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Theoretical Analysis of the Second-order Synchrosqueezing Transform

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Abstract

In this paper, we present a theoretical analysis of the synchrosqueezing transform adapted to multicomponent signals made of strongly frequency modulated modes, which was recently proposed in the short time Fourier transform framework [13]. Before dealing in detail with the theoretical aspect, we explain throughout numerical simulations why the hypotheses made on the modes making up the multicomponent signal must be different when one considers wavelet or STFT based synchrosqueezing. After having recalled the main results regarding the original synchrosqueezing transform applied to multicomponent signals made of modes with weak frequency modulation, we prove the same kind of results for the novel synchrosqueezing transform adapted to strongly frequency modulated modes defined in [13].

Keywords: time-frequency analysis; synchrosqueezing transform; multicomponent signals.

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1. Introduction

Most signals from the physical world are modeled as a sum of amplitude- and frequency-modulated (AM-FM) waves, called multicomponent signals. Multicomponent signals are encountered in a number of fields of practical interest such as meteorology, structural stability analysis, and medical studies - see, e.g. [4, 5, 9, 10]. Time-frequency (TF) analysis techniques have been extensively used to analyze and process these signals [11], the method to be used depending on the nature of the modes making up the signal.

Standard TF methods such as the short-time Fourier transform (STFT) and continuous wavelet transform (CWT) are the most commonly used linear methods to analyze such signals. In that context, the reassignment method (RM) was developed to improve the readability of these TF linear representations. Unfortunately, since RM applies to the amplitude on the studied TF transform, the method results in a loss of information from the underlying transform [1]. This also means that the improved representation given by RM cannot be used to recover the original signal. Recently, Daubechies et al. [6] showed interesting results on the so-called synchrosqueezing transform (SST) to represent multicomponent signals, which was introduced in the mid-1990s for audio signal analysis [7]. SST combines the localization and sparsity properties of RM with the invertibility of a traditional TF representation. Originally proposed as a post-processing method applied to the CWT, SST can alternatively be applied to STFT with minor changes [12]. However, what essentially limits the applicability of SST is the hypothesis of weak frequency modulation for the modes.

To better take into account the frequency modulation, SST applied to STFT (FSST) has recently been extended to the case modes with strong frequency modulation [13, 2]. However, in that paper, a theoretical analysis of the proposed new synchrosqueezing transform was not provided. In the present paper, we introduce a slightly different transform to the one proposed in [13] that allows for mathematical analysis and, we extend the results of [6] to modes with strong frequency modulation.

The outline of the paper is as follows: in Section 2, we recall some notation and definitions. Then, in Section 3, we introduce FSST and wavelet based SST (WSST) along with the corresponding approximation results. Section 4 is devoted to the study of the behaviors of FSST and WSST on different types of modes. This study justifies why the use of FSST is better for some

type of multicomponent signals than WSST and also motivates a posteriori the different hypotheses made on the modes in the theoretical results of Section 3. Then, after having introduced some necessary ingredients, we derive in Section 5 the approximation theorem for the synchrosqueezing transform adapted to modes having strong frequency modulation. A numerical comparison with the estimate provided in [13] concludes the paper in Section 6.

2. Definitions

2.1. Short-time Fourier transform

We denote by $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ the space of integrable, and square integrable functions. Consider a signal $f \in L^1(\mathbb{R})$, and take a window g in the Schwartz class, $\mathcal{S}(\mathbb{R})$, the space of smooth functions with fast decaying derivatives of any order; its Fourier transform is defined by:

$$\hat{f}(\eta) = \int_{\mathbb{R}} f(\tau) e^{-2i\pi\eta\tau} d\tau. \quad (1)$$

The need for time-localized frequency descriptors leads to short-time Fourier transform (STFT) which is obtained through the use of a sliding window $g \in L^2(\mathbb{R})$ defined by:

$$V_f^g(\eta, t) = \int_{\mathbb{R}} f(\tau) g(\tau - t) e^{-2i\pi\eta(\tau - t)} d\tau = \mathbb{A}(\eta, t) e^{2i\pi\Phi(\eta, t)}. \quad (2)$$

The representation of $|V_f^g(\eta, t)|^2$ in the TF plane is called the spectrogram of f . The STFT admits the following synthesis formula (provided the window g does not vanish and is continuous at 0):

$$f(t) = \frac{1}{g(0)} \int_{\mathbb{R}} V_f^g(\eta, t) d\eta. \quad (3)$$

Further, if the signal is analytic (i.e. $\eta \leq 0 \Rightarrow \hat{f}(\eta) = 0$) then the integral domain for η is restricted to \mathbb{R}_+ .

2.2. Continuous wavelet transform

Let us consider an admissible wavelet $\psi \in L^2(\mathbb{R})$, satisfying $0 < C_\psi = \int_0^\infty |\widehat{\psi}(\xi)|^2 \frac{d\xi}{\xi} < \infty$ and then define for any time t and scale $a > 0$, the *continuous wavelet transform* (CWT) of f by:

$$W_f(a, t) = \frac{1}{a} \int_{\mathbb{R}} f(\tau) \psi\left(\frac{\tau - t}{a}\right)^* d\tau, \quad (4)$$

where z^* denotes the complex conjugate of z . Assuming that ψ is analytic, i.e. $\text{Supp}(\widehat{\psi}) \subset [0, \infty[$, the CWT admits the following synthesis formula (Morlet formula):

$$f(t) = \frac{1}{C'_\psi} \int_0^\infty W_f(a, t) \frac{da}{a}, \quad (5)$$

where $C'_\psi = \int_0^\infty \widehat{\psi}^*(\xi) \frac{d\xi}{\xi}$.

2.3. Multicomponent signal

In the present paper, we analyze so-called *multicomponent signals* of the form,

$$f(t) = \sum_{k=1}^K f_k(t), \quad \text{with } f_k(t) = A_k(t) e^{2i\pi\phi_k(t)}, \quad (6)$$

for some finite K , where $A_k(t)$ and $\phi_k(t)$ are time-varying amplitude and phase functions respectively such that $A_k(t) > 0$, $\phi'_k(t) > 0$ and $\phi'_{k+1}(t) > \phi'_k(t)$ for all t . In the following, $\phi'_k(t)$ is often called instantaneous frequency (IF) of mode k and $A_k(t)$ its instantaneous amplitude. One of the goal of TF analysis is to recover the instantaneous frequencies (IFs) $\{\phi'_k(t)\}_{1 \leq k \leq K}$ and amplitude $\{A_k(t)\}_{1 \leq k \leq K}$, from a given TF representation of f .

3. Synchrosqueezing transform

The synchrosqueezing transform (SST) is an approach originally introduced in the context of audio signal analysis in [7] and was recently developed in [6]. It belongs to the family of TF reassignment methods and corresponds to a nonlinear operator that sharpens the TF representation of a signal while enabling mode reconstruction. Moreover, SST combines the localization and

sparsity properties of TF reassignment with the invertibility property of traditional TF transforms, and is robust to a variety of signal perturbations [14]. The key ingredient to SST is an IF estimates computed from the TF representation which we introduce now.

3.1. Computation of IF Estimate in the TF plane

In the STFT framework, the IF of signal f at time t and frequency η can be estimated by [1]:

$$\begin{aligned}\hat{\omega}_f(\eta, t) &:= \frac{1}{2\pi} \partial_t \arg V_f^g(\eta, t), \\ &= \Phi'(\eta, t) \text{ defined in (2),} \\ &= \frac{1}{2\pi} \mathcal{I} \left(\frac{\partial_t V_f^g(\eta, t)}{V_f^g(\eta, t)} \right),\end{aligned}\tag{7}$$

where \mathcal{I} is the imaginary part of a complex number. The quantity

$$\tilde{\omega}_f(\eta, t) = \frac{\partial_t V_f^g(\eta, t)}{2i\pi V_f^g(\eta, t)},\tag{8}$$

is useful to more theoretical investigations but should not be used to estimate real quantities such as IF. Similar quantities can be derived in the CWT case as follows:

$$\begin{aligned}\hat{\omega}_f(a, t) &= \frac{1}{2\pi} \mathcal{I} \left(\frac{\partial_t W_f(a, t)}{W_f(a, t)} \right) = \mathcal{R}(\tilde{\omega}_f(a, t)), \\ \text{with } \tilde{\omega}_f(a, t) &= \frac{1}{2i\pi} \frac{\partial_t W_f(a, t)}{W_f(a, t)}.\end{aligned}\tag{9}$$

Remark 1. In [6], the estimate $\tilde{\omega}_f(a, t)$ is used instead of $\hat{\omega}_f(a, t)$. However most theorems are valid only for a real estimate as will be shown later.

3.2. Fourier based synchrosqueezing transform

Definition 1. Let $\epsilon > 0$ and $\Delta \in (0, 1)$. The set $\mathcal{B}_{\Delta, \epsilon}$ of multicomponent signals with modulation ϵ and separation Δ is the set of all multicomponent

signals defined in (6) satisfying:

$$\begin{aligned} A_k &\in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \phi_k \in C^2(\mathbb{R}), \\ \sup_{t \in \mathbb{R}} \phi'_k(t) &< \infty, \phi'_k(t) > 0, A_k(t) > 0, \quad \forall t \\ |A'_k(t)| &\leq \epsilon, |\phi''_k(t)| \leq \epsilon \quad \forall t \in \mathbb{R}. \end{aligned} \quad (10)$$

Further, the f_k s are separated with resolution Δ , i.e., for all $k \in \{1, \dots, K-1\}$ and all t ,

$$\phi'_{k+1}(t) - \phi'_k(t) \geq 2\Delta. \quad (11)$$

Definition 2. Let h be a L^1 normalized positive function belonging to $\mathbb{C}_c^\infty(\mathbb{R})$, the space of compactly supported smooth function, $\gamma, \delta > 0$. The STFT-based synchrosqueezing (FSST) of f with threshold γ and accuracy δ is defined by:

$$T_f^{\delta, \gamma}(\omega, t) = \frac{1}{g(0)} \int_{|V_f^g(\eta, t)| > \gamma} V_f^g(\eta, t) \frac{1}{\delta} h\left(\frac{\omega - \widehat{\omega}_f(\eta, t)}{\delta}\right) d\eta. \quad (12)$$

If we make δ and γ tend to zero, then

$$T_f(\omega, t) = \frac{1}{g(0)} \int_{\mathbb{R}} V_f^g(\eta, t) \delta(\omega - \widehat{\omega}_f(\eta, t)) d\eta, \quad (13)$$

which we call FSST in the sequel.

Theorem 1. Consider $f \in \mathcal{B}_{\Delta, \epsilon}$ and put $\tilde{\epsilon} = \epsilon^{1/3}$. Let $g \in \mathcal{S}(\mathbb{R})$, the Schwartz class, be such that $\text{supp}(\widehat{g}) \subset [-\Delta, \Delta]$, and h a L^1 normalized positive function belonging to $\mathbb{C}_c^\infty(\mathbb{R})$. Then, if ϵ is small enough, the following holds:

(a) $|V_f^g(\eta, t)| > \tilde{\epsilon}$ only when there exists $k \in \{1, \dots, K\}$ such that $(\eta, t) \in Z_k := \{(\eta, t), \text{ s.t. } |\eta - \phi'_k(t)| < \Delta\}$.

(b) For all $k \in \{1, \dots, K\}$ and all $(\eta, t) \in Z_k$ such that $|V_f^g(\eta, t)| > \tilde{\epsilon}$, we have

$$|\widehat{\omega}_f(\eta, t) - \phi'_k(t)| \leq \tilde{\epsilon}. \quad (14)$$

(c) For all $k \in \{1, \dots, K\}$, there exists a constant C such that for all $t \in \mathbb{R}$,

$$\left| \lim_{\delta \rightarrow 0} \left(\int_{|\omega - \phi'_k(t)| < \tilde{\epsilon}} T_f^{\delta, \tilde{\epsilon}}(\omega, t) d\omega \right) - f_k(t) \right| \leq C\tilde{\epsilon}. \quad (15)$$

Proof. This theorem gives a strong approximation result for the class $\mathcal{B}_{\Delta, \epsilon}$. Since it ensures that the non-zero coefficients of the FSST are localized around the curves $(t, \phi'_k(t))$, and that the construction of the modes is easily obtained from the concentrated representation, for the detailed proof we refer to [12], but we recall the main ingredients which will help us in our investigations.

(a) For any $(\eta, t) \in Z_k$, one can write:

$$|V_f^g(\eta, t) - \sum_{l=1}^K f_l(t) \hat{g}(\eta - \phi'_l(t))| \leq \epsilon \Gamma_1(t), \quad (16)$$

where $\Gamma_1(t) = KI_1 + \pi I_2 \sum_{l=1}^K A_l(t)$, and $I_n = \int_{\mathbb{R}} |x|^n |g(x)| dx$.

For any ϵ sufficiently small and since \hat{g} is compactly supported in $[-\Delta, \Delta]$, for $(\eta, t) \notin \bigcup_{k=1}^K Z_k$, one can write:

$$|V_f^g(\eta, t)| \leq \epsilon \Gamma_1(t) \leq \tilde{\epsilon}. \quad (17)$$

(b) Since \hat{g} is compactly supported and because of (16), one has when $(\eta, t) \in Z_k$

$$|V_f^g(\eta, t) - f_k(t) \hat{g}(\eta - \phi'_k(t))| \leq \epsilon \Gamma_1(t). \quad (18)$$

Further, for any $(\eta, t) \in Z_k$, differentiating $V_f^g(\eta, t)$ with respect to t , one gets,

$$\partial_t V_f^g(\eta, t) = 2i\pi\eta V_f^g(\eta, t) - V_f^{g'}(\eta, t). \quad (19)$$

Since

$$|\hat{\omega}(\eta, f) - \phi'_k(t)| \leq \left| \frac{\partial_t V_f^g(\eta, t) - 2i\pi\phi'_k(t) f_k(t) \hat{g}(\eta - \phi'_k(t))}{2i\pi V_f^g(\eta, t)} \right| + \left| \frac{\phi'_k(t) (f_k(t) \hat{g}(\eta - \phi'_k(t)) - V_f^g(\eta, t))}{V_f^g(\eta, t)} \right|.$$

Using (18) and (19), the first part of the above inequality reads:

$$\begin{aligned}
& \left| \partial_t V_f^g(\eta, t) - 2i\pi f_k(t) \phi'_k(t) \hat{g}(\eta - \phi'_k(t)) \right|, \\
&= \left| 2i\pi \eta V_f^g(\eta, t) - V_f^{g'}(\eta, t) - 2i\pi f_k(t) \phi'_k(t) \hat{g}(\eta - \phi'_k(t)) \right|, \\
&\leq \left| V_f^{g'}(\eta, t) - \sum_{l=1}^K f_l(t) \hat{g}'(\eta - \phi'_l(t)) \right| \\
&\quad + 2\pi \left| \sum_{l=1}^K f_l(t) (\eta - \phi'_l(t)) \hat{g}(\eta - \phi'_l(t)) - \eta V_f^g(\eta, t) + f_l(t) \phi'_l(t) \hat{g}(\eta - \phi'_l(t)) \right|, \\
&\leq \epsilon (\Gamma_2(t) + 2\pi |\eta| \Gamma_1(t)), \\
&\leq \epsilon (\Gamma_2(t) + 2\pi (\Delta + \phi'_k(t)) \Gamma_1(t)), \tag{20}
\end{aligned}$$

where $\Gamma_2(t) = KI'_1 + \pi I'_2 \sum_{l=1}^K A_l(t)$ and $I'_n = \int_{\mathbb{R}} |t|^n |g'(t)| dt$.

