(\eta,t) - V_f^{tg'}(\eta,t)V_f^g(\eta,t)}, \\ t - \tilde{t}_f(\eta,t) &= -\frac{V_f^{tg}(\eta,t)}{V_f^g(\eta,t)}, \end{aligned}$$

Proof. The expressions for $\tilde{\omega}_f(\eta,t)$ and $t-\tilde{t}_f(\eta,t)$ are straightforward. Indeed, since g is in the Schwartz class, the STFT of f belongs to $C^\infty(\mathbb{R})$, and we have:

$$\begin{aligned} \partial_\eta V_f^g(\eta,t) &= -2i\pi V_f^{tg}(\eta,t) \\ \partial_t V_f^g(\eta,t) &= 2i\pi\eta V_f^g(\eta,t) - V_f^{g'}(\eta,t) \end{aligned} \tag{40}$$

As far as the expression for $\tilde{q}_f(\eta,t)$ is concerned, we remark that:

$$\tilde{q}_f(\eta,t) = \frac{V_f^g(\eta,t)\partial_{tt}^2 V_f^g(\eta,t) - (\partial_t V_f^g(\eta,t))^2}{2i\pi(V_f^g)^2 - V_f^g\partial_{\eta t}^2 V_f^g(\eta,t) + \partial_t V_f^g(\eta,t)\partial_\eta V_f^g(\eta,t)},$$

from which we derive the desired expression by noticing that:

$$\begin{aligned} \partial_{tt}^2 V_f^g(\eta,t) &= V_f^{g''}(\eta,t) - 4i\pi\eta V_f^{g'}(\eta,t) - 4\pi^2\eta^2 V_f^g(\eta,t) \\ \partial_{\eta t}^2 V_f^g(\eta,t) &= 2i\pi V_f^{tg'}(\eta,t) + 4\pi^2\eta V_f^{tg}(\eta,t), \end{aligned}$$

where $V_f^{g''}$ and $V_f^{tg'}$ are the STFT of f computed with windows g'' and $t \to tg'(t)$ respectively. □

In the following section, we use this new IF estimate $\hat{\omega}_f^{(2)}(\eta,t)$ to define a new synchrosqueezing transform for which we prove approximation results.

4.2. Definition of the new FSST and approximation results

In this section, we define another class of chirp-like functions larger than $\mathcal{B}_{\Delta,\varepsilon}$ and show that they can be sucessfully dealt with by means of second order FSST, which is defined in this section. So first, we define the new set of multicomponent signals we are studying:

Definition 7. *Let $\varepsilon > 0$ and $\Delta > 0$. The set $\mathcal{B}^{(2)}_{\Delta,\varepsilon}$ of multicomponent signals with second order modulation ε and separation Δ corresponds to the signals defined in* (6) *satisfying:*

(a) function f_k is such that A_k and ϕ_k satisfy the following conditions:

$$\begin{gathered} A_k(t) \in L^\infty(\mathbb{R}) \cap C^2(\mathbb{R}),\ \phi_k(t) \in C^3(\mathbb{R}), \\ \phi_k'(t),\ \phi_k''(t),\ \phi_k'''(t) \in L^\infty(\mathbb{R}), \\ A_k(t) > 0,\ \inf_{t\in\mathbb{R}} \phi_k'(t) > 0,\ \sup_{t\in\mathbb{R}} \phi_k'(t) < \infty, \\ |A_k'(t)| \leq \varepsilon,\ |A_k''(t)| \leq \varepsilon,\ \text{and } |\phi_k'''(t)| \leq \varepsilon, \end{gathered}$$

(b) the functions f_k's satisfy the following separation condition

$$\phi_{k+1}'(t) - \phi_k'(t) > 2\Delta \quad \forall t \in \mathbb{R} \quad , \quad \forall k \in \{1, \cdots, K-1\}. \tag{41}$$

Now, let us define the second order FSST as follows

Definition 8. *Let h be a positive L^1 normalized window belonging to $C_c^\infty(\mathbb{R})$, and consider $\gamma, \delta > 0$, the second order FSST of f with threshold γ and accuracy δ is defined by:*

$$\widetilde{T}_f^{\delta,\gamma}(\omega, t) = \frac{1}{g(0)} \int_{|V_f^g(\eta,t)|>\gamma} V_f^g(\eta, t) \frac{1}{\delta} h\left(\frac{\omega - \hat{\omega}_f^{(2)}(\eta, t)}{\delta}\right) d\eta. \tag{42}$$

In Section 3, we showed that for functions $f \in \mathcal{B}_{\Delta,\varepsilon}$ a good IF estimate was given by $\hat{\omega}_f(\eta, t)$ and the approximation theorem followed. Here, to assess the approximation property of the second order FSST we have just introduced, we consider a function $f \in \mathcal{B}^{(2)}_{\Delta,\varepsilon}$ for which we are going to prove that a good IF estimate is provided by $\hat{\omega}_f^{(2)}(\eta, t)$. The approximation theorem is as follows:

Theorem 4. *Consider $f \in \mathcal{B}^{(2)}_{\Delta,\varepsilon}$, set $\tilde{\varepsilon} = \varepsilon^{1/6}$. Let g be a window satisfying, for all t and all $k = 1, \cdots, K$, if $|\eta| \geq \Delta$, $|\widehat{g(\tau)e^{i\pi\phi_k''(t)\tau^2}}(\eta)| \leq K_3\varepsilon$, $|\widehat{\tau g(\tau)e^{i\pi\phi_k''(t)\tau^2}}(\eta)| \leq K_4\varepsilon$, and $|\widehat{\tau^2 g(\tau)e^{i\pi\phi_k''(t)\tau^2}}(\eta)| \leq K_5\varepsilon$, K_3, K_4 and K_5 being some constants. Furthermore if, for all $k = 1, \cdots, K$, $\int_{\{|\eta|>\Delta} |\widehat{g(\tau)e^{i\pi\phi_k''(t)\tau^2}}(\eta)|d\eta \leq K_6\tilde{\varepsilon}$, for some constant K_6, then, provided ε is sufficiently small, the following hold:*

(a) $|V_f^g(\eta, t)| \geq \tilde{\varepsilon}$ only when there exist $k \in \{1, ..., K\}$ such that $(\eta, t) \in Z_k$.

(b) For all $k \in \{1, ..., K\}$ and all $(\eta, t) \in Z_k$ such that $|V_f^g(\eta, t)| > \tilde{\varepsilon}$ and $|\partial_t \tilde{t}_f(\eta, t)| > \tilde{\varepsilon}$ we have

$$|\widehat{\omega}_f^{(2)}(\eta, t) - \phi_k'(t)| \leq \tilde{\varepsilon}. \tag{43}$$

(c) Moreover, for each $k \in \{1, ..., K\}$, there exists a constant D such that

$$\left| \left(\lim_{\delta \to 0} \frac{1}{g(0)} \int_{\mathcal{M}_{k,\tilde{\varepsilon}}} \widetilde{T}_f^{\delta,\tilde{\varepsilon}}(\omega, t) d\omega \right) - f_k(t) \right| \leq D\tilde{\varepsilon}, \tag{44}$$

where $\mathcal{M}_{k,\tilde{\varepsilon}} := \{\omega : |\omega - \phi_k'(t)| < \tilde{\varepsilon}\}$, provided the Lebesgue measure $\mu\{\eta,\ s.t.(\eta, t)\ \in Z_k, |\partial_t \tilde{t}_f(\eta, t)| \leq \tilde{\varepsilon}\} \leq \gamma\tilde{\varepsilon}$ for some constant γ.

