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# SCHUR AND FOURIER MULTIPLIERS OF AN AMENABLE GROUP ACTING ON NON-COMMUTATIVE $L^p$ -SPACES

MARTIJN CASPERS AND MIKAEL DE LA SALLE

ABSTRACT. Consider a completely bounded Fourier multiplier  $\phi$  of a locally compact group  $G$ , and take  $1 \leq p \leq \infty$ . One can associate to  $\phi$  a Schur multiplier on the Schatten classes  $\mathcal{S}_p(L^2G)$ , as well as a Fourier multiplier on  $L^p(\mathcal{L}G)$ , the non-commutative  $L^p$ -space of the group von Neumann algebra of  $G$ . We prove that the completely bounded norm of the Schur multiplier is not greater than the completely bounded norm of the  $L^p$ -Fourier multiplier. When  $G$  is amenable we show that equality holds, extending a result by Neuwirth and Ricard to non-discrete groups.

For a discrete group  $G$  and in the special case when  $p \neq 2$  is an even integer, we show the following. If there exists a map between  $L^p(\mathcal{L}G)$  and an ultraproduct of  $L^p(\mathcal{M}) \otimes \mathcal{S}_p(L^2G)$  that intertwines the Fourier multiplier with the Schur multiplier, then  $G$  must be amenable. This is an obstruction to extend the Neuwirth-Ricard result to non-amenable groups.

## 1. INTRODUCTION

This paper studies the close connection between Schur and Fourier multipliers of locally compact groups  $G$ . In particular, we are interested in the relation between the completely bounded norms of such multipliers.

Given a bounded function  $\phi : G \rightarrow \mathbb{C}$ , the associated Fourier multiplier  $T_\phi$ , when it exists, is the unique weak-\* continuous map on the von Neumann algebra  $\mathcal{L}G$  of  $G$  extending  $\lambda_g \mapsto \phi(g)\lambda_g$ . The associated Schur multiplier  $M_\phi$ , when it exists, is the unique weak-\* continuous map on  $B(L^2G)$  extending  $(a_{s,t})_{s,t \in G} \in \mathcal{S}_2(L^2G) \mapsto (\phi(st^{-1})a_{s,t})_{s,t \in G}$ . Bożejko and Fendler proved [2] that  $M_\phi$  indeed determines a bounded map if and only if  $T_\phi$  defines a completely bounded map, and in this case the completely bounded norms coincide. The function  $\phi$  is then called a completely bounded Fourier multiplier and the space of such  $\phi$  is denoted  $M_{\mathcal{CBA}}(G)$ . Note that  $T_\phi$  is the restriction of  $M_\phi$  to  $\mathcal{L}G \subset B(L^2G)$ .

In this paper we investigate the Bożejko-Fendler result for  $L^p$ -multipliers. When  $G$  is discrete and  $\phi \in M_{\mathcal{CBA}}(G)$ , there is no technical difficulty for defining the Fourier multiplier  $T_\phi^p : L^p(\mathcal{L}G) \rightarrow L^p(\mathcal{L}G)$  and the Schur multiplier  $M_\phi^p : \mathcal{S}_p(\ell^2G) \rightarrow \mathcal{S}_p(\ell^2G)$  (they are just the extension of  $T_\phi$ / restriction of  $M_\phi$ ). When  $G$  is not discrete, the definition of the Schur multiplier  $M_\phi^p : \mathcal{S}_p(L^2G) \rightarrow \mathcal{S}_p(L^2G)$  is known (see Subsection 3.3), and we define in Subsection 3.7 the  $L^p$ -Fourier multiplier  $T_\phi^p : L^p(\mathcal{L}G) \rightarrow L^p(\mathcal{L}G)$  spatially [5], [6].

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We are interested in the following question.

**Question 1.1.** *Let  $G$  be a locally compact group. Is it true that,*

$$\|T_\phi^p\|_{\mathcal{CB}(L^p(\mathcal{L}G))} = \|M_\phi^p\|_{\mathcal{CB}(\mathcal{S}_p(L^2G))}.$$

*for all completely bounded Fourier multipliers  $\phi$  of  $G$ ?*

There are several motivations to study this problem. One is that a positive answer would imply that for a discrete group  $G$ , the property  $\text{AP}_{p,cb}^{\text{Schur}}$  considered in [10], see also [9], is an invariant of the group von Neumann algebra of  $G$ . Furthermore, one is often interested in strict estimates of the norm of the transferred Fourier multiplier, see for example [4, Section 7].

In [11], Neuwirth and Ricard studied Question 1.1 for discrete groups. They noted that the inequality  $\geq$  always holds, and they proved the other inequality in the case  $G$  is amenable. This is an  $L^p$ -version of the Bożejko-Fendler result. Their proof relies on the fact that the amenability of  $G$  allows them to construct a completely isometric embedding of  $L^p(\mathcal{L}G)$  into an ultrapower of  $\mathcal{S}_p(\ell^2G)$  that intertwines Fourier and Schur multipliers.

Marius Junge (personal communication) pointed out to the second-named author that a positive answer to Question 1.1 would also follow from the existence of a completely isometric embedding of  $L^p(\mathcal{L}G)$  into an ultraproduct of  $L^p(\mathcal{M}_n) \otimes \mathcal{S}_p(L^2G)$  for some net of von Neumann algebras  $\mathcal{M}_n$ , that intertwines Fourier and Schur multipliers. He suggested that some weaker approximation property (as exactness) might provide such an embedding. Our first result (Theorem 2.1) partially answers his suggestion negatively: if  $p \neq 2$  is an even integer,  $G$  is a discrete group, and such an embedding exists, then  $G$  is amenable. This bad news was the motivation to study Question 1.1 for general locally compact groups.

The main part of the present paper is to extend the Neuwirth-Ricard result to arbitrary locally compact groups: we first prove in Theorem 4.2 by a transference technique that the inequality  $\geq$  always holds. We then prove the inequality  $\leq$  for  $G$  amenable in Theorem 5.2 and its Corollary 5.3. Our proof is close to the proof given in [11, Section 3], but we encounter several technical issues that we have to overcome, mainly in the case of non-unimodular groups.

Let us explain a bit more in details the technical complications that we meet. Recall [13] that to a von Neumann algebra  $\mathcal{M}$  one can associate non-commutative  $L^p$  spaces  $L^p(\mathcal{M})$  for  $1 \leq p \leq \infty$ . There are several different ways to define  $L^p(\mathcal{M})$ , but they all yield (completely) isometric spaces. When  $\mathcal{M} = B(\mathcal{H})$ ,  $L^p(\mathcal{M})$  is most naturally realized as the Schatten class  $\mathcal{S}_p(\mathcal{H})$  (which is contained in  $B(\mathcal{H})$ ). When  $\mathcal{M}$  carries a normal faithful finite trace  $\tau$  (for example  $\mathcal{M} = \mathcal{L}G$  for a discrete group  $G$ ), the natural choice is to see  $L^p(\mathcal{M})$  as the completion of  $\mathcal{M}$  for the norm  $\|x\|_p = \tau(|x|^p)^{1/p}$  ( $L^p(\mathcal{M})$  contains  $\mathcal{M}$ ). When  $G$  is a locally compact group, there is a natural weight on  $\mathcal{L}G$ , the Plancherel weight, which is a trace if and only if  $G$  is unimodular. Although  $\mathcal{L}G$  is often semi-finite, it is more natural to view the  $L^p$ -spaces of  $\mathcal{L}G$  as constructed from this weight. As is well-known, the non-commutative  $L^p$ -spaces then have the property that  $L^p$  and  $L^q$  do not naturally intersect for  $p \neq q$ . It turns out that the most convenient way to define the  $L^p$ -Fourier multiplier is by realizing  $L^p(\mathcal{L}G)$  in the Connes-Hilsum construction (Subsection 3.5) associated to the Plancherel weight on  $\mathcal{L}G' = \mathcal{R}G$  (the von

Neumann algebra of  $L^2G$  associated to the right translations). Indeed, this realizes  $L^p(\mathcal{L}G)$  as a space of unbounded operators on  $B(L^2G)$ , and the naive definition of  $T_\phi^p$  would just be the restriction of  $T_\phi$  to  $L^p(\mathcal{L}G)$ . Since in general (*i.e.* unless  $G$  is unimodular)  $L^p(\mathcal{L}G) \cap B(L^2(G)) = \{0\}$  this is a bit too naive, and the precise definition along these lines is given in the Definition-Proposition 3.5.

An important observation we make is that although  $L^p(\mathcal{L}G)$  might not contain nonzero bounded operators, its corners in fact lie in the Schatten class  $\mathcal{S}_p$ . More precisely, for  $F \subset G$  relatively compact we can (carefully) define the corner  $P_F x P_F$  of any element  $x \in L^p(\mathcal{L}G)$ , and we prove in Theorem 5.1 that  $P_F x P_F \in \mathcal{S}_p(L^2F)$  with norm at most  $|F|^{1/p} \|x\|_{L^p(\mathcal{L}G)}$ . When  $G$  is amenable, one then chooses a Følner net  $F_n \subset G$  and consider the maps  $x \mapsto |F_n|^{-1/p} P_{F_n} x P_{F_n}$ .

In all the definitions and statements we give,  $L^p(\mathcal{L}G)$  will be the Connes-Hilsum space, but the theory of interpolation [17], [7] of non-commutative  $L^p$ -spaces associated to a weight plays an important role in our proofs. This is a bit technical. Therefore, to make the paper readable we collect (and prove when necessary) all the results we need from interpolation in Theorem 6.2, and in the proofs we use this theorem as an abstract black box.

The structure of the paper is as follows. Section 2, which is independent from the rest of the paper, gives a sufficient (and necessary) condition for a discrete group to be amenable in terms of intertwiners of Schur and Fourier multipliers. Section 3 recalls the necessary results and fixes the notation. In Section 4, we prove that the inequality  $\geq$  in Question 1.1 holds for any locally compact group. Section 5 gives the affirmative answer to Question 1.1 for amenable groups. The last Section 6 on interpolation states and then proves all the results needed in the first sections on interpolation of non-commutative  $L^p$ -spaces.

## 2. CHARACTERIZING AMENABILITY BY INTERTWINING SCHUR AND FOURIER MULTIPLIERS

In this section we prove the following characterization of the amenability of a discrete group. It takes away our hope to give a direct positive answer to Question 1.1 for non-amenable discrete groups.

Recall that the Schur multiplier  $M_\phi^p$  and the Fourier multiplier  $T_\phi^p$  were defined in the introduction for discrete groups. We define them more generally in Section 3.

