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Pattern matching in (213, 231)-avoiding permutations

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Abstract. Given permutations $\sigma \in S_k$ and $\pi \in S_n$ with $k < n$, the *pattern matching* problem is to decide whether π matches σ as an order-isomorphic subsequence. We give a linear-time algorithm in case both π and σ avoid the two size-3 permutations 213 and 231. For the special case where only σ avoids 213 and 231, we present a $O(\max(kn^2, n^2 \log(\log(n))))$ time algorithm. We extend our research to bivincular patterns that avoid 213 and 231 and present a $O(kn^4)$ time algorithm. Finally we look at the related problem of the longest subsequence which avoids 213 and 231.

1 Introduction

A permutation π is said to match another permutation σ , in symbols $\sigma \preceq \pi$, if there exists a subsequence of elements of π that has the same relative order as σ . Otherwise, π is said to *avoid* the permutation σ . For example a permutation matches the pattern 123 (resp. 321) if it has an increasing (resp. decreasing) subsequence of length 3. As another example, 6152347 matches 213 but not 231. During the last decade, the study of the pattern matching on permutations has become a very active area of research [14] and a whole annual conference (PERMUTATION PATTERNS) is now devoted to this topic.

We consider here the so-called *pattern matching* problem (also sometimes referred to as the *pattern involvement problem*): Given two permutations σ and π , this problem is to decide whether $\sigma \preceq \pi$ (the problem is ascribed to Wilf in [5]). The permutation matching problem is known to be **NP**-hard [5]. It is, however, polynomial-time solvable by brute-force enumeration if σ has bounded size. Improvements to this algorithm were presented in [2] and [1], the latter describing a $O(|\pi|^{0.47|\sigma|+o(|\sigma|)})$ time algorithm. Bruner and Lackner [7] gave a fixed-parameter algorithm solving the pattern matching problem with an exponential worst-case runtime of $O(1.79^{\text{run}(\pi)})$, where $\text{run}(\pi)$ denotes the number of alternating runs of π . (This is an improvement upon the $O(2^{|\pi|})$ runtime required by brute-force search without imposing restrictions on σ and π .) A recent major

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step was taken by Marx and Guillemot [11]. They showed that the permutation matching problem is fixed-parameter tractable (FPT) for parameter $|\sigma|$.

A few particular cases of the pattern matching problem have been attacked successfully. The case of increasing patterns is solvable in $O(|\pi| \log \log |\sigma|)$ time in the RAM model [8], improving the previous 30-year bound of $O(|\pi| \log |\sigma|)$. Furthermore, the patterns 132, 213, 231, 312 can all be handled in linear-time by stack sorting algorithms. Any pattern of length 4 can be detected in $O(|\pi| \log |\pi|)$ time [2]. Algorithmic issues for 321-avoiding patterns matching has been investigated in [12]. The pattern matching problem is also solvable in polynomial-time for separable patterns [13, 5] (see also [6] for LCS-like issues of separable permutations). Separable permutations are those permutations that match neither 2413 nor 3142, and they are enumerated by the Schröder numbers (Notice that the separable permutations include as a special case the stack-sortable permutations, which avoid the pattern 231.)

There exists many generalisation of patterns that are worth considering in the context of algorithmic issues in pattern matching (see [14] for an up-to-date survey). *Vincular patterns*, also called *generalized patterns*, resemble (classical) patterns with the additional constraint that some of the elements in a matching must be consecutive in positions. Of particular importance in our context, Bruner and Lackner [7] proved that deciding whether a vincular pattern σ of length k can be match to a longer permutation π is $W[1]$ -complete for parameter k ; for an up to date survey of the $W[1]$ class and related material, see [9]. *Bivincular patterns* generalize classical patterns even further than vincular patterns by adding a constraint on values.

We focus in this paper on pattern matching issues for (213, 231)-avoiding permutations (*i.e.*, those permutations that avoid both 213 and 231). The number of n -permutations that avoid both 213 and 231 is $t_0 = 1$ for $n = 0$ and $t_n = 2^{n-1}$ for $n \geq 1$ [16]. On an individual basis, the permutations that do not match the permutation pattern 231 are exactly the *stack-sortable permutations* and they are counted by the Catalan numbers [15]. A stack-sortable permutation is a permutation whose elements may be sorted by an algorithm whose internal storage is limited to a single stack data structure. As for 213, it is well-known that if $\pi = \pi_1 \pi_2 \dots \pi_n$ avoids 132, then its complement $\pi' = (n+1-\pi_1)(n+1-\pi_2) \dots (n+1-\pi_n)$ avoids 312, and the reverse of π' avoids 213. This paper is organized as follows. In Section 2 the needed definitions are presented. Section 3 is devoted to presenting an online linear-time algorithm in case both permutations are (213, 231)-avoiding, whereas Section 4 focuses on the case where only the pattern is (213, 231)-avoiding. In Section 5 we give a polynomial-time algorithm for (213, 231)-avoiding bivincular patterns. In Section 6 we consider the problem of finding the longest (213, 231)-avoiding pattern in permutations.

2 Definitions

A *permutation* of length n is a one-to-one function from an n -element set to itself. We write permutations as words $\pi = \pi_1\pi_2 \dots \pi_n$, whose elements are distinct and usually consist of the integers $1 \dots n$, and we let $\pi[i]$ stand for π_i . For the sake of convenience, we let $\pi[i : j]$ stand for $\pi_i\pi_{i+1} \dots \pi_j$, $\pi[: j]$ stand for $\pi[1 : j]$ and $\pi[i :]$ stand for $\pi[i : n]$. As usual, we let S_n denote the set of all permutations of length n . It is convenient to use a geometric representation of permutation to ease the understanding of algorithms. The geometric representation corresponds to the set of point with coordinate $(i, \pi[i])$ (see figure 1).

A permutation π is said to *match* the permutation σ if there exists a subsequence of (not necessarily consecutive) element of π that has the same relative order as σ , and in this case π is said to match σ , written $\sigma \preceq \pi$. Otherwise, π is said to *avoid* the permutation σ . For example, the permutation $\pi = 391867452$ matches the pattern $\sigma = 51342$, as can be seen in the highlighted subsequence of $\pi = \mathbf{391867452}$ (or $\pi = \mathbf{391867452}$ or $\pi = \mathbf{391867452}$ or $\pi = \mathbf{391867452}$). Each subsequence 91674, 91675, 91672, 91452, in π is called a *matching* of σ . Since the permutation $\pi = 391867452$ contains no increasing subsequence of length four, π avoids 1234. Geometrically, π matches σ if there exists a set of point in π that is isomorph to the set of point of σ . In other word, if there exists a set of point in π with the same disposition as the set of point of σ , without regard to the distance (see figure 1).