Remark 5. The assumption $|\eta| > \Delta \Rightarrow |\widehat{g(\tau)e^{i\pi\phi_k''(t)\tau^2}}(\eta)| \leq K_3\varepsilon$ is somewhat complex, so we try here to explain what it really means. Let us fix the ridge number k and set $c = \phi_k''(t)$, the assumption ensures a sufficient decrease of the Fourier transform of:

$$l(\tau) = g(\tau)e^{i\pi c\tau^2}.$$

Indeed, remarking that for $g(t) = e^{-\pi t^2}$, $\hat{l}$ has the closed-form expression:

$$|\hat{l}(\eta)| = \left(1 + c^2\right)^{-\frac{1}{2}} e^{\frac{-\pi}{1+c^2}\eta^2},$$

it is clear that, if we take Δ large enough, we achieve the assumption. When g is not a Gaussian function, we cannot compute $\hat{l}$, but can still perform an estimation of the decay of $|\hat{l}|$ by means of the stationary phase approximation: the phase within the integral

$$\hat{l}(\eta) = \int_{\mathbb{R}} g(\tau)e^{i\pi c\tau^2}e^{-2i\pi\eta\tau}\, d\tau$$

admits a stationary point at $\tau_c = \frac{\eta}{c}$. The stationary phase theorem finally gives

$$|\hat{l}(\eta)| \approx \frac{1}{\sqrt{|c|}} \left| g\left(\frac{\eta}{c}\right) \right|.$$

Again, by assuming Δ large enough, one can satisfy the assumption. Now, if we consider all the ridges together, things are more complex, since choosing a large Δ implies the ridges need to be farther apart. But this is something that is expected: the frequency modulation makes the frequency support of the TF response associated with one mode larger, thus it changes the separation condition. In practice, the window's choice is driven by this trade-off between separation and localization.

The proof of Theorem 4 involves a number of estimates, which are shown hereafter, assuming all the hypotheses of Theorem 4 are satisfied. The main idea of the proof is to approximate the mode f_k around time t by the linear chirp

$$f_{k,2}(\tau) = A_k(t) e^{2i\pi[\phi_k(t)+\phi_k'(t)(\tau-t)+\frac{1}{2}\phi_k''(t)(\tau-t)^2]}. \tag{45}$$

Lemma 2. *For any (η, t), there is at most one $k \in \{1, \cdots, K\}$ for which $|\eta - \phi_k'(t)| \leq \Delta$.*

The proof is straightforward and is left to the reader. For our purpose, we need to analyze the behavior of the STFT on a linear chirp, for which we have the following proposition.

Proposition 3. *Let $h(\tau) = Ae^{2i\pi\phi(\tau)}$ be a linear chirp and consider V_h^g the STFT of h obtained with a window g as introduced in Theorem 4. If $|\eta - \phi'(t)| > \Delta$ then*

$$|V_h^g(\eta, t)| \leq \varepsilon K_3 A.$$

Proof. We know that $V_h^g(\eta, t) = h(t)g(\tau)\widehat{e^{i\pi\phi''(t)\tau^2}}(\eta - \phi'(t))$ so that, if $|\eta - \phi'(t)| > \Delta$, assumptions on g lead to $|V_h^g(\eta, t)| \leq AK_3\varepsilon$. □

Proposition 4. *For any $k \in \{1, \cdots, K\}$ and for any $(\eta, t) \in \mathbb{R}^2$ we have:*

$$|V_{f_k}^g(\eta, t) - V_{f_{k,2}}^g(\eta, t)| \leq \varepsilon E_k(t), \tag{46}$$

with $E_k(t) = I_1 + \frac{\pi}{3} I_3 A_k(t)$. if $(\eta, t) \notin Z_k$, we get

$$|V_{f_k}^g(\eta, t)| \leq \varepsilon(E_k(t) + K_3 A_k(t)), \tag{47}$$

and also, for any $(\eta, t) \in Z_k$*:*

$$|V_f^g(\eta,t) - V_{f_{k,2}}^g(\eta,t)| \;\leq\; \varepsilon\Omega_k(t), \tag{48}$$

where $\Omega_k(t) = \sum_{l=1}^{K} E_l(t) + \sum_{l\neq k} K_3 A_l(t)$.

Proof. First, we write

$$\begin{aligned} f_k(\tau) &= A_k(\tau)e^{2i\pi\phi_k(\tau)} = (A_k(\tau) - A_k(t))e^{2i\pi\phi_k(\tau)} \\ &+ A_k(t)e^{2i\pi[\phi_k(t)+\phi_k'(t)(\tau-t)+\frac{1}{2}\phi_k''(t)(\tau-t)^2]} \\ &+ A_k(t)[e^{2i\pi[\phi_k(t)+\phi_k'(t)(\tau-t)+\frac{1}{2}\phi_k''(t)(\tau-t)^2+\frac{1}{2}\int_t^\tau \phi_k'''(x)(\tau-x)^2dx]} - e^{2i\pi[\phi_k(t)+\phi_k'(t)(\tau-t)+\frac{1}{2}\phi_k''(t)(\tau-t)^2]}] \\ &= f_{k,1}(\tau) + f_{k,2}(\tau) + f_{k,3}(\tau). \end{aligned} \tag{49}$$

Then, for any (η, t),

$$\begin{aligned} |V_{f_{k,1}}^g(\eta,t)| &\leq \int_{\mathbb{R}} |A_k(\tau) - A_k(t)||g(\tau-t)|d\tau, \\ &\leq \int_{\mathbb{R}} \varepsilon|\tau-t||g(\tau-t)|d\tau \leq \varepsilon I_1. \end{aligned}$$

Now, we also have

$$\begin{aligned} |V_{f_{k,3}}^g(\eta,t)| &\leq \pi A_k(t) \int_{\mathbb{R}} \left(\int_t^\tau |\phi'''(x)||\tau-x|^2\,dx \right) |g(\tau-t)|\,d\tau \\ &\leq \frac{\pi}{3} A_k(t) \int_{\mathbb{R}} \varepsilon|\tau-t|^3|g(\tau-t)|\,d\tau \\ &\leq \frac{\pi}{3} A_k(t)\varepsilon I_3, \end{aligned}$$

leading to equation (46).
From this proposition, if $(\eta, t) \notin Z_k$, we trivially get

$$|V_{f_k}^g(\eta,t)| \leq \varepsilon\left(E_k(t) + K_3 A_k(t)\right),$$

and also, for any $(\eta, t) \in Z_k$:

$$|V_f^g(\eta,t) - V_{f_{k,2}}^g(\eta,t)| \;\leq\; \varepsilon\left(\sum_{l=1}^{K} E_l(t) + \sum_{l\neq k} K_3 A_l(t)\right) = \varepsilon\Omega_k(t),$$

where $\Omega_k(t) = \sum_{l=1}^{K} E_l(t) + \sum_{l\neq k} K_3 A_l(t)$. □

We now prove item (a) of Theorem 4. If $(\eta,t) \notin \bigcup_{l=1}^{K} Z_l$, we immediately get:

$$|V_f^g(\eta,t)| \leq \varepsilon \sum_{l=1}^{K} \left(E_l(t) + K_3 A_l(t)\right) \leq \tilde{\varepsilon}$$

When $\tilde{\varepsilon}$ is sufficiently small, which corresponds to item (a) of Theorem 4. Now, we introduce several propositions that are useful to prove item (b) of Theorem 4.

