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# Exact observability and controllability for linear neutral type systems <sup>☆</sup>

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## Abstract

The problem of exact observability is analyzed for a wide class of neutral type systems by an infinite dimensional approach. The duality with the exact controllability problem is the main tool. It is based on an explicit expression of a neutral type system which corresponding to the abstract adjoint system. A nontrivial relation is obtained between the initial neutral system and the system obtained via the adjoint abstract state operator. The characterization of the duality between controllability and observability is deduced, and then observability conditions are obtained.

*Keywords:* Observability, Controllability, Duality, Neutral type systems

*2010 MSC:* 93B05, 93B23

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## 1. Introduction

Approximate and spectral controllability and the corresponding dual notions of observability for delay systems of neutral type were widely investigated at the end of the last century (see books by [1] and [2] and references therein). The duality between these notions for systems of neutral type is not so trivial. The main reason is that the dual or adjoint system is not obtained directly by simple transposition. It is necessary to consider the duality using some hereditary product proposed first for retarded systems and later for neutral type systems (see [3, 4, 5] and [2] for example). In this context, the important technique of the so called structural operator was used. It enables some explicit formulations for duality between approximate controllability, spectral controllability and the same notions for observability and the characterizations of that concepts. We shall consider some of them in the context of our framework.

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The infinite dimensional setting has been developed essentially for exact controllability and often for neutral type systems without distributed delays. The exact observability problem has been less studied. In [6, 7] and [8] an approach is described based on the reconstruction of a part of the state for the case of a neutral type system with discrete delays. A duality condition with null controllability is given. The time of controllability (and of possible reconstruction of a part of the observed state) is estimated sufficiently large, without more precision.

The present paper is concerned with exact observability which is related to the notion of exact controllability developed in the paper of the authors [9] as an extension of other results obtained essentially for neutral type systems with discrete delay [10, 11]. The semigroup approach used by the authors in [9] is based on the model introduced in [12] in the product space  $M_2$  (see the definition below, in this section). In the infinite dimensional setting described in [9], exact controllability means reachability of the operator domain because reachability of all the state space is not possible by finite dimensional control. Hence, as it may be expected, the dual notion of observability is also adapted. Here, the approach using the structural operator is not used. Considering the adjoint system in the operator form, that is in an infinite dimensional framework, we construct a transposed neutral type system corresponding to the adjoint system in Hilbert space. This relation between the adjoint semigroup and the obtained neutral type system is different from that of the model given in [12].

The notions of exact controllability and observability are important because they imply exponential stabilizability or exponential convergences for possible estimators.

The results obtained in [9] use the approach of moment problems and allows the minimal time of exact controllability to be determined. The main contribution of our study is to specify how duality may be used in a nontrivial context and to deduce the characterization of exact observability and also the minimal time of observation.

We consider the neutral type system given by the equation

$$\dot{z}(t) = A_{-1}\dot{z}(t-1) + \int_{-1}^0 [A_2(\theta)\dot{z}(t+\theta) + A_3(\theta)z(t+\theta)] d\theta \quad (1)$$

where  $A_{-1}$  is a constant  $n \times n$ -matrix, and  $A_2, A_3$  are  $n \times n$ -matrices whose elements belong to  $L_2(-1, 0)$ .

If we introduce the linear operator  $L : H^1([-1, 0]; \mathbb{R}^n) \rightarrow \mathbb{R}^n$ , defined by

$$Lf = \int_{-1}^0 A_2(\theta)f'(\theta) + A_3(\theta)f(\theta) d\theta \quad (2)$$

then the system may be written concisely as

$$\dot{z}(t) = A_{-1}\dot{z}(t-1) + Lz_t, \quad z_t(\theta) = z(t+\theta).$$

This system may be represented, following the approach developed in [12], by an operator model in Hilbert space given by the equation

$$\dot{x} = \mathcal{A}x, \quad x(t) = \begin{pmatrix} v(t) \\ z_t(\cdot) \end{pmatrix}, \quad (3)$$

where  $\mathcal{A}$  is the infinitesimal generator of a  $C_0$ -semigroup  $S(t) = e^{\mathcal{A}t}$  given in the product space

$$M_2 = M_2(-1, 0; \mathbb{R}^n) \stackrel{\text{def}}{=} \mathbb{R}^n \times L_2(-1, 0; \mathbb{R}^n)$$

and defined by

$$\mathcal{A}x(t) = \mathcal{A} \begin{pmatrix} v(t) \\ z_t(\cdot) \end{pmatrix} = \begin{pmatrix} Lz_t(\cdot) \\ dz_t(\theta)/d\theta \end{pmatrix}, \quad (4)$$

with the domain  $D(\mathcal{A}) \subset M_2$  given by

$$\left\{ \begin{pmatrix} v \\ \varphi(\cdot) \end{pmatrix} : \varphi(\cdot) \in H^1, v = \varphi(0) - A_{-1}\varphi(-1) \right\}, \quad (5)$$

where  $H^1 = H^1([-1, 0]; \mathbb{R}^n)$ .

We consider the finite dimensional observation

$$y(t) = \mathcal{C}x(t) \quad (6)$$

where  $\mathcal{C}$  is a linear operator and  $y(t) \in \mathbb{R}^p$  is a finite dimensional output. There are several ways to design the output operator  $\mathcal{C}$  [13, 2, 6]. One of our goals in this paper is to investigate how to design a minimal output operator like

$$\mathcal{C}x(t) = Cz(t) \quad \text{or} \quad \mathcal{C}x(t) = Cz(t-1), \quad (7)$$

where  $C$  is a  $p \times n$  matrix. More general outputs, for example with several and/or distributed delays are not considered in this paper. We want to use some results on exact controllability in order to analyze, by duality, the exact observability property in the infinite dimensional setting like, for example, in [14].

The operator  $\mathcal{C}$  defined in (7) is linear but not bounded in  $M_2$ . However, in both cases it is admissible in the following sense:

$$\int_0^T \|\mathcal{C}S(t)x_0\|_{\mathbb{R}^p}^2 dt \leq \kappa^2 \|x_0\|_{M_2}^2, \quad \forall x_0 \in D(\mathcal{A}),$$

because it is bounded on  $D(\mathcal{A})$ . We recall that if  $x_0 \in D(\mathcal{A})$  then  $S(t)x_0 \in D(\mathcal{A})$ ,  $t \geq 0$  (see for example [15]). In fact,  $\mathcal{C}$  is admissible in the resolvent norm:

$$\|x_0\|_{-1} = \|R(\lambda, \mathcal{A})x_0\| = \|(\lambda I - \mathcal{A})^{-1}x_0\|_{M_2}, \quad \lambda \in \rho(\mathcal{A}).$$

This is a consequence of the fact that  $\mathcal{C}$  is a closed operator and takes value in a finite dimensional space (see [14, Def. 4.3.1] and comments on this Definition).