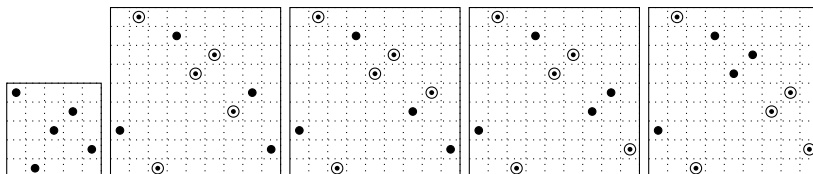


Fig. 1: The pattern $\sigma = 51342$ and four matchings of σ in 391867452.

Suppose P is a set of permutations. We let $\text{Av}_n(P)$ denote the set of all n -permutations avoiding each permutation in P . For the sake of convenience (and as it is customary [14]), we omit P 's braces thus having e.g. $\text{Av}_n(213, 231)$ instead of $\text{Av}_n(\{213, 231\})$. If $\pi \in \text{Av}_n(P)$, we also say that π is P -*avoiding*.

An *ascent* of a permutation $\pi \in S_n$ is any element $1 \leq i < n$ where the following value is bigger than the current one. That is, if $\pi = \pi_1\pi_2 \dots \pi_n$, then π_i is an ascent if $\pi_i < \pi_{i+1}$. For example, the permutation 3452167 has ascents 3, 4, 1 and 6. Similarly, a *descent* is any element $1 \leq i < n$ with $\pi_i > \pi_{i+1}$, so for every $1 \leq i < n$, π_i is either an ascent or is a descent of π .

A *left to right maxima* (abbreviate LRMax) of π is a element that does not have any element bigger than it on its right (see fig. 2). Formally, $\pi[i]$ is a LRMax if and only if $\pi[i]$ is the biggest element of $\pi[i :]$. Similarly $\pi[i :]$ is a

left to right minima (abbreviate LRMin) if and only if $\pi[i]$ is the smallest element of $\pi[i :]$.

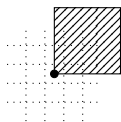


Fig. 2: The element is a LRMax if and only if the dashed area is empty.

A *bivincular pattern* σ of length k is a permutation in S_k written in two-line notation (that is the top row is $12 \dots k$ and the bottom row is a permutation $\sigma_1\sigma_2 \dots \sigma_k$). We have the following conditions on the top and bottom rows of σ , as see in [14] in definition 1.4.1:

- If the bottom line of σ contains $\sigma_i\sigma_{i+1} \dots \sigma_j$ then the elements corresponding to $\sigma_i\sigma_{i+1} \dots \sigma_j$ in a matching of σ in π must be adjacent, whereas there is no adjacency condition for non-underlined consecutive elements. Moreover if the bottom row of σ begins with $\lrcorner\sigma_1$ then any matching of σ in a permutation π must begin with the leftmost element of π , and if the bottom row of σ begins with $\sigma_k\lrcorner$ then any matching of σ in a permutation π must end with the rightmost element of π .
- If the top line of σ contains $\overline{i+1} \dots \overline{j}$ then the elements corresponding to $i, i+1, \dots, j$ in an matching of σ in π must be adjacent in values, whereas there is no value adjacency restriction for non-overlined elements. Moreover, if the top row of σ begins with $\overline{1}$ then any matching of σ is a permutation π must contain the smallest element of π , and if top row of σ ends with \overline{k} then any matching of σ is a permutation π must contain the largest element of π .

For example, let $\sigma = \begin{smallmatrix} \overline{1234} \\ \lrcorner 2143 \end{smallmatrix}$. In 3217845 , **3217845** is a matching of σ but **3217845** is not. The best general reference is [14].

Geometrically, We represent underlined and overlined elements by *forbidden areas*. A vertical area between two points indicates that the two matching of those points must be consecutive in positions, whereas a horizontal area between two points indicates that the two matching of those points must be consecutive in value. The forbidden areas can be understand as follow : in a matching, the forbidden areas must be empty. Thus, π matches a bivincular pattern σ if there exists a set of point in π that is isomorph to σ and if the forbidden areas are empty. (see figure 3).

3 Both π and σ are (213, 231)-avoiding

This section is devoted to presenting a fast algorithm for deciding if $\sigma \preceq \pi$ in case both π and σ are (213, 231)-avoiding. We begin with an easy but crucial structure lemma.

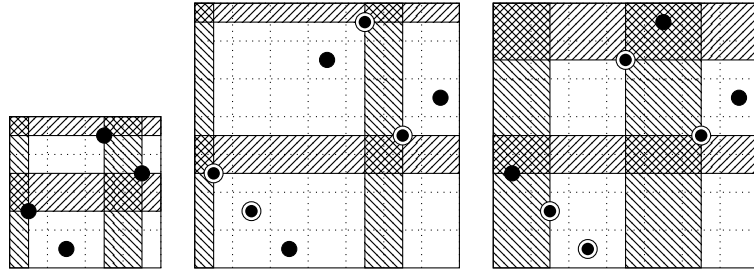


Fig. 3: From left to right, the bivincular pattern $\sigma = \begin{smallmatrix} \overline{1234} \\ \underline{2143} \end{smallmatrix}$, A matching of σ in 3216745, A matching of 2143 in 3216745 but not a matching of σ in 3216745 because the point (1, 3) and (5, 7) are in the forbidden area.

Lemma 1 (Folklore). *The first element of any (213, 231)-avoiding permutation must be either the minimal or the maximal element.*

Proof (of Lemma 1). Any other initial element would serve as a ‘2’ in either a 231 or 213 with 1 and n as the ‘1’ and ‘3’ respectively. \square

Corollary 1. $\pi \in \text{Av}_n(213, 231)$ if and only if for $1 \leq i < n$, $\pi[i]$ is a LRMax or a LRMin.

Corollary 2. Let $\pi \in \text{Av}_n(213, 231)$ and $1 \leq i < n$. Then, (1) $\pi[i]$ is an ascent element if and only if $\pi[i]$ is a LRMin and (2) $\pi[i]$ is a descent element if and only if $\pi[i]$ is a LRMax

Lemma 2 gives a bijection between $\text{Av}_n(213, 231)$ and the set of binary word of size $n - 1$. The bijected word w of π , is the word where each letter at position i represents if $\pi[i]$ is an ascent or descent element (or is a LRMax or a LRMin). We call this bijection B .