**Theorem 2.1.** *Let  $\Gamma$  be a discrete group, let  $\mathcal{H}$  be a Hilbert space and let  $p \geq 4$  be an even integer. Let  $\mathcal{U}$  be a non-trivial ultrafilter on some set. Assume that there exists an isometric embedding*

$$(1) \quad j_p : L^p(\mathcal{L}\Gamma) \rightarrow \prod_{\mathcal{U}} \mathcal{S}_p(\ell^2\Gamma \otimes \mathcal{H})$$

*such that for every  $\phi : \Gamma \rightarrow \mathbb{C}$  of finite support,  $j_p \circ T_\phi^p = (\prod_{\mathcal{U}} M_\phi^p \otimes id) \circ j_p$ . Then,  $\Gamma$  is amenable.*

*Conversely, if  $\Gamma$  is amenable such an embedding exists for all  $1 \leq p \leq \infty$  and we can take  $\mathcal{H} = \mathbb{C}$ .*

**Remark 2.2.** In fact the proof below shows more generally that the same conclusion holds if there is such an embedding of  $L^p(\mathcal{L}(\Gamma))$  into  $\prod_{\mathcal{U}} L^p(B(\ell^2\Gamma) \otimes \mathcal{M}_\alpha)$  with a net  $(\mathcal{M}_\alpha, \tau_\alpha)$  of semi-finite von Neumann algebras.

*Proof.* To keep the notation simple, we consider the case  $p = 4$  and write  $j$  for  $j_4$ . The same proof works for every  $p \in 2\mathbb{N}$ .

Fix  $\gamma \in \Gamma$ , and let  $(\Xi_\alpha)$  be a representative of  $j(\lambda(\gamma))$ . Express that  $j \circ T_\phi^4 = (\prod_{\mathcal{U}} M_\phi^p \otimes id) \circ j$  for  $\phi = \delta_\gamma$  a Dirac mass point. This implies that  $j(\lambda(\gamma)) = ((M_{\delta_\gamma}^4 \otimes id)(\Xi_\alpha))_\alpha$ , so that we can assume that  $(M_{\delta_\gamma} \otimes id)(\Xi_\alpha) = \Xi_\alpha$  for all  $\alpha$ . In other words, if we see  $\Xi_\alpha$  as an infinite matrix with values in  $\mathcal{S}_p(\mathcal{H})$ , its  $(s, t)$  entry is zero unless  $st^{-1} = \gamma$ . Equivalently, there is an element  $\xi \in \ell^p(\Gamma; \mathcal{S}_p(\mathcal{H})) \subset \mathcal{S}_p(\ell^2\Gamma \otimes \mathcal{H})$  such that  $\Xi_\alpha = (\lambda(\gamma) \otimes id)\xi$ . Let us denote this  $\xi$  by  $\xi_\gamma^{(\alpha)}$ . Using that  $1 = \|\lambda(\gamma)\|_{L^4(\mathcal{L}(G))} = \lim_{\alpha, \mathcal{U}} \|\xi_\gamma^{(\alpha)}\|_{\ell^4(\Gamma; \mathcal{S}_4(\mathcal{H}))}$ , we can as well assume that  $\|\xi_\gamma^{(\alpha)}\|_{\ell^4(\Gamma; \mathcal{S}_4(\mathcal{H}))} = 1$ . Since  $\gamma \in \Gamma$  was arbitrary, we can choose such a representative for all  $\gamma$ .

Let  $\eta_\gamma^{(\alpha)}$  be the unit vector in  $\ell^4\Gamma$  defined by  $\eta_\gamma^{(\alpha)} = (\|\xi_\gamma^{(\alpha)}(s)\|_{\mathcal{S}_4(\mathcal{H})})_{s \in \Gamma}$ . We claim that,

$$(2) \quad \forall \gamma, s \in \Gamma, \lim_{\alpha, \mathcal{U}} \|\lambda_s(\eta_\gamma^{(\alpha)})^2 - (\eta_\gamma^{(\alpha)})^2\|_{\ell^2\Gamma} = 0.$$

In particular,  $\lambda$  has almost invariant vectors in  $\ell^2\Gamma$ , which implies that  $\Gamma$  is amenable.

To prove the claim, take  $x = \sum_\gamma x_\gamma \lambda_\gamma$  arbitrary in the group algebra  $\mathbb{C}[\Gamma]$  with  $x_\gamma \geq 0$ . Then, we find  $x^*x = \sum_s (\sum_\gamma \overline{x_\gamma} x_{\gamma s}) \lambda_s$ , so that

$$\|x\|_{L^4(\mathcal{L}G)}^4 = \sum_{s, \gamma, \tilde{\gamma}} x_\gamma \overline{x_{\gamma s}} \overline{x_{\tilde{\gamma}}} x_{\tilde{\gamma} s}.$$

To compute  $\|j(x)\|_{\prod_{\mathcal{U}} \mathcal{S}_4(\ell^2\Gamma \otimes \mathcal{H})}^4$  and to simplify the notation, we shall denote, for  $s \in \Gamma$  and  $\xi \in \ell^p(\Gamma; \mathcal{S}_p(\mathcal{H}))$ ,  $s \cdot \xi = (\xi(s^{-1}t))_{t \in \Gamma}$ . We shall also use the relation  $s \cdot \xi = (\lambda_s \otimes id)\xi(\lambda_s^* \otimes id)$ . Hence,

$$j(x)^*j(x) = \left( \sum_s (\lambda_s \otimes id) \left( \sum_\gamma \overline{x_\gamma} x_{\gamma s} (s^{-1} \cdot \xi_\gamma^{(\alpha)})^* \xi_{\gamma s}^{(\alpha)} \right) \right)_\alpha,$$

so that

$$\|j(x)\|_{\prod_{\mathcal{U}} \mathcal{S}_4(\ell^2\Gamma \otimes \mathcal{H})}^4 = \lim_{\alpha, \mathcal{U}} \sum_{s, \gamma, \tilde{\gamma}} x_\gamma \overline{x_{\gamma s}} \overline{x_{\tilde{\gamma}}} x_{\tilde{\gamma} s} \text{Tr} \left( (\xi_{\gamma s}^{(\alpha)})^* s^{-1} \cdot (\xi_\gamma^{(\alpha)} \xi_{\tilde{\gamma}}^{(\alpha)*}) \xi_{\tilde{\gamma} s}^{(\alpha)} \xi_{\gamma s}^{(\alpha)*} \right).$$

Here,  $\text{Tr}$  is the natural (semi-finite) trace on  $\ell^\infty(\Gamma; B(\mathcal{H}))$ . By  $x_\gamma \geq 0$  and Hölder's inequality, we have

$$\begin{aligned} \|j(x)\|_{\prod_{\mathcal{U}} \mathcal{S}_4(\ell^2\Gamma \otimes \mathcal{H})}^4 &= \sum_{s, \gamma, \tilde{\gamma}} x_\gamma \overline{x_{\gamma s}} \overline{x_{\tilde{\gamma}}} x_{\tilde{\gamma} s} \lim_{\alpha, \mathcal{U}} \text{Re} \text{Tr} \left( s^{-1} \cdot (\xi_\gamma^{(\alpha)} \xi_{\tilde{\gamma}}^{(\alpha)*}) \xi_{\tilde{\gamma} s}^{(\alpha)} \xi_{\gamma s}^{(\alpha)*} \right) \\ &\leq \sum_{s, \gamma, \tilde{\gamma}} x_\gamma \overline{x_{\gamma s}} \overline{x_{\tilde{\gamma}}} x_{\tilde{\gamma} s} 1 = \|x\|_{L^4(\mathcal{L}G)}^4. \end{aligned}$$

Hence if  $\gamma, \tilde{\gamma}, \gamma s, \tilde{\gamma} s$  belong to the support of  $x$ ,

$$\lim_{\alpha, \mathcal{U}} \text{Re} \text{Tr} \left( s^{-1} \cdot (\xi_\gamma^{(\alpha)} \xi_{\tilde{\gamma}}^{(\alpha)*}) \xi_{\tilde{\gamma} s}^{(\alpha)} \xi_{\gamma s}^{(\alpha)*} \right) = 1,$$

so that by the uniform convexity Lemma 2.3 below

$$(3) \quad \lim_{\alpha, \mathcal{U}} \|\lambda_s(\eta_{\tilde{\gamma} s}^{(\alpha)})^2 - (\eta_{\tilde{\gamma} s}^{(\alpha)})^2\|_{\ell^2(\Gamma)} = 0.$$

Since  $x$  was arbitrary, (3) holds for all  $s, \gamma, \tilde{\gamma} \in \Gamma$ . This proves (2), and hence that  $\Gamma$  is amenable.

The converse was proved in [11].  $\square$

**Lemma 2.3.** *Let  $\Gamma$  be a discrete group and let  $\mathcal{H}$  be a Hilbert space. For all  $\epsilon > 0$ , there exists  $\delta > 0$  such that the following holds: Let  $s \in \Gamma$  and  $\xi_i \in \ell^4(\Gamma; \mathcal{S}_4(\mathcal{H}))$ ,  $i = 1 \dots 4$  be unit vectors, and denote by  $\eta_i$  the unit vector  $(\|\xi_i(t)\|_{\mathcal{S}_4(\mathcal{H})})_{t \in \Gamma}$  in  $\ell^4\Gamma$ . If*

$$\operatorname{Re} \operatorname{Tr} (s^{-1} \cdot (\xi_1 \xi_2) \xi_3 \xi_4) \geq 1 - \delta,$$

then  $\|\lambda_s \eta_3^2 - \eta_1^2\|_{\ell^2\Gamma} \leq \epsilon$ .

*Proof.* In the proof we write  $o(1)$  for a quantity that goes to 0 as  $\delta$  goes to 0. By Hölder's inequality in  $\mathcal{S}_4(\mathcal{H})$ ,

$$\begin{aligned} 1 - \delta \leq \operatorname{Re} \operatorname{Tr} (s^{-1} \cdot (\xi_1 \xi_2) \xi_3 \xi_4) &\leq \langle \lambda_{s^{-1}}(\eta_1 \eta_2), \eta_3 \eta_4 \rangle \\ &\leq \|\eta_1 \eta_2\|_{\ell^2\Gamma} \|\eta_3 \eta_4\|_{\ell^2\Gamma} \leq 1. \end{aligned}$$

In particular,  $\langle \eta_1^2, \eta_2^2 \rangle = \|\eta_1 \eta_2\|_{\ell^2\Gamma}^2 \geq (1 - \delta)^2$ , which clearly implies that  $\|\eta_1^2 - \eta_2^2\|_{\ell^2\Gamma} = o(1)$ . The elementary inequality  $\|a - b\|_{\ell^4\Gamma} \leq \sqrt{\|a^2 - b^2\|_{\ell^2\Gamma}}$  valid for all  $a, b : \Gamma \rightarrow \mathbb{R}_+$  in turn implies  $\|\eta_1 - \eta_2\|_{\ell^4\Gamma} = o(1)$ . Similarly,  $\|\eta_3 - \eta_4\|_{\ell^4\Gamma} = o(1)$ . By the first line of the series of inequalities above, we get  $\langle \lambda_{s^{-1}} \eta_1^2, \eta_3^2 \rangle = 1 + o(1)$ , which indeed implies  $\|\lambda_s \eta_3^2 - \eta_1^2\|_{\ell^2\Gamma} = o(1)$ .  $\square$

### 3. PRELIMINARIES

We collect the necessary results and fix the notation. In particular, we introduce  $L^p$ -Fourier multipliers for any locally compact group.