**Definition 1.1.** Let  $\mathcal{K}$  be the output operator

$$\mathcal{K} : M_2 \longrightarrow L_2(0, T; \mathbb{R}^p), \quad x_0 \longmapsto \mathcal{K}x_0 = \mathcal{C}S(t)x_0.$$

The system (1) is said to be approximately observable (or observable) if  $\ker \mathcal{K} = \{0\}$  and exactly observable (or continuously observable [2]) if

$$\|\mathcal{K}x_0\|_{L_2}^2 = \int_0^T \|\mathcal{C}S(t)x_0\|_{\mathbb{R}^p}^2 dt \geq \delta^2 \|x_0\|_{M_2}^2, \quad (8)$$

for some constant  $\delta$ .

This is the classic definition. In the case of a neutral type system with a finite dimensional output (7) the exact observability in this sense is not possible. It may be possible if we consider another topology for the initial states  $x_0$ .

Unlike approximate observability, which does not depend on the topology, exact observability depends essentially on the topology in the space. We can expect that, the given neutral type system is not exactly observable if we consider  $x_0 \in D(\mathcal{A})$ , with the norm of the graph and no longer in the topology of  $M_2$ . Taking in account the result on exact controllability, it seems that (8) must be changed by taking a weaker norm for  $x_0$ , namely the resolvent norm  $\|(\lambda I - \mathcal{A})^{-1}x_0\|$  and considering the extension of the operator  $\mathcal{K}$  to the completion of the space with this norm. In fact, we obtain the observability in the initial norm but we need some delay in the observation in the general case.

Exact observability can be investigated directly, but another way is to use the duality between exact observability and exact controllability. In [9] the conditions of exact controllability were given for the controlled system

$$\dot{z}(t) = A_{-1}z(t-1) + Lz_t + Bu(t).$$

In order to use the duality between observability and controllability, we need to compute the adjoint operator  $\mathcal{K}^*$  in the duality with respect to the pivot space  $M_2$  in the embedding

$$X_1 \subset X = M_2 \subset X_{-1}, \quad (9)$$

where  $X_1 = D(\mathcal{A})$  with the graph norm noted  $\|x\|_1$  and  $X_{-1}$  is the completion of the space  $M_2$  with respect to the resolvent norm  $\|x\|_{-1} = \|(\lambda I - \mathcal{A})^{-1}x\|_{M_2}$ . The duality relation is

$$\langle \mathcal{K}x_0, u(\cdot) \rangle_{L_2(0,T;\mathbb{R}^p)} = \langle x_0, \mathcal{K}^*u(\cdot) \rangle_{X_1, X_{-1}^d}, \quad (10)$$

where  $X_{-1}^d$  is constructed as  $X_{-1}$  with  $\mathcal{A}^*$  instead of  $\mathcal{A}$  (see [14] for example). Our purpose is to compute the adjoint operator

$$\mathcal{K}^* : L_2(0, T; \mathbb{R}^p) \rightarrow X_{-1}^d.$$

The abstract formulation is well known. Exact controllability is dual with exact observability in the corresponding spaces with the corresponding topologies. It is expected that the operator  $\mathcal{K}^*$  corresponds to a control operator for some adjoint system.

We then need the expression of the adjoint state operator  $\mathcal{A}^*$  and the corresponding adjoint system in the same class: the class of neutral type systems. As it will be shown, the situation is not so simple. This is the object of Section 2. In Section 3 we return to the duality relation with the explicit expression of the adjoint system after formulations of exact controllability results. As the adjoint neutral type system has a slightly different structure, we give an explicit relation between the new neutral type system and the original one. After that we can give the expression of duality between exact controllability and exact observability. This enables to formulate the characterization of exact observability and to give the minimal time of observability. Some illustrative examples are given.

For the sake of completeness, we recall some results on approximate controllability (from [2] and [11]) and formulate the duality with the corresponding notion of observability in our framework.

## 2. The adjoint system

In this section we give the expression of the adjoint system corresponding to the adjoint operator  $\mathcal{A}^*$  as the operator  $\mathcal{A}$  corresponds to the system (1). Let us recall first the expression of the adjoint operator  $\mathcal{A}^*$  and its spectrum  $\sigma(\mathcal{A}^*)$ .

**Proposition 2.1.** ([16]) *The adjoint operator  $\mathcal{A}^*$  is given by*

$$\mathcal{A}^* \begin{pmatrix} w \\ \psi(\cdot) \end{pmatrix} = \begin{pmatrix} A_2^*(0)w + \psi(0) \\ -\frac{d[\psi(\theta) + A_2^*(\theta)w]}{d\theta} + A_3^*(\theta)w \end{pmatrix}, \quad (11)$$

with the domain  $D(\mathcal{A}^*)$ :

$$\{(w, \psi(\cdot)) : \psi(\theta) + A_2^*(\theta)w \in H^1, A_{-1}^*(A_2^*(0)w + \psi(0)) = \psi(-1) + A_2^*(-1)w\}. \quad (12)$$

$\sigma(\mathcal{A}^*)$  consists of eigenvalues, roots of the equation  $\det \Delta^*(\lambda) = 0$ , where

$$\Delta^*(\lambda) = \lambda I - \lambda e^{-\lambda} A_{-1}^* - \int_{-1}^0 e^{\lambda s} [A_3^*(s) + \lambda A_2^*(s)] ds. \quad (13)$$

The adjoint operator  $\mathcal{A}^*$  in (11) seems to be different from a state operator generated by a neutral type system. However, we can construct a system of neutral type corresponding, in some sense, to the given adjoint operator.

**Theorem 2.2.** *Let  $x$  be a solution of the abstract equation*

$$\dot{x} = \mathcal{A}^* x, \quad x(t) = \begin{pmatrix} w(t) \\ \psi_t(\theta) \end{pmatrix}. \quad (14)$$

Then the function  $w(t)$  is the solution of the neutral type equation

$$\dot{w}(t+1) = A_{-1}^* \dot{w}(t) + \int_{-1}^0 [A_2^*(\tau) \dot{w}(t+1+\tau) + A_3^*(\tau) w(t+1+\tau)] d\tau. \quad (15)$$

**Proof.** Our purpose is to find the corresponding neutral type equation in  $\mathbb{R}^n$ . Equation (14) may be written as

$$\frac{\partial}{\partial t} \begin{pmatrix} w(t) \\ \psi_t(\theta) \end{pmatrix} = \begin{pmatrix} A_2^*(0)w(t) + \psi_t(0) \\ -\frac{\partial[\psi_t(\theta) + A_2^*(\theta)w(t)]}{\partial \theta} + A_3^*(\theta)w(t) \end{pmatrix}$$