A (213, 231)-avoiding permutation has a particular form. If we take only the descent elements, the points draw a north-east to south-west line and if we take only the ascent elements, the points draw a south-east to north-west line. This shape the permutation as a $>$. For convenience when drawing a random (213, 231)-avoiding permutation we will sometime represent a sequence of ascent/descent element by lines (see figure 4 and 5).

The following lemma is central to our algorithm.

Lemma 2. Let π and σ be two (213, 231)-avoiding permutations, Then, π matches σ if and only if there exists a subsequence t of π such $B(t) = B(\sigma)$.

Proof (of Lemma 2). The forward direction is obvious. We prove the backward direction by induction on the size of the pattern σ . The base case is a pattern of size 2. Suppose that $\sigma = 12$ and thus $B(\sigma) = \text{ascent}$. Let $t = \pi_{i_1} \pi_{i_2}$, $i_1 < i_2$, be a subsequence of π such that $B(t) = \text{ascent}$, this reduces to saying that $\pi_{i_1} < \pi_{i_2}$, and hence that t is a matching of $\sigma = 12$ in π . A similar argument

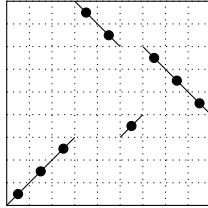


Fig. 4: The $(213, 231)$ -avoiding permutation 123984765, every point which is on a north-east to south-west line represents a descent element and every point which is on a south-east to north-west line represents an ascent element.

shows that the lemma holds true for $\sigma = 21$. Now, assume that the lemma is true for all patterns up to size $k \geq 2$. Let $\sigma \in \text{Av}_{k+1}(231, 213)$ and let t , be a subsequence of π of length $k+1$ such that $B(t) = B(\sigma)$. As $B(t)[2:] = B(\sigma)[2:]$ by the inductive hypothesis, it follows that $t[2:]$ is a matching of $\sigma[2:]$. Moreover $B(t)[1] = B(\sigma)[1]$ thus $t[1]$ and $\sigma[1]$ are both either the minimal or the maximal element of their respective subsequences. Therefore, t is a matching of σ in π . \square

We are now ready to solve the pattern matching problem in case both π and σ are $(213, 231)$ -avoiding.

Proposition 1. *Let π and σ be two $(213, 231)$ -avoiding permutations. One can decide whether π matches σ in linear time.*

Proof. According to Lemma 1 the problem reduces to deciding whether s_σ occurs as a subsequence in s_π . A straightforward greedy approach solves this issue in linear-time. \square

Thank to Corollary 2, we can compute the bijected words in the same time that we running the greedy algorithm, this gives us a on-line algorithm.

4 σ only is $(213, 231)$ -avoiding

This section focuses on the pattern matching problem in case only the pattern σ avoids 231 and 213. We need to consider a specific decomposition of σ into *factor* : we split the permutation into largest sequences of ascent element and descent element, respectively called an ascent factor and a descent factor. This correspond to split the permutation between every pair of ascent-descent element and descent-ascent element (see figure 5). For the special case of $(213, 231)$ -avoiding, this correspond to split the permutation into largest sequence of consecutive element. We will label the factors from right to left.

We introduce the notation $\text{LMEi}(s)$: Suppose that s is a subsequence of S , $\text{LMEi}(s)$ is the index of the left most element of s in S . Thus for every factor, $\text{LMEi}(\text{factor}(j))$ stand for the index in σ of the leftmost element of $\text{factor}(j)$. For example, $\sigma = 123984765$ is split as $123 - 98 - 4 - 765$. Hence $\sigma = \text{factor}(4)$

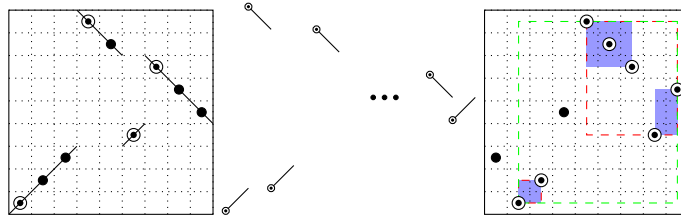


Fig. 5: The left figure is the $(213, 231)$ -avoiding permutation 123984765. Every line represents a factor, every circled point represents the left most element of each factor. The central figure is a generalisation of $(213, 231)$ -avoiding permutation. The right figure represents a matching of 1276534 in 312598746, the blue rectangles represent the matching, the red dotted rectangles represent the matching after the first replacement and the green rectangle represents the matching after the second replacement.

factor(3) factor(2) factor(1) with factor(4) = 123, factor(3) = 98, factor(2) = 4 and factor(1) = 765. Furthermore, $\text{LMEi}(\text{factor}(4)) = 1$, $\text{LMEi}(\text{factor}(3)) = 4$, $\text{LMEi}(\text{factor}(2)) = 6$ and $\text{LMEi}(\text{factor}(1)) = 7$. We represent elements matching an ascent (resp. descent) factor by a rectangle which has the left most matched point of the factor as the left bottom (resp. top) corner and the right most matched point of the factor as the right top (resp. bottom) corner. We can remove the two right most rectangles and replace it by the smallest rectangle that contains both of them and repeat this operation to represent part of a matching (see figure ??).

Remark 1. A factor is either an increasing or a decreasing sequence of element. Thus while matching a factor, it is enough to find an increasing or a decreasing subsequence of same size or bigger than the factor.

Corollary 3. *Given a permutation $\sigma \in \text{Av}(213, 231)$ and a suffix of its decomposition $\text{factor}(i) \text{factor}(i-1) \dots \text{factor}(1)$, if $\text{factor}(i)$ is an ascent (respectively descent) factor then the maximal (resp. minimal) element of $\text{factor}(i) \text{factor}(i-1) \dots \text{factor}(1)$ is the left most element of $\text{factor}(i-1)$*

This is a corollary of lemma 1. This states that given a permutation in $\text{Av}(213, 231)$ if the permutation starts with ascent (respectively descent) elements then the maximal (resp. minimal) element of this permutation is the first descent (resp. ascent) element (see figure 6).

We now define the set $S_{\sigma}^{\pi}(i, j)$ as the set of every subsequence s of $\pi[j :]$ that starts at $\pi[j]$ and that is a matching of $\text{factor}(i) \text{factor}(i-1) \dots \text{factor}(1)$ and .