**3.1. General notation.** For a Hilbert space  $\mathcal{H}$ ,  $\mathcal{S}_p(\mathcal{H})$  denotes the Schatten class, i.e. the non-commutative  $L^p$ -space associated with  $B(\mathcal{H})$ . We use brackets  $[\ ]$  to denote the closure of an operator and  $\cdot$  for the strong product of (unbounded) operators. Recall that  $a \cdot b$ , when it exists, is the closure of the operator defined on  $\{x \in \operatorname{dom}(b), b(x) \in \operatorname{dom}(a)\}$  by  $x \mapsto a(b(x))$ . For  $\varphi$  a normal, semi-finite, faithful weight on a von Neumann algebra  $\mathcal{M}$ , we set  $\mathfrak{n}_\varphi = \{x \in \mathcal{M} \mid \varphi(x^*x) < \infty\}$  and  $\mathfrak{m}_\varphi = \mathfrak{n}_\varphi^* \mathfrak{n}_\varphi$ .

**3.2. Integral operators.** Let  $(X, dx)$  be a measure space. A (bounded) integral operator  $A$  on  $L^2(X, dx)$  is a bounded operator for which there is a measurable function  $(s, t) \mapsto A_{s,t}$  (called the kernel of  $A$ ) on  $X \times X$  such that, for all  $\xi \in L^2(X, dx)$ ,

- $t \mapsto A_{s,t} \xi(t) \in L^1(X, dx)$  for almost every  $s \in X$ .
- $A\xi(s) = \int_X A_{s,t} \xi(t) dt$  for almost every  $s \in X$ .

By Fubini the kernel is only defined up to an almost everywhere zero function. The set of integral operators is a self-adjoint subspace of  $B(L^2(X))$ , and the kernel of the adjoint is given by  $(A^*)_{s,t} = \overline{A_{t,s}}$ . The Schatten  $\mathcal{S}_2$ -class consists of all integral operators with kernel belonging to  $L^2(X \times X)$ , and for such operator,

$$\|A\|_{\mathcal{S}_2(L^2X)} = \left( \iint |A_{s,t}|^2 ds dt \right)^{1/2}.$$

More generally if  $A, B \in \mathcal{S}^2(L^2X)$ ,

$$(4) \quad \text{Tr}(AB) = \iint A_{s,t} B_{t,s} dt ds.$$

**3.3. Schur multipliers.** Let  $(X, \mu)$  be a measure space and let  $\psi : X \times X \rightarrow \mathbb{C}$  be an essentially bounded measurable function. We identify  $\mathcal{S}_2(L^2X)$  linearly with  $L^2(X \times X)$  by the previous paragraph. Then, we obtain a bounded map

$$L^2(X \times X) \rightarrow L^2(X \times X) : (a_{x,y})_{x,y \in X} \mapsto (\psi(x,y)a_{x,y})_{x,y \in X},$$

which canonically determines a bounded map on  $\mathcal{S}_2(L^2X)$ . If it maps  $\mathcal{S}_2(L^2X) \cap \mathcal{S}_p(L^2X)$  to  $\mathcal{S}_p(L^2X)$  and it extends boundedly to  $\mathcal{S}_p(L^2X)$ , it will be called a (Schur) multiplier of  $\mathcal{S}_p(L^2X)$ .

We will use the following result, which is a minor modification of [10, Theorem 1.19].

**Theorem 3.1.** *Let  $\mu$  be a Radon measure on a locally compact space  $X$ , and  $\psi : X \times X \rightarrow \mathbb{C}$  a continuous function. Let  $1 \leq p \leq \infty$  and  $K > 0$ . The following are equivalent:*

- (i)  $\psi$  defines a bounded multiplier on  $\mathcal{S}_p(L^2(X, \mu))$  with norm less than  $K$ .
- (ii) For every  $\sigma$ -finite measurable subset  $X_0$  in  $X$ ,  $\psi$  restricts to a bounded multiplier on  $\mathcal{S}_p(L^2(X_0, \mu))$  with norm less than  $K$ .
- (iii) For any finite subset  $F = \{x_1, \dots, x_N\}$  in  $X$  belonging to the support of  $\mu$ , the multiplier  $(\psi(x_i, x_j))$  is bounded on  $\mathcal{S}_p(\ell^2 F)$  with norm less than  $K$ .

*Proof.* The equivalence of (i) and (iii) of Theorem 3.1 was proved in [10, Theorem 1.19] under the additional assumption that  $\mu$  is  $\sigma$ -finite. In the current theorem, this implies the equivalence of (ii) and (iii). The implication (i) implies (ii) is trivial. Assume (ii). Let  $x \in \mathcal{S}_p(L^2(X, \mu))$ . The supports of  $x$  and  $x^*$  are  $\sigma$ -finite projections in  $B(L^2X)$ . So, let  $X_0 \subset X$  be  $\sigma$ -finite such that the support projections of  $x$  and  $x^*$  project onto spaces contained in  $L^2(X_0, \mu)$ . Then, the multiplier  $\psi$  applied to  $x$  is equal to the restriction of  $\psi$  to  $X_0 \times X_0$  applied to  $x$ . Hence, (i) follows.  $\square$

Let  $G$  be a locally compact group. Let  $\phi \in M_{CB}A(G)$  and set  $\check{\phi} \in L^\infty(G \times G)$  by  $\check{\phi}(s, t) = \phi(st^{-1})$ . Then,  $\check{\phi}$  is a Schur multiplier acting on  $\mathcal{S}_p(L^2G)$ , see [10] for details. We will denote this map by  $M_\phi^p$ .

**3.4. The von Neumann algebra of  $G$ .** Let  $G$  be a locally compact group equipped with a left Haar measure, and denote by  $\Delta : G \rightarrow (0, \infty)$  the modular function. Recall (see [15, p. 65]) that  $\Delta$  is the group morphism satisfying the following equations for all compactly supported continuous function  $f : G \rightarrow \mathbb{C}$  :

$$(5) \quad \int f(ts) dt = \Delta(s)^{-1} \int f(t) dt,$$

$$(6) \quad \int f(t^{-1}) dt = \int f(t) \Delta(t)^{-1} dt.$$

Denote by  $\lambda$  and  $\rho$  the left and right regular representations of  $G$  : for  $s \in G$ ,  $\lambda_s$  and  $\rho_s$  are unitaries on  $L^2(G)$  given by  $\lambda_s \xi(t) = \xi(s^{-1}t)$  and  $\rho_s \xi(t) = \sqrt{\Delta(s)} \xi(ts)$ . The (left) von Neumann algebra of  $G$  is defined by  $\mathcal{L}G = \lambda(G)''$ . Its commutant is  $\rho(G)''$ , the right von Neumann algebra of  $G$ . If  $f \in L^1(G)$ , the formulas  $\langle \lambda(f)\xi, \eta \rangle =$

$\int f(s)\langle \lambda_s \xi, \eta \rangle ds$  and  $\langle \rho(f)\xi, \eta \rangle = \int f(s)\langle \rho_s \xi, \eta \rangle ds$  define operators  $\lambda(f) \in \mathcal{L}G$  and  $\rho(f) \in \rho(G)''$  that are integral operators on  $L^2(G)$  with kernel

$$(7) \quad (\lambda(f))_{s,t} = \Delta(t^{-1})f(st^{-1}), \quad (\rho(f))_{s,t} = \sqrt{\Delta(s^{-1}t)}f(s^{-1}t).$$

As a consequence,  $\lambda(f)^* = \lambda(f^*)$  and  $\rho(f)^* = \rho(f^*)$  where  $f^* \in L^1(G)$  is given by

$$(8) \quad f^*(s) = \overline{f(s^{-1})}\Delta(s^{-1}).$$

When  $G$  is discrete,  $\mathcal{L}G$  is finite and carries a natural trace. More generally when  $G$  is unimodular  $\mathcal{L}G$  carries a natural semifinite trace. In general, although  $\mathcal{L}G$  might be semifinite, it is not equipped with a natural trace, but rather with its natural weight, given by  $\varphi(x^*x) = \|f\|_{L^2G}^2$  if there is  $f \in L^2G$  such that  $x\xi = f * \xi$  for all  $\xi \in L^2G$  and  $\varphi(x^*x) = \infty$  otherwise. (it is an easy exercise to check that  $\varphi$  is a trace if and only if  $G$  is unimodular). It is convenient to work with this weight. A natural weight on the commutant  $\rho(G)''$  of  $\mathcal{L}G$  in  $B(L^2G)$  is given by  $\psi(x^*x) = \|f\|_{L^2G}^2$  if  $x = \rho(f)$  with  $f \in L^2G$  and  $\psi(x^*x) = \infty$  otherwise.

**3.5. The non-commutative  $L^p$ -space of  $\mathcal{L}G$ .** We recall the Connes-Hilsum construction [5], [6] of  $L^p(\mathcal{L}G)$  in the particular case when  $\mathcal{L}G$  is the von Neumann algebra of  $G$ . The space  $D(L^2G, \psi)$  of  $\psi$ -bounded elements of  $L^2G$  is the set of functions  $\xi \in L^2G$  such that  $f \mapsto \xi * f$  is bounded on  $L^2G$ . We will also denote this operator by  $\lambda(\xi)$ . Given a weight  $\omega$  on  $\mathcal{L}G$ , the spatial derivative  $d\omega/d\psi$  is the unique positive self-adjoint operator such that  $\omega(\lambda(\xi)\lambda(\xi)^*) = \|(d\omega/d\psi)^{1/2}\xi\|_{L^2G}^2$  for all  $\psi$ -bounded  $\xi$ . For example, (8) gives that  $d\varphi/d\psi = \Delta$ . The spatial derivative of an element  $\omega$  of the predual is defined by  $d\omega/d\psi = u| \omega|/d\psi$  where  $\omega = u| \omega|$  is the polar decomposition of  $\omega$ . The set  $L^1(\mathcal{L}G)$  defined as  $\{d\omega/d\psi, \omega \in \mathcal{L}G_*\}$  is then a linear space (the sum being the closure of the sum), and we denote  $\int (d\omega/d\psi)d\psi = \omega(1)$ . For an arbitrary  $1 \leq p < \infty$ ,  $L^p(\mathcal{L}G)$  is defined as the set of closed densely defined operators  $T$  with polar decomposition  $T = u|T|$  satisfying  $u \in \mathcal{L}G$ ,  $|T|^p \in L^1(\mathcal{L}G)$ , and one denotes  $\|T\|_{L^p(\mathcal{L}G)} = (\int |T|^p d\psi)^{1/p}$ . In general the non-zero elements of  $L^p(\mathcal{L}G)$  are not bounded operators. Hilsum [6] proved that the sum of two elements of  $L^p(\mathcal{L}G)$  is densely defined, closable, that its closure belongs to  $L^p(\mathcal{L}G)$ , and that for this linear structure if  $p \geq 1$ ,  $L^p(\mathcal{L}G)$  is a Banach space for the norm  $\|\cdot\|_{L^p(\mathcal{L}G)}$ . Moreover, for  $0 \leq p, q, r \leq \infty$  with  $1/r = 1/p + 1/q$  and  $a \in L^p(\mathcal{L}G)$  and  $b \in L^q(\mathcal{L}G)$ ,  $ab$  is closable and its closure (still denoted by  $ab$ ) belongs to  $L^r(\mathcal{L}G)$  and this product is associative. Lastly, if  $r = 1$ ,  $\int abd\psi = \int bad\psi$ , and if  $p \neq \infty$  the pairing  $\langle a, b \rangle = \int abd\psi$  realizes  $L^q(\mathcal{L}G)$  isometrically as the dual of  $L^p(\mathcal{L}G)$ .