Hence, if ϵ is sufficiently small, one obtains

$$|\hat{\omega}(\eta, f) - \phi'_k(t)| \leq \tilde{\epsilon}^2 \left(\frac{\Gamma_2(t)}{2\pi} + (2\phi'_k(t) + \Delta) \Gamma_1(t) \right) \leq \tilde{\epsilon}.$$

(c) Since

$$\begin{aligned}
& \left| \lim_{\delta \rightarrow 0} \int_{|\omega - \phi'_k(t)| < \tilde{\epsilon}} T_f^{\delta, \tilde{\epsilon}}(\omega, t) d\omega - f_k(t) \right| \\
&= \left| \lim_{\delta \rightarrow 0} \int_{|\omega - \phi'_k(t)| < \tilde{\epsilon}} \frac{1}{g(0)} \int_{|V_f^g(\eta, t)| > \tilde{\epsilon}} V_f^g(\eta, t) \frac{1}{\delta} h\left(\frac{\omega - \hat{\omega}_f(\eta, t)}{\delta}\right) d\eta d\omega - f_k(t) \right|, \\
&= \left| \lim_{\delta \rightarrow 0} \int_{|V_f^g(\eta, t)| > \tilde{\epsilon}} \frac{1}{g(0)} V_f^g(\eta, t) \int_{|\omega - \phi'_k(t)| < \tilde{\epsilon}} \frac{1}{\delta} h\left(\frac{\omega - \hat{\omega}_f(\eta, t)}{\delta}\right) d\omega d\eta - f_k(t) \right|, \\
&= \left| \frac{1}{g(0)} \int_{|V_f^g(\eta, t)| > \tilde{\epsilon}} V_f^g(\eta, t) \lim_{\delta \rightarrow 0} \int_{|\omega - \phi'_k(t)| < \tilde{\epsilon}} \frac{1}{\delta} h\left(\frac{\omega - \hat{\omega}_f(\eta, t)}{\delta}\right) d\omega d\eta - f_k(t) \right|, \\
&= \left| \frac{1}{g(0)} \int_{\{|V_f^g(\eta, t)| > \tilde{\epsilon}\} \cap \{|\hat{\omega}_f(\eta, t) - \phi'_k(t)| < \tilde{\epsilon}\}} V_f^g(\eta, t) d\eta - f_k(t) \right|, \\
&= \left| \frac{1}{g(0)} \int_{\{|V_f^g(\eta, t)| > \tilde{\epsilon}\} \cap \{|\eta - \phi'_k(t)| < \Delta\}} V_f^g(\eta, t) d\eta - f_k(t) \right|,
\end{aligned}$$

$$\begin{aligned}
&= \left| \frac{1}{g(0)} \int_{\{|\eta - \phi'_k(t)| < \Delta\}} V_f^g(\eta, t) d\eta - f_k(t) - \frac{1}{g(0)} \int_{\{|\eta - \phi'_k(t)| < \Delta\} \cap \{|V_f^g(\eta, t)| \leq \tilde{\epsilon}\}} V_f^g(\eta, t) d\eta \right|, \\
&\leq \left| \frac{1}{g(0)} \int_{\{|\eta - \phi'_k(t)| < \Delta\}} V_f^g(\eta, t) d\eta - f_k(t) \right| + \frac{2\Delta}{g(0)} \tilde{\epsilon}, \\
&\leq \left| \frac{1}{g(0)} \int_{\{|\eta - \phi'_k(t)| < \Delta\}} f_k(t) \hat{g}(\eta - \phi'_k(t)) d\eta - f_k(t) \right| \\
&\quad + \frac{1}{g(0)} \int_{\{|\eta - \phi'_k(t)| < \Delta\}} |V_f^g(\eta, t) - f_k(t) \hat{g}(\eta - \phi'_k(t))| d\eta + \frac{2\Delta}{g(0)} \tilde{\epsilon}, \\
&\leq 0 + \frac{2\Delta}{g(0)} \epsilon \Gamma_1(t) + \frac{2\Delta}{g(0)} \tilde{\epsilon}, \\
&= \left| \lim_{\delta \rightarrow 0} \int_{|\omega - \phi'_k(t)| < \tilde{\epsilon}} T_f^{\delta, \tilde{\epsilon}}(\omega, t) d\omega - f_k(t) \right| \leq \frac{4\Delta}{g(0)} \tilde{\epsilon},
\end{aligned}$$

because when ϵ is small enough, $\Gamma_1(t) \leq \tilde{\epsilon}^{-2}$.

□

Remark 2. For the above proof to be valid, it is necessary to have a real IF estimate, therefore $\tilde{\omega}_f(\eta, t)$ cannot be used.

Remark 3. Note also that the real estimate $\hat{\omega}_f(\eta, t)$ is always better than $\tilde{\omega}_f(\eta, t)$ since we trivially have:

$$|\hat{\omega}_f(\eta, t) - \phi'_k(t)| \leq |\tilde{\omega}_f(\eta, t) - \phi'_k(t)|. \quad (21)$$

Definition 3. A window $g_\sigma(t) = \sigma^{-1} g(\frac{t}{\sigma})$ is said to be with quadratic decay if $\hat{g}(\eta)$ behaves like $\frac{1}{\gamma\eta^2}$ for any $|\eta| \geq \Delta$.

Remark 4. If \hat{g} is not compactly supported but $g_\sigma(t)$ is defined as in Definition 3, we have the following proposition:

Proposition 1. Let us consider a window with quadratic decay defined in Definition 3. Then if $|\eta| \geq \Delta$, $\hat{g}_\sigma(\eta) \leq \tilde{\epsilon}^2$ if and only if $\sigma \tilde{\epsilon} \geq \frac{1}{\sqrt{|\gamma|\Delta}}$.

The proof is straightforward and is left to the reader.

Remark 5. To prove (a) of Theorem 1, assuming the STFT is computed with g_σ , a window with quadratic decay, we consider $\tilde{\epsilon} = \epsilon^{\frac{1}{4}}$, and then remark that

$$|V_f^{g_\sigma}(\eta, t) - \sum_{l=1}^K f_l(t) \hat{g}_\sigma(\eta - \phi'_l(t))| \leq \epsilon \Gamma_{1,\sigma}(t),$$

is valid for any (η, t) , with $\Gamma_{1,\sigma}(t) = K\sigma I_1 + \pi\sigma^2 I_2 \sum_{l=1}^K A_l(t)$, I_1 and I_2 corresponding to the moments of g_σ computed with $\sigma = 1$. When $(\eta, t) \notin \bigcup_{k=1}^K Z_k$, and if σ satisfies the conditions in Proposition 1, we then have:

$$\begin{aligned} |V_f^{g_\sigma}(\eta, t)| &\leq \left| \sum_{l=1}^K f_l(t) \hat{g}_\sigma(\eta - \phi'_l(t)) \right| + \epsilon \Gamma_{1,\sigma}(t), \\ &\leq \tilde{\epsilon}^2 \left(\sum_{l=1}^K A_l(t) + \tilde{\epsilon}^2 \Gamma_{1,\sigma}(t) \right). \end{aligned}$$

Now, due to the expression of $\Gamma_{1,\sigma}(t)$, $\tilde{\epsilon}^2 \Gamma_{1,\sigma}(t)$ is bounded if $\tilde{\epsilon}\sigma \leq C$, for some constant C . So, we get the following constraint on σ : $\frac{1}{\sqrt{|\eta|\Delta}} \leq \sigma \tilde{\epsilon} \leq C$. Now, if $\tilde{\epsilon}$ is chosen sufficiently small we get point (a).

To prove (b) of Theorem 1, when \hat{g} was supposed to be compactly supported, one used inequalities (18) and (20). Now, in the case a window \hat{g}_σ with quadratic decay, we only have:

$$\begin{aligned} |V_f^{g_\sigma}(\eta, t) - \sum_{l=1}^K f_l(t) \hat{g}_\sigma(\eta - \phi'_l(t))| &\leq \epsilon \Gamma_{1,\sigma}(t), \\ |\partial_t V_f^{g_\sigma}(\eta, t) - 2i\pi \sum_{l=1}^K f_l(t) \phi'_l(t) \hat{g}_\sigma(\eta - \phi'_l(t))| &\leq \epsilon(\Gamma_{2,\sigma}(t) + 2\pi|\eta|\Gamma_{1,\sigma}(t)), \end{aligned}$$

where $\Gamma_{2,\sigma}(t) = KI'_1 + \pi\sigma I'_2 \sum_{l=1}^K A_l(t)$, again I'_1 and I'_2 being computed with

$\sigma = 1$. From Proposition 1, we fortunately get

$$\begin{aligned} |V_f^{g\sigma}(\eta, t) - f_k(t)\widehat{g}_\sigma(\eta - \phi'_k(t))| &\leq \tilde{\epsilon}^2 \left(\sum_{l \neq k} A_l(t) + \tilde{\epsilon}^2 \Gamma_{1,\sigma}(t) \right), \\ |\partial_t V_f^{g\sigma}(\eta, t) - 2i\pi f_k(t)\phi'_k(t)\widehat{g}_\sigma(\eta - \phi'_k(t))| \\ &\leq \tilde{\epsilon}^2 \left(2\pi \sum_{l \neq k} \phi'_l(t)A_k(t) + \tilde{\epsilon}^2 \Gamma_{2,\sigma}(t) + 2\pi|\eta|\tilde{\epsilon}^2 \Gamma_{1,\sigma}(t) \right). \end{aligned}$$

Hence, if $\tilde{\epsilon}$ is small enough, the terms to the right can be made lower than $\tilde{\epsilon}$. So (b) is still valid in that case.

Regarding point (c) of Theorem 1, the only difference when one considers g_σ with quadratic decay lies in the fact we no longer have:

$$\frac{1}{g(0)} \int_{|\eta - \phi'_k(t)| < \Delta} \widehat{g}(\eta - \phi'_k(t)) d\eta = \frac{1}{g(0)} \int_{|\eta| < \Delta} \widehat{g}(\eta) d\eta = 1.$$

To obtain (c), we need to have the window g_σ satisfying:

$$\frac{1}{g_\sigma(0)} \int_{|\eta| \geq \Delta} \widehat{g}_\sigma(\eta) d\eta = \frac{\sigma}{g(0)} \int_{|\eta| \geq \Delta} \widehat{g}(\sigma\eta) d\eta = \frac{1}{g(0)} \int_{|\eta| \geq \sigma\Delta} \widehat{g}(\eta) d\eta \leq \tilde{\epsilon}.$$

Due to the decay of $\widehat{g}(\eta)$, which behaves like $\frac{1}{\gamma\eta^2}$ when $\eta \geq \Delta$, we get by integration that:

$$\left| \frac{1}{g(0)} \int_{|\eta| \geq \sigma\Delta} \widehat{g}(\eta) d\eta \right| \sim \frac{2}{|\gamma|\sigma\Delta} \leq \frac{2}{\sqrt{|\gamma|}} \tilde{\epsilon}.$$

So property (c) is also valid.

Remark 6. It is interesting to remark here that $f_k(t)\widehat{g}_\sigma(\eta - \phi'_k(t)) = V_{f_k^1}^{g\sigma}(\eta, t)$ with $f_k^1(\tau) = A_k e^{2i\pi(\phi_k(t) + (\tau-t)\phi'_k(t))}$. Then, the right hand side of equation (18) and (20) can be rewritten as $|V_f^{g\sigma}(\eta, t) - V_{f_k^1}^{g\sigma}(\eta, t)|$ and $|\partial_t V_f^{g\sigma}(\eta, t) - \partial_t V_{f_k^1}^{g\sigma}(\eta, t)|$ so that the whole theorem is based on the STFT of a constant amplitude and first order phase approximation of the k th mode.

3.3. Wavelet based synchrosqueezing transform

We now introduce the synchrosqueezing transform in the wavelet framework.

Definition 4. Let $\epsilon > 0$ and $d > 0$. The set $\mathcal{A}_{d,\epsilon}$ of multicomponent signals with modulation ϵ and separation d corresponds to signals defined in (6) with f satisfying:

$$\begin{aligned} A_k &\in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \phi \in C^2(\mathbb{R}), \\ \inf_{t \in \mathbb{R}} \phi'_k(t) &> 0, \sup_{t \in \mathbb{R}} \phi'_k(t) < \infty, A_k(t) > 0, \\ |A'_k(t)| &\leq \epsilon |\phi'_k(t)|, |\phi''_k(t)| \leq \epsilon |\phi'_k(t)| \quad \forall t \in \mathbb{R}. \end{aligned} \quad (22)$$

Further, the f_k s are separated with resolution d , i.e., for all $k \in \{1, \dots, K-1\}$ and all t

$$|\phi'_{k+1}(t) - \phi'_k(t)| \geq 2d(\phi'_{k+1}(t) + \phi'_k(t)). \quad (23)$$

Definition 5. Let h be a positive L^1 normalized window belonging to $\mathbb{C}_c^\infty(\mathbb{R})$, and consider $\gamma, \delta > 0$, the wavelet-based synchrosqueezing (WSST) of f with threshold γ and accuracy δ is defined by:

$$S_f^{\delta,\gamma}(\omega, t) := \int_{|W_f(a,t)| > \gamma} W_f(a,t) \frac{1}{\delta} h\left(\frac{\omega - \hat{\omega}_f(a,t)}{\delta}\right) \frac{da}{a}. \quad (24)$$

Theorem 2. Let $f \in \mathcal{A}_{d,\epsilon}$, and set $\tilde{\epsilon} = \epsilon^{\frac{1}{3}}$. Let h be a positive L^1 normalized window belonging to $\mathbb{C}_c^\infty(\mathbb{R})$, a wavelet $\psi \in \mathcal{S}(\mathbb{R})$ such that $\hat{\psi}$ is supported in $[1 - \Delta, 1 + \Delta]$, with $\Delta < d/(1+d)$ and set $\mathcal{R}_\psi = \sqrt{2\pi} \int \hat{\psi}(\zeta) \zeta^{-1} d\zeta$. Then, provided ϵ is sufficiently small, the following hold:

- (a) $|W_f(a,t)| \geq \tilde{\epsilon}$ only when, there exists $k \in \{1, \dots, K\}$, such that for each pair $(a,t) \in Z_k := \{(a,t), \text{ s.t. } |a\phi'_k(t) - 1| < \Delta\}$.
- (b) For each $k \in \{1, \dots, K\}$ and then for all $(a,t) \in Z_k$ for which holds $|W_f(a,t)| > \tilde{\epsilon}$, we have

$$|\hat{\omega}_f(a,t) - \phi'_k(t)| \leq \tilde{\epsilon}. \quad (25)$$

- (c) Moreover, for each $k \in \{1, \dots, K\}$, there exists a constant C such that for any $t \in \mathbb{R}$

$$\left| \lim_{\delta \rightarrow 0} \left(\mathcal{R}_\psi^{-1} \int_{|\omega - \phi'_k(t)| < \tilde{\epsilon}} S_f^{\delta,\tilde{\epsilon}}(\omega, t) d\omega \right) - A_k(t) e^{i\phi_k(t)} \right| \leq C\tilde{\epsilon}. \quad (26)$$

Proof. This theorem is proved in [6] and gives a strong approximation result for the class $\mathcal{A}_{d,\epsilon}$, since it ensures that the non-zero coefficients of the WSST are localized around the curves $(t, \frac{1}{\phi'_k(t)})$ in the time-scale space, and that the reconstruction of the modes is easily obtained from the concentrated representation. \square

Remark 7. It is worth noting here that similarly to what was said for the FSST in equation (21), the real estimate $\hat{\omega}_f(a, t)$ is always better than the complex one, i.e. $\tilde{\omega}_f(a, t)$.

4. Comparison between WSST and FSST

In the above section, we defined WSST and FSST, each of them behaving differently depending on the type of multicomponent signals they are applied to. We now investigate which kind of signals are better dealt with FSST or WSST. Pure waves obviously satisfy the assumptions of Theorems 1 and 2 for any ϵ . However, in these theorems, we made a series of hypotheses which suggest that the two transforms behave differently when the modes depart from pure waves.

4.1. FSST and WSST on a linear chirp

To illustrate the differences between the two transforms, we consider the SST of a linear chirp, i.e., a wave with linear instantaneous frequency. In practice, the STFT computed with a L^1 normalized Gaussian window with variance σ^2 , i.e. $g_\sigma(t) = \sigma^{-1}e^{-\pi\frac{t^2}{\sigma^2}}$ is often used even though the Fourier transform of the window is not compactly supported. To choose such a window is interesting for our purpose, because one obtains closed form expression for the STFT of a linear chirp. Furthermore, when one studies a single mode, the constraint on the decay of \hat{g} is not as important as in the multicomponent case. To analyze a linear chirp with WSST, we will make use of the so-called Morlet wavelet defined as follows: $\psi_\sigma(t) = \sigma^{-1}e^{-\pi\frac{t^2}{\sigma^2}}e^{2i\pi t}$.