Let us put  $r(\theta) = A_2^*(\theta)w + \psi(\theta)$  and

$$r(t, \theta) = A_2^*(\theta)w(t) + \psi_t(\theta) = A_2^*(\theta)w(t) + \psi_t(\theta). \quad (16)$$

Then the operator  $\mathcal{A}^*$  may be rewritten as

$$\mathcal{A}^* \begin{pmatrix} w \\ r(\theta) - A_2^*(\theta)w \end{pmatrix} = \begin{pmatrix} r(0) \\ -\frac{dr(\theta)}{d\theta} + A_3^*(\theta)w \end{pmatrix}, \quad (17)$$

and the differential equation  $\dot{x} = \mathcal{A}^* x$  as a system of two equations:

$$\frac{\partial}{\partial t} \begin{pmatrix} w(t) \\ r(t, \theta) - A_2^*(\theta)w(t) \end{pmatrix} = \begin{pmatrix} r(t, 0) \\ -\frac{\partial r(t, \theta)}{\partial \theta} + A_3^*(\theta)w(t) \end{pmatrix}. \quad (18)$$

The second equation of this system may be written as a partial differential equation:

$$\frac{\partial}{\partial t} r(t, \theta) + \frac{\partial}{\partial \theta} r(t, \theta) = A_2^*(\theta) \dot{w}(t) + A_3^*(\theta) w(t). \quad (19)$$

The general solution of this equation is

$$r(t, \theta) = f(t - \theta) + \int_0^\theta [A_2^*(\tau) \dot{w}(t - \theta + \tau) + A_3^*(\tau) w(t - \theta + \tau)] d\tau, \quad (20)$$

where  $f(t - \theta)$  is the solution of the homogeneous equation obtained from (19):

$$\frac{\partial}{\partial t} r(t, \theta) + \frac{\partial}{\partial \theta} r(t, \theta) = 0.$$

and the second term is a particular solution of (19).

The first equation of the system (18) gives

$$\dot{w}(t) = r(t, 0). \quad (21)$$

From (20) (obtained from the second equation), putting  $\theta = 0$ , we get with (21)

$$\dot{w}(t) = r(t, 0) = f(t). \quad (22)$$

Then (20) and (22) allow  $r(t, \theta)$  to be written as follows:

$$r(t, \theta) = \dot{w}(t - \theta) + \int_0^\theta [A_2^*(\tau) \dot{w}(t - \theta + \tau) + A_3^*(\tau) w(t - \theta + \tau)] d\tau. \quad (23)$$

From the definition of the domain  $D(\mathcal{A}^*)$  we obtain  $A_{-1}^* r(0) = r(-1)$ . For the function  $r(t, \theta)$ , this condition reads

$$r(t, -1) = A_{-1}^* r(t, 0) = A_{-1}^* \dot{w}(t) \quad (24)$$

and by (23) we have

$$r(t, -1) = \dot{w}(t + 1) - \int_{-1}^0 [A_2^*(\tau) \dot{w}(t + 1 + \tau) + A_3^*(\tau) w(t + 1 + \tau)] d\tau. \quad (25)$$

Finally, from (24) and (25), we obtain the dual equation

$$\dot{w}(t + 1) = A_{-1}^* \dot{w}(t) + \int_{-1}^0 [A_2^*(\tau) \dot{w}(t + 1 + \tau) + A_3^*(\tau) w(t + 1 + \tau)] d\tau. \quad (26)$$

On the other hand the solution of equation (18) is

$$e^{\mathcal{A}^* t} x_0 = \begin{pmatrix} w(t) \\ \psi_t(\theta) \end{pmatrix} = \begin{pmatrix} w(t) \\ r(t, \theta) - A_2^*(\theta) w(t) \end{pmatrix}, \quad (27)$$

where  $w(t)$  is the solution of equation (26). If  $x_0 \in X$  then it is a mild solution. ■

This result may also be formulated, by simple duality (transposition), in the following way.

**Theorem 2.3.** *Let  $x$  be a solution of the abstract equation*

$$\dot{x} = \tilde{\mathcal{A}} x, \quad x(t) = \begin{pmatrix} w(t) \\ \psi_t(\theta) \end{pmatrix},$$

where the operator  $\tilde{\mathcal{A}}$  is defined by

$$\tilde{\mathcal{A}} \begin{pmatrix} w \\ \psi(\cdot) \end{pmatrix} = \begin{pmatrix} A_2(0)w + \psi(0) \\ -\frac{d[\psi(\theta) + A_2(\theta)w]}{d\theta} + A_3(\theta)w \end{pmatrix},$$

with the domain

$$D(\tilde{\mathcal{A}}) = \{(w, \psi(\cdot)) : \psi(\theta) + A_2(\theta)w \in H^1, (A_{-1}A_2(0) - A_2(-1))w = \psi(-1) - A_{-1}\psi(0)\}.$$

Then the function  $w(t)$  is the solution of the neutral type equation

$$\dot{w}(t+1) = A_{-1}\dot{w}(t) + \int_{-1}^0 [A_2(\tau)\dot{w}(t+1+\tau) + A_3(\tau)w(t+1+\tau)] d\tau. \quad (28)$$

Let us now specify the relation between the solutions of neutral type equations (28) and (1). Let us put

$$\begin{pmatrix} w(t) \\ \psi_t(\theta) \end{pmatrix} = e^{\tilde{\mathcal{A}} t} \tilde{x}_0 = e^{\tilde{\mathcal{A}} t} \begin{pmatrix} w(0) \\ \psi_0(\theta) \end{pmatrix},$$

and

$$\begin{pmatrix} v(t) \\ z_t(\theta) \end{pmatrix} = \begin{pmatrix} w(t+1) - A_{-1}w(t) \\ w(t+1+\theta) \end{pmatrix} = e^{\tilde{\mathcal{A}} t} \begin{pmatrix} v(0) \\ z_0(\theta) \end{pmatrix} = e^{\tilde{\mathcal{A}} t} \xi_0,$$

where  $z_0(\theta) = w(t+1)$  and  $v(0) = z_0(0) - A_{-1}z_0(-1)$ . Our purpose is to give the explicit relation between the initial conditions  $\tilde{x}_0$  and  $\xi_0$ :

$$\tilde{x}_0 = \begin{pmatrix} w(0) \\ \psi_0(\theta) \end{pmatrix}, \quad \xi_0 = \begin{pmatrix} v(0) \\ z_0(\theta) \end{pmatrix}.$$

The formal relation between these vectors is

$$\tilde{x}_0 = \begin{pmatrix} w(0) \\ \psi_0(\theta) \end{pmatrix} = F \xi_0 = F \begin{pmatrix} w(1) - A_{-1}w(0) \\ w(\theta+1) \end{pmatrix}.$$