Proposition 5. *Let $h(\tau) = Ae^{2i\pi\phi(\tau)}$ be a linear chirp, g be a window as defined in Theorem 4. When $|\eta - \phi'(t)| > \Delta$, we have*

$$|V_h^{tg}(\eta,t)| \leq AK_4\varepsilon.$$

Proof. The proof is analogous to that with window g. □

Proposition 6. *Let f_k be a mode defined as in Definition 7, and consider the STFT of f_k computed using window tg, where g is a window of the type introduced in Theorem 4. Then, for any (η,t), we have*

$$|V_{f_k}^{tg}(\eta,t) - V_{f_{k,2}}^{tg}(\eta,t)| \leq \varepsilon F_k(t),$$

with $F_k(t) = I_2 + \frac{\pi}{3} I_4 A_k(t)$. Now, if $(\eta,t) \notin Z_k$, we get:

$$|V_{f_k}^{tg}(\eta,t)| \leq \varepsilon \left(F_k(t) + K_4 A_k(t)\right),$$

and, we also have, when $(\eta,t) \in Z_k$:

$$|V_f^{tg}(\eta,t) - V_{f_{k,2}}^{tg}(\eta,t)| \quad \leq \quad \varepsilon \Lambda_k(t), \tag{50}$$

where $\Lambda_k(t) = \sum_{l=1}^{K} F_l(t) + K_4 \sum_{l \neq k} A_l(t)$.

Proof. The proof is the same as that of Proposition 4 replacing g by tg. □

Proposition 7. *For any (η,t), assuming g satisfies the hypotheses of Theorem 4, we have:*

$$\left| \partial_t V_f^g(\eta,t) - 2i\pi \sum_{k=1}^{K} \left(\phi_k'(t) V_{f_{k,2}}^g(\eta,t) + \phi_k''(t) V_{f_{k,2}}^{tg}\right) \right| \leq \varepsilon G(t), \tag{51}$$

where $G(t) = KI_0 + 2\pi \sum_{k=1}^{K} \left(\|A_k\|_\infty \frac{I_2}{2} + \phi_k'(t) E_k(t) + |\phi_k''(t)| F_k(t)\right)$.

Proof. Differentiating the STFT of f with respect to t, we get:

$$\begin{aligned}
\partial_t V_f^g(\eta,t) &= \sum_{k=1}^K \int_{\mathbb{R}} A_k'(\tau) e^{2i\pi\phi_k(\tau)} g(\tau-t) e^{-2i\pi\eta(\tau-t)} d\tau \\
&+ \sum_{k=1}^K \int_{\mathbb{R}} A_k(\tau) 2i\pi\phi_k'(\tau) e^{2i\pi\phi_k(\tau)} g(\tau-t) e^{-2i\pi\eta(\tau-t)} d\tau \\
&= \sum_{k=1}^K \int_{\mathbb{R}} A_k'(\tau) e^{2i\pi\phi_k(\tau)} g(\tau-t) e^{-2i\pi\eta(\tau-t)} d\tau \\
&+ \sum_{k=1}^K \int_{\mathbb{R}} A_k(\tau) 2i\pi \left[\phi_k'(t) + (\tau-t)\phi_k''(t) + \int_t^\tau (\tau-u)\phi_k'''(u) du\right] e^{2i\pi\phi_k(\tau)} g(\tau-t) e^{-2i\pi\eta(\tau-t)} d\tau \\
&= \sum_{k=1}^K \int_{\mathbb{R}} A_k'(\tau) e^{2i\pi\phi_k(\tau)} g(\tau-t) e^{-2i\pi\eta(\tau-t)} d\tau + \sum_{k=1}^K 2i\pi\phi_k'(t) V_{f_k}^g(\eta,t) \\
&+ \sum_{k=1}^K 2i\pi\phi_k''(t) V_{f_k}^{tg}(\eta,t) \\
&+ \sum_{k=1}^K \int_{\mathbb{R}} A_k(\tau) 2i\pi \left(\int_t^\tau (\tau-u)\phi'''(u) du\right) e^{2i\pi\phi_k(\tau)} e^{-2i\pi\eta(\tau-t)} g(\tau-t) d\tau.
\end{aligned}$$

We may then write:

$$\begin{aligned}
&\left|\partial_t V_f^g(\eta,t) - 2i\pi \sum_{k=1}^K \left(\phi_k'(t) V_{f_k}^g(\eta,t) + \phi_k''(t) V_{f_k}^{tg}(\eta,t)\right)\right| \\
&\quad \leq \sum_{k=1}^K \int_{\mathbb{R}} |A_k'(\tau)||g(\tau-t)| d\tau + 2\pi \sum_{k=1}^K \int_{\mathbb{R}} A_k(\tau) \left(\int_t^\tau |\tau-u||\phi'''(u)| du\right) |g(\tau-t)| d\tau \\
&\quad \leq \varepsilon \left((KI_0 + \sum_{k=1}^K \|A_k\|_\infty \pi I_2\right).
\end{aligned}$$

Finally, from Propositions 4 and 6 we have:

$$\begin{aligned}
&\left|\partial_t V_f^g(\eta,t) - 2i\pi \sum_{k=1}^K \left(\phi_k'(t) V_{f_{k,2}}^g(\eta,t) + \phi_k''(t) V_{f_{k,2}}^{tg}(\eta,t)\right)\right| \\
&\qquad \leq \left|\partial_t V_f^g(\eta,t) - 2i\pi \sum_{k=1}^K \left(\phi_k'(t) V_{f_k}^g(\eta,t) + \phi_k''(t) V_{f_k}^{tg}(\eta,t)\right)\right| \\
&\qquad + 2\pi\varepsilon \sum_{k=1}^K \left(\phi_k'(t) E_k(t) + |\phi_k''(t)| F_k(t)\right),
\end{aligned}$$

hence (51). □

From this proposition, we get that, if $(\eta, t) \in Z_k$:

$$\begin{aligned}
&\left|\partial_t V_f^g(\eta,t) - 2i\pi\left(\phi_k'(t)V_{f_{k,2}}^g(\eta,t) + \phi_k''(t)V_{f_{k,2}}^{tg}(\eta,t)\right)\right| \\
&\qquad \leq \left|\partial_t V_f^g(\eta,t) - 2i\pi\sum_{l=1}^{K}\left(\phi_l'(t)V_{f_{l,2}}^g(\eta,t) + \phi_l''(t)V_{f_{l,2}}^{tg}(\eta,t)\right)\right| \\
&\qquad + \left|2\pi\sum_{l\neq k}\left(\phi_l'(t)V_{f_{l,2}}^g(\eta,t) + \phi_l''(t)V_{f_{l,2}}^{tg}(\eta,t)\right)\right| \\
&\qquad \leq \left|\partial_t V_f^g(\eta,t) - 2i\pi\sum_{l=1}^{K}\left(\phi_l'(t)V_{f_{l,2}}^g(\eta,t) + \phi_l''(t)V_{f_{l,2}}^{tg}(\eta,t)\right)\right| \\
&\qquad + 2\pi\varepsilon\sum_{l\neq k} A_l(t)\left(K_3\phi_l'(t) + K_4|\phi_l''(t)|\right) \\
&\qquad \leq \varepsilon G_k(t),
\end{aligned}$$