To deal with the FSST case, we need the following two propositions:

Proposition 2. *Let $V_f^{g_\sigma}$ be the STFT of f computed using the L^1 normalized Gaussian window g with variance σ^2 . Then $\hat{\omega}_f(\eta, t)$ can be expressed as:*

$$\hat{\omega}_f(\eta, t) = \eta + \frac{1}{2\pi\sigma^2} \frac{\partial_\eta |V_f^{g_\sigma}(\eta, t)|}{|V_f^{g_\sigma}(\eta, t)|}. \quad (27)$$

Proof. We may write $\widehat{\omega}_f(\eta, t)$, when g_σ is a L^1 normalized Gaussian window, as

$$\begin{aligned}\widehat{\omega}_f(\eta, t) &= \frac{1}{2\pi} \mathcal{I} \left(\frac{\partial_t V_f^{g_\sigma}(\eta, t)}{V_f^{g_\sigma}(\eta, t)} \right), \\ &= \frac{1}{2\pi} \mathcal{I} \left(\frac{2i\pi\eta V_f^{g_\sigma}(\eta, t) - V_f^{g_\sigma'}(\eta, t)}{V_f^{g_\sigma}(\eta, t)} \right), \\ &= \eta + \frac{1}{\sigma^2} \mathcal{I} \left(\frac{V_f^{tg_\sigma}(\eta, t)}{V_f^{g_\sigma}(\eta, t)} \right).\end{aligned}$$

Now we can write for any window $g \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned}\frac{\partial_\eta |V_f^{g_\sigma}(\eta, t)|}{|V_f^{g_\sigma}(\eta, t)|} &= \frac{\partial_\eta (V_f^{g_\sigma}(\eta, t) e^{-2i\pi\Phi(\eta, t)})}{V_f^{g_\sigma}(\eta, t) e^{-2i\pi\Phi(\eta, t)}} = 2i\pi \partial_\eta \Phi(\eta, t) + \frac{\partial_\eta V_f^{g_\sigma}(\eta, t)}{V_f^{g_\sigma}(\eta, t)}, \\ &= 2i\pi \partial_\eta \Phi(\eta, t) - 2i\pi \frac{V_f^{tg_\sigma}(\eta, t)}{V_f^{g_\sigma}(\eta, t)} = 2\pi \mathcal{I} \left(\frac{V_f^{tg_\sigma}(\eta, t)}{V_f^{g_\sigma}(\eta, t)} \right).\end{aligned}$$

Hence

$$\widehat{\omega}_f(\eta, t) = \eta + \frac{1}{2\pi\sigma^2} \frac{\partial_\eta |V_f^{g_\sigma}(\eta, t)|}{|V_f^{g_\sigma}(\eta, t)|}.$$

□

Let us now introduce the linear chirp $h_c(\tau) = Ae^{2i\pi\phi(\tau)}$. To start with, we remark that $h_c(\tau)$ can be written in the following form:

$$h_c(\tau) = h_c(t) e^{2i\pi[\phi'(t)(\tau-t) + \frac{1}{2}\phi''(t)(\tau-t)^2]}. \quad (28)$$

First, we seek a simple expression of the modulus of the STFT of h_c , for which we need the following lemma:

Lemma 1. *Consider the function $u(t) = e^{-\pi z t^2}$, where $z = re^{i\theta}$ with $\cos \theta > 0$, so that u is integrable. Then its Fourier transform reads*

$$\widehat{u}(\xi) = r^{-\frac{1}{2}} e^{-i\frac{\theta}{2}} e^{-\frac{\pi}{re^{i\theta}} \xi^2}. \quad (29)$$

Proof. This result is straightforward as one can proceed as in the case z real. □

From this, we get the second proposition:

Proposition 3. *The magnitude of the STFT of h_c , computed using the L^1 normalized Gaussian window g_σ with variance σ^2 , admits the following closed-form expression:*

$$|V_{h_c}^{g_\sigma}(\eta, t)| = A(1 + \sigma^4 \phi''(t)^2)^{-\frac{1}{4}} e^{-\frac{\pi \sigma^2 (\eta - \phi'(t))^2}{1 + \sigma^4 \phi''(t)^2}}. \quad (30)$$

This shows that the magnitude of the STFT of a linear chirp computed with a Gaussian window is also a Gaussian function centered at $\eta = \phi'(t)$.

Proof. First, we remark that

$$V_{h_c}^{g_\sigma}(\eta, t) = h_c(t) \sigma^{-1} \widehat{e^{-\pi(\frac{1}{\sigma^2} - i\phi''(t))\tau^2}}(\eta - \phi'(t)) = h_c(t) \sigma^{-1} \widehat{e^{-\pi z \tau^2}}(\eta - \phi'(t)),$$

where $z = \frac{1}{\sigma^2} - i\phi''(t) = r e^{i\theta}$. Lemma 1 then gives

$$V_{h_c}^{g_\sigma}(\eta, t) = h_c(t) \sigma^{-1} z^{-\frac{1}{2}} e^{-\pi \frac{(\eta - \phi'(t))^2}{z}} = h_c(t) \sigma^{-1} z^{-\frac{1}{2}} e^{-\pi \frac{(\eta - \phi'(t))^2}{r}} e^{-i\theta}, \quad (31)$$

with $r = (\frac{1}{\sigma^4} + \phi''(t)^2)^{\frac{1}{2}}$ and $\theta = \text{atan}(-\phi''(t)\sigma^2)$. Using the identity $\cos(\text{atan}(x)) = \frac{1}{\sqrt{1+x^2}}$, one then gets the result. \square

Using this proposition, we can assess the quality of the IF estimate given by $\widehat{\omega}_{h_c}(\eta, t)$ which conditions the quality of FSST. Indeed, we have the following result:

Theorem 3. *Let h_c be a linear chirp and assume g_σ is the L^1 normalized Gaussian window with variance σ^2 . If $|\eta - \phi'(t)| < \epsilon$ then*

$$|\widehat{\omega}_{h_c}(\eta, t) - \phi'(t)| \leq \epsilon \left| 1 - \frac{1}{1 + \sigma^4 \phi''(t)^2} \right|. \quad (32)$$

Proof. Since

$$|V_{h_c}^{g_\sigma}(\eta, t)| = A(1 + \sigma^4 \phi''(t)^2)^{-\frac{1}{4}} e^{-\frac{\pi \sigma^2 (\eta - \phi'(t))^2}{1 + \sigma^4 \phi''(t)^2}},$$

equation (27) becomes

$$\begin{aligned}\widehat{\omega}_{h_c}(\eta, t) &= \eta + \frac{1}{2\pi\sigma^2} \left(\frac{-2\pi\sigma^2(\eta - \phi'(t))}{1 + \sigma^4\phi''(t)^2} \right), \\ \Leftrightarrow \widehat{\omega}_{h_c}(\eta, t) - \phi'(t) &= (\eta - \phi'(t)) - \frac{\eta - \phi'(t)}{1 + \sigma^4\phi''(t)^2}, \\ \Rightarrow |\widehat{\omega}_{h_c}(\eta, t) - \phi'(t)| &= |\eta - \phi'(t)| \left| 1 - \frac{1}{1 + \sigma^4\phi''(t)^2} \right|.\end{aligned}$$

If $|\eta - \phi'(t)| < \epsilon$ then

$$|\widehat{\omega}_{h_c}(\eta, t) - \phi'(t)| \leq \epsilon \left| 1 - \frac{1}{1 + \sigma^4\phi''(t)^2} \right|.$$

□

Remark 8. Here we see that the quality of the estimate $\widehat{\omega}_{h_c}(\eta, t)$ close to the curve $(t, \phi'(t))$ only depends on the magnitude of $\phi''(t)$: if the latter goes to zero, the estimate tends to $\phi'(t)$. This justifies a posteriori, why in Theorem 1 we assume the modes satisfy $\phi''(t) \leq \epsilon$: what matters is how small the modulation is, $\phi''(t)$ playing no role.

Similarly to what we have been doing for FSST, we study the WSST applied to the linear chirp which is tied to the quality of the estimate $\widehat{\omega}_{h_c}(a, t)$, for which we have the following result:

Theorem 4. *Let h_c be a linear chirp and assume the wavelet is the Morlet wavelet. If $|\frac{1}{a} - \phi'(t)| \leq \epsilon$ then*

$$|\widehat{\omega}_{h_c}(a, t) - \phi'(t)| \leq \epsilon \left| 1 - \frac{1}{1 + \sigma^4 a^4 \phi''(t)^2} \right|. \quad (33)$$

Proof. In such a case, we have:

$$\begin{aligned}
W_{h_c}(a, t) &= \int_{\mathbb{R}} A e^{2i\pi\phi(\tau)} \sigma^{-1} e^{-\frac{\pi}{\sigma^2} \left(\frac{\tau-t}{a}\right)^2} e^{-2i\pi\left(\frac{\tau-t}{a}\right)} d\tau, \\
&= h_c(t) a \int_{\mathbb{R}} e^{2i\pi \left[a\phi'(t)u + \frac{a^2}{2}\phi''(t)u^2 \right]} \sigma^{-1} e^{-\frac{\pi}{\sigma^2}u^2} e^{-2i\pi u} du, \quad \text{where } u = \frac{\tau-t}{a}, \\
&= h_c(t) a \int_{\mathbb{R}} e^{2i\pi \left[a\phi'(t)-1 \right]u} \sigma^{-1} e^{-\pi \left[\frac{1}{\sigma^2} - ia^2\phi''(t) \right]u^2} d\tau, \\
&= h_c(t) a \sigma^{-1} e^{-\pi \left[\frac{1}{\sigma^2} - ia^2\phi''(t) \right]u^2} (1 - a\phi'(t)), \\
&= h_c(t) a \sigma^{-1} \left(\frac{1}{\sigma^2} - ia^2\phi''(t) \right)^{-\frac{1}{2}} e^{\left[\frac{-\pi\sigma^2(1-a\phi'(t))^2}{1-i\sigma^2a^2\phi''(t)} \right]}.
\end{aligned}$$

With that expression, one can compute the estimate, bearing in mind that $\phi''(t)$ is constant:

$$\begin{aligned}
\widehat{\omega}_{h_c}(a, t) &= \frac{1}{2\pi} \mathcal{I} \left[\frac{\partial_t W_{h_c}(a, t)}{W_{h_c}(a, t)} \right], \\
&= \frac{1}{2\pi} \mathcal{I} \left[\frac{h_c'(t)}{h_c(t)} + \frac{2\pi\sigma^2 a\phi''(t)(1-a\phi'(t))}{1-i\sigma^2a^2\phi''(t)} \right], \\
&= \frac{1}{2\pi} \mathcal{I} \left[2i\pi\phi'(t) \right] + \left(\frac{\sigma^2 a\phi''(t)(1-a\phi'(t))}{1+\sigma^4 a^4\phi''(t)^2} (\sigma^2 a^2\phi''(t)) \right). \\
\Rightarrow |\widehat{\omega}_{h_c}(a, t) - \phi'(t)| &= \left| \frac{\sigma^4 a^3\phi''(t)^2(1-a\phi'(t))}{1+\sigma^4 a^4\phi''(t)^2} \right| = \left| \frac{\sigma^4 a^4\phi''(t)^2 \left(\frac{1}{a} - \phi'(t)\right)}{1+\sigma^4 a^4\phi''(t)^2} \right|.
\end{aligned}$$

If $\left| \frac{1}{a} - \phi'(t) \right| \leq \epsilon$ then

$$\begin{aligned}
|\widehat{\omega}_{h_c}(a, t) - \phi'(t)| &\leq \epsilon \left| \frac{\sigma^4 a^4\phi''(t)^2}{1+\sigma^4 a^4\phi''(t)^2} \right|, \\
&\leq \epsilon \left| 1 - \frac{1}{1+\sigma^4 a^4\phi''(t)^2} \right|.
\end{aligned}$$

□

The quality of the estimate is thus dependent on the scale a , a phenomenon that did not occur with the estimate based on STFT (the quality

of the estimate was then independent from the frequency). To illustrate this, we display on Figure 1 both transforms for a linear chirp with phase $\phi(t) = 10t + 100t^2$. It is clear that the quality of the representation given by the FSST remains constant with time. For the WSST however, the representation is very concentrated around $t = 1$, but it is of poor quality for low t . This is in accordance with the results of Theorems 3 and 4.

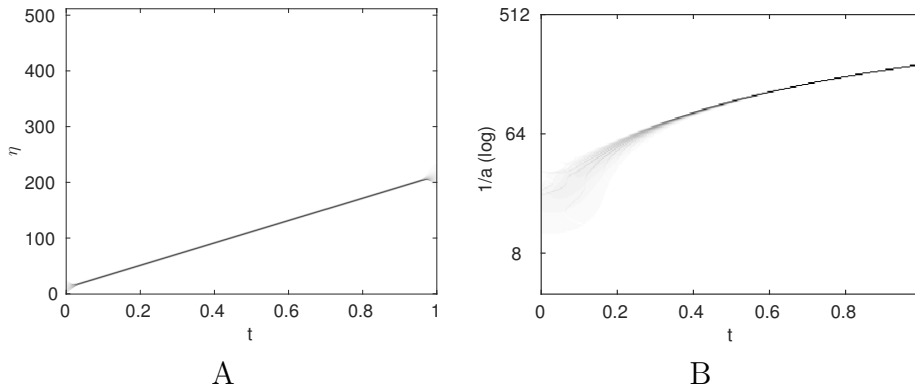


Figure 1: Comparison of the methods on a linear chirp. A: FSST and B WSST

4.2. WSST and FSST of an hyperbolic chirp

In this section, we show that WSST is well behaved when applied to an hyperbolic chirp. Numerical simulations, at the end of the section, will show that it is, on the contrary, not the case with FSST.

Theorem 5. *Let f be an hyperbolic chirp defined by $f(t) = (-it)^{2i\pi\alpha} = e^{\pi^2\alpha} e^{2i\pi\alpha \log(t)} = e^{\pi^2\alpha} e^{2i\pi\alpha \log(t)}$, with $\alpha < 1$, and consider the estimate $\widehat{\omega}_f(a, t)$ computed using the Cauchy wavelet with parameter 1. Then, if $|\frac{1}{a} - \phi'(t)| \leq \epsilon$, we have*

$$|\widehat{\omega}_f(a, t) - \phi'(t)| \leq \left| \frac{\epsilon}{1+\alpha^2} \right| + O(\epsilon^2), \quad (34)$$

where $O(x)$ is the Landau notation (meaning the quantity is of the order of x).

Proof. First, let us recall the definition of the Cauchy wavelet

$$g_{2\pi\beta}(t) = \frac{1}{2\pi} \Gamma(1 + 2\pi\beta) (1 - it)^{-(1+2\pi\beta)}, \quad (35)$$

which is analytic, and admits the following Fourier transform $\hat{g}_{2\pi\beta}(\eta) = \eta^{2\pi\beta} e^{-\eta} H(\eta)$ with H the Heaviside function. So, even if $\hat{g}_{2\pi\beta}$ is not compactly supported, it has a fast decay. Now, we can write for $\beta > \alpha$ [8],

$$\begin{aligned} W_f(a, t) &= \frac{1}{a} \int_{\mathbb{R}} (-i\tau)^{2i\pi\alpha} g_{\beta}^* \left(\frac{\tau - t}{a} \right) d\tau, \\ &= ca^{2\pi\beta} (a - it)^{2\pi(i\alpha - \beta)}, \end{aligned}$$

where $c = \frac{1}{i\pi} \sinh(2\pi\alpha\pi) \Gamma(1 + 2i\pi\alpha) \Gamma(2\pi\beta - 2i\pi\alpha)$. Using that expression for the wavelet transform, we may then write:

$$\begin{aligned} \hat{\omega}_f(a, t) &= \frac{1}{2\pi} \mathcal{I} \left[\frac{\partial_t W_f(a, t)}{W_f(a, t)} \right] = \mathcal{I} \left[-i(i\alpha - \beta)(a - it)^{-1} \right] \\ &= \mathcal{I} \left[\frac{(\alpha + i\beta)(a + it)}{a^2 + t^2} \right] = \frac{a\beta + \alpha t}{a^2 + t^2}, \\ &= \frac{\alpha}{t} + \alpha \left(\frac{t}{a^2 + t^2} - \frac{1}{t} \right) + \frac{a\beta}{a^2 + t^2}, \\ \Rightarrow \left| \hat{\omega}_f(a, t) - \frac{\alpha}{t} \right| &= \left| \frac{a\beta t - \alpha a^2}{t(a^2 + t^2)} \right| \Leftrightarrow |\hat{\omega}_f(a, t) - \phi'(t)| = \left| \frac{\frac{\beta}{a} - \frac{\alpha}{t}}{1 + \left(\frac{t}{a}\right)^2} \right|. \end{aligned}$$

Let us consider $\beta = 1$ and, since $\phi'(t) = \frac{\alpha}{t}$, we get when $\alpha < 1$,

$$|\hat{\omega}_f(a, t) - \phi'(t)| = \left| \frac{\frac{1}{a} - \phi'(t)}{1 + \left(\frac{t}{a}\right)^2} \right|.$$

Finally, if $|\frac{1}{a} - \phi'(t)| \leq \epsilon$, we obtain

$$\begin{aligned} |\hat{\omega}_f(a, t) - \phi'(t)| &\leq \epsilon \left| \frac{1}{1 + t^2(\phi'(t) - \epsilon)^2} \right| = \epsilon \left| \frac{1}{1 + t^2\left(\frac{\alpha^2}{t^2} - 2\epsilon\frac{\alpha}{t} + \epsilon^2\right)} \right|, \\ &= \epsilon \left| \left(\frac{1}{1 + \alpha^2} \right) \left(\frac{1}{1 + \frac{-2\epsilon\alpha t + \epsilon^2 t^2}{1 + \alpha^2}} \right) \right|, \\ &= \left| \frac{\epsilon}{1 + \alpha^2} + \frac{\epsilon}{1 + \alpha^2} \left(\frac{\epsilon\alpha t}{1 + \alpha^2} \right) \right| + O(\epsilon^2), \\ &= \left| \frac{\epsilon}{1 + \alpha^2} \left(1 + \frac{\epsilon\alpha t}{1 + \alpha^2} \right) \right| + O(\epsilon^2) = \left| \frac{\epsilon}{1 + \alpha^2} \right| + O(\epsilon^2). \end{aligned}$$

□

The above estimate does not depend on the scale a . On the contrary, no such expression is available with estimate $\hat{\omega}_f(\eta, t)$ for an hyperbolic chirp. Nevertheless, we display on Figure 2, the FSST and WSST of an hyperbolic chirp showing that to use WSST for that type of mode is more relevant.