**Theorem 2.4.** *The operator  $F$  representing the relation between initial conditions  $\tilde{x}_0$  and  $\xi_0$  corresponding to the neutral type systems (1) and (28) is linear bounded and bounded invertible from  $X_1$  to  $M_2$ .*

**Proof.** Let us calculate the explicit expression for the linear operator  $F$ . From (23) and (16), taking in account that we consider here the operator  $\tilde{\mathcal{A}}$  instead of  $\mathcal{A}^*$ , we obtain

$$\begin{aligned} r(0, \theta) &= \psi_0(\theta) + A_2(\theta) w(0) \\ &= \dot{w}(-\theta) + \int_0^\theta [A_2(\tau) \dot{w}(\tau - \theta) + A_3(\tau) w(\tau - \theta)] d\tau, \end{aligned} \quad (29)$$

which can be written as

$$r(0, \theta) = \dot{w}(-\theta) + \int_0^\theta [A_2(\theta - s) \dot{w}(-s) + A_3(\theta - s) w(-s)] ds.$$

Putting  $w(-s) = \int_0^s \dot{w}(-\sigma) d\sigma + w(0)$ , we get

$$\begin{aligned} r(0, \theta) &= \int_0^\theta A_3(\theta - s) ds \cdot w(0) \\ &= \dot{w}(-\theta) + \int_0^\theta \left[ A_2(\theta - s) \dot{w}(-s) + A_3(\theta - s) \int_0^s \dot{w}(-\sigma) d\sigma \right] ds. \end{aligned} \quad (30)$$

This may be represented by the expression

$$r(0, \theta) - \int_0^\theta A_3(\theta - s) ds \cdot w(0) = (I + \mathcal{V}) \dot{w}(-s), \quad (31)$$

where  $\mathcal{V}$  is the Volterra operator defined by

$$\mathcal{V} \mu(\cdot) = \int_0^\theta \left[ A_2(\theta - s) \mu(s) + A_3(\theta - s) \int_0^s \mu(\sigma) d\sigma \right] ds.$$

The operator  $\mathcal{V}$  is a compact linear operator from  $L_2(-1, 0; \mathbb{R}^n)$  to  $L_2(-1, 0; \mathbb{R}^n)$  with a spectrum  $\sigma(\mathcal{V}) = \{0\}$ . This implies that the operator  $I + \mathcal{V}$  is bounded invertible on  $L_2(-1, 0; \mathbb{R}^n)$ .

Let us now represent the operator  $F$  as a composition of operators according to the following commutative diagram

$$\begin{array}{ccc} \begin{pmatrix} v(0) \\ z_0(\theta) \end{pmatrix} & \xrightarrow{P} & \begin{pmatrix} w(0) \\ \dot{w}(-\theta) \end{pmatrix} \\ \downarrow F & & \downarrow Q \\ \begin{pmatrix} w(0) \\ \psi_0(\theta) \end{pmatrix} & \xleftarrow{R} & \begin{pmatrix} w(0) \\ (I + \mathcal{V}) \dot{w}(-s) \end{pmatrix} \end{array}$$

where, as explained above,  $\begin{pmatrix} v(0) \\ z_0(\theta) \end{pmatrix} = \begin{pmatrix} w(1) - A_{-1} w(0) \\ w(\theta + 1) \end{pmatrix}$  and

$$(I + \mathcal{V}) \dot{w}(-s) = r(0, \theta) + \int_0^{-\theta} A_3(\theta + s) ds \cdot w(0),$$

where  $r(0, \theta)$  is given as in (29). The operators  $P : X_1 \rightarrow M_2$  and  $R : M_2 \rightarrow M_2$  are bounded invertible. Moreover, as  $I + \mathcal{V} : L_2(-1, 0; \mathbb{R}^n) \rightarrow L_2(-1, 0; \mathbb{R}^n)$  is bounded invertible, then  $Q : M_2 \rightarrow M_2$  is also bounded invertible. ■

We also need the following property of the bounded operator  $F^{-1}$ .

**Proposition 2.5.** For  $\lambda \neq \sigma(\mathcal{A})$ , the operator

$$F^{-a} = F^{-1}(\lambda I - \mathcal{A})$$

can be extended to a bounded (and bounded invertible) operator from  $M_2$  to  $M_2$ .

**Proof.** We need to prove that

$$\|F^{-1}(\lambda I - \mathcal{A})\tilde{x}_0\| \leq C\|\tilde{x}_0\|, \quad \tilde{x}_0 \in D(\mathcal{A}), C > 0, \quad (32)$$

where  $\|\cdot\|$  is the initial norm in  $M_2$ . Let  $L_0$  and  $D_0$  be the subspaces

$$L_0 = \{(0, \psi(\cdot)) : \psi(\cdot) \in L_2(-1, 0; \mathbb{R}^n)\}, \quad D_0 = L_0 \cap D(\mathcal{A}).$$

It is clear that  $D_0$  is of finite co-dimension  $n$ , and this implies that it is enough to prove the relation (32) for  $\tilde{x}_0 \in D_0$ . Let  $\tilde{x}_0 = (0, \psi_0(\cdot)) \in D_0$ . The action of the operator  $F^{-a} = F^{-1}(\lambda I - \mathcal{A})$  may be decomposed according to the following diagram

$$\begin{array}{ccc} \begin{pmatrix} 0 \\ \psi_0(\cdot) \end{pmatrix} & \xrightarrow{(\lambda I - \mathcal{A})} & \begin{pmatrix} -\psi_0(0) \\ \lambda\psi_0(\cdot) + \dot{\psi}_0(\cdot) \end{pmatrix} & \xrightarrow{R^{-1}} & \begin{pmatrix} -\psi_0(0) \\ \Psi_0(\cdot) \end{pmatrix} \\ \downarrow F^{-a} & & & & \downarrow Q^{-1} \\ \begin{pmatrix} w(1) - A_{-1}w(0) \\ w(\theta + 1) \end{pmatrix} & \xleftarrow{P^{-1}} & & & \begin{pmatrix} \psi_0(0) \\ \dot{w}(-\theta) \end{pmatrix} \end{array}$$

where

$$\Psi_0(\theta) = \lambda\psi_0(\cdot) + \dot{\psi}_0(\cdot) + \int_0^{-\theta} A_3(\theta + s)ds \cdot \psi_0(0),$$

and the function  $w(\cdot)$  is determined from the equation (obtained from (29)):

$$\lambda\psi_0(\theta) + \dot{\psi}_0(\theta) - A_2(\theta)\psi_0(0) = \dot{w}(-\theta) + \int_0^\theta [A_2(\tau)\dot{w}(\tau - \theta) + A_3(\tau)w(\tau - \theta)]d\tau,$$

with the initial condition  $w(0) = -\psi_0(0)$ . Integrating the last equation from 0 to  $-1 - \theta$  and taking in account the initial condition, we obtain

$$\begin{aligned} \psi_0(-1 - \theta) + \lambda \int_0^{-1 - \theta} \psi_0(\tau)d\tau - \int_0^{-1 - \theta} A_2(\tau)d\tau \cdot \psi_0(0) = \\ -w(1 + \theta) + \int_0^{-1 - \theta} \left( \int_0^\tau A_2(s)\dot{w}(s - \tau) + A_3(s)w(s - \tau)ds \right) d\tau. \end{aligned}$$