with $G_k(t) = G(t) + 2\pi\sum\limits_{l\neq k} A_l(t)\left(K_3\phi_l'(t) + K_4|\phi_l''(t)|\right)$. From this, we finally deduce:

$$\begin{aligned}
&\left|\partial_t V_f^g(\eta,t) - 2i\pi\left(\phi_k'(t)V_f^g(\eta,t) + \phi_k''(t)V_f^{tg}(\eta,t)\right)\right| \\
&\qquad \leq \varepsilon\left(G_k(t) + 2\pi(\phi_k'(t)\Omega_k(t) + |\phi_k''(t)|\Lambda_k(t))\right) := \varepsilon B_{1,k}(t). \qquad (52)
\end{aligned}$$

Proposition 8. *Let $h(t) = Ae^{2i\pi\phi(t)}$ be a linear chirp, and assume g is a window defined as in Theorem 4. When $|\eta - \phi'(t)| > \Delta$, one has*

$$|V_h^{t^2 g}(\eta,t)| \leq AK_5\varepsilon. \qquad (53)$$

Proof. The proof is identical to that with window g. □

Proposition 9. *Let f_k be a mode defined as in Definition 7 and consider the STFT of f_k computed using window $t^2 g$, where g is defined as in Theorem 4. In such a case, one has:*

$$|V_{f_k}^{t^2 g}(\eta,t) - V_{f_{k,2}}^{t^2 g}(\eta,t)| \leq \varepsilon H_k(t),$$

with $H_k(t) = I_3 + \frac{\pi}{3}I_5 A_k(t)$. Now, if $(\eta,t) \notin Z_k$, we get:

$$|V_{f_k}^{t^2 g}(\eta,t)| \leq \varepsilon\left(H_k(t) + K_5 A_k(t)\right).$$

From this, we immediately deduce that, for $(\eta,t)\in Z_k$, we have:

$$|V_f^{t^2g}(\eta,t)-V_{f_{k,2}}^{t^2g}(\eta,t)|\leq\varepsilon\Psi_k(t),$$

where $\Psi_k(t)=\sum_{l=1}^{K}H_k(t)+K_5\sum_{l\neq k}A_l(t)$.

Proof. The proof is similar to that of Proposition 4, replacing g by t^2g. □

Proposition 10. *For any (η,t) one has:*

$$\begin{aligned}\Big|\partial_{tt}^2V_f^g(\eta,t)-2i\pi\sum_{l=1}^{K}\Big((\phi_l''(t)+2i\pi\phi_l'(t)^2)V_{f_l}^g(\eta,t)\\ +4i\pi\phi_l'(t)\phi_l''(t)V_{f_l}^{tg}(\eta,t)+2i\pi\phi_l''(t)^2V_{f_l}^{t^2g}(\eta,t)\Big)\Big|\leq\varepsilon J(t),\end{aligned}\tag{54}$$

where

$$\begin{aligned}J(t) &= KI_0+2\pi I_1\sum_{l=1}^{K}\|A_l\|_\infty+4\pi^2I_2\sum_{l=1}^{K}\phi_l'(t)\|A_l\|_\infty\\ &\quad+2\pi\sum_{l=1}^{K}\phi_l'(t)(I_0+2\pi\|A_l\|_\infty I_2)+|\phi_l''(t)|(I_1+2\pi\|A_l\|_\infty I_3).\end{aligned}$$

Proof. Since

$$\begin{aligned}\partial_{tt}^2V_f^g(\eta,t)&=\sum_{l=1}^{K}\int_{\mathbb{R}}(A_l''(\tau)+2i\pi\phi_l''(\tau)A_l(\tau)+2i\pi\phi_l'(\tau)A_l'(\tau)\\ &\qquad-4\pi^2\phi_l'(\tau)^2A_l(\tau)\Big)e^{2i\pi\phi_l(\tau)}g(\tau-t)e^{-2i\pi\eta\tau-t}d\tau\\ &=\sum_{l=1}^{K}\int_{\mathbb{R}}A_l''(\tau)e^{2i\pi\phi_l(\tau)}g(\tau-t)e^{-2i\pi\eta(\tau-t)}d\tau\\ &\quad+2i\pi\sum_{l=1}^{K}\int_{\mathbb{R}}\phi_l''(\tau)A_l(\tau)e^{2i\pi\phi_l(\tau)}g(\tau-t)e^{-2i\pi\eta(\tau-t)}d\tau\\ &\quad+2i\pi\sum_{l=1}^{K}\int_{\mathbb{R}}\phi_l'(\tau)f_l'(\tau)g(\tau-t)e^{-2i\pi\eta(\tau-t)}d\tau,\end{aligned}$$

we have, for the first part of $\partial^2_{tt} V^g_f(\eta,t)$,

$$\left|\sum_{l=1}^{K}\int_{\mathbb{R}} A_l''(\tau)e^{2i\pi\phi_l(\tau)}g(\tau-t)e^{-2i\pi\eta(\tau-t)}d\tau\right| \leq K\varepsilon I_0. \tag{55}$$

Then, for the second part of $\partial^2_{tt} V^g_f(\eta,t)$, we may write

$$\begin{aligned}
&\left|2i\pi\sum_{l=1}^{K}\int_{\mathbb{R}}\phi_l''(\tau)A_l(\tau)g(\tau-t)e^{-2i\pi\eta(\tau-t)}d\tau - 2i\pi\sum_{l=1}^{K}\phi_l''(t)V^g_{f_l}(\eta,t)\right| \\
&\quad\leq 2\pi\sum_{l=1}^{K}\int_{\mathbb{R}}\left(\int_t^\tau|\phi_l'''(u)|du\right)\ A_l(\tau)|g(\tau-t)|d\tau| \\
&\quad\leq 2\pi\varepsilon\sum_{l=1}^{K}\|A_l\|_\infty I_1.
\end{aligned}$$