Lemma 3. *Let σ be a permutation, $\text{factor}(i)$ be an ascent (respectively descent) factor, s be a subsequence of π such that $s \in S_{\sigma}^{\pi}(i, j)$ and that minimizes (resp. maximizes) the match of the left most element of $\text{factor}(i-1)$. For all subsequences $s' \in S_{\sigma}^{\pi}(i, j)$ and for all subsequences t of π , such that $t = t's'$, if t is a*

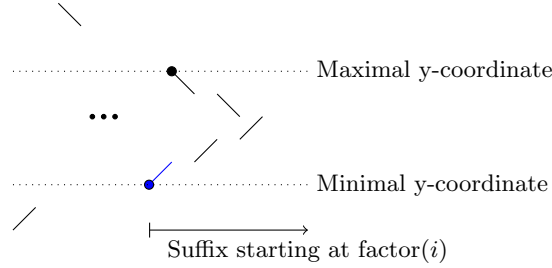


Fig. 6: The maximal element of the suffix starting at factor(i) (represented by the blue line) is the left most element of factor($i - 1$) (represented by the black dot). $LM_{\sigma}^{\pi}(i, j)$ is the minimal value of the matching of the maximal element (the black dot) in all the matchings of the suffix starting at LMEi(factor(i)) (the blue dot) in $\pi[j :]$.

matching of $\sigma[\text{LMEi}(\text{factor}(i+1)) :]$ such that the left most element of factor(i) is matched to $\pi[j]$ then the subsequence t 's is a matching of $\sigma[\text{LMEi}(\text{factor}(i+1)) :]$ such that the left most element of factor(i) is matched to $\pi[j]$.

This lemma states that given any matching of factor($i + 1$) factor(i) ... factor(1), where factor(i) is an ascent (resp. descent) factor, we can replace the part of the match that match factor(i) ... factor(1) by any match that minimise (resp. maximise) the left most element of factor($i - 1$). Indeed the left most element of factor($i - 1$) is the maximal (resp. minimal) element of factor(i) ... factor(1) (see figure 7).

Proof (of Lemma 3). By definition s is a matching of $\sigma[\text{LMEi}(\text{factor}(i)) :]$. To prove that ts is an matching of $\sigma[\text{LMEi}(\text{factor}(i + 1)) :]$ we need to prove that every element of t is larger than every element of s . If ts' is a matching of $\sigma[\text{LMEi}(\text{factor}(i + 1)) :]$ then every element of t is larger than every element of s' . Moreover the maximal element of s is smaller than the maximal element of s' so every element of s is smaller than every element of s' thus every element of s is smaller than every element of t . We use a similar argument if factor(i) is a descent factor. \square

Corollary 4. *Let σ be a permutation, factor(i) be an ascent (respectively descent) factor and s be a subsequence of π such that $s \in S_{\sigma}^{\pi}(i, j)$ and that minimizes (resp. maximizes) the match of the left most element of factor($i - 1$). These following statements are equivalent :*

- *There exists a matching of σ in π with the left most element of factor(i) is matched to $\pi[j]$.*
- *There exists a matching t of $\sigma[: \text{LMEi}(\text{factor}(i)) - 1]$ in $\pi[: j - 1]$ such that ts is a matching of σ in π with the left most element of factor(i) matched to $\pi[j]$.*

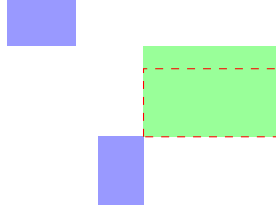


Fig. 7: In a matching, we can replace the green area by the red dashed area.

This corollary takes a step further from the previous one, it states that if there is no matching using any match that maximise (resp. minimise) the left most element of $\text{factor}(i-1)$ then there does not exist any matching at all. This is central to the algorithm because it allows to test only the matching that maximise (resp. minimise) the left most element of $\text{factor}(i-1)$ (see figure 7).

Proposition 2. *Let $\sigma \in \text{Av}_k(213, 231)$ and $\pi \in S_n$. One can decide in $O(\max(kn^2, n^2 \log(\log(n))))$ time and $O(kn^2)$ space if π matches σ .*

Proof. We first introduce a set of values needed to our proof. Let $LIS_\pi(i, j, bound)$ (resp. $LDS_\pi(i, j, bound)$) be the longest increasing (resp. decreasing) sequence in $\pi[i : j]$ starting at i , with every element of this sequence smaller (resp. bigger) than $bound$. LIS_π and LDS_π can be computed in $O(n^2 \log(\log(n)))$ time (see [3]). As stated before, those values allow us to find a matching of a factor.

Now consider the following set of values (see figure 8) :

$$LM_\sigma^\pi(i, j) = \left\{ \begin{array}{l} \begin{array}{l} \text{The match of } LMEi(\text{factor}(i-1)) \\ \text{in a matching of } \sigma[LMEi(\text{factor}(i)) :] \text{ in } \pi[j :] \\ \text{and that minimizes the match} \\ LMEi(\text{factor}(i-1)) \\ \text{and starts with } \pi[j] \\ \text{Or } \infty \text{ if no such matching exists} \end{array} & \begin{array}{l} \text{If } \text{factor}(i) \text{ is} \\ \text{an ascent factor} \end{array} \\ \\ \begin{array}{l} \text{The match of } LMEi(\text{factor}(i-1)) \\ \text{in a matching of } \sigma[LMEi(\text{factor}(i)) :] \text{ in } \pi[j :] \\ \text{and that maximizes the match} \\ LMEi(\text{factor}(i-1)) \\ \text{and starts with } \pi[j] \\ \text{Or } \infty \text{ if no such matching exists} \end{array} & \begin{array}{l} \text{If } \text{factor}(i) \text{ is} \\ \text{a descent factor} \end{array} \end{array} \right.$$

Clearly there exists a matching of σ in π if and only if there exists a $1 \leq i \leq n$ such that $LM(n_{factors}, i) \neq 0$ and $LM(n_{factors}, i) \neq \infty$ with $n_{factors}$ the number of factor in σ . We show how to compute recursively those values.

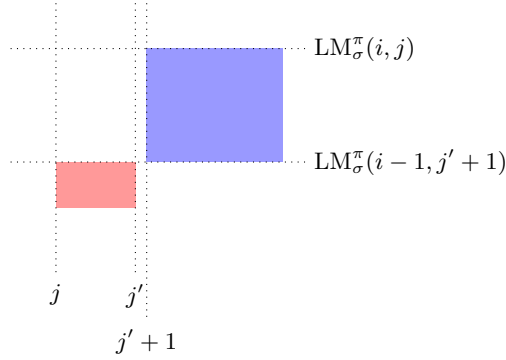


Fig. 8: When looking for a matching of $\sigma[\text{LMEi}(\text{factor}(i)) :]$ (red and blue areas) starting at $\pi[j :]$, (1) the red area is a matching of $\text{factor}(i)$ ie. if and only if it contains an increasing subsequence of size equal or bigger than $|\text{factor}(i)|$. (2) the red and blue areas have to be "compatible" (their x-coordinates and y-coordinates have to be disjoint and the red area has to be before the blue area in the x-coordinate and y-coordinate). $\text{AF}_\sigma^\pi(i, j)$ is the set of every top y-coordinate of every blue area which is compatible. $\text{LM}_\sigma^\pi(i, j)$ is the minimal value of $\text{AF}_\sigma^\pi(i, j)$ if the set is not void.