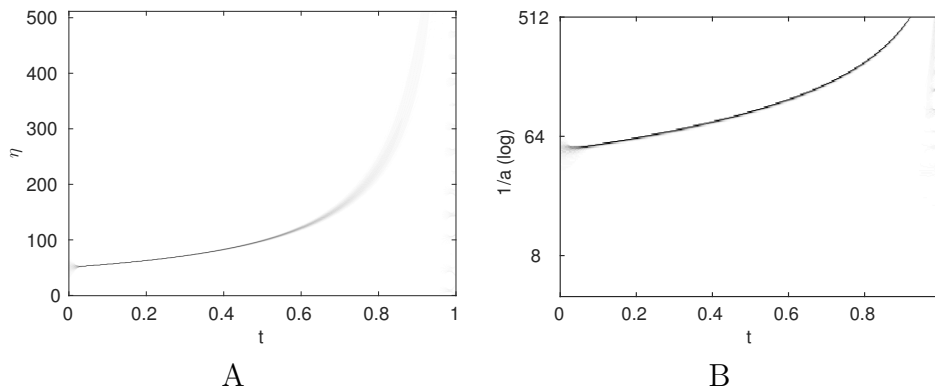


Figure 2: Comparison of the methods on an hyperbolic chirp. A:FSST and B:WSST

5. New FSST based on second order approximation of the phase

The original FSST assumes $\phi''(t)$ is negligible. When it is not the case, Theorems 1 and 2 are no longer valid, because they apply to the wrong type of modes. Recently, an extension of FSST based on a second order approximation of the phase was proposed [13], but to our knowledge no theoretical analysis of the proposed estimate is available. We propose to bridge that gap in this section.

5.1. Computation of the new IF estimate

Remember that $\tilde{\omega}_f(\eta, t) = \frac{\partial_t V_f^g(\eta, t)}{2i\pi V_f^g(\eta, t)}$ and then introduce:

$$\tilde{t}_f(\eta, t) = t - \frac{\partial_\eta V_f^g(\eta, t)}{2i\pi V_f^g(\eta, t)}. \quad (36)$$

Similarly to $\hat{\omega}_f(\eta, t) = \mathcal{R}(\tilde{\omega}_f(\eta, t))$, we define $\hat{t}_f(\eta, t) = \mathcal{R}(\tilde{t}_f(\eta, t))$. First, we recall an estimate of the frequency modulation introduced in [13].

Definition 6. Let $f \in L^2(\mathbb{R})$ and consider when $V_f^g(\eta, t) \neq 0$ and $\frac{\partial_t(\partial_\eta V_f^g(\eta, t))}{V_f^g(\eta, t)} \neq 2i\pi$ the quantity

$$\tilde{q}_f(\eta, t) = \frac{\partial_t \tilde{\omega}_f(\eta, t)}{\partial_t \tilde{t}_f(\eta, t)} = \frac{\partial_t \left(\frac{\partial_t V_f^g(\eta, t)}{V_f^g(\eta, t)} \right)}{2i\pi - \partial_t \left(\frac{\partial_\eta V_f^g(\eta, t)}{V_f^g(\eta, t)} \right)}. \quad (37)$$

An estimate of the frequency modulation is then defined by

$$\hat{q}_f(\eta, t) = \mathcal{R}(\tilde{q}_f(\eta, t)). \quad (38)$$

The estimate (38) was used in [13] where it was proved that $\hat{q}_f(\eta, t) = \phi''(t)$, when f is a Gaussian modulated linear chirp, i.e. a chirp where both ϕ and $\log(A)$ are quadratic. Further, a new estimate of $\phi'(t)$ was also derived there, namely:

$$\phi'(t) = \hat{\omega}_f(\eta, t) + \hat{q}_f(\eta, t)(t - \hat{t}_f(\eta, t)), \quad (39)$$

which is exact for constant amplitude linear chirp. Here we propose a slightly different IF estimate than the one introduced in [13], which allows for mathematical study.

Definition 7. Let $f \in L^2(\mathbb{R})$, we define the second order IF complex estimate of f as:

$$\tilde{\omega}_f^{(2)}(\eta, t) = \begin{cases} \tilde{\omega}_f(\eta, t) + \tilde{q}_f(\eta, t)(t - \tilde{t}_f(\eta, t)) & \text{if } \partial_t \tilde{t}_f(\eta, t) \neq 0 \\ \tilde{\omega}_f(\eta, t) & \text{otherwise,} \end{cases} \quad (40)$$

and then its real part:

$$\hat{\omega}_f^{(2)}(\eta, t) = \begin{cases} \mathcal{R}(\tilde{\omega}_f(\eta, t) + \tilde{q}_f(\eta, t)(t - \tilde{t}_f(\eta, t))) & \text{if } \partial_t \tilde{t}_f(\eta, t) \neq 0 \\ \hat{\omega}_f(\eta, t) & \text{otherwise.} \end{cases} \quad (41)$$

Proposition 4. Let $f \in L^2(\mathbb{R})$, then the IF estimate $\hat{\omega}_f^{(2)}(\eta, t)$ can be expressed as

$$\begin{aligned} \hat{\omega}_f^{(2)}(\eta, t) &= \hat{\omega}_f(\eta, t) + \mathcal{R}(\tilde{q}_f(\eta, t)) \mathcal{R}(t - \tilde{t}_f(\eta, t)) - \mathcal{I}(\tilde{q}_f(\eta, t)) \mathcal{I}(t - \tilde{t}_f(\eta, t)), \\ &= \hat{\omega}_f(\eta, t) - \hat{q}_f(\eta, t) \mathcal{R}\left(\frac{V_f^{tg}(\eta, t)}{V_f^g(\eta, t)}\right) + \mathcal{I}(\tilde{q}_f(\eta, t)) \mathcal{I}\left(\frac{V_f^{tg}(\eta, t)}{V_f^g(\eta, t)}\right), \\ &= \hat{\omega}_f(\eta, t) + \hat{q}_f(\eta, t)(t - \hat{t}_f(\eta, t)) + \mathcal{I}(\tilde{q}_f(\eta, t)) \mathcal{I}\left(\frac{V_f^{tg}(\eta, t)}{V_f^g(\eta, t)}\right), \end{aligned} \quad (42)$$

where,

$$\begin{aligned}\hat{\omega}_f(\eta, t) &= \eta - \frac{1}{2\pi} \mathcal{I} \left(\frac{V_f^{g'}(\eta, t)}{V_f^g(\eta, t)} \right), \\ \tilde{q}_f(\eta, t) &= \frac{1}{2i\pi} \frac{V_f^{g''} V_f^g - (V_f^{g'})^2}{(V_f^g)^2 + V_f^{tg} V_f^{g'} - V_f^{tg'} V_f^g}, \\ t - \hat{t}_f(\eta, t) &= -\frac{V_f^{tg}(\eta, t)}{V_f^g(\eta, t)},\end{aligned}$$

meaning that the IF estimate can be computed by means of five different STFTs.

Proof. Indeed, from Definition 7 of $\hat{\omega}_f^{(2)}(\eta, t)$, we may write

$$\begin{aligned}\hat{\omega}_f^{(2)}(\eta, t) &= \mathcal{R}(\tilde{\omega}_f(\eta, t)) + \mathcal{R}(\tilde{q}_f(\eta, t)(t - \tilde{t}_f(\eta, t))), \\ &= \hat{\omega}_f(\eta, t) + \hat{q}_f(\eta, t)(t - \hat{t}_f(\eta, t)) - \mathcal{I}(\tilde{q}_f(\eta, t)) \mathcal{I}(t - \tilde{t}_f(\eta, t)), \\ &= \hat{\omega}_f(\eta, t) + \hat{q}_f(\eta, t)(t - \hat{t}_f(\eta, t)) + \mathcal{I}(\tilde{q}_f(\eta, t)) \mathcal{I}\left(\frac{V_f^{tg}(\eta, t)}{2\pi V_f^g(\eta, t)}\right).\end{aligned}$$

This expression is obtained using $\partial_\eta V_f^g(\eta, t) = -2i\pi V_f^{tg}(\eta, t)$, while the expressions for $\tilde{q}_f(\eta, t)$ and $\hat{\omega}_f(\eta, t)$ use the other following equalities:

$$\begin{aligned}\partial_t V_f^g(\eta, t) &= 2i\pi\eta V_f^g(\eta, t) - V_f^{g'}(\eta, t), \\ \partial_{tt}^2 V_f^g(\eta, t) &= V_f^{g''}(\eta, t) - 4i\pi\eta V_f^{g'}(\eta, t) - 4\pi^2\eta^2 V_f^g(\eta, t), \\ \partial_{\eta t}^2 V_f^g(\eta, t) &= 2i\pi V_f^{tg'}(\eta, t) + 4\pi^2\eta V_f^{tg}(\eta, t),\end{aligned}\tag{43}$$

where $V_f^{g''}$ and $V_f^{tg'}$ are the STFT of f with windows g'' and $t \rightarrow tg'(t)$. \square

At this stage, we shall remark that the estimate $\hat{\omega}_f^{(2)}(\eta, t)$ is different from the one proposed in [13] due to the addition of the product of the two imaginary parts. Also, as was remarked when dealing with FSST, the real estimate $\hat{\omega}_f^{(2)}(\eta, t)$ should be used and not the complex one. In the following section, we prove theoretical results regarding the former IF estimate.

5.2. Definition of the new FSST

In this section, we define another class of chirp-like functions larger than $\mathcal{B}_{\Delta,\epsilon}$ and show that they can be successfully dealt with by means of second order FSST which is defined in this section. So first, we define the new set of multicomponent signals we are studying:

Definition 8. Let $\epsilon > 0$ and $\Delta \in (0, 1)$. The set $\mathcal{B}_{\Delta,\epsilon}^{(2)}$ of multicomponent signals with second order modulation ϵ and separation Δ corresponds to the signals defined in (6) satisfying:

(a) function f_k is such that A_k and ϕ_k satisfy the following conditions:

$$\begin{aligned} A_k(t) &\in L^\infty(\mathbb{R}) \cap \mathcal{C}^2(\mathbb{R}), \quad \phi_k(t) \in \mathcal{C}^3(\mathbb{R}), \\ \phi_k'(t), \phi_k''(t), \phi_k'''(t) &\in L^\infty(\mathbb{R}), \\ A_k(t) > 0, \quad \inf_{t \in \mathbb{R}} \phi_k'(t) > 0, \quad \sup_{t \in \mathbb{R}} \phi_k'(t) < \infty, \\ |A_k'(t)| \leq \epsilon, \quad |A_k''(t)| \leq \epsilon, \quad \text{and} \quad |\phi_k'''(t)| \leq \epsilon, \end{aligned}$$

(b) the functions f_k s satisfy the following separation condition

$$\phi_{k+1}'(t) - \phi_k'(t) > 2\Delta \quad \forall t \in \mathbb{R} \quad , \quad \forall k \in \{1, \dots, K-1\}. \quad (44)$$

Now, let us define the second order FSST as follows

Definition 9. Let h be a positive L^1 normalized window belonging to $\mathcal{C}_c^\infty(\mathbb{R})$, and consider $\gamma, \delta > 0$, the second order FSST of f with threshold γ and accuracy δ is defined by:

$$\tilde{T}_f^{\delta,\gamma}(\omega, t) = \frac{1}{g(0)} \int_{|V_f^{g\sigma}(\eta, t)| > \gamma} V_f^{g\sigma}(\eta, t) \frac{1}{\delta} h\left(\frac{\omega - \hat{\omega}_f^{(2)}(\eta, t)}{\delta}\right) d\eta. \quad (45)$$

In Section 3, we showed that for functions $f \in \mathcal{B}_{\Delta,\epsilon}$ a good IF estimate was given by $\hat{\omega}_f(\eta, t)$ and the approximation theorem followed. Here, to assess the approximation property of the second order FSST we have just introduced, we consider a function $f \in \mathcal{B}_{\Delta,\epsilon}^{(2)}$ for which we are going to prove that a good IF estimate is provided by $\hat{\omega}_f^{(2)}(\eta, t)$ computing from $V_f^{g\sigma}(\eta, t)$. The approximation theorem is as follows:

Theorem 6. Consider $f \in \mathcal{B}_{\Delta, \epsilon}^{(2)}$, set $\tilde{\epsilon} = \epsilon^{1/6}$ and define a L^1 normalized positive function h . Let g be a window satisfying for all t and all $k = 1, \dots, K$, $\overline{g(\tau)e^{i\pi\sigma^2\phi_k''(t)\tau^2}}$ behaves like $\frac{1}{\gamma\eta^2}$ when $|\eta| \geq \Delta$. Define $g_\sigma(t) = \sigma^{-1}g(\frac{t}{\sigma})$. Then, provided ϵ is sufficiently small and that $\frac{1}{\sqrt{|\gamma|\Delta}} \leq \sigma\tilde{\epsilon} \leq C$ for some constant C , then the following hold:

(a) $|V_f^{g_\sigma}(\eta, t)| \geq \tilde{\epsilon}$ only when there exist $k \in \{1, \dots, K\}$ such that $(\eta, t) \in Z_k$.

(b) For all $k \in \{1, \dots, K\}$ and all $(\eta, t) \in Z_k$ such that $|V_f^{g_\sigma}(\eta, t)| > \tilde{\epsilon}$, we have

$$|\hat{\omega}_f^{(2)}(\eta, t) - \phi_k'(t)| \leq \tilde{\epsilon}. \quad (46)$$

(c) Moreover, for each $k \in \{1, \dots, K\}$, there exists a constant C such that

$$\left| \left(\lim_{\delta \rightarrow 0} \frac{1}{2\pi g(0)} \int_{\mathcal{M}_{k, \tilde{\epsilon}}} \tilde{T}_f^{\delta, \tilde{\epsilon}}(\omega, t) d\omega \right) - A_k(t)e^{i\phi_k(t)} \right| \leq C\tilde{\epsilon}, \quad (47)$$

where $\mathcal{M}_{k, \tilde{\epsilon}} := \{\omega : |\omega - \phi_k'(t)| < \tilde{\epsilon}\}$.

The proof of Theorem 6 involves a number of estimates, which are shown hereafter, assuming all the hypotheses of Theorem 6 are satisfied.

Lemma 2. For any (η, t) , there is at most one $k \in \{1, \dots, K\}$ for which $|\eta - \phi_k'(t)| \leq \Delta$.

The proof is straightforward and is left to the reader.

For our purpose, we need to analyze the behavior of the STFT on a linear chirp:

Proposition 5. Let $h_c(\tau) = Ae^{2i\pi\phi_k(\tau)}$ be a linear chirp and consider $V_{h_c}^{g_\sigma}$ the STFT of h_c obtained with a window g_σ as introduced in Theorem 6. If $\frac{1}{\sqrt{|\gamma|\Delta}} \leq \sigma\tilde{\epsilon}$ and if $(\eta, t) \notin Z_k$ then

$$|V_{h_c}^{g_\sigma}(\eta, t)| \leq \tilde{\epsilon}^2 A,$$

Proof. We know that $V_{h_c}^{g_\sigma}(\eta, t) = \overline{h_c(t)g_\sigma(\tau)e^{i\pi\phi_k''(t)\tau^2}}(\eta - \phi_k'(t))$ so that if $(\eta, t) \notin Z_k$ $|V_{h_c}^{g_\sigma}(\eta, t)| \leq \frac{A}{|\gamma|\sigma^2\Delta^2} \leq A\tilde{\epsilon}^2$. \square

Remark 9. In the original FSST, one considered a constant amplitude and first order approximation of the phase for the mode that is, $h(\tau) = Ae^{2i\pi(\phi(t)+(\tau-t)\phi'(t)}$ for which we had $V_h^{g\sigma}(\eta, t) = h(t)\widehat{g_\sigma}(\eta - \phi'(t))$ so that if $\widehat{g_\sigma}$ is compactly supported, $V_h^{g\sigma}(\eta, t)$ was also compactly supported for each t . In our case, note that $V_{h_c}^{g\sigma}(\eta, t)$ corresponds, modulo the multiplication by a constant, to $\widehat{g_\sigma(\tau)e^{i\pi\phi''(t)\tau^2}}(\eta - \phi'(t))$, t being fixed, for which we have the following proposition:

Proposition 6. *Even if $\widehat{g_\sigma}$ is compactly supported, $\widehat{g_\sigma(\tau)e^{i\pi\phi''(t)\tau^2}}$ is never compactly supported.*

Proof. Since $\widehat{g_\sigma}$ is compactly supported it belongs to $\mathcal{E}'(\mathbb{R})$, the space of compactly supported distribution, and $e^{i\pi\phi''(t)\tau^2}$ to $\mathcal{S}'(\mathbb{R})$, the space of tempered distributions, since the latter belongs to $L^\infty(\mathbb{R})$. Now we have the following property in $\mathcal{S}'(\mathbb{R})$:

$$g_\sigma(\tau)e^{i\pi\phi''(t)\tau^2} = \overline{\mathcal{F}}\left(\widehat{g_\sigma} * \widehat{e^{i\pi\phi''(t)\tau^2}}\right),$$

where $\overline{\mathcal{F}}$ is inverse Fourier transform, which is equivalent to

$$\widehat{g_\sigma(\tau)e^{i\pi\phi''(t)\tau^2}} = \widehat{g_\sigma} * \widehat{e^{i\pi\phi''(t)\tau^2}}.$$

Hence $\widehat{g_\sigma(\tau)e^{i\pi\phi''(t)\tau^2}}$ compactly supported if and only if $\widehat{e^{i\pi\phi''(t)\tau^2}}$ is compactly supported. Since the IF of $e^{2i\pi\phi''(t)\tau^2}$ is $2\phi''(t)\tau$, i.e. all the frequencies are present in that signal when τ varies so $\widehat{e^{i\pi\phi''(t)\tau^2}}$ is not compactly supported. \square

This shows that to impose $\widehat{g_\sigma}$ compactly supported does not help in our context, contrary to what happened in the original FSST case.