Finally, we write for the third part of $\partial^2_{tt} V^g_f(\eta,t)$:

$$\begin{aligned}
&\left|2i\pi\sum_{l=1}^{K}\int_{\mathbb{R}}\phi_l'(\tau)f_l'(\tau)g(\tau-t)e^{-2i\pi\eta(\tau-t)}d\tau - 2i\pi\sum_{l=1}^{K}\phi_l'(t)\partial_t V^g_{f_l}(\eta,t)+\phi_l''(t)\partial_t V^{tg}_{f_l}(\eta,t)\right| \\
&=\left|2i\pi\sum_{l=1}^{K}\int_{\mathbb{R}}\int_t^\tau(t-\tau)\phi_l'''(u)du\ \left(A_l'(\tau)+2i\pi\phi_l'(\tau)A_l(\tau)\right)e^{2i\pi\phi_l(\tau)}g(\tau-t)e^{-2i\pi\eta(\tau-t)}d\tau\right| \\
&\leq 4\pi^2\varepsilon\sum_{l=1}^{K}\|A_l(t)\|_\infty\phi_l'(t)I_2
\end{aligned}$$

if ε is sufficiently small. From this, we get:

$$\left|\partial^2_{tt}V^g_f(\eta,t) - 2i\pi\sum_{l=1}^{K}\left(\phi_l''(t)V^g_{f_l}(\eta,t)+\phi_l'(t)\partial_t V^g_{f_l}(\eta,t)+\phi_l''(t)\partial_t V^{tg}_{f_l}(\eta,t)\right)\right| \leq \varepsilon J^0(t), \tag{56}$$

where $J^0(t) = KI_0 + 2\pi\sum_{l=1}^{K}\|A_l(t)\|_\infty I_1 + 4\pi^2\sum_{l=1}^{K}\|A_l(t)\|_\infty\phi_l'(t)I_2$.
Now, recalling that:

$$\begin{aligned}
|\partial_t V^g_{f_l}(\eta,t) - 2i\pi\left(\phi_l'(t)V^g_{f_l}(\eta,t)+\phi_l''(t)V^{tg}_{f_l}(\eta,t)\right)| &\leq \varepsilon\left(I_0+2\pi\|A_l\|_\infty I_2\right), \\
|\partial_t V^{tg}_{f_l}(\eta,t) - 2i\pi\left(\phi_l'(t)V^{tg}_{f_l}(\eta,t)+\phi_l''(t)V^{t^2g}_{f_l}(\eta,t)\right)| &\leq \varepsilon(I_1+2\pi\|A_l\|_\infty I_3),
\end{aligned}$$

we get,

$$\left|\partial_{tt}^2 V_f^g(\eta,t) - 2i\pi \sum_{l=1}^{K} \Big((\phi_l''(t) + 2i\pi\phi_l'(t)^2) V_{f_l}^g(\eta,t) \right.$$
$$\left. + 4i\pi\phi_l'(t)\phi_l''(t) V_{f_l}^{tg}(\eta,t) + 2i\pi\phi_l''(t)^2 V_{f_l}^{t^2 g}(\eta,t) \Big) \right| \leq \varepsilon J(t),$$

where $J(t) = J^0(t) + 2\pi \sum_{l=1}^{K} \phi_l'(t)(I_0 + 2\pi\|A_l\|_\infty I_2) + |\phi_l''(t)|(I_1 + 2\pi\|A_l\|_\infty I_3)$.

□

Now, since we have for any l:

$$\begin{aligned} |V_{f_l}^g(\eta,t) - V_{f_{l,2}}^g(\eta,t)| &\leq \varepsilon E_l(t), \\ |V_{f_l}^{tg}(\eta,t) - V_{f_{l,2}}^{tg}(\eta,t)| &\leq \varepsilon F_l(t), \\ |V_{f_l}^{t^2 g}(\eta,t) - V_{f_{l,2}}^{t^2 g}(\eta,t)| &\leq \varepsilon H_l(t), \end{aligned}$$

we obtain:

$$\left|\partial_{tt}^2 V_f^g(\eta,t) - 2i\pi \sum_{l=1}^{K} \Big((\phi_l''(t) + 2i\pi\phi_l'(t)^2) V_{f_{l,2}}^g(\eta,t) \right.$$
$$\left. + 4i\pi\phi_l'(t)\phi_l''(t) V_{f_{l,2}}^{tg}(\eta,t) + 2i\pi\phi_l''(t)^2 V_{f_{l,2}}^{t^2 g}(\eta,t) \Big) \right| \leq \varepsilon J^1(t),$$

with

$$J^1(t) = J(t) + 2\pi \sum_{l=1}^{K} (|\phi_l''(t)| + 2\pi\phi_l'(t)^2) E_l(t) + 4\pi\phi_l'(t)|\phi_l''(t)| F_l(t) + 2\pi\phi_l''(t)^2 H_l(t).$$

Now, if $(\eta,t) \in Z_k$, we may write, using Propositions 3, 6, 9:

$$\left|\partial_{tt}^2 V_f^g(\eta,t) - 2i\pi \Big((\phi_k''(t) + 2i\pi\phi_k'(t)^2) V_{f_{k,2}}^g(\eta,t) \right.$$
$$\left. + 4i\pi\phi_k'(t)\phi_k''(t) V_{f_{k,2}}^{tg}(\eta,t) + 2i\pi\phi_k''(t)^2 V_{f_{k,2}}^{t^2 g}(\eta,t) \Big) \right| \leq \varepsilon J_k(t),$$

where

$$J_k(t) = J^1(t) + \pi \sum_{l \neq k} A_l(t) \left(2K_3|\phi_l''(t)| + 4\pi K_3 \phi_l'(t)^2 + 4\pi\phi_l'(t)|\phi_l''(t)| K_4 + \phi_l''(t)^2 K_5 \right).$$

Finally, we may write:

$$\left|\partial_{tt}^2 V_f^g(\eta,t) - 2i\pi\left((\phi_k''(t)+2i\pi\phi_k'(t)^2)V_f^g(\eta,t)+4i\pi\phi_k'(t)\phi_k''(t)V_f^{tg}(\eta,t)+2i\pi\phi_k''(t)^2V_f^{t^2g}(\eta,t)\right)\right|$$
$$\leq \tilde{\varepsilon}^2\left(J_k(t)+(\phi_k''(t)+2\pi\phi_k'(t)^2)\Omega_k(t)+4\pi\phi_k'(t)|\phi_k''(t)|\Lambda_k(t)+2\pi\phi_k''(t)^2\Psi_k(t)\right) := \varepsilon B_{2,k}(t), \quad (57)$$

the last inequality being obtained using Proposition 4, 6 and 9.

Proposition 11. *For any $(\eta,t)\in Z_k$, such that $|V_f^g(\eta,t)|>\tilde{\varepsilon}$ and $|\partial_t\tilde{t}_f(\eta,t)|>\tilde{\varepsilon}$ one has:*

$$|\tilde{q}_f(\eta,t)-\phi_k''(t)|\leq\tilde{\varepsilon}. \quad (58)$$

Proof. Before we start, we need to remark that if $(\eta,t)\in Z_k$:

$$\begin{aligned}
&\left|\partial_t V_f^g(\eta,t) - 2i\pi(\phi_k'(t)V_f^g(\eta,t)+\phi_k''(t)V_f^{tg}(\eta,t))\right| \\
&\leq \left|\sum_{l=1}^K \partial_t V_{f_l}^{tg}(\eta,t) - 2i\pi(\phi_l'(t)V_{f_l}^g(\eta,t)+\phi_l''(t)V_{f_l}^{tg}(\eta,t))\right| \\
&\qquad + \left|\sum_{l\neq k}(\phi_k'(t)-\phi_l'(t))V_{f_l}^g(\eta,t)+(\phi_k''(t)-\phi_l''(t))V_{f_l}^{tg}(\eta,t)\right| \\
&\leq \varepsilon\left(KI_0+\sum_{l=1}^K\|A_l\|_\infty\pi I_2+2\max_l\|\phi_l'\|_\infty\sum_{l\neq k}E_l(t)+K_3A_l(t)\right. \\
&\qquad \left.+2\max_l\|\phi_l''\|_\infty F_l(t)+K_4A_l(t)\right) := \varepsilon B_{3,k}(t) \quad (59)
\end{aligned}$$