Using a transformation in the double integration and the initial condition, we get

$$\begin{aligned} \psi_0(-1 - \theta) + \lambda \int_0^{-1 - \theta} \psi_0(\tau)d\tau = \\ -w(1 + \theta) + \int_0^{-1 - \theta} \left( A_2(\tau)w(1 + \theta + \tau) + \int_0^\tau A_3(s)w(s - \tau)ds \right) d\tau, \end{aligned}$$

and this can be written as  $(I + \mathcal{V}_1)\psi_0(-1 - \theta) = (-I + \mathcal{V}_2)w(1 + \theta)$ ,  $\theta \in [-1, 0]$ , where  $\mathcal{V}_1$  and  $\mathcal{V}_2$  are Volterra operators from  $L_2(-1, 0; \mathbb{R}^n)$  to  $L_2(-1, 0; \mathbb{R}^n)$ . Both operators have spectra concentrated at  $\{0\}$ . Then

$$w(1 + \theta) = (-I + \mathcal{V}_2)^{-1}(I + \mathcal{V}_1)\psi_0(-1 - \theta) = \mathcal{W}\psi_0(-1 - \theta),$$

where  $\mathcal{W}$  is a bounded invertible operator on  $L_2(-1, 0; \mathbb{R}^n)$ . This enables the final expression for the operator  $F^{-a} = F^{-1}(\lambda I - \mathcal{A})$  on the set  $D_0$  to be obtained:

$$F^{-a} \begin{pmatrix} 0 \\ \psi_0(\theta) \end{pmatrix} = \begin{pmatrix} \mathcal{W}\psi_0(-1 - \theta)|_{\theta=0} - A_{-1}\mathcal{W}\psi_0(-1 - \theta)|_{\theta=-1} \\ \mathcal{W}\psi_0(-1 - \theta) \end{pmatrix} = \begin{pmatrix} w(1) - A_{-1}w(0) \\ w(1 + \theta) \end{pmatrix}.$$

Taking in account  $A_{-1}w(0) = -A_{-1}\psi_0(0) = -\psi_0(-1)$ , we can rewrite

$$\mathcal{W}\psi_0(-1 - \theta)|_{\theta=0} - A_{-1}\mathcal{W}\psi_0(-1 - \theta)|_{\theta=-1} = \mathcal{W}\psi_0(-1 - \theta)|_{\theta=0} + \psi_0(-1).$$

As a Volterra operator is quasinilpotent, we can write:  $(-I + \mathcal{V}_2)^{-1} = -\sum_{k=0}^{\infty} \mathcal{V}_2^k$  and  $\mathcal{W} = -\sum_{k=0}^{\infty} \mathcal{V}_2^k(I + \mathcal{V}_1)$ . This gives

$$\begin{aligned} \mathcal{W}\psi_0(-1 - \theta)|_{\theta=0} + \psi_0(-1) &= -\left(\sum_{k=1}^{\infty} \mathcal{V}_2^k(I + \mathcal{V}_1) + \mathcal{V}_1\right)\psi_0(-1 - \theta)\Big|_{\theta=0} \\ &= G\psi_0(-1 - \theta), \end{aligned}$$

and  $G$  is a linear operator from a dense set of  $L_2(-1, 0; \mathbb{R}^n)$  to  $\mathbb{R}^n$ . Since equalities

$$\begin{aligned} \mathcal{V}_1\varphi(-1 - \theta)\Big|_{\theta=0} &= -\int_{-1}^0 \varphi(\tau) d\tau, \\ \mathcal{V}_2\varphi(-1 - \theta)\Big|_{\theta=0} &= -\int_{-1}^0 \left( A_2(\tau)\varphi(-1 - \tau) \int_0^\tau A_2(s)\varphi(\tau - s) ds \right) d\tau \end{aligned}$$

define bounded operators from  $L_2(-1, 0; \mathbb{R}^n)$  to  $\mathbb{R}^n$ , we conclude that the operator  $G$  can be extended by continuity to a linear bounded operator from  $L_2(-1, 0; \mathbb{R}^n)$  to  $\mathbb{R}^n$ . As a consequence, the operator  $F^{-a}$  is extended to a bounded (in the norm of  $M_2$ ) operator, defined on  $L_0$  by the formula

$$F^{-a} \begin{pmatrix} 0 \\ \psi(\theta) \end{pmatrix} = \begin{pmatrix} G\psi(-1 - \theta) \\ \mathcal{W}\psi(-1 - \theta) \end{pmatrix}. \quad (33)$$

Let us observe that the subspace  $L_0$  as well as its image  $F^{-a}L_0$  have codimension  $n$  in the space  $M_2$  (a complement subspace for them is  $\mathbb{R}^n \times \{0\} \subset M_2$ ). Moreover, the mapping  $L_0 \xrightarrow{F^{-a}} F^{-a}L_0$  is bijective. On the other hand, the subspace  $D_0$  has codimension equals  $n$  in  $D(\mathcal{A})$ ,  $F^{-a}$  is defined on  $D(\mathcal{A})$  by the formula

$$F^{-a} = F^{-1}(\lambda I - \mathcal{A}), \quad (34)$$

and the mapping  $D(\mathcal{A}) \xrightarrow{F^{-a}} D(\mathcal{A})$  is also bijective. Besides,  $D(\mathcal{A}) + L_0 = D(\mathcal{A}) + F^{-a}L_0 = M_2$ . Comparing all these facts, we conclude that the operator  $F^{-a}$ , given by formulas (33), (34), can be extended to a bounded bijective operator on  $M_2$ . ■

A direct consequence of this proposition is the following corollary.

**Corollary 2.6.** For all  $x \in D(\mathcal{A})$ , for  $\lambda \neq \sigma(\tilde{\mathcal{A}})$ , we have

$$c\|x\| \leq \|(\lambda I - \tilde{\mathcal{A}})^{-1}Fx\| \leq C\|x\|,$$

where  $\|\cdot\|$  is the norm of the space  $M_2$ .