Apart from what we just recalled, we will use some facts from [6, Proposition 11] that we collect in the following proposition.

**Proposition 3.2.** *We have the following properties.*

- (1) *We have  $uD(L^2G, \psi) \subset D(L^2G, \psi)$  for every  $u \in \mathcal{L}G$ .*
- (2) *If  $x \in L^p(\mathcal{L}G)$  with  $2 \leq p \leq \infty$ , then  $D(L^2G, \psi) \subset \text{dom}(x)$ .*
- (3) *Let  $1 < p \leq \infty$ . For any  $\xi, \eta \in D(L^2G, \psi)$  there is a bounded linear map  $\omega_{\xi, \eta}^p : L^p(\mathcal{L}G) \rightarrow \mathbb{C}$  satisfying  $\omega_{\xi, \eta}^p(x) = \langle |x|^{1/2}\xi, |x|^{1/2}u^*\eta \rangle$  if  $x = u|x|$  is the polar decomposition of  $x \in L^p(\mathcal{L}G)$ .*

As a direct consequence, using the inclusion  $L^2F \subset D(L^2G, \psi)$  and the closed graph theorem we get the following.



**Proposition 3.3.** *Let  $F \subset G$  be a relatively compact Borel subset with positive measure, and  $P_F : L^2G \rightarrow L^2F$  the orthogonal projection.*

- (1) *If  $p \geq 2$ ,  $L^2F \subset \text{dom}(x)$  for every  $x \in L^p(\mathcal{L}G)$ , and  $x \in L^p(\mathcal{L}G) \rightarrow xP_F \in B(L^2G)$  is a linear bounded map.*
- (2) *If  $1 \leq p \leq \infty$  there is a bounded linear map  $L^p(\mathcal{L}G) \rightarrow B(L^2F)$ , that maps an element  $x = u|x| \in L^p(\mathcal{L}G)$  to  $(|x|^{1/2}u^*P_F)^*|x|^{1/2}P_F$ . We abusively denote this map by  $x \mapsto P_FxP_F$ .*

**3.6. Completely bounded maps between von Neumann algebras.** Let  $\mathcal{M} \subset B(\mathcal{H})$  a von Neumann algebra,  $\psi$  a weight on  $\mathcal{M}'$  and  $L^p(\mathcal{M}, \psi)$  the spatial non-commutative  $L^p$ -space. The commutant of  $M_n \otimes \mathcal{M}$  in  $M_n \otimes B(\mathcal{H})$  is  $1 \otimes \mathcal{M}'$ , and is therefore equipped with the weight  $1 \otimes \psi$ . This naturally identifies  $\mathcal{S}_p^n \otimes L^p(\mathcal{M}, \psi)$  with  $L^p(M_n \otimes \mathcal{M}, 1 \otimes \psi)$ . We use this identification to define a Banach space structure on  $\mathcal{S}_p^n \otimes L^p(\mathcal{M}, \psi)$ .

If  $\mathcal{M} \subset B(\mathcal{H}), \mathcal{N} \subset B(\mathcal{H}')$  are von Neumann algebras and  $\psi, \psi'$  are weights on  $\mathcal{M}', \mathcal{N}'$ , a bounded map  $u : L^p(\mathcal{M}, \psi) \rightarrow L^p(\mathcal{N}, \psi')$  between the non-commutative  $L^p$ -spaces (Connes-Hilsum construction) is called completely bounded if  $\|u\|_{cb} := \sup_n \|u_n\| < \infty$ . Here  $u_n = id \otimes u : \mathcal{S}_p^n \otimes L^p(\mathcal{M}, \psi) \rightarrow \mathcal{S}_p^n \otimes L^p(\mathcal{N}, \psi')$ . When  $\mathcal{M}$  and  $\mathcal{N}$  are semifinite, one can check that this definition agrees with the natural operator space structure on non-commutative  $L^p$ -spaces given in [12].

**3.7. Definition of  $L^p$  Fourier multipliers.** Recall that the Fourier algebra  $A(G) = \{\varphi : s \mapsto \langle \lambda_s \xi, \eta \rangle, \xi, \eta \in L^2G\}$  coincides with the predual of  $\mathcal{L}G$  through the pairing  $\langle \varphi, \lambda(f) \rangle = \int \varphi(s)f(s)ds$  for every  $f \in L^1(G)$ . We record here the explicit isomorphism between the Fourier algebra  $A(G)$  and  $L^1(\mathcal{L}G)$ .

**Lemma 3.4.** *An element  $\varphi \in A(G)$  corresponds (isometrically) to  $x_\varphi \in L^1(\mathcal{L}G)$  that satisfies  $P_Fx_\varphi P_F = (\varphi(ts^{-1}))_{s,t \in F}$ .*

*Proof.* Start by considering the case when  $\varphi$  is positive definite, so that the corresponding element of  $\mathcal{L}G_*$  is positive. By the definition of  $L^1(\mathcal{L}G)$  and the spatial derivative,  $x_\varphi$  is the positive self-adjoint operator characterized by  $\|x_\varphi^{1/2}\xi\|^2 = \langle \varphi, \lambda(\xi)\lambda(\xi)^* \rangle$  for all  $\xi \in L^2G$  that are  $\psi$ -bounded. By definition of  $P_Fx_\varphi P_F$ , and the fact that  $L^2F \subseteq L^1G$ , we get

$$\langle P_Fx_\varphi P_F \xi, \xi \rangle = \iint_{F \times F} \varphi(ts^{-1})\xi(t)\overline{\xi(s)}dsdt,$$

and by polarization

$$\langle P_Fx_\varphi P_F \xi, \eta \rangle = \iint_{F \times F} \varphi(ts^{-1})\xi(t)\overline{\eta(s)}dsdt.$$

This proves the lemma when  $\varphi$  is positive definite. The general case follows by the linearity of  $\varphi \mapsto P_Fx_\varphi P_F$ .  $\square$

**3.8. Fourier multipliers.** Given a function  $\phi \in M_{CB}A(G)$ , we would like to define the Fourier multiplier on  $L^p(\mathcal{L}G)$  as the restriction to  $L^p(\mathcal{L}G)$  of  $M_\phi$ . This is possible when  $G$  is unimodular because  $L^p(\mathcal{L}G) \cap B(L^2G)$  is dense in  $L^p(\mathcal{L}G)$ . When  $G$  is not unimodular (and  $p \neq \infty$ ) this is more problematic:  $L^p(\mathcal{L}G) \cap B(L^2G) = \{0\}$ , whereas  $M_\phi$  is only defined on  $B(L^2G)$ . The definition is therefore done with the help of Proposition 3.3.

**Definition-Proposition 3.5.** *Let  $\phi \in M_{\mathcal{CB}A}(G)$ . There is a unique completely bounded linear map  $T_\phi^p : L^p(\mathcal{L}G) \rightarrow L^p(\mathcal{L}G)$  that satisfies  $P_F T_\phi^p(x) P_F = M_\phi(P_F x P_F)$ . It has completely bounded norm less than  $\|\phi\|_{M_{\mathcal{CB}A}(G)}$ . This map is called the Fourier multiplier with symbol  $\phi$ .*

*Proof.* Let us first prove that  $T_\phi^p$  exists and is bounded. For  $p = \infty$  this is obvious. Assume  $p = 1$ . Consider  $\tilde{\phi}(s) = \phi(s^{-1})$ . Then  $\|\tilde{\phi}\|_{M_{\mathcal{CB}A}(G)} = \|\phi\|_{M_{\mathcal{CB}A}(G)}$ . Take  $\varphi \in A(G) \rightarrow x_\varphi \in L^1(\mathcal{L}G)$  the isometry described in Lemma 3.4. Define  $T_\phi^1$  by  $T_\phi^1(x_\varphi) = x_{\tilde{\phi}\varphi}$ . It is a map of norm at most  $\|\phi\|_{M_{\mathcal{CB}A}(G)}$ , and by Lemma 3.4 it satisfies  $P_F T_\phi^1(x) P_F = M_\phi(P_F x P_F)$ .

The general case follows by interpolation. We claim that the maps  $T_\phi^1$  and  $T_\phi^\infty$  are compatible with respect to the pair  $(A_0, A_1)$  given by Theorem 6.2. Indeed, let  $a \in A_0 \cap A_1$ . We have to show that  $j_1^{-1}(T_\phi^1(j_1(a))) = j_\infty^{-1}(T_\phi^\infty(j_\infty(a)))$ . By (iv) it suffices to have that  $u_F(j_1^{-1}(T_\phi^1(j_1(a)))) = u_F(j_\infty^{-1}(T_\phi^\infty(j_\infty(a))))$  for every relatively compact  $F \subseteq G$ . But by (ii), this equality is equivalent to  $j_{1,F}^{-1}(M_\phi(j_{1,F}(u_F(a)))) = j_{\infty,F}^{-1}(M_\phi(j_{\infty,F}(u_F(a))))$ , which holds by (i).

By interpolation, there is therefore a map  $T_\phi^p : L^p(\mathcal{L}G) \rightarrow L^p(\mathcal{L}G)$  of norm at most  $\|\phi\|_{M_{\mathcal{CB}A}(G)}$ . It satisfies  $P_F T_\phi^p(x) P_F = M_\phi(P_F x P_F)$  by (ii) and (i) again.

The fact that  $T_\phi^p$  is completely bounded follows from the argument in [3, Theorem 1.6] (see the proof of Theorem 5.2 for details): apply the preceding to the function  $\phi(s, k) = \phi(s)$  on  $G \times SU(2)$  and use that  $\mathcal{L}SU(2) \simeq \bigoplus_{n \geq 1} M_n$ .  $\square$

#### 4. TRANSFERENCE

This section is devoted to the equality  $\geq$  of Question 1.1. The proof relies on transference techniques [11, Lemma 2.4], which we adapt to non-discrete groups.

**Lemma 4.1.** *Let  $G$  be a locally compact group and let  $1 < p < \infty$  be such that the conjugate exponent  $q$  satisfies  $p/q \in \mathbb{Q}$ . Then there is a net  $x_\alpha \in L^p(\mathcal{L}G)$  and  $y_\alpha \in L^q(\mathcal{L}G)$  such that  $\|x_\alpha\|_p = \|y_\alpha\|_q = 1$  and for all  $\phi \in M_{\mathcal{CB}A}(G)$ ,*

$$\lim_\alpha \langle T_\phi^p x_\alpha, y_\alpha^* \rangle = \phi(e).$$

*Proof.* The proof relies on Theorem 6.2. Since  $p/q \in \mathbb{Q}$  there exists  $r < \infty$  and  $k, k' \in \mathbb{N}$  such that  $\frac{1}{p} = \frac{k}{r}$  and  $\frac{1}{q} = \frac{k'}{r}$ .