Proposition 7. *Consider a window g_σ as introduced in Theorem 6. If $\frac{1}{\sqrt{|\gamma|\Delta}} \leq \sigma\tilde{\epsilon}$, we have for any (η, t) :*

$$|V_{f_k}^{g\sigma}(\eta, t) - V_{f_{k,2}}^{g\sigma}(\eta, t)| \leq \epsilon E_k(t), \quad (48)$$

with $f_{k,2}(t) = A_k(t)e^{2i\pi[\phi_k(t)+\phi'_k(t)(\tau-t)+\frac{1}{2}\phi''_k(t)(\tau-t)^2]}$ and where $E_k(t) = \sigma I_1 + \frac{\pi}{3}\sigma^3 I_3 A_k(t)$. If $(\eta, t) \notin Z_k$ we get

$$|V_{f_k}^{g\sigma}(\eta, t)| \leq \epsilon E_k(t) + \tilde{\epsilon}^2 A_k(t). \quad (49)$$

As a consequence, we also have, for any $(\eta, t) \in Z_k$:

$$|V_f^{g\sigma}(\eta, t) - V_{f_{k,2}}^{g\sigma}(\eta, t)| \leq \epsilon \sum_{l=1}^K E_l(t) + \tilde{\epsilon}^2 \sum_{l \neq k} A_k(t), \quad (50)$$

$$= \tilde{\epsilon}^2 \left(\tilde{\epsilon}^4 \sum_{l=1}^K E_l(t) + \sum_{l \neq k} A_k(t) \right) = \tilde{\epsilon}^2 \Omega_k(t), \quad (51)$$

where $\Omega_k(t) = \tilde{\epsilon}^4 \sum_{l=1}^K E_l(t) + \sum_{l \neq k} A_k(t)$.

Proof. First, we write

$$\begin{aligned} f_k(\tau) &= A_k(\tau) e^{2i\pi\phi_k(\tau)} = (A_k(\tau) - A_k(t)) e^{2i\pi\phi_k(\tau)} \\ &\quad + A_k(t) e^{2i\pi[\phi_k(t) + \phi'_k(t)(\tau-t) + \frac{1}{2}\phi''_k(t)(\tau-t)^2]} \\ &\quad + A_k(t) [e^{2i\pi[\phi_k(t) + \phi'_k(t)(\tau-t) + \frac{1}{2}\phi''_k(t)(\tau-t)^2 + \frac{1}{2} \int_t^\tau \phi'''_k(x)(\tau-x)^2 dx]} \\ &\quad \quad - e^{2i\pi[\phi_k(t) + \phi'_k(t)(\tau-t) + \frac{1}{2}\phi''_k(t)(\tau-t)^2]}], \\ &= f_{k,1}(\tau) + f_{k,2}(\tau) + f_{k,3}(\tau). \end{aligned} \quad (52)$$

Then, for any (η, t) ,

$$\begin{aligned} |V_{f_{k,1}}^{g\sigma}(\eta, t)| &\leq \int_{\mathbb{R}} |A_k(\tau) - A_k(t)| |g_\sigma(\tau - t)| d\tau, \\ &\leq \int_{\mathbb{R}} \epsilon |\tau - t| |g_\sigma(\tau - t)| d\tau \leq \epsilon \sigma I_1. \end{aligned}$$

Now, we also have, $|V_{f_{k,3}}^{g\sigma}(\eta, t)| \leq \epsilon \frac{\pi}{3} \sigma^3 I_3 A_k(t)$, leading to:

$$|V_{f_k}^{g\sigma}(\eta, t) - V_{f_{k,2}}^{g\sigma}(\eta, t)| \leq \epsilon E_k(t)$$

Now, if $(\eta, t) \notin Z_k$, we write:

$$|V_{f_k}^{g\sigma}(\eta, t)| \leq \epsilon E_k(t) + \tilde{\epsilon}^2 A_k(t),$$

which leads to the second inequality. The last inequality is then straightforward. \square

We now prove item (a) of Theorem 6. If $(\eta, t) \notin \bigcup_{l=1}^K Z_l$, we immediately get:

$$|V_f^{g_\sigma}(\eta, t)| \leq \tilde{\epsilon}^2 \sum_{l=1}^K (\tilde{\epsilon}^4 E_l(t) + A_l(t)),$$

if we take $\tilde{\epsilon}$ sufficiently small, we get $\tilde{\epsilon} \sum_{l=1}^K (\tilde{\epsilon}^4 E_l(t) + A_l(t)) \leq 1$, since $\sigma \tilde{\epsilon} \leq C$. Now, we introduce several propositions that are useful to prove item (b) of Theorem 6.

Proposition 8. *Let $h_c(t) = Ae^{2i\pi\phi_k(t)}$ be a linear chirp, g_σ be a window as defined in Theorem 6. When $(\eta, t) \notin Z_k$, we have*

$$|V_{h_c}^{tg_\sigma}(\eta, t)| \leq \frac{A}{\pi\Delta} \tilde{\epsilon}^2.$$

Proof. Since $\partial_\eta V_{h_c}^{g_\sigma}(\eta, t) = -2i\pi V_{h_c}^{tg_\sigma}(\eta, t)$ and since $V_{h_c}^{g_\sigma}(\eta, t)$ behaves like $\frac{A}{\gamma\sigma^2\eta^2}$ for $|\eta| \geq \Delta$, we get that $|V_{h_c}^{tg_\sigma}(\eta, t)| \leq \frac{A}{\pi|\gamma|\sigma^2\Delta^3} \leq \frac{A}{\pi\Delta} \tilde{\epsilon}^2$ for $(\eta, t) \notin Z_k$. \square

Proposition 9. *Let f_k be a mode defined as in Definition 8 and consider the STFT of f_k computed using window tg_σ where g_σ is a window of the type introduced in Theorem 6 with $\frac{1}{\sqrt{|\gamma|\Delta}} \leq \sigma\tilde{\epsilon}$. Then, for any (η, t) , we have*

$$|V_{f_k}^{tg_\sigma}(\eta, t) - V_{f_{k,2}}^{tg_\sigma}(\eta, t)| \leq \epsilon F_k(t),$$

where with $F_k(t) = \sigma^2 I_2 + \frac{\pi}{3} \sigma^4 I_4 A_k(t)$. Now, if $(\eta, t) \notin Z_k$, we get:

$$|V_{f_k}^{tg_\sigma}(\eta, t)| \leq \epsilon F_k(t) + \tilde{\epsilon}^2 \frac{A_k(t)}{\pi\Delta},$$

and, we also have, when $(\eta, t) \in Z_k$:

$$\begin{aligned} |V_f^{tg_\sigma}(\eta, t) - V_{f_{k,2}}^{tg_\sigma}(\eta, t)| &\leq \epsilon \sum_{l=1}^K F_l(t) + \tilde{\epsilon}^2 \sum_{l \neq k} \frac{A_l(t)}{\pi\Delta}, \\ &= \tilde{\epsilon}^2 \left(\tilde{\epsilon}^4 \sum_{l=1}^K F_l(t) + \sum_{l \neq k} \frac{A_l(t)}{\pi\Delta} \right) = \tilde{\epsilon}^2 \Lambda_k(t), \end{aligned} \quad (53)$$

where $\Lambda_k(t) = \tilde{\epsilon}^4 \sum_{l=1}^K F_l(t) + \sum_{l \neq k} \frac{A_l(t)}{\pi\Delta}$.

Proof. Using the same decomposition as in the previous proposition, we may write:

$$\begin{aligned}
|V_{f_k}^{tg_\sigma}(\eta, t) - V_{f_{k,2}}^{tg_\sigma}(\eta, t)| &\leq \int_{\mathbb{R}} |f_{k,1}(\tau)(\tau - t)g_\sigma(\tau - t)e^{-2i\pi\eta(\tau-t)}| d\tau, \\
&\quad + \int_{\mathbb{R}} |f_{k,3}(\tau)(\tau - t)g_\sigma(\tau - t)e^{-2i\pi\eta(\tau-t)}| d\tau, \\
&\leq \epsilon(\sigma^2 I_2 + \frac{\pi}{3}\sigma^4 I_4), \\
&= \epsilon F_k(t).
\end{aligned}$$

Now if $(\eta, t) \notin Z_k$, from Proposition 8, we get:

$$\begin{aligned}
|V_{f_k}^{tg_\sigma}(\eta, t)| &\leq \epsilon F_k(t) + |V_{f_{k,2}}^{tg_\sigma}(\eta, t)|, \\
&\leq \tilde{\epsilon}^2(\tilde{\epsilon}^4 F_k(t) + \frac{1}{\pi\Delta} A_k(t)).
\end{aligned}$$

The last inequality is then straightforward. \square

Then we may write:

Proposition 10. *For any (η, t) , assuming g_σ satisfies the hypothesis of Theorem 6 and assuming $\frac{1}{\sqrt{|\eta|\Delta}} \leq \sigma\tilde{\epsilon}$. Then, we have:*

$$\left| \partial_t V_f^{g_\sigma}(\eta, t) - 2i\pi \sum_{l=1}^K \left(\phi_l'(t) V_{f_{l,2}}^{g_\sigma}(\eta, t) + \phi_l''(t) V_{f_{l,2}}^{tg_\sigma}(\eta, t) \right) \right| \leq \epsilon G(t), \quad (54)$$

where $G(t) = KI_0 + 2\pi \sum_{l=1}^K \left(A_l(t) \frac{\sigma^2 I_2}{2} + \frac{\sigma^3}{2} I_3 + \phi_l'(t) E_l(t) + |\phi_l''(t)| F_l(t) \right)$.

If $(\eta, t) \in Z_k$ we get:

$$\begin{aligned}
\left| \partial_t V_f^{g_\sigma}(\eta, t) - 2i\pi \left(\phi_k'(t) V_{f_{k,2}}^{g_\sigma}(\eta, t) + \phi_k''(t) V_{f_{k,2}}^{tg_\sigma}(\eta, t) \right) \right| &\leq \tilde{\epsilon}^2 G_k(t), \\
\left| \partial_t V_f^{g_\sigma}(\eta, t) - 2i\pi \left(\phi_k'(t) V_f^{g_\sigma}(\eta, t) + \phi_k''(t) V_f^{tg_\sigma}(\eta, t) \right) \right| & \quad (55)
\end{aligned}$$

$$\begin{aligned}
&\leq \tilde{\epsilon}^2 \left(G_k(t) + 2\pi(\phi_k'(t)\Omega_k(t) + |\phi_k''(t)|\Lambda_k(t)) \right), \\
&= \tilde{\epsilon}^2 B_{1,k}(t). \quad (56)
\end{aligned}$$

where $G_k(t) = \tilde{\epsilon}^4 G(t) + 2\pi \sum_{l \neq k} A_l(t) \left(\phi_l'(t) + \frac{|\phi_l''(t)|}{\pi\Delta} \right)$.

Proof. Differentiating the STFT of f with respect to t we get:

$$\begin{aligned}
\partial_t V_f^{g_\sigma}(\eta, t) &= \sum_{l=1}^K \int_{\mathbb{R}} A_l'(\tau) e^{2i\pi\phi_l(\tau)} g_\sigma(\tau - t) e^{-2i\pi\eta(\tau-t)} d\tau \\
&\quad + \sum_{l=1}^K \int_{\mathbb{R}} A_l(\tau) 2i\pi\phi_l'(\tau) e^{2i\pi\phi_l(\tau)} g_\sigma(\tau - t) e^{-2i\pi\eta(\tau-t)} d\tau, \\
&= \sum_{l=1}^K \int_{\mathbb{R}} A_l'(\tau) e^{2i\pi\phi_l(\tau)} g_\sigma(\tau - t) e^{-2i\pi\eta(\tau-t)} d\tau \\
&\quad + \sum_{l=1}^K \int_{\mathbb{R}} A_l(\tau) 2i\pi \left[\phi_l'(t) + (\tau - t)\phi_l''(t) \right. \\
&\quad \left. + \int_t^\tau (\tau - u)\phi_l'''(u) du \right] e^{2i\pi\phi_l(\tau)} g_\sigma(\tau - t) e^{-2i\pi\eta(\tau-t)} d\tau, \\
&= \sum_{l=1}^K \int_{\mathbb{R}} A_l'(\tau) e^{2i\pi\phi_l(\tau)} g_\sigma(\tau - t) e^{-2i\pi\eta(\tau-t)} d\tau + \sum_{l=1}^K 2i\pi\phi_l'(t) V_{f_l}^{g_\sigma}(\eta, t) \\
&\quad + \sum_{l=1}^K 2i\pi\phi_l''(t) V_{f_l}^{tg_\sigma}(\eta, t) \\
&\quad + 2 \sum_{l=1}^K \int_{\mathbb{R}} A_l(\tau) i\pi \int_t^\tau (\tau - u)\phi_l'''(u) du e^{2i\pi\phi_l(\tau)} e^{-2i\pi\eta(\tau-t)} g_\sigma(\tau - t) d\tau.
\end{aligned}$$

We may then write:

$$\begin{aligned}
&\left| \partial_t V_f^{g_\sigma}(\eta, t) - 2i\pi \sum_{l=1}^K \left(\phi_l'(t) V_{f_l}^{g_\sigma}(\eta, t) + \phi_l''(t) V_{f_l}^{tg_\sigma}(\eta, t) \right) \right| \\
&\leq \left| \sum_{l=1}^K \int_{\mathbb{R}} A_l'(\tau) e^{2i\pi\phi_l(\tau)} g_\sigma(\tau - t) e^{-2i\pi\eta(\tau-t)} d\tau \right| \\
&\quad + \left| \sum_{l=1}^K 2 \int_{\mathbb{R}} A_l(\tau) i\pi \int_t^\tau (\tau - u)\phi_l'''(u) du e^{2i\pi\phi_l(\tau)} e^{-2i\pi\eta(\tau-t)} g_\sigma(\tau - t) d\tau \right|.
\end{aligned}$$

We can write for the first part,

$$\left| \sum_{l=1}^K \int_{\mathbb{R}} A_l'(\tau) e^{2i\pi\phi_l(\tau)} g_\sigma(\tau - t) e^{-2i\pi\eta(\tau-t)} d\tau \right| \leq \epsilon K I_0, \quad (57)$$

and for the second:

$$\begin{aligned} & \left| \sum_{l=1}^K 2 \int_{\mathbb{R}} A_l(\tau) i\pi \int_t^\tau (\tau - u) \phi'''(u) du e^{2i\pi\phi_l(\tau)} e^{-2i\pi\eta(\tau-t)} g_\sigma(\tau - t) d\tau \right| \\ & \leq \epsilon \left(\sum_{l=1}^K A_l(t) \pi \sigma^2 I_2 + \epsilon \pi \sigma^3 I_3 \right) \leq \epsilon \left(\sum_{l=1}^K A_l(t) \pi \sigma^2 I_2 + \pi \sigma^3 I_3 \right). \end{aligned}$$