BASE :

$$\text{LM}_\sigma^\pi(1, j) = \begin{cases} \min_{j < j'} \{\infty\} \cup \{\pi[j'] \mid j' \text{ such that} \\ |\text{factor}(1)| \leq \text{LIS}(j, j', \pi[j'] + 1)\} & \text{If factor}(i) \text{ is} \\ & \text{an ascent factor} \\ \max_{j < j'} \{0\} \cup \{\pi[j'] \mid j' \text{ such that} \\ |\text{factor}(1)| \leq \text{LDS}(j, j', \pi[j'] - 1)\} & \text{If factor}(i) \text{ is} \\ & \text{a descent factor} \end{cases}$$

In the base case, one is looking for a matching of the first factor.

STEP :

$$\text{LM}_\sigma^\pi(i, j) = \begin{cases} \min\{\infty\} \cup \text{AF}_\sigma^\pi(i, j) & \text{If factor}(i) \text{ is an ascent factor} \\ \max\{0\} \cup \text{DF}_\sigma^\pi(i, j) & \text{If factor}(i) \text{ is a descent factor} \end{cases}$$

where $\text{AF}_\sigma^\pi(i, j)$ and $\text{DF}_\sigma^\pi(i, j)$ are the sets of elements matching the left most element of $\text{factor}(i-1)$ in a match of $\sigma[\text{LMEi}(\text{factor}(i)) :]$ starting at $\pi[j :]$. Suppose that $\text{factor}(i)$ is an ascent (resp descent) factor, index j' and matching t exists if and only if $\pi[j : j']$ contains a matching of $\text{factor}(i)$ "compatibles" with a matching in $\pi[j' + 1 :]$ of $\sigma[\text{LMEi}(\text{factor}(i-1)) :]$. It is enough to assure that every element of the matching of $\text{factor}(i)$ are smaller (resp. bigger) than the element of the matching of $\sigma[\text{LMEi}(\text{factor}(i-1)) :]$.

Thus we can compute $AF_\sigma^\pi(i, j)$ and $DF_\sigma^\pi(i, j)$ as follows:

$$\begin{aligned} AF_\sigma^\pi(i, j) &= \{\pi[j' + 1] \mid j < j' < n \text{ and } LM_\sigma^\pi(i - 1, j' + 1) \neq 0 \text{ and} \\ &\quad | \text{factor}(i) | \leq LIS_\pi(j, j', LM_\sigma^\pi(i - 1, j' + 1))\} \\ DF_\sigma^\pi(i, j) &= \{\pi[j' + 1] \mid j < j' < n \text{ and } LM_\sigma^\pi(i - 1, j' + 1) \neq \infty \text{ and} \\ &\quad | \text{factor}(i) | \leq LDS_\pi(j, j', LM_\sigma^\pi(i - 1, j' + 1))\} \end{aligned}$$

The number of factor is bound by k . Every instance of LIS_π and LDS_π can be computed in $O(n^2 \log(\log(n)))$. There are n base cases that can be computed $O(n)$ time, thus computing every base cases takes $O(n^2)$ time. There are kn different instance of AF and each one of them take $O(n)$ time to compute, thus computing every instance of AF takes $O(kn^2)$ time. There are kn different instance of LM and each one of them take $O(n)$, because the size of an AF is bounded by n , thus computing every LM takes $O(kn^2)$ time. Thus computing all the values takes $O(\max(kn^2, n^2 \log(\log(n))))$. Every value takes $O(1)$ space, thus the whole problem takes $O(kn^2)$ space. \square

5 (213, 231)-avoiding bivincular patterns

This section is devoted to the pattern matching problem with (213, 231)-avoiding bivincular pattern. Recall that a bivincular pattern generalises a permutation pattern by being able to force elements to be consecutive in value or in position. Hence, bivincular pattern is stronger in constraint than permutation pattern, intuitively we can not use the previous algorithm. As in a (213, 231)-avoiding permutation, we can describe structural property of a (213, 231)-avoiding bivincular pattern.

Lemma 4. *Given σ a (213, 231)-avoiding bivincular pattern, If $\overline{\sigma[i]\sigma[j]}$ (this implies that $\sigma[j] = \sigma[i] + 1$) and if $\sigma[i]$ is an ascent (resp. descent) element and $\sigma[i] + 1$ is an ascent (resp. descent) element then :*

- $i < j$ (resp. $j > i$)
- For every l , $i < l < j$ (resp. $j > l > i$), $\pi[l]$ is a descent (resp. ascent) element.

This lemma states that if two ascent (resp. descent) elements need to be matched to consecutive elements in value then every element between those two elements (if any) is a descent (resp. ascent) element.

Proof (of Lemma 4). Suppose that there exists l , $i < l < j$, such that $\sigma[l]$ is ascent. Ascent elements are increasing so $\sigma[i] < \sigma[l] < \sigma[j]$ which is in contradiction with $\sigma[j] = \sigma[i] + 1$. We use a similar argument if $\sigma[i]$ is a descent element \square

Proposition 3. *Let σ be a (213, 231)-avoiding bivincular of length k and $\pi \in S_n$. One can decide in $O(kn^4)$ time and $O(kn^3)$ space if π matches σ .*

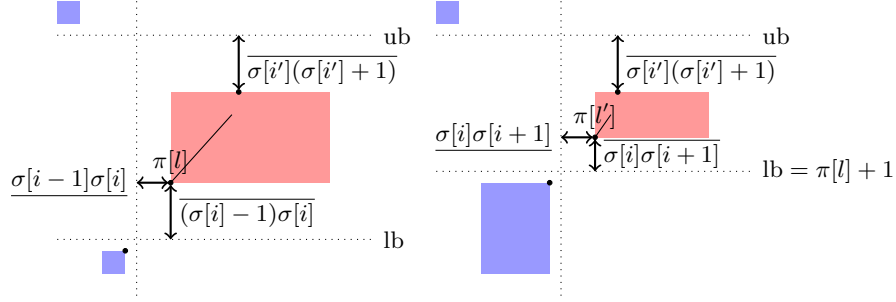


Fig. 9: When matching an ascent element, the value of ub does not change, and the lb is equal to the last match plus one. Note that there is only ascent element before $\sigma[i']$

Proof. We consider the following set of boolean : Given σ a (213, 231)-avoiding bivincular pattern, and a text $\pi \in Av_n(231, 213)$,

$$PM_{\sigma}^{\pi, lb, ub}(i, j) = \begin{cases} true & \text{If } \pi[j :] \text{ matches } \sigma[i :] \\ & \text{with every element of the matching is in } [lb, ub] \\ & \text{and starting } \pi[j] \\ false & \text{otherwise} \end{cases}$$

The argument lb (respectively ub) stands for the match of the last ascent (resp. descent) element matched plus (resp. minus) one. We now show how to compute recursively those boolean (see fig. 9).