Take a net of  $\mathbb{R}^+$ -valued functions  $h_\alpha \in A_c(G)$  such that the support  $S_\alpha$  of  $h_\alpha$  converge to  $\{e\}$ . This means that every neighbourhood of  $e$  in  $G$  contains  $S_\alpha$  for all  $n$  large enough. By (iii) in Theorem 6.2,  $j_\infty^{-1}(\lambda(h_\alpha))$  belongs to  $A_0 \cap A_1$ , we can therefore define  $a_\alpha = j_{2r}(j_\infty^{-1}(\lambda(h_\alpha))) \in L_{2r}(\mathcal{L}G)$ . Normalise  $h_\alpha$  so that  $\|a_\alpha\|_{2r} = 1$ . Consider the unit vectors  $x_\alpha \in L_p(\mathcal{L}G)$  and  $y_\alpha \in L_q(\mathcal{L}G)$  defined by  $x_\alpha = |a_\alpha|^{2k}$  and  $y_\alpha = |a_\alpha|^{2k'}$ . By (v) in Theorem 6.2, we can write  $x_\alpha = j_p(j_\infty^{-1}(\lambda(f_\alpha)))$  for a non-negative function  $f_\alpha \in A_c(G)$  with support contained in  $(S_\alpha^{-1} S_\alpha)^k$ . Similarly  $j_q^{-1}(y_\alpha)$  belongs to  $A_0 \cap A_1$ , and  $(j_1(j_q^{-1}(y_\alpha)))^* \in L_1(\mathcal{L}G)$  corresponds to  $g_\alpha \in A(G)$  satisfying  $g_\alpha(s) \geq 0$  for every  $s \in S$ .

For every  $\phi \in M_{\mathcal{CB}A}(G)$ , since the maps  $T_\phi^p$  are compatible with respect to the interpolation given by Theorem 6.2, we can write by (10),

$$\langle T_\phi^p x_\alpha, y_\alpha^* \rangle_{L^p, L^q} = \langle T_\phi^\infty(\lambda(f_\alpha)), (j_1(j_q^{-1}(y_\alpha)))^* \rangle_{L^\infty, L^1} = \int_G \phi(s) f_\alpha(s) g_\alpha(s) ds.$$

Taking  $\phi = 1$  we get  $\int f_\alpha(s)g_\alpha(s)ds = \langle x_\alpha, y_\alpha \rangle = \|a_\alpha\|_{2^r}^2 = 1$ . Hence,

$$\begin{aligned} |\langle T_\phi^p x_\alpha, y_\alpha^* \rangle - \phi(e)| &= \left| \int (\phi(s) - \phi(e))f_\alpha(s)g_\alpha(s)ds \right| \\ &\leq \sup_{s \in \text{support}(f_\alpha)} |\phi(s) - \phi(e)|. \end{aligned}$$

But  $\phi$  is continuous and the support of  $f_\alpha$  converges to  $\{e\}$ . This proves the lemma.  $\square$

**Theorem 4.2.** *Let  $G$  be a locally compact group and let  $1 \leq p \leq \infty$ . Let  $\phi \in M_{\mathcal{CB}A}(G)$ . Then,*

$$\|M_\phi^p\|_{\mathcal{CB}(\mathcal{S}_p(L^2G))} \leq \|T_\phi^p\|_{\mathcal{CB}(L^p(\mathcal{L}G))}.$$

*Proof.* Let  $q$  be the conjugate exponent of  $p$ . The values of  $p$  for which  $p/q \in \mathbb{Q}$  form a dense subset of  $[1, \infty]$ . It suffices to prove the theorem for such a  $p$ . Indeed, since  $M_\phi^p$  is defined by means of interpolation, it follows from [1, Theorem 4.1.2 and Section 2.4 (6)] and the re-iteration theorem that the logarithm of  $\|M_\phi^p\|_{\mathcal{CB}(\mathcal{S}_p(L^2G))}$  is a convex, hence continuous function in  $p$ . Similarly,  $\|T_\phi^p\|_{\mathcal{CB}(\mathcal{S}_p(L^2G))}$  is continuous in  $p$ .

Let  $F = \{s_1, \dots, s_k\} \subseteq G$  be a finite subset, and consider two matrices  $a, b \in M_k(\mathbb{C})$ . We claim that

$$\left| \sum_{i,j} \phi(s_i s_j^{-1}) a_{i,j} b_{j,i} \right| \leq \|T_\phi^p\|_{\mathcal{CB}(L^p(\mathcal{L}G))} \|a\|_{\mathcal{S}_p^k} \|b\|_{\mathcal{S}_q^k}.$$

By Theorem 3.1 this implies that  $\|M_\phi^p\|_{\mathcal{CB}(\mathcal{S}_p(L^2G))} \leq \|T_\phi^p\|_{\mathcal{CB}(L^p(\mathcal{L}G))}$ . The inequality  $\|M_\phi^p\|_{\mathcal{CB}(\mathcal{S}_p(L^2G))} \leq \|T_\phi^p\|_{\mathcal{CB}(L^p(\mathcal{L}G))}$  follows similarly, by taking  $a_{i,j} \in \mathcal{S}_p^n$ ,  $b_{i,j} \in \mathcal{S}_q^n$  instead of  $\mathbb{C}$ .

Take  $x_\alpha, y_\alpha$  given by Lemma 4.1, and consider the element  $A_\alpha \in \mathcal{S}_p^k \otimes L^p(\mathcal{L}G)$  and  $B_\alpha \in \mathcal{S}_q^k \otimes L^q(\mathcal{L}G)$  defined by

$$A_\alpha = (a_{i,j} \lambda(s_i) x_\alpha \lambda(s_j^{-1}))_{i,j \leq k}, B_\alpha = (b_{i,j} \lambda(s_i) y_\alpha \lambda(s_j^{-1}))_{i,j \leq k}.$$

$A_\alpha$  is obtained from  $a \otimes x_\alpha$  by conjugating by the unitary  $\sum_i e_{i,i} \otimes \lambda(s_i)$ , so that  $\|A_\alpha\|_p = \|a \otimes x_\alpha\|_p = \|a\|_{\mathcal{S}_p^k}$ . Similarly  $\|B_\alpha\|_q = \|b\|_{\mathcal{S}_q^k}$ . Notice now that

$$\lambda(s_i^{-1}) T_\phi^p(\lambda(s_i) x_\alpha \lambda(s_j)^{-1}) \lambda(s_j) = T_{\phi_{i,j}}^p(x_\alpha)$$

where  $\phi_{i,j}(s) = \phi(s_i s s_j^{-1})$ . In particular  $\phi_{i,j} \in M_{\mathcal{CB}A}(G)$  and  $\phi_{i,j}(e) = \phi(s_i s_j^{-1})$ . Lemma 4.1 gives

$$\begin{aligned} \sum_{i,j} \phi(s_i s_j^{-1}) a_{i,j} b_{j,i} &= \lim_\alpha \sum_{i,j} a_{i,j} b_{j,i} \langle T_{\phi_{i,j}}^p(x_\alpha), y_\alpha \rangle \\ &= \lim_\alpha \langle (id \otimes T_\phi^p)(A_\alpha), B_\alpha \rangle. \end{aligned}$$

The claim follows from the inequality

$$\begin{aligned} |\langle (id \otimes T_\phi^p)(A_\alpha), B_\alpha \rangle| &\leq \|T_\phi^p\|_{\mathcal{CB}(L^p(\mathcal{L}G))} \|A_\alpha\|_p \|B_\alpha\|_q \\ &= \|T_\phi^p\|_{\mathcal{CB}(L^p(\mathcal{L}G))} \|a\|_{\mathcal{S}_p^k} \|b\|_{\mathcal{S}_q^k}, \end{aligned}$$

valid for every  $n$ .  $\square$

## 5. AMENABLE GROUPS

We prove that the answer to Question 1.1 is affirmative for amenable locally compact groups. This generalizes beyond (amenable) discrete groups the result by Neuwirth and Ricard [11].

**Theorem 5.1.** *Let  $F \subset G$  be a relatively compact Borel subset with positive measure, and  $P_F : L_2(G) \rightarrow L_2(F)$  the orthogonal projection. Let  $1 \leq p \leq \infty$ . For any  $x \in L^p(\mathcal{L}G)$ ,  $P_F x P_F \in \mathcal{S}_p(L_2G)$  and  $\|P_F x P_F\|_{\mathcal{S}_p(L_2F)} \leq |F|^{1/p} \|x\|_{L^p(\mathcal{L}G)}$ .*

*Proof.* When  $p = \infty$  this is obvious.

Assume  $p = 1$ , and take  $x \in L_1(\mathcal{L}G)$ . Let  $\phi \in A(G)$  corresponding to  $x$ , and write  $\phi(s) = \langle \lambda(s)\xi, \eta \rangle$  for  $\xi, \eta \in L^2(G)$  with  $\|\xi\| \|\eta\| = \|x\|_{L^1(\mathcal{L}G)}$ . By Lemma 3.4,  $P_F x P_F = (\langle \lambda(s^{-1})\xi, \lambda(t^{-1})\eta \rangle)_{s,t \in F} = AB^*$  where  $A = (\xi(st))_{s \in F, t \in G}$  and  $B = (\eta(st))_{s \in F, t \in G}$ . But  $\|A\|_{\mathcal{S}_2(L_2G)}^2 = \iint |\xi(st)|^2 \mathbf{1}_{s \in F} dt ds = |F| \|\xi\|^2$ , and similarly  $\|B\|_{\mathcal{S}_2(L_2G)} = |F| \|\eta\|^2$ , so that  $\|P_F x P_F\|_{\mathcal{S}_1(L_2G)} \leq \|A\|_{\mathcal{S}_2(L_2G)} \|B\|_{\mathcal{S}_2(L_2G)} = |F| \|x\|_{L^1(\mathcal{L}G)}$ . This proves the case  $p = 1$ .