Finally, we may write:

$$\begin{aligned} & \left| \partial_t V_f^{g_\sigma}(\eta, t) - 2i\pi \sum_{l=1}^K \left(\phi'_l(t) V_{f_{l,2}}^{g_\sigma}(\eta, t) + \phi''_l(t) V_{f_{l,2}}^{tg_\sigma}(\eta, t) \right) \right| \\ & \leq \left| \partial_t V_f^{g_\sigma}(\eta, t) - 2i\pi \sum_{l=1}^K \left(\phi'_l(t) V_{f_l}^{g_\sigma}(\eta, t) + \phi''_l(t) V_{f_l}^{tg_\sigma}(\eta, t) \right) \right| \\ & \quad + 2\pi\epsilon \sum_{l=1}^K \phi'_l(t) E_l(t) + |\phi''_l(t)| F_l(t), \end{aligned}$$

hence (54). To get (55), it suffices to remark that when $(\eta, t) \in Z_k$:

$$\begin{aligned} & \left| \partial_t V_f^{g_\sigma}(\eta, t) - 2i\pi \left(\phi'_k(t) V_{f_{k,2}}^{g_\sigma}(\eta, t) + \phi''_k(t) V_{f_{k,2}}^{tg_\sigma}(\eta, t) \right) \right| \\ & \leq \left| \partial_t V_f^{g_\sigma}(\eta, t) - 2i\pi \sum_{l=1}^K \left(\phi'_l(t) V_{f_{l,2}}^{g_\sigma}(\eta, t) + \phi''_l(t) V_{f_{l,2}}^{tg_\sigma}(\eta, t) \right) \right| \\ & \quad + \left| 2\pi \sum_{l \neq k} \left(\phi'_l(t) V_{f_{l,2}}^{g_\sigma}(\eta, t) + \phi''_l(t) V_{f_{l,2}}^{tg_\sigma}(\eta, t) \right) \right|, \\ & \leq \left| \partial_t V_f^{g_\sigma}(\eta, t) - 2i\pi \sum_{l=1}^K \left(\phi'_l(t) V_{f_{l,2}}^{g_\sigma}(\eta, t) + \phi''_l(t) V_{f_{l,2}}^{tg_\sigma}(\eta, t) \right) \right| \\ & \quad + 2\pi\tilde{\epsilon}^2 \sum_{l \neq k} A_l(t) \left(\phi'_l(t) + \frac{|\phi''_l(t)|}{\pi\Delta} \right). \end{aligned}$$

The last in equality is straightforward. \square

Proposition 11. *Let $h_c(t) = Ae^{2i\pi\phi_k(t)}$ be a linear chirp, and assume g_σ is a window defined in Definition 3 and $\frac{1}{\sqrt{|\gamma|\Delta}} \leq \sigma\tilde{\epsilon}$, when $(\eta, t) \notin Z_k$, then one has*

$$|V_{h_c}^{t^2 g_\sigma}(\eta, t)| \leq \frac{3A}{4\pi^3 \Delta^2} \tilde{\epsilon}^2. \quad (58)$$

Proof. We have

$$V_{h_c}^{t^2 g_\sigma}(\eta, t) = -\frac{1}{4\pi^2} \partial_\eta^2 V_{h_c}^{g_\sigma}(\eta, t). \quad (59)$$

Since $h_c(\tau) = A e^{2i\pi[\phi_k(t) + \phi'_k(t)(t-\tau) + \frac{1}{2}\phi''_k(t)(t-\tau)^2]}$ and $V_{h_c}^{g_\sigma}(\eta, t)$ behaves like $\frac{A}{\gamma\sigma^2\eta^2}$, then $\partial_\eta^2 V_{h_c}^{g_\sigma}(\eta, t)$ behaves like $\frac{6A}{\gamma\sigma^2\eta^4}$ for $|\eta| \geq \Delta$. Now for $(\eta, t) \notin Z_k$, we can write

$$\left| V_{h_c}^{t^2 g_\sigma}(\eta, t) \right| \leq \frac{6A}{4\pi^2 |\gamma| \sigma^2 \Delta^4} \leq \frac{3A}{2\pi^2 \Delta^2} \tilde{\epsilon}^2. \quad (60)$$

□

Proposition 12. *Let f_k be a mode defined as in Definition 8 and consider the STFT of f_k computed using window $t^2 g_\sigma$, where g_σ is a defined as in Definition 3, again assume $\frac{1}{\sqrt{|\gamma|\Delta}} \leq \sigma \tilde{\epsilon}$, in such a case, we have:*

$$\left| V_{f_k}^{t^2 g_\sigma}(\eta, t) - V_{f_{k,2}}^{t^2 g_\sigma}(\eta, t) \right| \leq \epsilon H_k(t),$$

with $H_k(t) = \sigma^3 I_3 + \frac{\pi}{3} \sigma^5 I_5 A_k(t)$. Now, if $(\eta, t) \notin Z_k$, we get:

$$\left| V_{f_k}^{t^2 g_\sigma}(\eta, t) \right| \leq \epsilon H_k(t) + \frac{3A_k(t)}{2\pi^2 \Delta^2} \tilde{\epsilon}^2.$$

From this, we immediately deduce that, for $(\eta, t) \in Z_k$, we have:

$$\left| V_f^{t^2 g_\sigma}(\eta, t) - V_{f_{k,2}}^{t^2 g_\sigma}(\eta, t) \right| \leq \epsilon \sum_{l=1}^K H_k(t) + \tilde{\epsilon}^2 \sum_{l \neq k} \frac{3A_l(t)}{2\pi^2 \Delta^2} = \tilde{\epsilon}^2 \Psi_k(t),$$

where $\Psi_k(t) = \tilde{\epsilon}^4 \sum_{l=1}^K H_k(t) + \sum_{l \neq k} \frac{3A_l(t)}{2\pi^2 \Delta^2}$.

Proof. The proof is similar to that of Proposition 9 and is therefore not displayed here. □

Proposition 13. *For any (η, t) one has:*

$$\left| \partial_{tt}^2 V_f^{g_\sigma}(\eta, t) - 2i\pi \sum_{l=1}^K \left((\phi_l''(t) + 2i\pi \phi_l'(t)^2) V_{f_{l,2}}^{g_\sigma}(\eta, t) + 4i\pi \phi_l'(t) \phi_l''(t) V_{f_{l,2}}^{t^2 g_\sigma}(\eta, t) + 2i\pi \phi_l''(t)^2 V_{f_{l,2}}^{t^2 g_\sigma}(\eta, t) \right) \right| \leq \epsilon J(t). \quad (61)$$

where

$$\begin{aligned}
J(t) = & K(I_0 + 2\pi\sigma^2 I_2 + \frac{\pi}{2}\sigma^4 I_4) + \pi \sum_{l=1}^K [2A_l(t)\sigma I_1 + \phi'_l(t) (A_l(t)(1 + \pi)\sigma^2 I_2 + I_0) \\
& + |\phi''_l(t)| (\sigma^3 I_3 + 2\pi E_l(t) + \sigma I_1 + \pi A_l(t)\sigma^3 I_3) \\
& + 2\pi\phi'(t)^2 E_l(t) + 4\pi\phi'(t)|\phi''_l(t)|F_l(t) + 2\pi|\phi''(t)|^2 H_l(t)].
\end{aligned}$$

Then, if $(\eta, t) \in Z_k$ one obtains:

$$\begin{aligned}
& \left| \partial_{tt}^2 V_f^{g\sigma}(\eta, t) - 2i\pi \left((\phi''_k(t) + 2i\pi\phi'_k(t)^2) V_{f_{k,2}}^{g\sigma}(\eta, t) \right. \right. \\
& \quad \left. \left. + 4i\pi\phi'_k(t)\phi''_k(t) V_{f_{k,2}}^{tg\sigma}(\eta, t) + 2i\pi\phi''_k(t)^2 V_{f_{k,2}}^{t^2g\sigma}(\eta, t) \right) \right| \leq \tilde{\epsilon}^2 J_k(t), \text{ and} \\
& \left| \partial_{tt}^2 V_f^{g\sigma}(\eta, t) - 2i\pi \left((\phi''_k(t) + 2i\pi\phi'_k(t)^2) V_f^{g\sigma}(\eta, t) \right. \right. \\
& \quad \left. \left. + 4i\pi\phi'_k(t)\phi''_k(t) V_f^{tg\sigma}(\eta, t) + 2i\pi\phi''_k(t)^2 V_f^{t^2g\sigma}(\eta, t) \right) \right| \\
= & \left| \partial_{tt}^2 V_f^{g\sigma}(\eta, t) - 2i\pi \left(\phi''_k(t) V_f^{g\sigma}(\eta, t) + \phi'_k(t) \partial_t V_f^{g\sigma}(\eta, t) + \phi''_k(t) \partial_t V_f^{tg\sigma}(\eta, t) \right) \right| \\
& \leq \tilde{\epsilon}^2 \left(J_k(t) + (\phi''_k(t) + 2i\pi\phi'_k(t)^2) \Omega_k(t) + 4i\pi\phi'_k(t)\phi''_k(t) \Lambda_k(t) \right. \\
& \quad \left. + 2i\pi\phi''_k(t)^2 \Psi_k(t) \right) \leq \tilde{\epsilon}^2 B_{2,k}(t),
\end{aligned}$$

where

$$J_k(t) = \tilde{\epsilon}^4 J(t) + \pi \sum_{l \neq k} A_l(t) \left(2|\phi''_l(t)| + 4\pi\phi'_l(t)^2 + \frac{8}{\Delta} \phi'_l(t)|\phi''_l(t)| + \frac{3}{\pi^2 \Delta^2} \phi''_l(t)^2 \right).$$

Proof. Since

$$\begin{aligned}
\partial_{tt}^2 V_f^{g\sigma}(\eta, t) &= \sum_{l=1}^K \int_{\mathbb{R}} \left(A_l''(\tau) + 2i\pi\phi_l''(\tau)A_l(\tau) + 2i\pi\phi_l'(\tau)A_l'(\tau) \right. \\
&\quad \left. - 4\pi^2\phi_l'(\tau)^2 A_l(\tau) \right) e^{2i\pi\phi_l(\tau)} g_\sigma(\tau - t) e^{-2i\pi\eta\tau - t} d\tau, \\
&= \sum_{l=1}^K \int_{\mathbb{R}} A_l''(\tau) e^{2i\pi\phi_l(\tau)} g_\sigma(\tau - t) e^{-2i\pi\eta(\tau - t)} d\tau \\
&\quad + 2i\pi \sum_{l=1}^K \int_{\mathbb{R}} \phi_l''(\tau) A_l(\tau) e^{2i\pi\phi_l(\tau)} g_\sigma(\tau - t) e^{-2i\pi\eta(\tau - t)} d\tau \\
&\quad + 2i\pi \sum_{l=1}^K \int_{\mathbb{R}} \phi_l'(\tau) \left(A_l'(\tau) + 2i\pi\phi_l'(\tau)A_l(\tau) \right) e^{2i\pi\phi_l(\tau)} g_\sigma(\tau - t) e^{-2i\pi\eta(\tau - t)} d\tau.
\end{aligned}$$

We have for the first part of $\partial_{tt}^2 V_f^{g\sigma}(\eta, t)$

$$\left| \sum_{l=1}^K \int_{\mathbb{R}} A_l''(\tau) e^{2i\pi\phi_l(\tau)} g_\sigma(\tau - t) e^{-2i\pi\eta(\tau - t)} d\tau \right| \leq K\epsilon I_0. \quad (62)$$

Then, for the second part of $\partial_{tt}^2 V_f^{g\sigma}(\eta, t)$ we may write

$$\begin{aligned}
&2i\pi \sum_{l=1}^K \int_{\mathbb{R}} \phi_l''(\tau) A_l(\tau) e^{2i\pi\phi_l(\tau)} g_\sigma(\tau - t) e^{-2i\pi\eta(\tau - t)} d\tau, \\
&= 2i\pi \sum_{l=1}^K \int_{\mathbb{R}} \left(\phi_l''(t) + \int_t^\tau \phi_l'''(u) du \right) A_l(\tau) e^{2i\pi\phi_l(\tau)} g_\sigma(\tau - t) e^{-2i\pi\eta(\tau - t)} d\tau, \\
&= 2i\pi \sum_{l=1}^K \phi_l''(t) V_{f_l}^{g\sigma}(\eta, t) + 2i\pi \sum_{l=1}^K \int_{\mathbb{R}} \int_t^\tau \phi_l'''(u) du A_l(\tau) e^{2i\pi\phi_l(\tau)} g_\sigma(\tau - t) e^{-2i\pi\eta(\tau - t)} d\tau.
\end{aligned} \quad (63)$$

We then deduce that:

$$\begin{aligned}
&\left| 2i\pi \sum_{l=1}^K \left(\int_{\mathbb{R}} \phi_l''(\tau) A_l(\tau) e^{2i\pi\phi_l(\tau)} g_\sigma(\tau - t) e^{-2i\pi\eta(\tau - t)} d\tau - \phi_l''(t) V_{f_l}^{g\sigma}(\eta, t) \right) \right| \\
&\leq 2\pi\epsilon \sum_{l=1}^K (A_l(t)\sigma I_1 + \epsilon\sigma^2 I_2) \leq 2\pi\epsilon \sum_{l=1}^K (A_l(t)\sigma I_1 + \sigma^2 I_2).
\end{aligned} \quad (64)$$

We finally write for the third part of $\partial_{tt}^2 V_f^{g\sigma}(\eta, t)$:

$$\begin{aligned}
& 2i\pi \sum_{l=1}^K \int_{\mathbb{R}} \phi_l'(\tau) \left(A_l'(\tau) + 2i\pi\phi_l'(\tau)A(\tau) \right) e^{2i\pi\phi_l(\tau)} g_\sigma(\tau-t) e^{-2i\pi\eta(\tau-t)} d\tau \\
&= 2i\pi \sum_{l=1}^K \int_{\mathbb{R}} \left(\phi_l'(t) + (\tau-t)\phi_l''(t) + \int_t^\tau (\tau-u)\phi_l'''(u)du \right) \\
&\quad \left(A_l'(\tau) + 2i\pi\phi_l'(\tau)A_l(\tau) \right) e^{2i\pi\phi_l(\tau)} g_\sigma(\tau-t) e^{-2i\pi\eta(\tau-t)} d\tau, \\
&= 2i\pi \sum_{l=1}^K \left(\phi_l'(t)\partial_t V_{f_l}^{g\sigma}(\eta, t) + \phi_l''(t)\partial_t V_{f_l}^{tg\sigma}(\eta, t) \right) \\
&\quad + 2i\pi \sum_{l=1}^K \int_{\mathbb{R}} \int_t^\tau (\tau-u)\phi_l'''(u)du \left(A_l'(\tau) + 2i\pi\phi_l'(\tau)A(\tau) \right) e^{2i\pi\phi_l(\tau)} g_\sigma(\tau-t) e^{-2i\pi\eta(\tau-t)} d\tau,
\end{aligned} \tag{65}$$

for which we remark that

$$\begin{aligned}
& \left| 2i\pi \sum_{l=1}^K \int_{\mathbb{R}} \int_t^\tau (t-\tau)\phi_l'''(u)du \left(A_l'(\tau) + 2i\pi\phi_l'(\tau)A_l(\tau) \right) e^{2i\pi\phi_l(\tau)} g_\sigma(\tau-t) e^{-2i\pi\eta(\tau-t)} d\tau \right|, \\
& \leq \pi\epsilon \left[\sum_{l=1}^K A_l(t) \left(\phi_l'(t)\sigma^2 I_2 + |\phi_l''(t)|\sigma^3 I_3 + \epsilon\sigma^4 \frac{I_4}{2} \right) \right. \\
& \quad \left. + \epsilon \left(\sigma^2 I_2 + \sigma^3 I_3 \phi_l'(t) + \sigma^4 I_4 |\phi_l''(t)| + \epsilon\sigma^5 \frac{I_5}{2} \right) \right], \\
& \leq \pi\epsilon \sum_{l=1}^K A_l(t) \left(\phi_l'(t)\sigma^2 I_2 + |\phi_l''(t)|\sigma^3 I_3 + \sigma^4 \frac{I_4}{2} \right),
\end{aligned} \tag{66}$$

if ϵ is sufficiently small. From this, we get:

$$\left| \partial_{tt}^2 V_f^{g\sigma}(\eta, t) - 2i\pi \sum_{l=1}^K \left(\phi_l''(t)V_{f_l}^{g\sigma}(\eta, t) + \phi_l'(t)\partial_t V_{f_l}^{g\sigma}(\eta, t) + \phi_l''(t)\partial_t V_{f_l}^{tg\sigma}(\eta, t) \right) \right| \leq \epsilon J_0(t), \tag{67}$$

where

$$\begin{aligned}
J_0(t) &= KI_0 + 2\pi \sum_{l=1}^K (A_l(t)\sigma I_1 + \sigma^2 I_2) + \pi \sum_{l=1}^K A_l(t) \left(\phi_l'(t)\sigma^2 I_2 + |\phi_l''(t)|\sigma^3 I_3 + \sigma^4 \frac{I_4}{2} \right), \\
&= K(I_0 + 2\pi\sigma^2 I_2 + \frac{\pi}{2}\sigma^4 I_4) + \pi \sum_{l=1}^K A_l(t) \left(2\sigma I_1 + \phi_l'(t)\sigma^2 I_2 + |\phi_l''(t)|\sigma^3 I_3 \right).
\end{aligned}$$