Similarly we have:

$$\begin{aligned}
&\left|\partial_t V_f^{tg}(\eta,t) - 2i\pi(\phi_k'(t)V_f^{tg}(\eta,t)+\phi_k''(t)V_f^{t^2g}(\eta,t))\right| \\
&\leq \varepsilon\left(KI_1+\sum_{l=1}^K\|A_l\|_\infty\pi I_3+2\max_l\|\phi_l'\|_\infty\sum_{l\neq k}F_l(t)+K_4A_l(t)\right. \\
&\qquad \left.+2\max_l\|\phi_l''\|_\infty H_l(t)+K_5A_l(t)\right) := \varepsilon B_{4,k}(t) \quad (60)
\end{aligned}$$

With these two inequalities we can prove the proposition. For the sake of

clarity, we omit to write (η, t) in the proof that follows.

$$
\begin{aligned}
|\phi_k''(t) - \tilde{q}_f(\eta,t)| &= \left| \phi_k''(t) - \frac{V_f^g \partial_{tt}^2 V_f^g - (\partial_t V_f^g)^2}{2i\pi (V_f^g)^2 - V_f^g \partial_{\eta t}^2 V_f^g + \partial_t V_f^g \partial_\eta V_f^g} \right| \\
&= \left| \frac{\partial_t V_f^g \left[\partial_t V_f^g + \phi_k''(t) \partial_\eta V_f^g \right]}{2i\pi (V_f^g)^2 - V_f^g \partial_{\eta t}^2 V_f^g + \partial_t V_f^g \partial_\eta V_f^g} - \frac{V_f^g \left[\partial_{tt}^2 V_f^g + \phi_k''(t) \partial_{\eta t}^2 V_f^g - 2i\pi \phi_k''(t) V_f^g \right]}{2i\pi (V_f^g)^2 - V_f^g \partial_{\eta t}^2 V_f^g + \partial_t V_f^g \partial_\eta V_f^g} \right| \\
&= \left| \frac{\partial_t V_f^g [\partial_t V_f^g - 2i\pi \phi_k''(t) V_f^{tg} - 2i\pi \phi_k'(t) V_f^g]}{2i\pi (V_f^g)^2 + 2i\pi V_f^g \partial_t V_f^{tg} - 2i\pi \partial_t V_f^g V_f^{tg}} \right| \\
&\quad + \left| \frac{V_f^g [\partial_{tt}^2 V_f^g - 2i\pi \phi_k''(t) \partial_t V_f^{tg} - 2i\pi \phi_k''(t) V_f^g - 2i\pi \phi_k'(t) \partial_t V_f^g]}{2i\pi (V_f^g)^2 + 2i\pi V_f^g \partial_t V_f^{tg} - 2i\pi \partial_t V_f^g V_f^{tg}} \right| \\
&\leq \varepsilon B_{1,k}(t) \left| \frac{\partial_t V_f^g}{2i\pi [(V_f^g)^2 + V_f^g \partial_t V_f^{tg} - \partial_t V_f^g V_f^{tg}]} \right| \\
&\quad + \varepsilon B_{2,k}(t) \left| \frac{V_f^g}{2i\pi [(V_f^g)^2 + V_f^g \partial_t V_f^{tg} - \partial_t V_f^g V_f^{tg}]} \right| \\
&\quad + \varepsilon \frac{|V_f^g| (|\phi_k''(t)| B_{4,k}(t) + \phi_k'(t) B_{3,k}(t))}{\left| (V_f^g)^2 + V_f^g \partial_t V_f^{tg} - \partial_t V_f^g V_f^{tg} \right|} \\
&\leq \tilde{\varepsilon}^3 \left(B_{1,k}(t) |\partial_t V_f^g| + |V_f^g| (B_{2,k}(t) + |\phi_k''(t)| B_{4,k}(t) + \phi_k'(t) B_{3,k}(t)) \right) \leq \tilde{\varepsilon},
\end{aligned}
$$

if $\tilde{\varepsilon}$ is sufficiently small. Note that the above result was proved using (52), (57), (59) and (60). □

Lemma 3. *For all $k \in \{1, ..., K\}$ and any $(\eta, t) \in Z_k$ such that $|V_f^g(\eta, t)| > \tilde{\varepsilon}$ and $|\partial_t \tilde{t}_f(\eta, t)| > \tilde{\varepsilon}$, we have*

$$
|\widetilde{\omega}_f^{(2)}(\eta, t) - \phi_k'(t)| \leq \tilde{\varepsilon}. \tag{61}
$$

Proof. According to definition of $\widetilde{\omega}_f^{(2)}(\eta, t)$ in (39), we have

$$
\widetilde{\omega}_f^{(2)}(\eta, t) = \tilde{\omega}_f(\eta, t) + \tilde{q}_f(\eta, t)(t - \tilde{t}_f(\eta, t)).
$$

It follows that

$$\begin{aligned}\left|\widetilde{\omega}_f^{(2)}(\eta,t)-\phi_k'(t)\right| &= \left|\frac{\partial_t V_f^g(\eta,t)+\tilde{q}_f(\eta,t)\partial_\eta V_f^g(\eta,t)}{2i\pi V_f^g(\eta,t)}-\phi_k'(t)\right| \\ &= \left|\frac{\partial_t V_f^g(\eta,t)-2i\pi\tilde{q}_f(\eta,t)V_f^{tg}(\eta,t)}{2i\pi V_f^g(\eta,t)}-\phi_k'(t)\right| \\ &\leq \varepsilon\frac{B_{1,k}(t)}{\left|2i\pi V_f^g(\eta,t)\right|}+\frac{\left|\tilde{q}_f(\eta,t)-\phi_k''(t)\right|\left|V_f^{tg}(\eta,t)\right|}{\left|V_f^g(\eta,t)\right|} \\ &\leq \varepsilon\frac{B_{1,k}(t)}{\left|2\pi V_f^g(\eta,t)\right|} \\ &+\tilde{\varepsilon}^3\frac{\left[B_{1,k}(t)|\partial_t V_f^g|+|V_f^g|\left(B_{2,k}(t)+|\phi_k''(t)|B_{4,k}(t)+\phi_k'(t)B_{3,k}(t)\right)\right]|V_f^{tg}(\eta,t)|}{|V_f^g(\eta,t)|}\leq\tilde{\varepsilon}\end{aligned}$$

when $\tilde{\varepsilon}$ is sufficiently small. □

To prove (c) of Theorem 4, is exactly the same as the proof in the weak modulation case, except that we use at the very end of the proof the hypothesis:

$$\left|\frac{1}{g(0)}\int_{|\eta|>\Delta}\widehat{g(\tau)e^{i\pi\phi_l''(t)\tau^2}}(\eta)d\eta\right|\leq\frac{K_6\tilde{\varepsilon}}{|g(0)|}\text{ for any }l,$$

and

$$\frac{1}{|g(0)|}\left|\int_{\{|\eta-\phi_k'(t)|<\Delta\}\bigcap\{\{|V_f^g(\eta,t)|\leq\tilde{\varepsilon}\}\bigcup\{|\partial_t\tilde{t}_f(\eta,t)|\leq\tilde{\varepsilon}\}\}}V_f^g(\eta,t)d\eta\right|\leq\left(\frac{2\Delta}{|g(0)|}+\gamma|A_k(t)|\right)\tilde{\varepsilon}$$

to conclude.