The general case follows by the interpolation Theorem 6.2: the cases  $p = 1$  and  $p = \infty$  above say that  $u_F$  maps  $A_0 \rightarrow A_{0,F}$  with norm 1 and  $A_1 \rightarrow A_{1,F}$  with norm  $|F|$ . Hence it maps  $A_{1/p} \rightarrow A_{1/p,F}$  with norm less than  $|F|^{1/p}$ . By (ii) this is exactly saying that  $P_F x P_F \in \mathcal{S}_p(L^2F)$  and  $\|P_F x P_F\|_{\mathcal{S}_p(L^2F)} \leq |F|^{1/p} \|x\|_{L^p(\mathcal{L}G)}$ .  $\square$

**Theorem 5.2.** *Let  $G$  be an amenable locally compact group and  $1 < p < \infty$ . Then there is an ultrafilter  $\mathcal{U}$  on some set and a completely isometric embedding  $i : L^p(\mathcal{L}G) \rightarrow \prod_{\mathcal{U}} \mathcal{S}_p(L^2G)$  that intertwines Fourier and Schur multipliers.*

*Proof.* By amenability there exists a Følner net  $(F_\alpha)_\alpha : F_\alpha$  are compact subsets with positive measure such that  $\lim_\alpha |F_\alpha \cap gF_\alpha|/|F_\alpha| = 1$  for all  $g \in G$ . Choose an ultrafilter  $\mathcal{U}$  refining the net  $\alpha$ , denote by  $P_\alpha$  the orthogonal projection on  $L^2(F_\alpha)$  and consider the map  $i_p : L^p(\mathcal{L}G) \rightarrow \prod_{\mathcal{U}} \mathcal{S}_p(L^2F_\alpha)$  defined by  $i_p(x) = (P_\alpha x P_\alpha / |F_\alpha|^{1/p})_\alpha$ . By Theorem 5.1,  $i_p$  is a well defined contraction, and it intertwines Fourier and Schur multipliers by definition of Fourier multipliers. We use a duality argument to prove that it is isometric. To do this take  $q$  the conjugate exponent of  $p$  and consider  $x \in L^p(\mathcal{L}G)$  and  $y \in L^q(\mathcal{L}G)$ . We claim that

$$(9) \quad \langle i_p(x), i_q(y)^* \rangle = \langle x, y^* \rangle.$$

where on the left-side, the pairing is given by  $\langle (x_\alpha)_\alpha, (y_\alpha)_\alpha \rangle = \lim_{\mathcal{U}} \text{Tr}(x_\alpha y_\alpha)$  and on the right-side the pairing is given by  $\langle x, y \rangle = \int xy d\psi$ . Since  $L^q$  is isometrically the dual of  $L^p$  for this pairing, this clearly implies that  $i_p$  (and  $i_q$ ) are isometries.

Let us first prove (9) in the particular case that  $p = \infty, q = 1, x = \lambda(f)$  with  $f \in L^1(G)$  and  $y^*$  corresponds to  $\varphi \in A(G)$ . We compute  $\langle i_\infty(x), i_1(y)^* \rangle$ , using

Lemma 3.4, (7) and then (4),

$$\begin{aligned}
& \langle i_\infty(x), i_1(y)^* \rangle \\
&= \lim_{\alpha, \mathcal{U}} \frac{1}{|F_\alpha|} \text{Tr} (P_\alpha \lambda(f) P_\alpha y^* P_\alpha) \\
&= \lim_{\alpha, \mathcal{U}} \frac{1}{|F_\alpha|} \text{Tr} ((f(st^{-1})\Delta(t^{-1})_{s,t \in F_\alpha}(\varphi(ts^{-1}))_{s,t \in F_\alpha}) \\
&= \lim_{\alpha, \mathcal{U}} \frac{1}{|F_\alpha|} \int_{F_\alpha} \int_{F_\alpha} \Delta(t^{-1}) f(st^{-1}) \varphi(st^{-1}) ds dt \\
&= \lim_{\alpha, \mathcal{U}} \frac{1}{|F_\alpha|} \int_G f(u) \varphi(u) \left( \int_{F_\alpha} \chi_{F_\alpha}(u^{-1}s) ds \right) du \\
&= \lim_{\alpha, \mathcal{U}} \int_G f(u) \varphi(u) \frac{|F_\alpha \cap uF_\alpha|}{|F_\alpha|} du \\
&= \int_G f(u) \varphi(u) du = \langle x, y^* \rangle.
\end{aligned}$$

On the fifth line we made the change of variable  $u = st^{-1}$ , and the last line is justified by the dominated convergence theorem.

We now use the interpolation Theorem 6.2 and its notation. We claim that  $\langle i_p(j_p(a)), i_q(j_q(b))^* \rangle = \langle j_p(a), j_q(b)^* \rangle$  holds for all  $p, q$  with  $1/p + 1/q = 1$  and all  $a, b \in A_0 \cap A_1$ . This will conclude the proof of (9) since by general interpolation theory,  $j_p(A_0 \cap A_1)$  is dense in  $L^p(\mathcal{L}G)$  for every  $1 < p < \infty$ . By the interpolation equation (10) and (ii) in Theorem 6.2 neither  $\langle i_p(j_p(a)), i_q(j_q(b))^* \rangle$  nor  $\langle j_p(a), j_q(b)^* \rangle$  depend on  $p$ , and we know (case  $p = \infty$ ) that they are equal when  $a \in E$ . These quantities are therefore also (case  $p = 1$ ) equal when  $b \in E$ , and by the norm density of  $j_1(E)$  in  $L^1(\mathcal{L}G)$ , they are equal for every  $b$ . This proves (9).

To prove that  $i_p$  is completely isometric, the same proof can be applied. Otherwise we can use a classical argument [3, Theorem 1.6] and consider  $K = SU(2)$ . This is very convenient because as is well-known  $\mathcal{L}K = \bigoplus_{n \geq 1} M_n$ , but all we need is that  $K$  is a compact group with irreducible unitary representations of arbitrarily large dimension. Remark that  $G \times K$  is amenable and that  $F_\alpha \times K$  is a Følner net. By what we just proved,  $i_p \otimes id : L^p(\mathcal{L}(G \times K)) \rightarrow \prod_{\mathcal{U}} \mathcal{S}_p(L^2(G \times K))$  is isometric. But since  $\mathcal{L}(G \times K) = \mathcal{L}G \otimes \mathcal{L}K \simeq \bigoplus_n M_n(\mathcal{L}G)$ , this means that

$$id \otimes i_p : \ell^p \{ \mathcal{S}_p^n \otimes L^p(\mathcal{L}G), n \geq 1 \} \rightarrow \prod_{\mathcal{U}} \ell^p \{ \mathcal{S}_p^n \otimes \mathcal{S}_p(L^2G), n \geq 1 \}$$

is isometric, which implies that  $i_p$  was completely isometric.  $\square$

As a straightforward consequence.

**Corollary 5.3.** *Let  $G$  be an amenable group and let  $1 \leq p \leq \infty$ . Let  $\phi \in M_{CB}A(G)$ . Then,*

$$\begin{aligned}
\|T_\phi^p\|_{B(L^p(\mathcal{L}G))} &\leq \|M_\phi^p\|_{B(\mathcal{S}_p(L^2G))}, \\
\|T_\phi^p\|_{CB(L^p(\mathcal{L}G))} &\leq \|M_\phi^p\|_{CB(\mathcal{S}_p(L^2G))}.
\end{aligned}$$

## 6. INTERPOLATION

In this section, we collect the necessary results from [17] and [6] in order to interpret non-commutative  $L^p$ -spaces as interpolation spaces. This is done in Theorem 6.2. Using this theorem, we are able to prove our results in a self-contained way.

The theorem is technical in nature and its conceptual consequences are stated in the earlier sections.

Let  $\mathcal{M}$  be a von Neumann algebra, and consider  $L_p(\mathcal{M})$  ( $1 \leq p \leq \infty$ ) the associated non-commutative  $L_p$ -spaces together with a bilinear duality bracket  $\langle \cdot, \cdot \rangle_{L_p, L_q}$  for  $1/p + 1/q = 1$ . For simplicity we will only use this notion in two particular explicit cases: the first is when  $\mathcal{M} = B(\mathcal{H})$  for a Hilbert space  $\mathcal{H}$ , and in this case  $L_p(\mathcal{M})$  is  $\mathcal{S}_p(\mathcal{H})$  and the duality  $\langle A, B \rangle_{L_p, L_q} = \text{Tr}(AB)$ . The second case is when  $\mathcal{M}'$  is equipped with a normal faithful weight  $\psi$ ,  $L^p(\mathcal{M})$  is the associated Connes-Hilsum space, and the duality is  $\langle a, b \rangle = \int abd\psi = \int bad\psi$ . The special case  $\mathcal{M} = \mathcal{L}G$  is recalled in Section 3.5. The special case  $\mathcal{M} = B(\mathcal{H})$  and  $\psi$  is the canonical trace on  $B(\mathcal{H})' \simeq \mathbb{C}$  will then yield  $L^p(\mathcal{M}) = \mathcal{S}_p(\mathcal{H})$ .

**Definition 6.1.** Let  $\mathcal{M}, L^p(\mathcal{M})$  be as above. An *interpolation scale* for  $\{(L_p(\mathcal{M}), 1 \leq p \leq \infty)\}$  is a compatible pair  $\overline{A} = (A_0, A_1)$  and a family of isometric isomorphisms  $j_p : A_{1/p} \rightarrow L_p(\mathcal{M})$  such that for every  $a, b \in A_0 \cap A_1$  and every  $1 \leq p, q \leq \infty$  satisfying  $1/p + 1/q = 1$ ,

$$(10) \quad \langle j_p(a), j_q(b)^* \rangle_{L_p, L_q} = \langle j_1(a), j_\infty(b)^* \rangle_{L_1, L_\infty}.$$

In this definition  $A_\theta \subset A_0 + A_1$  stands for the complex interpolation space between  $A_0$  and  $A_1$  with parameter  $\theta$ . We write  $\Sigma(\overline{A}) = A_0 + A_1$ .

**Theorem 6.2.** *Let  $F \subset G$  be a relatively compact Borel subset. There exist interpolation scales  $\{\overline{A} = (A_0, A_1), j_p : A_{1/p} \rightarrow L_p(\mathcal{L}G)\}$ , and  $\{\overline{A}_F = (A_{0,F}, A_{1,F}), j_{p,F} : A_{1/p,F} \rightarrow \mathcal{S}_p(L^2(F))\}$  with  $A_{1,F} \subset A_{0,F}$  and a bounded map  $u_F : \Sigma(\overline{A}) \rightarrow \Sigma(\overline{A}_F) = A_{0,F}$  such that*

- (i) *for all  $1 \leq p \leq \infty$ ,  $j_{p,F}$  extends to a continuous isomorphism  $j_{p,F} : A_{0,F} \rightarrow B(L^2F)$ , and  $j_{p,F} \circ j_{q,F}^{-1}$  commutes with Schur multipliers for every  $p, q$ .*
- (ii) *The following diagram commutes.*

$$(11) \quad \begin{array}{ccc} L_p(\mathcal{L}G) & \xleftarrow{j_p} & A_{1/p} \\ \downarrow x \mapsto P_F x P_F & & \downarrow u_F \\ B(L^2F) & \xleftarrow{j_{p,F}} & A_{0,F} \end{array}$$

- (iii) *The space  $E = \text{span}\{j_\infty^{-1}(\lambda(f)^* \lambda(g)), f, g \in C_c(G)\}$  is contained in  $A_0 \cap A_1$ . Furthermore,  $\{j_1(a), a \in E\}$  is norm dense in  $L^1(\mathcal{L}G)$ .*
- (iv) *Let  $a \in \Sigma(\overline{A})$ . If  $u_F(a) = 0$  for every  $F \subseteq G$  relatively compact, then  $a = 0$ .*
- (v) *For every  $a = j_\infty^{-1}(\lambda(f_a)), b = j_\infty^{-1}(\lambda(f_b)) \in E$ , every  $1 \leq p, q, r \leq \infty$  such that  $1/p + 1/q = 1/r$ , there exist  $c = j_\infty^{-1}(\lambda(f_c)), d = j_\infty^{-1}(\lambda(f_d)) \in E$  such that  $j_p(a)^* = j_p(c)$  and  $j_p(a)j_q(b) = j_r(d)$ . Moreover,  $\text{support}(f_c) = \text{support}(f_a)^{-1}$ ,  $\text{support}(f_d) \subset \text{support}(f_a)\text{support}(f_b)$ , and if  $f_a$  and  $f_b$  are non-negative so are  $f_c$  and  $f_d$ . If  $f_a$  is non-negative,  $j_1(a) \in L^1(\mathcal{L}G)$  corresponds to a non-negative (i.e. point-wise non-negative) element in  $A(G)$ .*

The proof will use Terp's construction [17] of non-commutative  $L^p$ -spaces as interpolation spaces, that we now recall.