Now recalling that:

$$\begin{aligned}
|\partial_t V_{f_l}^{g\sigma}(\eta, t) - 2i\pi \left(\phi_l'(t)V_{f_l}^{g\sigma}(\eta, t) + \phi_l''(t)V_{f_l}^{tg\sigma}(\eta, t) \right)| &\leq \epsilon (I_0 + \pi A_l(t)\sigma^2 I_2), \\
|\partial_t V_{f_l}^{tg\sigma}(\eta, t) - 2i\pi \left(\phi_l'(t)V_{f_l}^{tg\sigma}(\eta, t) + \phi_l''(t)V_{f_l}^{t^2g\sigma}(\eta, t) \right)| &\leq \epsilon\sigma (I_1 + \pi A_l(t)\sigma^2 I_3),
\end{aligned}$$

we get,

$$\begin{aligned}
\left| \partial_{tt}^2 V_f^{g\sigma}(\eta, t) - 2i\pi \sum_{l=1}^K \left((\phi_l''(t) + 2i\pi\phi_l'(t)^2)V_{f_l}^{g\sigma}(\eta, t) \right. \right. \\
\left. \left. + 4i\pi\phi_l'(t)\phi_l''(t)V_{f_l}^{tg\sigma}(\eta, t) + 2i\pi\phi_l''(t)^2V_{f_l}^{t^2g\sigma}(\eta, t) \right) \right| \leq \epsilon J_{00}(t),
\end{aligned}$$

$$\text{where } J_{00}(t) = J_0(t) + 2\pi \sum_{l=1}^K \phi_l'(t)(I_0 + \pi A_l(t)\sigma^2 I_2) + \sigma |\phi_l''(t)|(I_1 + \pi A_l(t)\sigma^2 I_3).$$

Now, we recall that we have, for any l :

$$\begin{aligned}
|V_{f_l}^{g\sigma}(\eta, t) - V_{f_{l,2}}^{g\sigma}(\eta, t)| &\leq \epsilon E_l(t), \\
|V_{f_l}^{tg\sigma}(\eta, t) - V_{f_{l,2}}^{tg\sigma}(\eta, t)| &\leq \epsilon F_l(t), \\
|V_{f_l}^{t^2g\sigma}(\eta, t) - V_{f_{l,2}}^{t^2g\sigma}(\eta, t)| &\leq \epsilon H_l(t),
\end{aligned}$$

so that we can conclude:

$$\begin{aligned}
\left| \partial_{tt}^2 V_f^{g\sigma}(\eta, t) - 2i\pi \sum_{l=1}^K \left((\phi_l''(t) + 2i\pi\phi_l'(t)^2)V_{f_{l,2}}^{g\sigma}(\eta, t) \right. \right. \\
\left. \left. + 4i\pi\phi_l'(t)\phi_l''(t)V_{f_{l,2}}^{tg\sigma}(\eta, t) + 2i\pi\phi_l''(t)^2V_{f_{l,2}}^{t^2g\sigma}(\eta, t) \right) \right| \leq \epsilon J(t).
\end{aligned}$$

$$\begin{aligned}
\text{with } J(t) = J_{00}(t) + 2\pi \sum_{l=1}^K (|\phi_l''(t)| + 2\pi\phi_l'(t)^2)E_l(t) + 4\pi\phi_l'(t)|\phi_l''(t)|F_l(t) + \\
2\pi\phi_l''(t)^2H_l(t).
\end{aligned}$$

Rearranging the terms, we obtained the expected expression.
Now if $(\eta, t) \in Z_k$ we finally have, using Propositions 5, 9, 12:

$$\left| \partial_{tt}^2 V_f^{g\sigma}(\eta, t) - 2i\pi \left((\phi_k''(t) + 2i\pi\phi_k'(t)^2) V_{f_{k,2}}^{g\sigma}(\eta, t) + 4i\pi\phi_k'(t)\phi_k''(t) V_{f_{k,2}}^{tg\sigma}(\eta, t) + 2i\pi\phi_k''(t)^2 V_{f_{k,2}}^{t^2g\sigma}(\eta, t) \right) \right| \leq \tilde{\epsilon}^2 J_k(t),$$

where

$$J_k(t) = \tilde{\epsilon}^4 J(t) + \pi \sum_{l \neq k} A_l(t) \left(2|\phi_l''(t)| + 4\pi\phi_l'(t)^2 + \frac{8}{\Delta}\phi_l'(t)|\phi_l''(t)| + \frac{3}{\pi^2\Delta^2}\phi_l''(t)^2 \right).$$

Using Proposition 7, 9 and 12, one obtains the last inequality. \square

Proposition 14. *For any $(\eta, t) \in Z_k$ one has:*

$$|\tilde{q}_f(\eta, t) - \phi_k''(t)| \leq \tilde{\epsilon}. \quad (68)$$

Proof.

$$\begin{aligned} & |\phi_k''(t) - \tilde{q}_f(\eta, t)| \\ = & \left| \phi_k''(t) - \frac{V_f^{g\sigma}(\eta, t)\partial_{tt}^2 V_f^{g\sigma}(\eta, t) - (\partial_t V_f^{g\sigma}(\eta, t))^2}{2i\pi(V_f^{g\sigma}(\eta, t))^2 - V_f^{g\sigma}(\eta, t)\partial_{\eta t}^2 V_f^{g\sigma}(\eta, t) + \partial_t V_f^{g\sigma}(\eta, t)\partial_\eta V_f^{g\sigma}(\eta, t)} \right|, \\ = & \left| \frac{\partial_t V_f^{g\sigma}(\eta, t) [\partial_t V_f^{g\sigma}(\eta, t) + \phi_k''(t)\partial_\eta V_f^{g\sigma}(\eta, t)]}{2i\pi(V_f^{g\sigma}(\eta, t))^2 - V_f^{g\sigma}(\eta, t)\partial_{\eta t}^2 V_f^{g\sigma}(\eta, t) + \partial_t V_f^{g\sigma}(\eta, t)\partial_\eta V_f^{g\sigma}(\eta, t)} \right. \\ & \left. - \frac{V_f^{g\sigma}(\eta, t) [\partial_{tt}^2 V_f^{g\sigma}(\eta, t) + \phi_k''(t)\partial_{\eta t}^2 V_f^{g\sigma}(\eta, t) - 2i\pi\phi_k''(t)V_f^{g\sigma}(\eta, t)]}{2i\pi(V_f^{g\sigma}(\eta, t))^2 - V_f^{g\sigma}(\eta, t)\partial_{\eta t}^2 V_f^{g\sigma}(\eta, t) + \partial_t V_f^{g\sigma}(\eta, t)\partial_\eta V_f^{g\sigma}(\eta, t)} \right|, \\ = & \left| \frac{\partial_t V_f^{g\sigma}(\eta, t) [\partial_t V_f^{g\sigma}(\eta, t) - 2i\pi\phi_k''(t)V_f^{tg\sigma}(\eta, t) - 2i\pi\phi_k'(t)V_f^{g\sigma}(\eta, t)]}{2i\pi(V_f^{g\sigma}(\eta, t))^2 - 2i\pi V_f^{g\sigma}(\eta, t)\partial_t V_f^{tg\sigma}(\eta, t) - 2i\pi\partial_t V_f^{g\sigma}(\eta, t)V_f^{tg\sigma}(\eta, t)} \right| \\ & + \left| \frac{V_f^{g\sigma}(\eta, t) [\partial_{tt}^2 V_f^{g\sigma}(\eta, t) - 2i\pi\phi_k''(t)\partial_t V_f^{tg\sigma}(\eta, t) - 2i\pi\phi_k''(t)V_f^{g\sigma}(\eta, t) - 2i\pi\phi_k'(t)\partial_t V_f^{g\sigma}(\eta, t)]}{2i\pi(V_f^{g\sigma}(\eta, t))^2 - 2i\pi V_f^{g\sigma}(\eta, t)\partial_t V_f^{tg\sigma}(\eta, t) - 2i\pi\partial_t V_f^{g\sigma}(\eta, t)V_f^{tg\sigma}(\eta, t)} \right|, \\ \leq & \tilde{\epsilon}^2 B_{1,k}(t) \left| \frac{\partial_t V_f^{g\sigma}(\eta, t)}{2i\pi[(V_f^{g\sigma}(\eta, t))^2 - V_f^{g\sigma}(\eta, t)\partial_t V_f^{tg\sigma}(\eta, t) - \partial_t V_f^{g\sigma}(\eta, t)V_f^{tg\sigma}(\eta, t)]} \right|, \\ & + \tilde{\epsilon}^2 B_{2,k}(t) \left| \frac{V_f^{g\sigma}(\eta, t)}{2i\pi[(V_f^{g\sigma}(\eta, t))^2 - V_f^{g\sigma}(\eta, t)\partial_t V_f^{tg\sigma}(\eta, t) - \partial_t V_f^{g\sigma}(\eta, t)V_f^{tg\sigma}(\eta, t)]} \right|. \end{aligned}$$

Let

$$M_1(t) = \max_{\eta \in Z_k} \left| \frac{\partial_t V_f^{g\sigma}(\eta, t)}{2i\pi[(V_f^{g\sigma}(\eta, t))^2 - V_f^{g\sigma} \partial_t V_f^{tg\sigma}(\eta, t) - \partial_t V_f^{g\sigma}(\eta, t) V_f^{tg\sigma}(\eta, t)]} \right|,$$

$$M_2(t) = \max_{\eta \in Z_k} \left| \frac{V_f^{g\sigma}(\eta, t)}{2i\pi[(V_f^{g\sigma}(\eta, t))^2 - V_f^{g\sigma} \partial_t V_f^{tg\sigma}(\eta, t) - \partial_t V_f^{g\sigma}(\eta, t) V_f^{tg\sigma}(\eta, t)]} \right|.$$

Then, we can write

$$|\phi_k''(t) - \tilde{q}_f(\eta, t)| \leq \tilde{\epsilon}^2 (B_{1,k}(t)M_1(t) + B_{2,k}(t)M_2(t)). \quad (69)$$

Now, if ϵ is sufficiently small, we can get, for any k and time t :

$$B_{1,k}(t)M_1(t) + B_{2,k}(t)M_2(t) \leq \tilde{\epsilon}^{-1},$$

and Finally that

$$|\tilde{q}_f(\eta, t) - \phi_k''(t)| \leq \tilde{\epsilon} \quad (70)$$

□

Lemma 3. For all $k \in \{1, \dots, K\}$ and any $(\eta, t) \in Z_k$ such that $|V_f^{g\sigma}(\eta, t)| > \tilde{\epsilon}$, we have

$$|\tilde{\omega}_f^{(2)}(\eta, t) - \phi_k'(t)| \leq \tilde{\epsilon}. \quad (71)$$

Proof. According to definition of $\tilde{\omega}_f^{(2)}(\eta, t)$ in (40), we have

$$\tilde{\omega}_f^{(2)}(\eta, t) = \tilde{\omega}_f(\eta, t) + \tilde{q}_f(\eta, t)(t - \tilde{t}_f(\eta, t)),$$

its follows that

$$\begin{aligned} \left| \tilde{\omega}_f^{(2)}(\eta, t) - \phi_k'(t) \right| &= \left| \frac{\partial_t V_f^{g\sigma}(\eta, t) + \tilde{q}_f(\eta, t) \partial_\eta V_f^{g\sigma}(\eta, t)}{2i\pi V_f^{g\sigma}(\eta, t)} - \phi_k'(t) \right|, \\ &= \left| \frac{\partial_t V_f^{g\sigma}(\eta, t) - 2i\pi \tilde{q}_f(\eta, t) V_f^{tg\sigma}(\eta, t)}{2i\pi V_f^{g\sigma}(\eta, t)} - \phi_k'(t) \right|, \\ &\leq \tilde{\epsilon}^2 \frac{B_{1,k}(t)}{|2i\pi V_f^{g\sigma}(\eta, t)|} + \frac{|\tilde{q}_f(\eta, t) - \phi_k''(t)| |V_f^{tg\sigma}(\eta, t)|}{|V_f^{g\sigma}(\eta, t)|}, \\ &\leq \tilde{\epsilon}^2 \frac{B_{1,k}(t)}{|2i\pi V_f^{g\sigma}(\eta, t)|} + \frac{\tilde{\epsilon}^2 [B_{1,k}(t)M_1(t) + B_{2,k}(t)M_2(t)] |V_f^{tg\sigma}(\eta, t)|}{|V_f^{g\sigma}(\eta, t)|}, \\ &\leq \tilde{\epsilon} E_r(t), \end{aligned}$$

where

$$E_r(t) = \tilde{\epsilon} \left(\frac{B_{1,k}(t)}{|2\pi V_f^{g\sigma}(\eta, t)|} + \frac{(B_{1,k}(t)M_1(t) + B_{2,k}(t)M_2(t)) |V_f^{tg\sigma}(\eta, t)|}{|V_f^{g\sigma}(\eta, t)|} \right).$$

It is clear again that since $V_f^{g\sigma}(\eta, t) \geq \tilde{\epsilon}$, by imposing ϵ sufficient small we get:

$$|\tilde{\omega}_f^{(2)}(\eta, t) - \phi'_k(t)| \leq \tilde{\epsilon}.$$

□

Lemma 4. Let $\tilde{T}_f^{\delta, \gamma}(\omega, t)$ be the FSST of a signal $f \in \mathcal{B}_{\Delta, \epsilon}^{(2)}$ with threshold γ and accuracy δ and h be a positive window with unit integral belong to $\mathbb{C}_c^\infty(\mathbb{R})$:

$$\tilde{T}_f^{\delta, \gamma}(\omega, t) = \frac{1}{g_\sigma(0)} \int_{|V_f^{g\sigma}(\eta, t)| > \gamma} V_f^{g\sigma}(\eta, t) \frac{1}{\delta} h \left(\frac{\omega - \hat{\omega}_f^{(2)}(\eta, t)}{\delta} \right) d\eta. \quad (72)$$

Then for each $k \in \{1, \dots, K\}$, there exists a constant C such that

$$\left| \left(\lim_{\delta \rightarrow 0} \frac{1}{2\pi g_\sigma(0)} \int_{\mathcal{M}_{k, \tilde{\epsilon}}} \tilde{T}_f^{\delta, \tilde{\epsilon}}(\omega, t) d\omega \right) - A_k(t) e^{i\phi_k(t)} \right| \leq C\tilde{\epsilon}, \quad (73)$$

where $\mathcal{M}_{k, \tilde{\epsilon}} := \{\omega : |\omega - \phi'_k(t)| < \tilde{\epsilon}\}$, provided σ is chosen such that:

$$\left| \frac{1}{g_\sigma(0)} \int_{|\eta| \leq \Delta} \overline{g_\sigma(\tau) e^{i\pi \phi'_k(t) \tau^2}}(\eta) d\eta \right| \leq (1 + \tilde{\epsilon}) \text{ for any } k.$$