BASE:

$$PM_{\sigma}^{\pi, lb, ub}(k, j) = \begin{cases} true & \text{if } \pi[j] \in [lb, ub] \\ & \text{and if } \sigma[k]_{\downarrow} \text{ then } j = n \\ & \text{and if } \sigma[k]_{\uparrow} \text{ then } \pi[j] = ub = k \\ & \text{and if } \overline{\sigma[k]} \text{ then } \pi[j] = lb = 1 \\ & \text{and if } \overline{(\sigma[k] - 1)\sigma[k]} \text{ then } \pi[j] = lb \\ & \text{and if } \overline{(\sigma[k]\sigma[k] - 1)} \text{ then } \pi[j] = ub \\ false & \text{otherwise} \end{cases}$$

The base case finds an matching for the rightmost element of the pattern. If the last element does not have any restriction on positions and on values, then $PM_{\sigma}^{\pi, lb, ub}(k, j)$ is true if and only if $\sigma[k]$ is matched $\pi[j]$. Which is true if $\pi[j] \in [lb, ub]$. If $\sigma[k]_{\downarrow}$ then $\sigma[k]$ must be matched to the right most element of π thus j must be the n . If $\sigma[k]_{\uparrow}$ then $\sigma[k]$ must be matched to the largest element which is k . If $\overline{\sigma[k]}$ then $\sigma[k]$ must be matched to the smallest element which is 1. If $\overline{(\sigma[k] - 1)\sigma[k]}$ then the matched element of $\sigma[k]$ and $\sigma[k] - 1$ must be consecutive in value, by recursion the value of the matched element of $\sigma[k] - 1$ will be recorded in lb and by adding 1 to it thus $\sigma[k]$ must be matched to lb . If

$(\overline{\sigma[k]\sigma[k]} - 1)$ then the matched element of $\sigma[k]$ and $\sigma[k] - 1$ must be consecutive in value, by recursion the value of the matched element of $\sigma[k] - 1$ will be recorded in ub and by removing 1 to it thus $\sigma[k]$ must be matched to ub .

STEP:

We need to consider 3 cases for the problem $\text{PM}_\sigma^{\pi, \text{lb}, \text{ub}}(i, j)$:

- If $\pi[j] \notin [\text{lb}, \text{ub}]$ then :

$$\text{PM}_\sigma^{\pi, \text{lb}, \text{ub}}(i, j) = \text{false}$$

which is immediate from the definition. If $\pi[j] \notin [\text{lb}, \text{ub}]$ then it can not be part of a matching of $\sigma[i :]$ in $\pi[i :]$ with every matched element in $[\text{lb}, \text{ub}]$.

- If $\pi[j] \in [\text{lb}, \text{ub}]$ and $\sigma[i]$ is an ascent element then :

$$\text{PM}_\sigma^{\pi, \text{lb}, \text{ub}}(i, j) = \begin{cases} \bigcup_{j < l} \text{PM}_\sigma^{\pi, \pi[j]+1, \text{ub}}(i+1, l) & \text{if } \sigma[i] \text{ is not underlined} \\ & \text{and } \sigma[i] \text{ is not overlined} \\ \bigcup_{j < l} \text{PM}_\sigma^{\pi, \pi[j]+1, \text{ub}}(i+1, l) & \text{if } \sigma[i] \text{ is not underlined} \\ & \text{and } (\overline{\sigma[i] - 1}\sigma[i]) \text{ or } \lceil \sigma[i] \\ & \text{and } \pi[j] = \text{lb} \\ \text{PM}_\sigma^{\pi, \pi[j]+1, \text{ub}}(i+1, j+1) & \text{if } \underline{\sigma[i]\sigma[i+1]} \\ & \text{and } (\overline{\sigma[i] - 1}\sigma[i]) \text{ or } \lceil \sigma[i] \\ & \text{and } \pi[j] = \text{lb} \\ \text{PM}_\sigma^{\pi, \pi[j]+1, \text{ub}}(i+1, j+1) & \text{if } \underline{\sigma[i]\sigma[i+1]} \\ & \text{and } \sigma[i] \text{ is not overlined} \\ \text{false} & \text{otherwise} \end{cases}$$

Remark that $\pi[j]$ can be matched to $\sigma[i]$ because $\pi[j] \in [\text{lb}, \text{ub}]$. Thus if $\pi[j+1 :]$ matches $\sigma[i+1 :]$ with every element of the matching in $[\sigma[i] + 1, \text{ub}]$ then $\pi[j :]$ matches $\sigma[i :]$. The last condition correspond to know if there exists $l, j < l$ such that $\text{PM}_\sigma^{\pi, \pi[j]+1, \text{ub}}(i+1, l)$ is true. The first case correspond to a matching without restriction on position and on value. The second case asks for the match of $\sigma[i] - 1$ and $\sigma[i]$ to be consecutive in value, but the match of $\sigma[i] - 1$ is $\text{lb} - 1$ thus we want $\pi[j] = \text{lb}$. The fourth case asks for the match of $\sigma[i]$ and $\sigma[i+1]$ to be consecutive in index, thus the match of $\sigma[i+1]$ must be $j+1$. The third case is an union of the second and fourth case. Note that if $\pi[j]$ is an ascent element we can not have the condition that the match of $\sigma[i]$ and $\sigma[i] + 1$ have to consecutive in value.