The construction starts from a triple  $(\mathcal{M}, \varphi, \psi)$ . Here,  $\mathcal{M}$  is a von Neumann algebra.  $\varphi$  is a normal, semi-finite, faithful weight on  $\mathcal{M}$  and  $\psi$  is a normal, semi-finite, faithful weight on the commutant  $\mathcal{M}'$ . Associated with  $\varphi$ , we have a GNS-triple

$(\mathcal{H}, \pi, \Lambda)$ , a modular conjugation  $J$ , modular operator  $\Delta$  and modular automorphism group  $\sigma$  (see [15, p 92]). We identify  $\mathcal{M}$  with  $\pi(\mathcal{M})$  and omit  $\pi$  in the notation.

We turn  $(\mathcal{M}, \mathcal{M}_*)$  into a compatible couple of Banach spaces as follows. Let  $\mathcal{M} \cap \mathcal{M}_*$  be the set of  $x \in \mathcal{M}$ , for which there exists a  $\varphi_x \in \mathcal{M}_*$  such that

$$\forall y, z \in \mathfrak{n}_\varphi : \langle \varphi_x, z^* y \rangle = \langle Jx^* J\Lambda(y), \Lambda(z) \rangle.$$

For  $x \in \mathcal{M} \cap \mathcal{M}_*$ , we set the norm  $\|x\|_{\mathcal{M} \cap \mathcal{M}_*} = \max\{\|x\|_{\mathcal{M}}, \|\varphi_x\|_{\mathcal{M}_*}\}$ , in which it becomes a Banach space. Naturally, we find two embeddings  $i_\infty : \mathcal{M} \cap \mathcal{M}_* \rightarrow \mathcal{M} : x \mapsto x$  and  $i_1 : \mathcal{M} \cap \mathcal{M}_* : x \mapsto \varphi_x$ . Dualizing the embeddings (and restricting to  $\mathcal{M}_*$ ) we obtain a commutative diagram:

$$(12) \quad \begin{array}{ccc} & \mathcal{M}_* & \\ i_1 \nearrow & & \searrow i_\infty^* \\ \mathcal{M} \cap \mathcal{M}_* & & (\mathcal{M} \cap \mathcal{M}_*)^* \\ i_\infty \searrow & & \nearrow i_1^* \\ & \mathcal{M} & \end{array}$$

which turns  $(\mathcal{M}, \mathcal{M}_*)$  into a compatible couple of Banach spaces. Moreover,  $i_\infty^* i_1(\mathcal{M} \cap \mathcal{M}_*)$  is the intersection of  $i_\infty^*(\mathcal{M}_*)$  and  $i_1^*(\mathcal{M})$ , so that our notation is justified. We have  $\mathfrak{n}_\varphi \subseteq \mathcal{M} \cap \mathcal{M}_*$ .

We write  $L^p(\mathcal{M})$  for the Connes-Hilsum  $L^p$ -space [6]. We let  $d = d\varphi/d\psi$  be the spatial derivative. For any  $x \in \mathfrak{n}_\varphi$ , we have

$$[x d^{\frac{1}{2p}}] \in L^{2p}(\mathcal{M}) \text{ and } d^{\frac{1}{2p}} x^* \in L^{2p}(\mathcal{M})$$

c.f. [17, Theorem 26]. The main result of [17] states that for  $1 \leq p \leq \infty$ , there exists a unique isometry  $j_p : (\mathcal{M}, \mathcal{M}_*)_{1/p} \rightarrow L^p(\mathcal{M})$  such that for  $x, y \in \mathfrak{n}_\varphi$ , we have

$$(13) \quad j_p(x^* y) = d^{\frac{1}{2p}} x^* \cdot [y d^{\frac{1}{2p}}] \in L^{2p}(\mathcal{M}) \cdot L^{2p}(\mathcal{M}) \subseteq L^p(\mathcal{M}).$$

Moreover, for  $1 \leq p < \infty$ ,  $j_p$  is an isomorphism ( $j_\infty$  is just the identity map from  $\mathcal{M}$  to  $\mathcal{M}$ ). The following proposition is exactly [17, Eqn. (56)].

**Proposition 6.3.** *Let  $(\mathcal{M}, \varphi, \psi)$  be a triple as above.  $\{(\mathcal{M}, \mathcal{M}_*), j_p : (\mathcal{M}, \mathcal{M}_*)_{1/p} \rightarrow L^p(\mathcal{M})\}$  is an interpolation scale.*

*Remark 6.4.* In [8, 7] it is shown that depending on a *complex interpolation parameter*  $z \in \mathbb{C}$ , the triple  $(\mathcal{M}, \varphi, \psi)$  gives a compatible couple  $(\mathcal{M}, \mathcal{M}_*)$  by taking different embeddings of (subsets of)  $\mathcal{M}$  into  $\mathcal{M}_*$ . These all give rise to an interpolation scale, the special case treated here being  $z = 0$ . Since the Schur and Fourier multipliers in this paper commute with  $\sigma$ , there is in fact no preference for the choice of the interpolation parameter.

*Remark 6.5.* In the next proof we use the following standard fact [15, Chapter VII.3] on the Plancherel weight  $\varphi$  on  $\mathcal{L}G$ . For  $\xi \in L^1 G \cap L^2 G$  we have  $\lambda(\xi) \in \mathfrak{n}_\varphi$  and the mapping  $\lambda(L^1 G \cap L^2 G) \rightarrow L^2 G : \lambda(\xi) \mapsto \xi$  is a  $\sigma$ -weak/norm core for the GNS-map of  $\varphi$ .

*Proof of Theorem 6.2.* Consider the triple  $(\mathcal{L}G, \varphi, \psi)$ , see Sections 3.4 and 3.5 for notation. We will describe precisely the GNS-triple and modular conjugation  $J$  in the proof of (iv). Terp's construction gives a compatible couple  $\overline{A} = (A_0, A_1)$  and an interpolation scale  $\{\overline{A}, j_p : A_{1/p} \rightarrow L^p(\mathcal{L}G)\}$  satisfying (13).

Consider also  $(B(L^2F), \text{Tr}_\Delta, \text{Tr}')$ . Here,

$$\text{Tr}_\Delta(x) = \text{Tr}(\Delta_F^{\frac{1}{2}} x \Delta_F^{\frac{1}{2}}), \quad x \in B(L^2F)^+,$$

where  $\Delta_F$  is the bounded operator given by  $\Delta$  restricted to  $L^2F$ . Furthermore,  $\text{Tr}'$  is unique state on  $B(L^2F)' = \mathbb{C}$ . In this case,  $L^p(B(L^2F)) = \mathcal{S}_p(L^2F)$  and we have spatial derivative  $d\text{Tr}_\Delta/d\text{Tr}' = \Delta_F$ . We define the map,

$$j_{p,F} : B(L^2F) \rightarrow B(L^2F) : x \mapsto \Delta_F^{\frac{1}{2p}} x \Delta_F^{\frac{1}{2p}}.$$

Then, the restriction of  $j_{p,F}$  to  $A_{1/p,F}$  is an isometric isomorphism onto  $\mathcal{S}_p(L^2F)$  (by Terp's construction). In particular, we find an interpolation scale  $\{\overline{A}_F = (A_{0,F}, A_{1,F}), j_{p,F} : A_{1/p,F} \rightarrow \mathcal{S}_p(L^2(F))\}$ . The map  $j_{p,F} \circ j_{q,F}^{-1}$ , being a Schur multiplier, commutes with Schur multipliers. In all, we conclude (i).

(ii) Take  $x, y \in \mathfrak{n}_\varphi$ . Then, for all  $1 \leq p \leq \infty$  we have from (13)

$$(14) \quad j_{p,F}^{-1}(P_F j_p(j_\infty^{-1}(x^*y))P_F) = \Delta_F^{-\frac{1}{2p}}(P_F \Delta_F^{\frac{1}{2p}} x^* \cdot [y \Delta_F^{\frac{1}{2p}}] P_F) \Delta_F^{-\frac{1}{2p}} = P_F x^* y P_F.$$

In the case  $p = 1$ , [17, Theorem 8] implies that the equation  $j_{1,F}^{-1}(P_F j_1(z)P_F) = P_F j_\infty(z)P_F$  holds for all  $z \in A_0 \cap A_1$ . We can therefore define a bounded map  $u_F : A_0 + A_1 \rightarrow A_{0,F} = B(L^2F)$  by  $u_F(x_0 + x_1) = P_F j_\infty(x_0)P_F + j_{1,F}^{-1}(P_F j_1(x_1)P_F)$  if  $x_0 \in A_0, x_1 \in A_1$ . By the definition of  $u_F$ , (ii) then commutes for  $p = 1$  or  $p = \infty$ .

For  $1 < p < \infty$ , (14) proves that  $j_{p,F}^{-1}(P_F j_p(z)P_F) = u_F(z)$  for all  $z \in j_\infty^{-1}(\mathfrak{m}_\varphi)$ . Recall that  $j_p(j_\infty^{-1}(\mathfrak{m}_\varphi))$  is dense in  $L^p(\mathcal{L}G)$  (by [17, Theorem 26], Eqn. (13) and Hölder's inequality). This concludes (ii).