### 3. The control system and duality

Consider the controlled neutral type system

$$\dot{z}(t) = A_{-1}\dot{z}(t-1) + Lz_t + Bu(t), \quad (35)$$

where  $u(t) \in L_2(0, T; \mathbb{R}^m)$  is a  $m$ -dimensional control vector-function. This system may be represented by an operator model in Hilbert space given by the equation

$$\dot{x} = \mathcal{A}x + \mathcal{B}u(t), \quad x(t) = \begin{pmatrix} v(t) \\ z_t(\cdot) \end{pmatrix}, \quad (36)$$

where  $\mathcal{B}u = (Bu, 0)$  is linear and bounded from  $\mathbb{R}^m$  to  $M_2$ . We can note that  $\mathcal{B}$  is not bounded from  $M_2$  to  $X_1$  because  $\mathcal{B}u = (Bu, 0) \notin D(\mathcal{A})$  if  $Bu \neq 0$ .

#### 3.1. Exact controllability

Let us denote by  $R_T \subset M_2$  the reachable subspace of the system (36):

$$R_T = \left\{ \mathcal{R}_T u(\cdot) = \int_0^T e^{\mathcal{A}t} \mathcal{B}u(t) dt : u(t) \in L_2(0, T; \mathbb{R}^m) \right\},$$

where  $\mathcal{R}_T : L_2 \rightarrow M_2$  is a linear bounded operator. As was pointed out in [17] and [9],  $R_T \subset D(\mathcal{A})$  for all  $T > 0$ . This implies that exact controllability may be defined as follows.

**Definition 3.1.** The system (36) is exactly controllable if  $R_T = D(\mathcal{A})$ .

The abstract condition of exact controllability is (see [18] for example)

$$\int_0^T \|\mathcal{B}^* e^{\mathcal{A}^* t} x\|_{\mathbb{R}^m}^2 dt \geq \delta^2 \|x\|_{X_{-1}^d}^2, \quad \forall x_0 \in D(\mathcal{A}^*), \quad (37)$$

which means that the operator  $\mathcal{R}_T : L_2 \rightarrow X_1$  is onto. Here the space  $X_{-1}^d$  is the completion of the space  $X = M_2$  with respect to the norm

$$\|x\|_{X_{-1}^d} = \|(\lambda I - \mathcal{A}^*)^{-1}x\|_{M_2}, \quad \lambda \notin \sigma(\mathcal{A}^*).$$

For the system (35) the condition of exact controllability is given by the following theorem (see [9]).

**Theorem 3.2.** The system (35) is exactly controllable at time  $T$  if and only if, for all  $\lambda \in \mathbb{C}$ , the following two conditions are verified

$$i) \text{ rank} \left( \lambda I - \lambda e^{-\lambda} A_{-1} - \int_{-1}^0 e^{\lambda s} [\lambda A_2(s) + A_3(s)] ds \quad B \right) = n,$$

ii)  $\text{rank}(\lambda I - A_{-1} \ B) = n$ .

The time of controllability is  $T > n_1(A_{-1}, B)$ .

The integer  $n_1(A_{-1}, B)$  is the controllability index of the pair  $(A_{-1}, B)$  (see [19]). If the delay is  $h$ , then the critical time is  $n_1 h$ .

Let us now consider the dual notion of observability for the adjoint system. The condition (37) is equivalent to the exact observability of the observed system

$$\begin{cases} \dot{x} &= \mathcal{A}^* x, \\ y &= \mathcal{B}^* x \end{cases} \quad (38)$$

and the corresponding neutral type system is the system (15). Then the conditions (i)–(ii) of Theorem 3.2 are necessary and sufficient for the exact controllability of the adjoint system (38). But what is the corresponding property for the associate neutral type system (26)? This question will be investigated in the following paragraph.

### 3.2. Duality

Consider the transposed controlled neutral type system

$$\dot{z}(t) = A_{-1}^* \dot{z}(t-1) + L^* z_t + C^* u(t), \quad (39)$$

where  $L^* f = \int_{-1}^0 A_2^*(\theta) f'(\theta) + A_3^*(\theta) f(\theta) d\theta$ . Let  $\mathcal{A}^\dagger$  be the generator of the semi-group  $e^{\mathcal{A}^\dagger t}$  generated by this equation (39). We cannot consider  $\mathcal{A}^*$  for this system because this operator does not correspond directly to this system as infinitesimal generator of the semigroup of solutions. The domain  $D(\mathcal{A}^\dagger)$  of the operator  $\mathcal{A}^\dagger$  is given by

$$\left\{ \begin{pmatrix} v \\ z(\cdot) \end{pmatrix} : z \in H^1([-1, 0]; \mathbb{C}^n), v = z(0) - A_{-1}^* z(-1) \right\}.$$

The spectrum of  $\mathcal{A}^\dagger$  is  $\sigma(\mathcal{A}^\dagger) = \{\lambda : \Delta^*(\lambda) = 0\} = \sigma(\mathcal{A}^*)$ . Let  $X_1^\dagger$  be  $D(\mathcal{A}^\dagger)$  with the norm

$$\|x\|_{X_1^\dagger} = \left\| (\lambda I - \mathcal{A}^\dagger)x \right\|_{M_2}, \quad \lambda \notin \sigma(\mathcal{A}^\dagger),$$

which is equivalent to the graph norm. Consider now the reachability operator for this system

$$\mathcal{R}_T^\dagger u(t) = \int_0^T e^{\mathcal{A}^\dagger t} \begin{pmatrix} C^* \\ 0 \end{pmatrix} u(t) dt.$$

From the properties of the operator  $\mathcal{R}_T$ , we can deduce that  $\mathcal{R}_T^\dagger$  is linear, bounded from  $L_2(0, T; \mathbb{R}^l)$  to  $X_1^\dagger$ . The exact controllability for the system (39) can be formulated as

$$R_T^\dagger = \text{Im} \mathcal{R}_T^\dagger = X_1^\dagger.$$

The conditions of exact controllability for this system (39) may be obtained directly from Theorem 3.2.

Let us now consider the corresponding space  $X_{-1}^\dagger$  of linear functionals on  $X_1^\dagger$  as the completion of the space  $X = M_2$  with respect to the norm

$$\|x\|_{X_{-1}^\dagger} = \left\| (\lambda I - \mathcal{A}^\dagger)^{-1} x \right\|_{M_2}, \quad \lambda \notin \sigma(\mathcal{A}^\dagger).$$

We then have the embedding

$$X_1^\dagger \subset X = M_2 \subset X_{-1}^\dagger. \quad (40)$$

Then, for  $x \in X_1^\dagger$  and  $y \in X_{-1}^\dagger$ , the functional acts as

$$\langle x, y \rangle_{X_1^\dagger, X_{-1}^\dagger} = \left\langle (\lambda I - \mathcal{A}^\dagger)^{-1} x, (\lambda I - \mathcal{A}^{\dagger*})^{-1} y \right\rangle_X, \quad (41)$$

where  $\mathcal{A}^{\dagger*}$  is the adjoint of the operator  $\mathcal{A}^\dagger$  in  $M_2$ , and the space  $X_{-1}^{\dagger d}$  is constructed as  $X_{-1}^{\dagger d}$  with  $\mathcal{A}^{\dagger*}$  instead of  $\mathcal{A}^\dagger$  (see [14] for example).