- If $\pi[j] \in [\text{lb}, \text{ub}]$ and $\sigma[i]$ is a descent element then :

$$\text{PM}_\sigma^{\pi, \text{lb}, \text{ub}}(i, j) = \begin{cases} \bigcup_{j < l} \text{PM}_\sigma^{\pi, \text{lb}, \pi[j]-1}(i+1, l) & \text{if } \sigma[i] \text{ is not underlined} \\ & \text{and } \sigma[i] \text{ is not overlined} \\ \bigcup_{j < l} \text{PM}_\sigma^{\pi, \text{lb}, \pi[j]-1}(i+1, l) & \text{if } \sigma[i] \text{ is not underlined} \\ & \text{and } \overline{\sigma[i](\sigma[i]+1)} \text{ or } \sigma[i]^\lceil \\ & \text{and } \pi[j] = \text{ub} \\ \text{PM}_\sigma^{\pi, \text{lb}, \pi[j]-1}(i+1, j+1) & \text{if } \overline{\sigma[i]\sigma[i+1]} \\ & \text{and } \overline{\sigma[i](\sigma[i]+1)} \text{ or } \sigma[i]^\lceil \\ & \text{and } \pi[j] = \text{ub} \\ \text{PM}_\sigma^{\pi, \text{lb}, \pi[j]-1}(i+1, j+1) & \text{if } \overline{\sigma[i]\sigma[i+1]} \\ & \text{and } \sigma[i] \text{ is not overlined} \\ \text{false} & \text{otherwise} \end{cases}$$

The same remark as the last case holds.

Clearly if $\bigcup_{0 < j} \text{PM}_\sigma^{\pi, 1, n}(1, j)$ is true then π matches σ . We now discuss the position and value constraints.

Position Constraint. There are 3 types of position constraints that can be added by underlined elements.

- If $\lfloor \sigma[1]$ then the leftmost element of σ must be matched to the leftmost element of π ($\sigma[1]$ is matched to $\pi[1]$ on a matching of σ in π). This constraint is satisfied by requiring that the matching starts at the left most element of π : if $\text{PM}_\sigma^{\pi, 1, n}(1, 1)$ is true.
- If $\sigma[k] \rfloor$ then the rightmost element σ must be matched the rightmost element of π ($\sigma[k]$ is matched to $\pi[n]$ on a matching of σ in π). This constraint is checked in the base case.
- If $\overline{\sigma[i]\sigma[i+1]}$ then the index of the matched elements of $\sigma[i]$ and $\sigma[i+1]$ must be consecutive. In other word, if $\sigma[i]$ is matched to $\pi[j]$ then $\sigma[i+1]$ must be matched to $\pi[j+1]$. We assure this restriction by recursion by requiring that the matching of $\sigma[i+1]$ to start at index $j+1$ (see figure 9).

Value Constraint. There are 3 types of position constraints that can be added by overlined elements.

- If $\lceil \sigma[i]$ (and thus $\sigma[i] = 1$) then the minimal value of σ must be matched to the minimal value of π .
 - If $\sigma[i]$ is an ascent element, then remark that every problem $\text{PM}_\sigma^{\pi, \text{lb}, *}(i, *)$ is true if $\sigma[i]$ is matched to element with value lb (by recursion) thus it is enough to require that $\text{lb} = 1$. Now remark that $\sigma[i]$ is the leftmost ascent element, indeed if not, then there exists an ascent element $\sigma[i']$, $i' < i$ and by definition $\sigma[i'] < \sigma[i]$ which is not possible as $\sigma[i]$ must be the minimal element. As a consequence $\sigma[1], \dots, \sigma[i-1]$ are descent

elements. Moreover the recursive call from a descent element does not modified the lower bound thus for every $\text{PM}_\sigma^{\pi, \text{lb}, *}(i, *)$, $\text{lb} = 1$ (see figure 9).

- If $\sigma[i]$ is a descent element then $i = k$ ($\sigma[i]$ is the right most element). Thus every $\text{PM}_\sigma^{\pi, *, *}(i, *)$ is a base case and is true if $\sigma[i]$ is matched to 1.
- If $\sigma[i]^{-1}$ (and thus $\sigma[i] = k$) then the maximal value of σ must be matched to the maximal value of π .
 - If $\sigma[i]$ is an descent element, then remark that every recursive call $\text{PM}_\sigma^{\pi, *, \text{ub}}(i, *)$ is true if $\sigma[i]$ is matched to element with value ub (by recursion) thus it is enough to require that $\text{ub} = n_\pi$. Now remark that $\sigma[i]$ is the leftmost descent element, indeed if not, then there exists an descent element $\sigma[i']$, $i' < i$ and by definition $\sigma[i'] > \sigma[i]$ which is not possible as $\sigma[i]$ must be the maximal element. As a consequence $\sigma[1], \dots, \sigma[i-1]$ are ascent elements. Moreover the recursive call from a ascent element does not modified the upper bound thus for every $\text{PM}_\sigma^{\pi, *, \text{ub}}(i, *)$, $\text{ub} = n$ (see figure 9).
 - If $\sigma[i]$ is an ascent element then $\sigma[i]$ then $i = k$ ($\sigma[i]$ is the right most element). Thus every $\text{PM}_\sigma^{\pi, *, *}(i, *)$ is a base case and is true if $\sigma[i]$ is matched to n_π .
- If $\overline{\sigma[i]\sigma[i']}$, (which implies that $\sigma[i'] = \sigma[i] + 1$) then if $\sigma[i]$ is matched to $\pi[j]$ then $\sigma[i']$ must be matched to $\pi[j] + 1$.
 - The case $\sigma[i]$ is a descent element, $\sigma[i']$ is an ascent element and $i < i'$ (remark that this case is equivalent to the case $\sigma[i]$ is an ascent element, $\sigma[i']$ is a descent element and $i' < i$) is not possible. Indeed $\sigma[i]$ is the maximal element of $\sigma[i :]$ thus $\sigma[i] > \sigma[i']$ which is in contradiction with $\sigma[i'] = \sigma[i] + 1$.
 - If $\sigma[i]$ is an ascent element, $\sigma[i']$ is a descent element and $i < i'$ (remark that this case is symmetric to the case $\sigma[i]$ is a descent element, $\sigma[i']$ is an ascent element and $i' < i$), then remark that every recursive call $\text{PM}_\sigma^{\pi, \text{lb}, *}(i', *)$ is true if $\sigma[i']$ is matched to the element with lb (by recursion) thus it is enough to require that $\text{lb} = \pi[j] + 1$. Now remark that $\sigma[i]$ is the right most ascent element and $\sigma[i']$ is the right most element (or $\sigma[i'] \neq \sigma[i] + 1$). As a consequence $\sigma[i+1], \sigma[i+2], \dots, \sigma[i'-1]$ are descent elements. Moreover the recursive call from a descent element does not modified the lower bound and $\text{PM}_\sigma^{\pi, \text{lb}, *}(i, *)$ will put the lower bound to $\pi[j] + 1$ thus for every $\text{PM}_\sigma^{\pi, \text{lb}, *}(i', *)$, $\text{lb} = \pi[j] + 1$ (see figure 9).
 - If $\sigma[i]$ is an ascent element and $\sigma[i']$ is an ascent element then first remark that every recursive call $\text{PM}_\sigma^{\pi, *, \text{ub}}(i', *)$ is true if $\sigma[i']$ is matched to element with value lb . Now remark that $i < i'$ and there is no ascent element between $\sigma[i]$ and $\sigma[i']$ (lemma 4), As a consequence $\sigma[i+1], \sigma[i+2], \dots, \sigma[i'-1]$ are descent elements. Moreover the recursive call from a descent element does not modified the lower bound and $\text{PM}_\sigma^{\pi, \text{lb}, *}(i, *)$ will put the lower bound to $\pi[j] + 1$ thus for every $\text{PM}_\sigma^{\pi, \text{lb}, *}(i', *)$, $\text{lb} = \pi[j] + 1$ (see figure 9).