5. Numerical Results

In order to illustrate the behaviour of our new second order synchrosqueezing transform, we apply it two synthetic test signals. One (signal 1) is made of low-order polynomial chirps, that behave locally as linear chirps, and the other one (signal 2) contains strongly nonlinear sinusoidal frequency modulations. On Fig. 1, STFT, FSST, corresponding to the original synchrosqueezing transform and, our new technique called *new VSST* for both signals 1 and 2 are depicted. We choose this denomination after the so-called VSST technique which is the other transform based on the second order approximation of the phase, which was introduced in [13]. Note that in our simulations, we use 1024 time samples over $[0,1]$ and a Gaussian window with parameter $\sigma=0.005$, defined by $g(t)=\sigma^{-\frac{1}{2}}e^{-\frac{\pi t^2}{\sigma^2}}$.

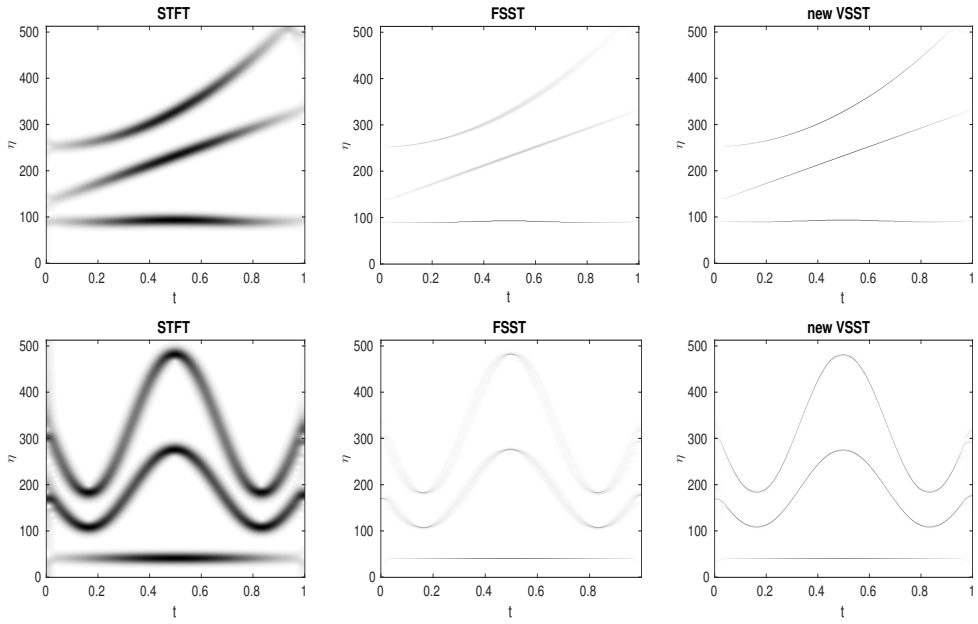


Figure 1: first row: magnitude of STFT, FSST and new VSST for test-signal 1; second row: same computation but for test-signal 2

5.1. Quality of representation

In order to quantify the quality of the representation given by the new VSST, we propose to measure the amount of information contained in the coefficients with the largest amplitude. In this regard, one way to compare the different transformations is to compute the normalized energy associated with the first coefficients with the largest amplitude: the faster the growth of this energy towards 1, the sharper the representation. On Fig. 2 A and B, these normalized energies are displayed with respect to the number of coefficients kept divided by the length of the signal, for both test-signals, and for the three representations, namely VSST proposed in [13], new VSST and FSST. In our context, the normalized energy is computed as the cumulative sum of the squared sorted coefficients over the sum of all the squared coefficients. The first remark is that both VSST and new VSST behave similarly for test-signal 1 while slightly better results are obtained when the latter is used on test-signal 2, which contains stronger frequency modulations. The second remark is that one needs only 3 coefficients per time instant to recover

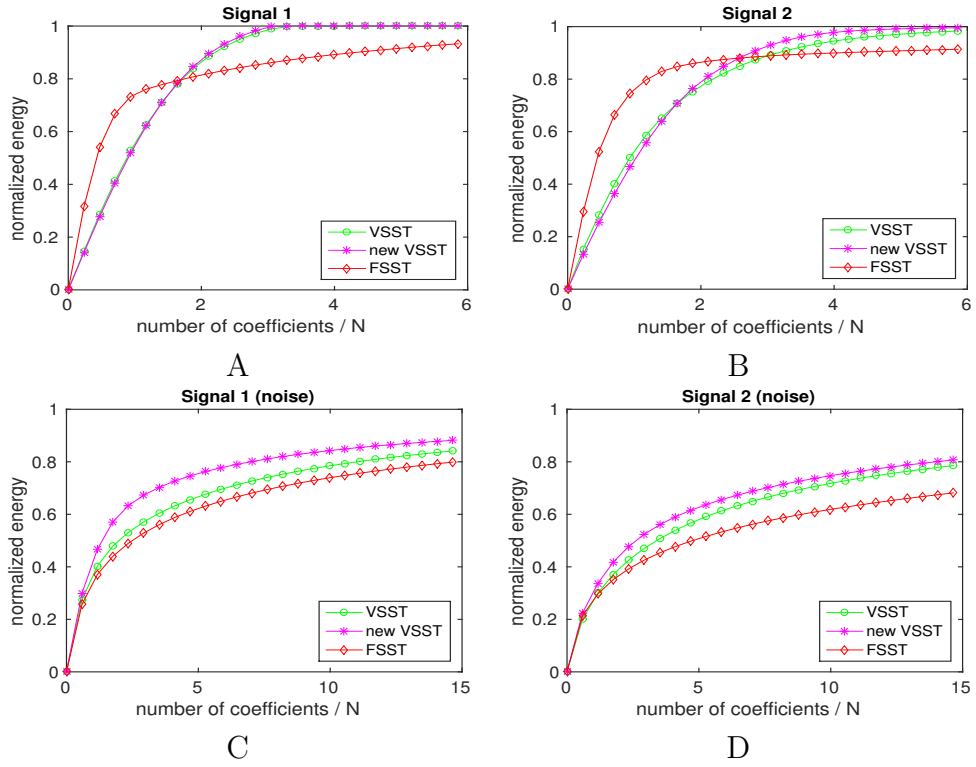


Figure 2: Normalized energy as a function of the number of sorted coefficients for test-signals 1 (A) and 2 (B). Abscissa gives the number of coefficients kept over the length N of the signal, i.e., the mean number of coefficients kept in each column of the TF plane. C and D: idem as A and B but for noisy signals (0 dB).

the signal energy, which is consistent with the three modes making up the test-signal. In order to investigate the influence of noise on the sparsity of the representation, we carry out the same experiments when the test-signals are contaminated by white Gaussian noise (noise level 0 dB). The results displayed on Fig. 2 C and D exhibit a slower increase of the normalized energy, since the coefficients corresponding to noise are spread over the whole TF plane. However, our proposed new VSST still behaves better than both VSST and FSST in this noisy situation.