(iii) For  $\xi \in L^1G \cap L^2G$  we have  $\lambda(\xi) \in \mathfrak{n}_\varphi$ . Hence  $\lambda(\xi)^* \lambda(\eta) \in \mathfrak{m}_\varphi \subseteq A_0 \cap A_1$  for  $\xi, \eta \in C_c(G)$ . Let  $x \in \mathcal{L}G$  be such that  $\langle j_1 \circ j_\infty^{-1}(\lambda(\xi)\lambda(\eta)), x \rangle = 0$  for all  $\xi, \eta \in C_c(G)$ . Then, using the notation (8) and [17, Eqn. (38)] in the fourth equality,

$$\begin{aligned} 0 &= \langle j_1 \circ j_\infty^{-1}(\lambda(\xi)\lambda(\eta)), x \rangle = \langle \Delta^{\frac{1}{2}} \lambda(\xi) \cdot [\lambda(\eta) \Delta^{\frac{1}{2}}], x \rangle \\ &= \int [\lambda(\xi \Delta^{\frac{1}{2}}) \Delta^{\frac{1}{2}}] \cdot \Delta^{\frac{1}{2}} \lambda(\eta \Delta^{-\frac{1}{2}}) \cdot x d\psi = \varphi(\lambda(\eta \Delta^{-\frac{1}{2}}) x \lambda(\xi \Delta^{\frac{1}{2}})) \\ &= \langle x \Delta^{\frac{1}{2}} \xi, \Delta^{\frac{1}{2}} \eta^* \rangle \end{aligned}$$

This implies that  $x = 0$ , since  $C_c(G)$  is dense in  $L^2G$ . Hence  $j_1(E)$  is dense in  $L^1(\mathcal{L}G)$ .

(iv) Let  $a = a_0 + a_1 \in A_0 + A_1$  be such that  $u_F(a) = 0$  for all  $F \subseteq G$  relatively compact. Write  $j_\infty(a_0) = x \in \mathcal{L}G$  and  $j_1(a_1) = \omega_\phi \in \mathcal{L}G_*$  corresponding to  $\phi \in A(G)$  and  $x_\phi \in L^1(\mathcal{L}G)$ . Firstly, let  $y = \lambda(\xi), z = \lambda(\eta)$ , with  $\xi, \eta \in C_c(G)$ . We will use the notation (8). We have  $\Lambda(\lambda(\xi)) = \xi \in L^2G$  and  $J\Lambda(\lambda(\xi)) = \xi^* \Delta^{\frac{1}{2}} \in L^2G$ . Furthermore,  $y^* z \in \mathfrak{m}_\varphi \subseteq j_\infty(A_0 \cap A_1)$ . By Terp's construction,  $\Sigma(\overline{A}) \subseteq (A_0 \cap A_1)^*$ . We first claim that

$$\langle a, j_\infty^{-1}(y^* z) \rangle_{(A_0 \cap A_1)^*, A_0 \cap A_1} = 0.$$



Indeed, taking  $F$  that contains the supports of  $\eta^*$ ,  $\xi^*$  and  $\xi^* * \eta$ ,

$$\begin{aligned}
& \langle a, j_\infty^{-1}(y^* z) \rangle_{(A_0 \cap A_1)^*, A_0 \cap A_1} \\
&= \langle Jx^* J\Lambda(z), \Lambda(y) \rangle + \langle \phi, y^* z \rangle_{A(G), \mathcal{L}G} \\
(15) \quad &= \langle xJ\Lambda(y), J\Lambda(z) \rangle + \int_F \phi(s) (\xi^* * \eta)(s) ds \\
&= \langle x\Delta^{\frac{1}{2}} \xi^*, \Delta^{\frac{1}{2}} \eta^* \rangle + \int_F \phi(ts^{-1}) \overline{\xi(t^{-1})} \Delta(t^{-1}) \eta(s^{-1}) \Delta(s^{-1}) ds.
\end{aligned}$$

We continue the equation using Lemma 3.4,

$$\begin{aligned}
& \langle a, j_\infty^{-1}(y^* z) \rangle_{(A_0 \cap A_1)^*, A_0 \cap A_1} \\
&= \langle \Delta^{\frac{1}{2}} P_F x P_F \Delta^{\frac{1}{2}} \xi^*, \eta^* \rangle + \langle P_F x_\phi P_F \xi^*, \eta^* \rangle \\
&= \langle (P_F x P_F + \Delta^{-\frac{1}{2}} P_F x_\phi P_F \Delta^{-\frac{1}{2}}) \Delta^{\frac{1}{2}} \xi^*, \Delta^{\frac{1}{2}} \eta^* \rangle \\
&= \langle u_F(a) \Delta^{\frac{1}{2}} \xi^*, \Delta^{\frac{1}{2}} \eta^* \rangle = 0.
\end{aligned}$$

We claim that it follows that in fact

$$\langle a, w \rangle_{(A_0 \cap A_1)^*, A_0 \cap A_1} = 0,$$

for every  $w \in A_0 \cap A_1$ . By [17, Theorem 8] it is enough to show this when  $w \in \mathfrak{n}_\varphi$ . By linearity we may assume that  $w = y^* z$  with  $y, z \in \mathfrak{n}_\varphi$ . We thus need to show that

$$(16) \quad \langle Jx^* J\Lambda(z), \Lambda(y) \rangle + \langle \phi, y^* z \rangle_{A(G), \mathcal{L}G} = 0,$$

c.f. the first three lines of (15). Since  $\lambda(L^1 G \cap L^2 G)$  is a  $\sigma$ -weak/norm core for  $\Lambda$ , there are nets  $\xi_i, \eta_i$  in  $L^1 G \cap L^2 G$  such that  $\lambda(\xi_i) \rightarrow y$   $\sigma$ -weakly and  $\|\xi_i - \Lambda(y)\|_{L^2 G} \rightarrow 0$  and similarly,  $\lambda(\eta_i) \rightarrow z$   $\sigma$ -weakly and  $\|\eta_i - \Lambda(z)\|_{L^2 G} \rightarrow 0$ . In fact, we may take  $\xi_i, \eta_i \in C_c(G)$ . Then, clearly (16) follows from what we have proved.

(v) We claim that the required functions are given by:

$$f_c(s) = \overline{f_a(s^{-1})} \Delta(s^{-1}), \quad f_d(s) = (f_a \Delta^{-\frac{1}{2q}}) * (f_b \Delta^{-\frac{1}{2p}}),$$

so that  $c = a^*$  and  $d = \sigma_{\frac{i}{2q}}(a) \sigma_{\frac{i}{2p}}(b)$ . For notational convenience, suppose that  $a = u^* v$ , with  $u, v \in \mathfrak{n}_\varphi$ . In the general case one considers a linear combination of such a decomposition. Recall that for any two unbounded operators  $x$  and  $y$  we have  $(xy)^* \supseteq x^* y^*$ . Then, using (13) and [6, Theorem 4.(1)] in the third equation,

$$j_p(c) = j_p(a^*) = \Delta^{\frac{1}{2p}} v^* \cdot [u \Delta^{\frac{1}{2p}}] = \left( \Delta^{\frac{1}{2p}} u^* \cdot [v \Delta^{\frac{1}{2p}}] \right)^* = j_p(a)^*.$$

Furthermore, write still  $a = u^* v$  and also  $b = x^* y$  with  $u, v, x, y \in \mathfrak{n}_\varphi$ . We claim that we have the following equalities,

$$\begin{aligned}
j_r(d) &= j_r(\sigma_{\frac{i}{2q}}(a) \sigma_{\frac{i}{2p}}(b)) = \Delta^{\frac{1}{2r}} \sigma_{\frac{i}{2q}}(a) \cdot [\sigma_{\frac{i}{2p}}(b) \Delta^{\frac{1}{2r}}] \\
&\stackrel{?}{=} \Delta^{\frac{1}{2p}} u^* \cdot [v \Delta^{\frac{1}{2p}}] \cdot \Delta^{\frac{1}{2q}} x^* \cdot [y \Delta^{\frac{1}{2q}}] = j_p(a) j_q(b).
\end{aligned}$$

Only  $\stackrel{?}{=}$  needs justification, the other equalities are immediate from (13). Firstly, using [17, Lemma 22],

$$\Delta^{\frac{1}{2r}} \sigma_{\frac{i}{2q}}(a) \supseteq \Delta^{\frac{1}{2p}} a \Delta^{\frac{1}{2q}} = \Delta^{\frac{1}{2p}} u^* v \Delta^{\frac{1}{2q}}.$$

Using that the left hand side is closed and taking the closure on the right hand side,  $\Delta^{\frac{1}{2r}} \sigma_{\frac{i}{2q}}(a) \supseteq \Delta^{\frac{1}{2p}} u^* \cdot [v \Delta^{\frac{1}{2q}}]$ . Since (use (13) for the right-hand side) both

sides are in  $L^{2r}(\mathcal{L}G)$  we in fact have equality by [6, Theorem 4.(1)]. Also, using the same argument in the second equality,

$$[\sigma_{\frac{i}{2p}}(b)\Delta^{\frac{1}{2r}}]^* = \Delta^{\frac{1}{2r}}\sigma_{\frac{i}{2p}}(b) = \Delta^{\frac{1}{2q}}y^* \cdot [x\Delta^{\frac{1}{2p}}] = (\Delta^{\frac{1}{2p}}x^* \cdot [y\Delta^{\frac{1}{2q}}])^*.$$

So,  $[\sigma_{\frac{i}{2p}}(b)\Delta^{\frac{1}{2r}}] = \Delta^{\frac{1}{2p}}x^* \cdot [y\Delta^{\frac{1}{2q}}]$ . Furthermore,

$$[v\Delta^{\frac{1}{2q}}] \cdot \Delta^{\frac{1}{2p}}x^* \supseteq v \cdot \Delta^{\frac{1}{2r}}x^* \subseteq [v\Delta^{\frac{1}{2q}}] \cdot \Delta^{\frac{1}{2p}}x^*.$$

Since each instance of this line is in  $L^2(\mathcal{L}G)$ , the inclusions are equalities by [6, Theorem 4.(1)]. In all, we have proved that

$$\begin{aligned} \Delta^{\frac{1}{2r}}\sigma_{\frac{i}{2q}}(a) \cdot [\sigma_{\frac{i}{2p}}(b)\Delta^{\frac{1}{2r}}] &= \Delta^{\frac{1}{2p}}u^* \cdot [v\Delta^{\frac{1}{2q}}] \cdot \Delta^{\frac{1}{2p}}x^* \cdot [y\Delta^{\frac{1}{2q}}] \\ &= \Delta^{\frac{1}{2p}}u^* \cdot [v\Delta^{\frac{1}{2p}}] \cdot \Delta^{\frac{1}{2q}}x^* \cdot [y\Delta^{\frac{1}{2q}}], \end{aligned}$$

so that  $f_c$  and  $f_d$  have the right properties.

The claims about the supports now follow automatically. For the non-negativity, write  $a = \lambda(f)\lambda(g)$ ,  $f, g \in C_c(G)$ . Let  $F \subseteq G$  be compact. We have  $P_F j_1(a) P_F = (\lambda(f)^* \Delta^{\frac{1}{2}} P_F)^* (\lambda(g) \Delta^{\frac{1}{2}} P_F) = (\phi(t^{-1}s))_{s,t \in F}$  for some  $\phi \in A(G)$ , see Lemma 3.4. But  $(\lambda(f)^* \Delta^{\frac{1}{2}} P_F)^* (\lambda(g) \Delta^{\frac{1}{2}} P_F)_{s,t} = \Delta^{\frac{1}{2}}(st^{-1})f_a(st^{-1})$  is nonnegative.  $\square$

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