There are n^3 base cases that can be computed in constant time. There are kn^3 different cases. Each case takes up to $O(n)$ time to compute. Thus computing all the cases take $O(kn^4)$ time. Each case take $O(1)$ space, thus we need $O(kn^3)$ space. \square

6 Computing the longest (213, 231)-avoiding pattern

This section is focused on a problem related to the pattern matching problem, finding the longest (213, 231)-avoiding subsequences in permutations: Given a set of permutations, find a longest (213, 231)-avoiding that can be matched by each input permutation. This problem is known to be NP-Hard for an arbitrary number of permutations and we do not hope it to be solvable in polynomial time even with the constraint of the subsequence must avoid (213, 231). Thus we focus on the cases where only one or two permutations are given in input. We start with the easiest case where we are given just one input permutation. We need the set of descent elements and the set of ascent elements. $A(\pi) = \{i | \pi[i] \text{ is an ascent element}\} \cup \{n\}$ and $D(\pi) = \{i | \pi[i] \text{ is a descent element}\} \cup \{n\}$.

Proposition 4. *If s is a longest (213, 231)-avoiding subsequence with last element at index f in π then $A(\pi)$ is a longest increasing subsequence with last element at index f and $D(\pi)$ is a longest decreasing subsequence with last element at index f .*

Proof (of Proposition 4). Let s be a longest subsequence avoiding (213, 231) with last element at index f in π , suppose that $P(s)$ is not a longest increasing subsequence with last element at index f . Let s_m be a longest increasing subsequence with last element f . Thus $|s_m| > |A(s)|$, clearly the sequence $s_m \cup D(s)$ is (213, 231)-avoiding and is longer than s , this is a contradiction. The same idea can be used to show that $D(\pi)$ is the longest decreasing subsequence. \square

Proposition 5. *Let π be a permutation. One can compute the longest (213, 231)-avoiding subsequence that can be matched in π in $O(n \log(\log(n)))$ time and in $O(n)$ space.*

Proof (of Proposition 5). The proposition 4 lead to an algorithm where one has to compute longest increasing and decreasing subsequence ending at every index. Then finding the maximal sum of longest increasing and decreasing subsequence ending at the same index. Computing the longest increasing subsequence and the longest decreasing subsequence can be done in $O(n \log(\log(n)))$ time and $O(n)$ space (see [3]), then finding the maximal can be done in linear time. \square

We now consider the case where the input is composed of two permutations.

Proposition 6. *Given two permutations π_1 and π_2 , one can compute the longest common (213, 231)-avoiding subsequence in $O(|\pi_1|^3 |\pi_2|^3)$ time and space.*

Proof. Consider the following problem that computes the longest stripe common to π_1 and π_2 : Given two permutations π_1 and π_2 , we define $\text{LCS}_{\pi_1, \text{lb}_1, \text{ub}_1}^{\pi_2, \text{lb}_2, \text{ub}_2}(i_1, i_2)$

$$= \max \{ |s| \mid s \text{ can be matched } \pi_1[i_1:] \text{ with every element of the match in } [\text{lb}_1, \text{ub}_1] \text{ and } s \text{ can be matched } \pi_2[i_2:] \text{ with every element of the match in } [\text{lb}_2, \text{ub}_2] \}$$

We show how to solve this problem by dynamic programming.

BASE:

$$\text{LCS}_{\pi_1, \text{lb}_1, \text{ub}_1}^{\pi_2, \text{lb}_2, \text{ub}_2}(|\pi_1|, |\pi_2|) = \begin{cases} 1 & \text{if } \text{lb}_1 \leq \pi_1[j] \leq \text{ub}_1 \\ & \text{and } \text{lb}_2 \leq \pi_2[j] \leq \text{ub}_2 \\ 0 & \text{otherwise} \end{cases}$$

STEP:

$$\text{LCS}_{\pi_1, \text{lb}_1, \text{ub}_1}^{\pi_2, \text{lb}_2, \text{ub}_2}(i_1, i_2) = \max \begin{cases} \text{LCS}_{\pi_1, \text{lb}_1, \text{ub}_1}^{\pi_2, \text{lb}_2, \text{ub}_2}(i_1, i_2 + 1) \\ \text{LCS}_{\pi_1, \text{lb}_1, \text{ub}_1}^{\pi_2, \text{lb}_2, \text{ub}_2}(i_1 + 1, i_2) \\ \text{M}_{\pi_1, \text{lb}_1, \text{ub}_1}^{\pi_2, \text{lb}_2, \text{ub}_2}(i_1, i_2) \end{cases}$$

with

$$\text{M}_{\pi_1, \text{lb}_1, \text{ub}_1}^{\pi_2, \text{lb}_2, \text{ub}_2}(i_1, i_2) = \begin{cases} 1 + \text{LCS}_{\pi_1, \pi_1[i_1]+1, \text{ub}_1}^{\pi_2, \pi_2[i_2]+1, \text{ub}_2}(i_1, i_2 + 1) & \pi_1[i_1] < \text{lb}_1 \\ & \text{and } \pi_2[i_2] < \text{lb}_2 \\ 1 + \text{LCS}_{\pi_1, \text{lb}_1, \pi_1[i_1]-1}^{\pi_2, \text{lb}_2, \pi_2[i_2]-1}(i_1 + 1, i_2) & \pi_1[i_1] > \text{ub}_1 \\ & \text{and } \pi_2[i_2] > \text{ub}_2 \\ 0 & \text{otherwise} \end{cases}$$

For every pair i, j we either ignore the element of π_1 , or we ignore the element of π_2 , or we match as the same step (if possible). These relations lead to a $O(|\pi_1|^3|\pi_2|^3)$ time and $O(|\pi_1|^3|\pi_2|^3)$ space algorithm. Indeed there is $|\pi_1|^3|\pi_2|^3$ cases possible for the problem and each case is solved in constant time. \square

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