Let us note that the operator  $\mathcal{A}^{\dagger*}$  is in fact the operator  $\tilde{\mathcal{A}}$  defined in Section 2 (see Theorem 2.3 and later). We shall use the properties obtained for this operator.

Let us now consider the adjoint  $\mathcal{R}_T^{\dagger*}$  of  $\mathcal{R}_T^\dagger$  with respect to the duality induced by  $X_1^\dagger$  and  $X_{-1}^{\dagger d}$  with the pivot space  $X = M_2$ . Let  $x_0 \in X$ , then

$$\begin{aligned} \left\langle \mathcal{R}_T^\dagger u(\cdot), x_0 \right\rangle_{X_1^\dagger, X_{-1}^{\dagger d}} &= \left\langle \mathcal{R}_T^\dagger u(\cdot), x_0 \right\rangle_X \\ &= \left\langle \int_0^T e^{\mathcal{A}^\dagger t} \begin{pmatrix} C^* \\ 0 \end{pmatrix} u(t) dt, x_0 \right\rangle_X \\ &= \int_0^T \left\langle \begin{pmatrix} C^* \\ 0 \end{pmatrix} u(t), e^{\mathcal{A}^{\dagger*} t} x_0 \right\rangle_X dt. \end{aligned}$$

Suppose now that  $x_0 = \begin{pmatrix} w(0) \\ \psi_0(\theta) \end{pmatrix} \in D(\mathcal{A}^{\dagger*})$ .

Then, as a consequence of the results in Section 2, namely from (27) but for the operator  $\mathcal{A}^{\dagger*} = \tilde{\mathcal{A}}$ , we obtain

$$e^{\mathcal{A}^{\dagger*} t} x_0 = \begin{pmatrix} w(t) \\ \psi_t(\theta) \end{pmatrix} = \begin{pmatrix} w(t) \\ r(t, \theta - A_2(\theta) w(t)) \end{pmatrix}, \quad t \geq 0.$$

Hence,

$$\begin{aligned} \left\langle \mathcal{R}_T^\dagger u(\cdot), x_0 \right\rangle_X &= \int_0^T \langle u(t), C w(t) \rangle_{\mathbb{R}^p} dt \\ &= \left\langle u(\cdot), \mathcal{R}_T^{\dagger*} x_0 \right\rangle_{L_2}. \end{aligned}$$

On the other hand, we can write  $x_0 = F \xi_0$  (cf. Proposition 2.5), where

$$\xi_0 = \begin{pmatrix} v(0) \\ z_0(\theta) \end{pmatrix} = \begin{pmatrix} z_0(0) - A_{-1} z_0(-1) \\ z_0(\theta) \end{pmatrix} = \begin{pmatrix} w(1) - A_{-1} w(0) \\ w(\theta + 1) \end{pmatrix},$$

and then

$$e^{\mathcal{A} t} \xi_0 = \begin{pmatrix} w(t+1) - A_{-1} w(t) \\ w(t+1+\theta) \end{pmatrix}.$$

Let  $\mathcal{K}$  be the output operator introduced in Definition 1.1. Then

$$\begin{aligned} \langle u(\cdot), \mathcal{K} \xi_0 \rangle_{L_2} &= \left\langle u(\cdot), \mathcal{C} e^{\mathcal{A}t} \xi_0 \right\rangle_{L_2} = \\ &= \begin{cases} \int_0^T \langle u(t), Cw(t) \rangle_{\mathbb{R}^p} dt & \text{if } \mathcal{C}x(t) = Cz(t-1) \\ \int_0^T \langle u(t), Cw(t+1) \rangle_{\mathbb{R}^p} dt & \text{if } \mathcal{C}x(t) = Cz(t) \end{cases}, \end{aligned}$$

which implies for all  $x_0 \in X$ :

$$\mathcal{K} F^{-1} x_0 = \begin{cases} \mathcal{R}_T^{\dagger*} x_0 & \text{if } \mathcal{C}x(t) = Cz(t-1), \\ e^{\mathcal{A}^{\dagger*}} \mathcal{R}_T^{\dagger*} x_0 & \text{if } \mathcal{C}x(t) = Cz(t). \end{cases} \quad (42)$$

We can now formulate our main result on duality between exact controllability and exact observability.

**Theorem 3.3.** 1. *The system (1) with the output*

$$y(t) = \mathcal{C}x(t) = Cz(t-1)$$

*is exactly observable in the interval  $[0, T]$ , i.e.*

$$\|\mathcal{K} x_0\|_{L_2}^2 = \int_0^T \left\| \mathcal{C} e^{\mathcal{A}t} x_0 \right\|_{\mathbb{R}^p}^2 dt \geq \delta^2 \|x_0\|_{M_2}^2$$

*if and only if the adjoint system (39) is exactly controllable at time  $T$ , i.e.*

$$R_T^\dagger = \mathcal{R}_T^\dagger(L_2(0, T; \mathbb{R}^p)) = X_1^\dagger = D(\mathcal{A}^\dagger).$$

2. *If  $\det A_{-1} \neq 0$ , Assertion 1 of the theorem is verified for the output*

$$y(t) = \mathcal{C}x(t) = Cz(t),$$

*and for the same time  $T$ .*

**Proof.** Let us recall that the exact controllability of the system (39) may be formulated by the equality  $\text{Im } \mathcal{R}_T^\dagger = X_1^\dagger$ . Then, taking in account the embedding (40) and the duality product (41), we can write the condition of exact controllability as (see [18] for example):

$$\|\mathcal{R}_T^{\dagger*} x_0\|_{L_2} \geq \delta \left\| (\lambda I - \mathcal{A}^{\dagger*})^{-1} x_0 \right\|_{M_2}, \quad \forall x_0 \in X. \quad (43)$$

Let  $\xi_0 = F^{-1} x_0 \in D(\mathcal{A})$ , where  $F : D(\mathcal{A}) \rightarrow M_2$  is the bounded invertible operator defined in Section 2. Let us remember that we have, from (42):

$$\mathcal{R}_T^{\dagger*} x_0 = \mathcal{K} F^{-1} x_0 = \mathcal{K} \xi_0.$$

Then the inequality (43) is equivalent to

$$\|\mathcal{K} \xi_0\|_{L_2} \geq \delta \left\| (\lambda I - \mathcal{A}^{\dagger*})^{-1} F \xi_0 \right\|_{M_2}, \quad \xi_0 \in D(\mathcal{A}). \quad (44)$$