5.2. Reconstruction of the modes

The main advantage of synchrosqueezing techniques over traditional reassignment techniques lies in its invertibility. To improve the accuracy of

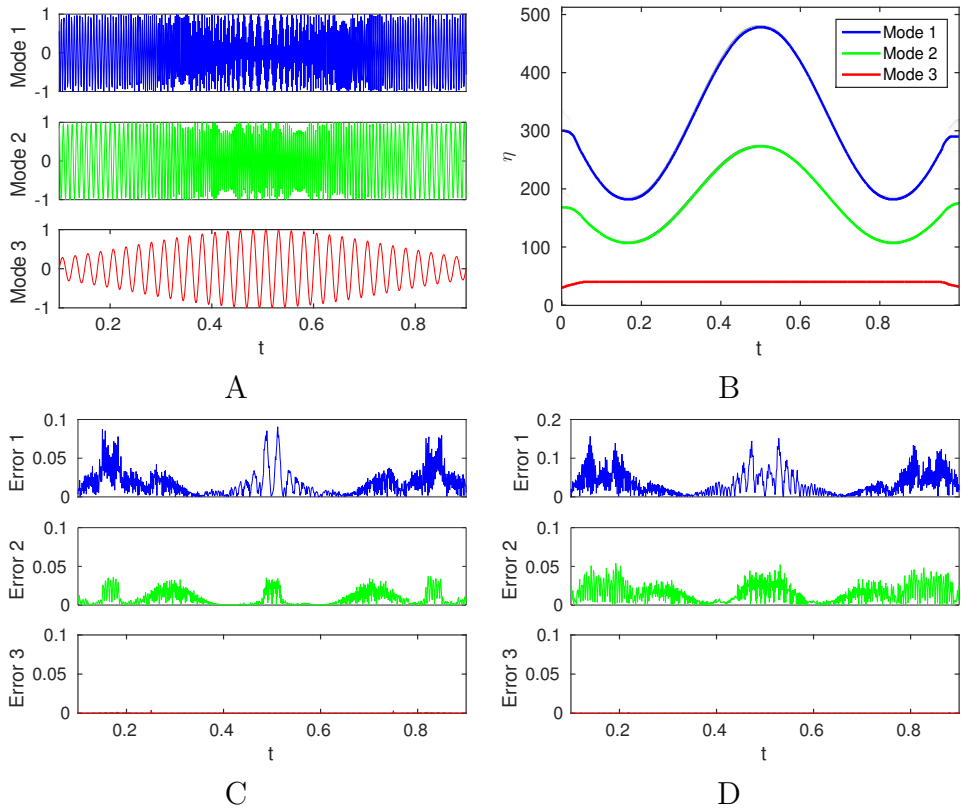


Figure 3: A:mode retrieval based on new VSST for test-signal 2; B: Estimated ridges superimposed on the new VSST of test-signal 2. We use $d = 5$ in the reconstruction, and $\beta = 0, \lambda = 0.02$ for the ridge extraction; C:Reconstruction errors associated with each mode of test-signal 2 using new VSST; D: same as C but using VSST technique

the reassignment step in synchrosqueezing techniques by using new VSST naturally leads to better reconstruction results. The reconstruction formula used to retrieve the kth mode, assuming $\hat{\phi}'_k(t)$ is an estimate of $\phi'_k(t)$, is as follows:

$$f_k(t) \approx \int_{|\omega - \hat{\phi}'_k(t)| < d} T_f(\omega, t) d\omega, \tag{62}$$

as explained in more details in [13]. The parameter d is here to compensate for the fact that $\hat{\phi}'_k(t)$ is an estimate of $\phi'_k(t)$ and not the true value. Provided d is of the same order of magnitude as the estimation error, this reconstruction formula ensures an asymptotically perfect reconstruction. To compute an estimation of the ridges $(t, \phi'_k(t))$, knowing the number K of modes, we

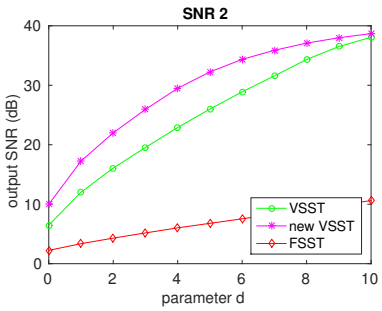
use the algorithm introduced in [3], which computes a local minimum of the functional:

$$E_f((\varphi_k)_{k=1,\cdots,K}) = \sum_{k=1}^{K} - \int_{\mathbb{R}} |T_f(t,\varphi_k(t))|^2 dt + \int_{\mathbb{R}} (\lambda \varphi_k'(t)^2 + \beta \varphi_k''(t)^2) dt, \quad (63)$$

λ and β being two positive tuning parameters. Then $(\hat{\phi}_k'(t))_{k=1,\cdots,K} = \operatorname{argmin} E_f((\varphi_k)_{k=1,\cdots,K})$.

	FSST	VSST	New VSST
Test-signal 2	2.3	7.6	**10.0**
mode 1	0.93	6	**7.8**
mode 2	1.4	7.2	**10.2**
mode 3	**56.6**	56.6	54.3

A



B

Figure 4: A: Mode retrieval is expressed in SNR for $d = 0$. The best results are displayed using a bold font, B: SNR associated with the reconstruction of test-signal 2 as a function of d

To illustrate the behaviour of our new VSST technique, we first display the reconstruction process associated with test-signal 2 on Fig. 3 A, the ridges used for reconstruction being depicted on Fig. 3 B. Then, to compare the proposed new VSST with the alternative technique proposed in [13], we display, on Fig. 3 C and D, the reconstruction error at each time instant for the three modes and for both VSST and new VSST. The improvement brought by using new VSST instead of VSST appears to be the most significant when the modulation is strong (i.e., in the mode associated with the highest frequency in the studied case). To further quantify the accuracy of mode reconstruction in terms of ouput SNR for test-signal 2, we display on Figure 4 A and for $d = 0$, the ouput SNR computed by using formula $||f||^2/||f - \hat{f}||^2$, where $||\hat{f}||$ is the reconstructed signal. As expected, new VSST behaves better than other synchrosqueezing techniques on that kind of signals. Finally, we study the quality of the reconstruction with respect to the parameter d used in the integration formula (62) and still for test-signal 2. The results depicted on Figure 4 B confirm that the sparser representation provided by new VSST naturally leads to a reconstruction procedure that

requires fewer coefficients than other synchrosqueezing techniques to reach the same quality of reconstruction.

6. Conclusion

In this paper, we developed a novel synchrosqueezing transform for analyzing multicomponent signals made of strongly frequency modulated modes, which appeared to be particularly efficient to denoise multicomponent signals. We slightly modified the IF estimate proposed in [13] so as to make the mathematical analysis of the corresponding synchrosqueezing transform possible. In this regard, we proved a novel approximation theorem involving the proposed new synchrosqueezing transform. Numerical experiments showed the benefits of taking into account phase modulation for better mode reconstruction. However, a limitation of our new synchrosqueezing transform is that its applicability is restricted to modes with almost constant amplitude. To improve this aspect is a work currently underway. Nevertheless, this novel transform can already be used without any restriction to denoise monocomponent signals as demonstrated by numerical experiments.

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