Proof. For each $k \in \{1, 2, \dots, K\}$, and pair $(\eta, t) \in Z_k$, we can write

$$\begin{aligned}
& \left| \lim_{\delta \rightarrow 0} \int_{|\omega - \phi'_k(t)| < \tilde{\epsilon}} \tilde{T}_f^{\delta, \tilde{\epsilon}}(\omega, t) d\omega - f_k(t) \right| \\
&= \left| \lim_{\delta \rightarrow 0} \int_{|\omega - \phi'_k(t)| < \tilde{\epsilon}} \frac{1}{g_\sigma(0)} \int_{|V_f^{g_\sigma}(\eta, t)| > \tilde{\epsilon}} V_f^{g_\sigma}(\eta, t) \frac{1}{\delta} h\left(\frac{\omega - \hat{\omega}_f^{(2)}(\eta, t)}{\delta}\right) d\eta d\omega - f_k(t) \right|, \\
&= \left| \lim_{\delta \rightarrow 0} \int_{|V_f^{g_\sigma}(\eta, t)| > \tilde{\epsilon}} \frac{1}{g_\sigma(0)} V_f^{g_\sigma}(\eta, t) \int_{|\omega - \phi'_k(t)| < \tilde{\epsilon}} \frac{1}{\delta} h\left(\frac{\omega - \hat{\omega}_f^{(2)}(\eta, t)}{\delta}\right) d\omega d\eta - f_k(t) \right|, \\
&= \left| \frac{1}{g_\sigma(0)} \int_{|V_f^{g_\sigma}(\eta, t)| > \tilde{\epsilon}} V_f^{g_\sigma}(\eta, t) \lim_{\delta \rightarrow 0} \int_{|\omega - \phi'_k(t)| < \tilde{\epsilon}} \frac{1}{\delta} h\left(\frac{\omega - \hat{\omega}_f^{(2)}(\eta, t)}{\delta}\right) d\omega d\eta - f_k(t) \right|, \\
&= \left| \frac{1}{g_\sigma(0)} \int_{\{|V_f^{g_\sigma}(\eta, t)| > \tilde{\epsilon}\} \cap \{|\hat{\omega}_f^{(2)}(\eta, t) - \phi'_k(t)| < \tilde{\epsilon}\}} V_f^{g_\sigma}(\eta, t) d\eta - f_k(t) \right|, \\
&= \left| \frac{1}{g_\sigma(0)} \int_{\{|V_f^{g_\sigma}(\eta, t)| > \tilde{\epsilon}\} \cap \{|\eta - \phi'_k(t)| < \Delta\}} V_f^{g_\sigma}(\eta, t) d\eta - f_k(t) \right|, \\
&= \left| \frac{1}{g_\sigma(0)} \int_{\{|\eta - \phi'_k(t)| < \Delta\}} V_f^{g_\sigma}(\eta, t) d\eta - f_k(t) - \frac{1}{g_\sigma(0)} \int_{\{|\eta - \phi'_k(t)| < \Delta\} \cap \{|V_f^{g_\sigma}(\eta, t)| \leq \tilde{\epsilon}\}} V_f^{g_\sigma}(\eta, t) d\eta \right|, \\
&\leq \left| \frac{1}{g_\sigma(0)} \int_{\{|\eta - \phi'_k(t)| < \Delta\}} V_f^{g_\sigma}(\eta, t) - V_{f_{k,2}}^{g_\sigma}(\eta, t) d\eta \right| \\
&\quad + \left| \frac{1}{g_\sigma(0)} \int_{\{|\eta - \phi'_k(t)| < \Delta\}} V_{f_{k,2}}^{g_\sigma}(\eta, t) d\eta - f_k(t) \right| \frac{2\Delta}{g_\sigma(0)} \tilde{\epsilon}, \\
&\leq \left| f_k(t) \left(\frac{1}{g_\sigma(0)} \int_{|\eta| < \Delta} \overline{g_\sigma(\tau) e^{i\pi\phi_k''(t)\tau^2}}(\eta) d\eta - 1 \right) \right| \\
&\quad + \frac{1}{g_\sigma(0)} \int_{|\eta - \phi'_k(t)| < \Delta} \left| V_f^{g_\sigma}(\eta, t) - V_{f_{k,2}}^{g_\sigma}(\eta, t) \right| d\eta + \frac{2\Delta}{g_\sigma(0)} \tilde{\epsilon} \\
&\leq A_k(t) \tilde{\epsilon} + \frac{2\Delta}{g_\sigma(0)} \epsilon \Lambda_k(t) + \frac{2\Delta}{g_\sigma(0)} \tilde{\epsilon}, \\
&= \left| \lim_{\delta \rightarrow 0} \int_{|\omega - \phi'_k(t)| < \tilde{\epsilon}} \tilde{T}_f^{\delta, \tilde{\epsilon}}(\omega, t) d\omega - f_k(t) \right| \leq \left(A_k(t) + \frac{4\Delta}{g_\sigma(0)} \right) \tilde{\epsilon}.
\end{aligned}$$

□

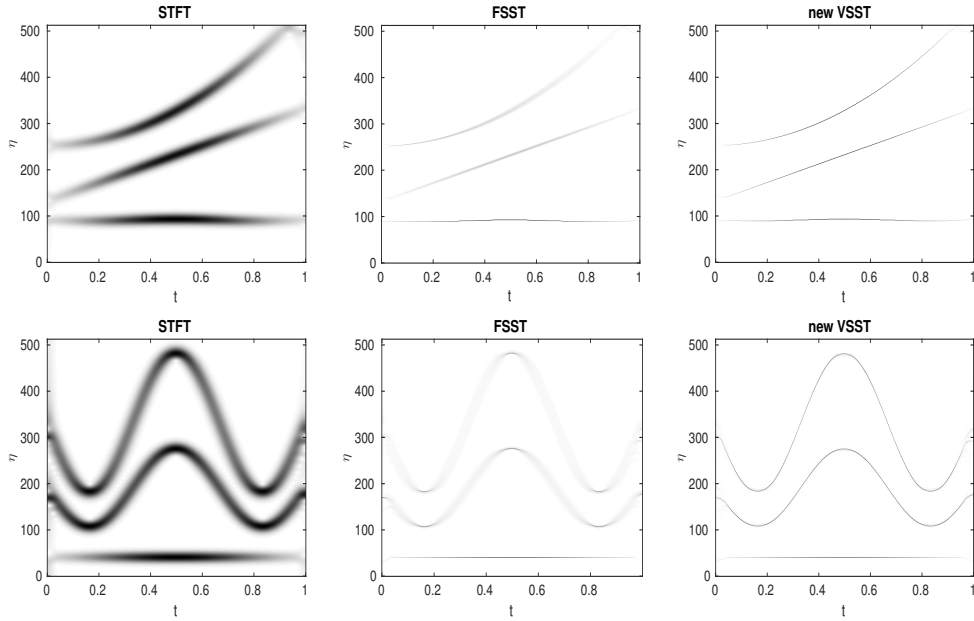


Figure 3: first row: magnitude of STFT, FSST and new VSST for test-signal 1; second row: same computation but for test-signal 2

6. Numerical Results

In order to illustrate the behavior of our new second order synchrosqueezing transform, we apply it two synthetic test signals. One (signal 1) is made of low-order polynomial chirps, that behave locally as linear chirps, and the other one (signal 2) contains strongly nonlinear sinusoidal frequency modulations. On Fig. 3, STFT, FSST, corresponding to the original synchrosqueezing transform and, our new technique called *new VSST* for both signals 1 and 2 are depicted. We choose this denomination after the so-called VSST technique which is the other transform based on the second order approximation of the phase, which was introduced in [13]. Note that in our simulations, we use 1024 time samples over $[0, 1]$ and a Gaussian window with parameter $\sigma = 0.005$, defined by $g_\sigma(t) = \sigma^{-\frac{1}{2}} e^{-\frac{\pi t^2}{\sigma^2}}$. Even though such a Gaussian window does not strictly obey the theoretical requirements on the window choice, it enables us to illustrate the improvement in terms of quality of representation and mode reconstruction brought about our new technique.

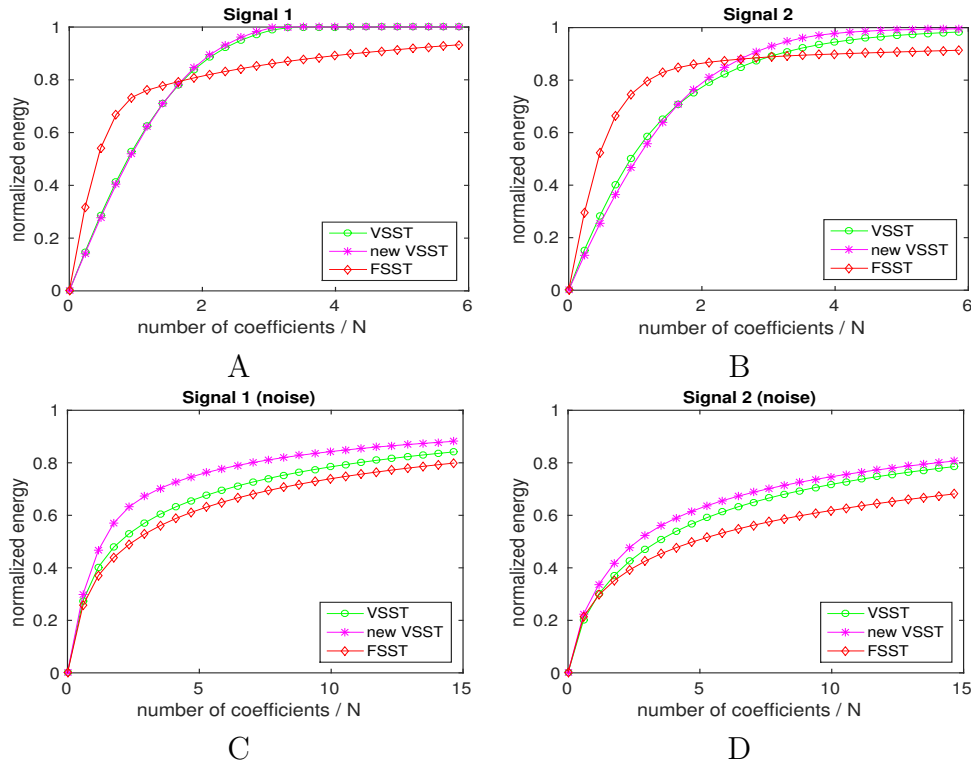


Figure 4: Normalized energy as a function of the number of sorted coefficients for test-signals 1 (A) and 2 (B). Abscissa gives the number of coefficients kept over the length N of the signal, i.e., the mean number of coefficients kept in each column of the TF plane. C and D: idem as A and B but for noisy signals (0 dB).

6.1. Quality of representation

In order to quantify the quality of the representation given by the new VSST, we propose here to measure the amount of information contained in the coefficients with the largest amplitude. In this regard, one way to compare the different transformations is to compute the normalized energy associated with the first coefficients with the largest amplitude: the faster the growth of this energy towards 1, the sharper the representation. On Fig. 4 A and B, these normalized energies are displayed with respect to the number of coefficients kept divided by the length of the signal for both test-signals and for the three representations, namely VSST proposed in [13], new VSST and FSST. In our context, the normalized energy is computed as the cumulative sum of the squared sorted coefficients over the sum of all the squared coeffi-

cients. The first remark is that both VSST and new VSST behave similarly for test-signal 1 while slightly better results are obtained when the latter is used on test-signal 2 which contains stronger nonlinear frequency modulations. The second remark is that one needs only 3 coefficients per time instant to recover the signal energy, which is consistent with the three modes making up the test-signal. In order to investigate the influence of noise on the sparsity of the representation, we carry out the same experiments when the test-signals are contaminated by white Gaussian noise (noise level 0 dB). The results displayed on Fig. 4 C and D exhibit a slower increase of the normalized energy since the coefficients corresponding to noise are spread over the whole TF plane. However, our proposed new VSST still behaves better than both VSST and FSST in this noisy situation.

6.2. Reconstruction of the modes

The main advantage of synchrosqueezing techniques over traditional reassignment techniques lies in its invertibility. To improve the accuracy of the reassignment step in synchrosqueezing techniques by using new VSST naturally leads to better reconstruction results. The reconstruction formula used to retrieve the k th mode, assuming $\hat{\phi}'_k(t)$ is an estimate of $\phi'_k(t)$, is as follows:

$$f_k(t) \approx \int_{|\omega - \hat{\phi}'_k(t)| < d} T_f(\omega, t) d\omega, \quad (74)$$

as explained in more details in [13]. The parameter d is here to compensate for the fact that $\hat{\phi}'_k(t)$ is an estimate of $\phi'_k(t)$ and not the true value. Provided d is of the same order of magnitude as the estimation error, this reconstruction formula ensures an asymptotically perfect reconstruction. To compute an estimation of the ridges $(t, \phi'_k(t))$, knowing the number K of modes, we use the algorithm introduced in [3], which computes a local minimum of the functional:

$$E_f((\varphi_k)_{k=1, \dots, K}) = \sum_{k=1}^K - \int_{\mathbb{R}} |T_f(t, \varphi_k(t))|^2 dt + \int_{\mathbb{R}} (\lambda \varphi'_k(t)^2 + \beta \varphi''_k(t)^2) dt, \quad (75)$$

λ and β being two positive tuning parameters. Then $(\hat{\phi}'_k(t))_{k=1, \dots, K} = \operatorname{argmin} E_f((\varphi_k)_{k=1, \dots, K})$.

To illustrate the behavior of our new VSST technique we first display the reconstruction process associated with test-signal 2 on Fig. 5 A, the ridges

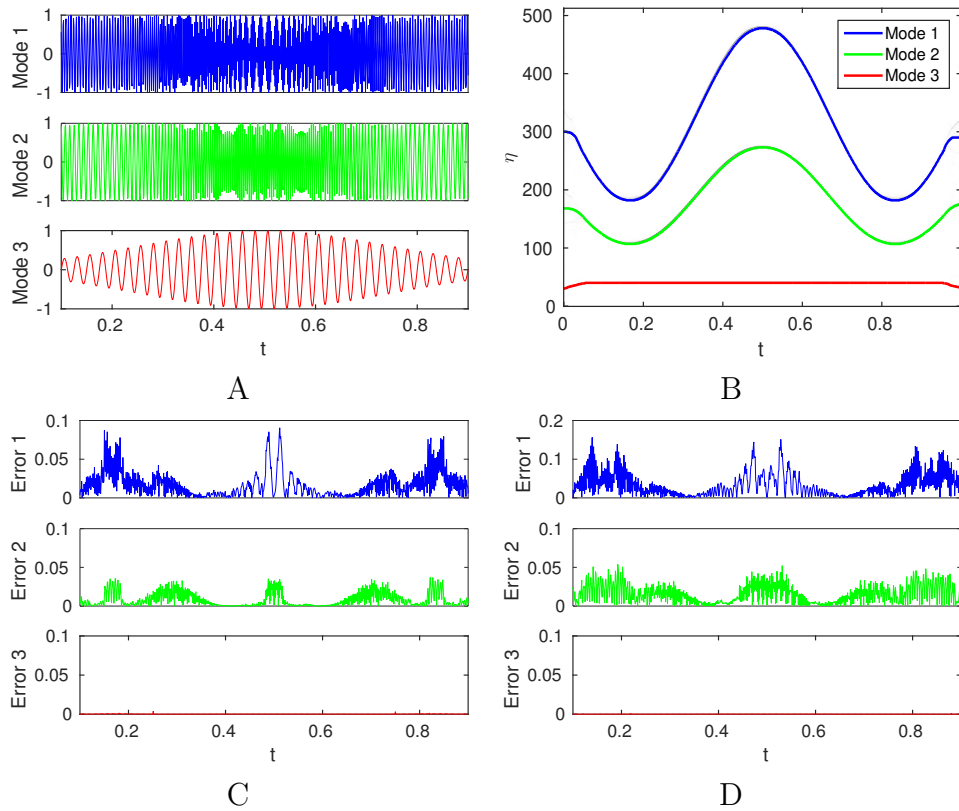
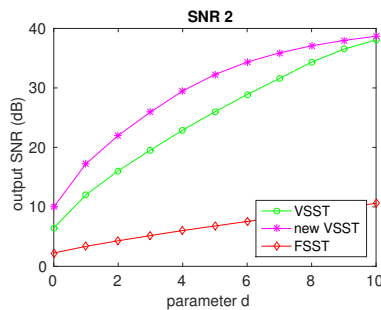


Figure 5: A:mode retrieval based on new VSST for test-signal 2; B: Estimated ridges superimposed on the new VSST of test-signal 2. We use $d = 5$ in the reconstruction, and $\beta = 0, \lambda = 0.02$ for the ridge extraction; C:Reconstruction errors associated with each mode of test-signal 2 using new VSST; D: same as C but using VSST technique

used for reconstruction being depicted on Fig. 5 B. Then, to compare the proposed new VSST with the alternative technique proposed in [13], we display, on Fig. 5 C and D, the reconstruction error associated with each time instant for each of the three modes and for both VSST and new VSST. The improvement brought by using new VSST instead of VSST appears to be the most significant when the modulation is strong (i.e., in the mode associated with the highest frequency in the studied case). To further quantify the accuracy of mode reconstruction in terms of output SNR for test-signal 2, we display in Figure 6 A and for $d = 0$, the output SNR computed by using formula $\|f\|^2/\|f - \hat{f}\|^2$, where $\|\hat{f}\|$ is the reconstructed signal. As expected, new

	FSST	VSST	New VSST
Test-signal 2	2.3	7.6	10.0
mode 1	0.93	6	7.8
mode 2	1.4	7.2	10.2
mode 3	56.6	56.6	54.3

A



B

Figure 6: A: Mode retrieval is expressed in SNR for $d = 0$. The best results are displayed using a bold font, B: SNR associated with the reconstruction of test-signal 2 as a function of d

VSST behaves better than other synchrosqueezing techniques on that kind of signals. Finally we study the quality of the reconstruction with respect to the parameter d used in the integration formula (74) and still for test-signal 2. The results depicted on Figure 6 B confirm that the sparser representation provided by new VSST naturally leads to a reconstruction procedure that requires fewer coefficients than other synchrosqueezing techniques to reach the same quality of reconstruction.

7. Conclusion

We have developed a novel synchrosqueezing transform for analysing multicomponent signals made of strongly frequency modulated modes which appears to be particularly efficient to denoise multicomponent signals. Here we slightly modified the IF estimate proposed in [13] so that the mathematical analysis of the corresponding synchrosqueezing transform is made possible. Further we have also discussed why the hypotheses made on the modes making up the multicomponent signal must be different when one considered wavelet or STFT based synchrosqueezing. A limitation of our new VSST is that it requires a window with quadratic decay and is restricted to modes with almost constant amplitude. To improve these aspects is a work currently underway. Nevertheless, this novel transform can already be used without any restriction to denoise monocomponent signals with numerical experiments demonstrated the effectiveness of the approach for the denoising of multicomponent signals.

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