Suppose now that the relation (44) is verified for all  $\xi_0 \in D(\mathcal{A})$ , then from Corollary 2.6 we obtain

$$\|\mathcal{K}\xi_0\|_{L_2} \geq \delta \left\| (\lambda I - \mathcal{A}^{\dagger*})^{-1} F \xi_0 \right\|_{M_2} \geq \underbrace{\delta C}_{=\delta_1} \|\xi_0\|_{M_2}, \quad \xi_0 \in D(\mathcal{A}).$$

This inequality can be extended by continuity to  $\xi_0 \in M_2$ :

$$\|\mathcal{K}\xi_0\|_{L_2} \geq \delta_1 \|\xi_0\|_{M_2}, \quad \forall \xi_0 \in M_2$$

Conversely, suppose that the preceding relation is verified. For  $\xi_0 \in D(\mathcal{A})$ , and from Corollary 2.6, we get

$$\|\xi_0\| \geq \frac{1}{C} \left\| (\lambda I - \mathcal{A}^{\dagger*})^{-1} F \xi_0 \right\|_{M_2},$$

and then

$$\|\mathcal{K}\xi_0\|_{L_2} \geq \frac{\delta_1}{C} \left\| (\lambda I - \mathcal{A}^{\dagger*})^{-1} F \xi_0 \right\|_{M_2}.$$

This is the relation (44) with  $\delta = \delta_1/C$ . As the relations (43) and (44) are equivalent, the first assertion of the theorem is proved.

To prove item 2 of the theorem, it is sufficient to remark that the condition  $\det A_{-1} \neq 0$  is equivalent to the fact that the operator  $e^{\mathcal{A}t}$  is bounded invertible ( $e^{\mathcal{A}t}$  is a group), and then the relations (43) and (44) are equivalent. ■

From this result and from Theorem 3.2 we can formulate the condition of exact observability.

**Theorem 3.4.** 1. *The system (1) with the output  $y = Cz(t-1)$  is exactly observable over  $[0, T]$  if and only if*

- (i) *For all  $\lambda \in \mathbb{C}$ ,  $\text{rank} \begin{pmatrix} \Delta^*(\lambda) & C^* \end{pmatrix} = n$ , where  $\Delta^*(\lambda)$  is defined in (13).*
  - (ii) *For all  $\lambda \in \mathbb{C}$ ,  $\text{rank} \begin{pmatrix} \lambda I - A_{-1}^* & C^* \end{pmatrix} = n$ ,*
  - (iii)  *$T > n_1(A_{-1}^*, C^*)$ , where  $n_1$  is the index of controllability for the pair  $(A_{-1}^*, C^*)$ .*
2. *If  $\det A_{-1} \neq 0$ , then Assertion 1 is verified for the output  $y(t) = Cz(t)$ .*

#### 4. Approximate controllability and observability

Let us now formulate the result on approximate controllability and observability. For more general duality relations between approximate controllability and observability of neutral type, we refer to the book [2]. We give here a precise formulation in the light of our results on adjoint systems. Let us first recall the definition of approximate controllability.

**Definition 4.1.** *The system (36) is approximately controllable at time  $T$  if  $\text{cl } R_T = M_2$ , where  $\text{cl } R_T$  is the closure of the attainable set  $R_T$  at time  $T$ .*

Sometimes approximate controllability is defined as

$$\text{cl } \bigcup_{T>0} R_T = M_2,$$

however for neutral type systems this notion of approximate controllability (as exact controllability) means that there is an universal time of controllability  $T_0 > 0$  (see [2]), i.e. such that:

$$\text{cl } R_{T_0} = \text{cl } \bigcup_{T>0} R_T.$$

According to the relation (42) and the definition of observability we obtain the following result on duality between approximate controllability and observability.

**Theorem 4.2.** *1. The system (1) with the output  $y(t) = \mathcal{C}x(t) = Cz(t-1)$  is approximately observable in the interval  $[0, T]$ , i.e.  $\mathcal{K} = \{0\}$  if and only if the adjoint system (39) is approximately controllable at time  $T$ , i.e.  $\text{cl } R_T^\dagger = M_2$ .*

*2. If  $\det A_{-1} \neq 0$ , Assertion 1 of the theorem is verified for the output  $y(t) = \mathcal{C}x(t) = Cz(t)$ , and for the same time  $T$ .*

**Proof.** The proof is a direct consequence of the definitions and (42). ■

The conditions of approximate observability may be obtained from the conditions of approximate controllability by duality. For our system, such conditions were obtained in [11] in the space  $W_1^2([-1, 0]; \mathbb{R}^n)$ . In our notations, this means that the reachability set  $R_{T_0}$  for the system (36) is dense in  $D(\mathcal{A})$  with the norm of the graph. It is equivalent to the density of  $R_{T_0}$  in the space  $M_2$ . In [2], it is shown that approximate observability and approximate controllability for such neutral type systems are dual and this does not depend on the state space, and then duality holds in the space  $M_2$  [2, Corollary 4.2.10].

From the necessary and sufficient conditions in [11, Th. 2] and Theorem 4.2, it is easy to see that approximate observability holds if the following two conditions are verified:

1.  $\forall \lambda \in \mathbb{C}, \text{rank} \begin{pmatrix} \Delta^*(\lambda) & C^* \end{pmatrix} = n,$
2.  $\text{rank} \begin{pmatrix} A_{-1}^* & C^* \end{pmatrix} = n.$

Note that the second condition is not necessary. The two conditions are verified when exact observability holds (the second condition is the condition (ii) of Theorem 3.4 for the particular case  $\lambda = 0$ ). This emphasizes the difference between the concepts of exact and approximate observability.

## 5. Examples

Let us give some simple examples to illustrate our results.

*Example 1.* Consider the system

$$\dot{z}(t) = \dot{z}(t-1),$$

where  $z(t) \in \mathbb{R}^n$ ,  $n > 1$ , with two possible outputs

$$y_0(t) = \mathcal{C}_0 x(t) = z(t), \quad y_1(t) = \mathcal{C}_1 x(t) = z(t-1).$$

The conditions of observability are verified, and the system is exactly observable for the output  $y_0$  or  $y_1$ .

*Example 2.* Consider the system

$$\begin{cases} \dot{z}_1(t) &= 0 \\ \dot{z}_2(t) &= \dot{z}_2(t-1) + z_1(t-1), \end{cases}$$

where  $z(t) = (z_1(t), z_2(t)) \in \mathbb{R}^2$ , with two possible output

$$y_0(t) = \mathcal{C}_0 x(t) = z(t), \quad y_1(t) = \mathcal{C}_1 x(t) = z(t-1).$$

The system with the output  $y_1$  is exactly observable for the time  $T > 1$  and not observable for  $T = 1$ . The system with the output  $y_0$  is not observable for any time  $T > 0$ .

## 6. Conclusion

For a large class of linear neutral type systems which include distributed delays we give the duality relation between exact controllability and exact observability. The characterization of exact observability is deduced.

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