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BILLIARD COMPLEXITY IN RATIONAL POLYHEDRA

NICOLAS BEDARIDE

ABSTRACT. We give a new proof for the directional billiard complexity in the cube, which was conjectured in [?] and proven in [?]. Our technique gives us a similar theorem for some rational polyhedra.

1. INTRODUCTION

A billiard ball, i.e. a point mass, moves inside a polyhedron P with unit speed along a straight line until it reaches the boundary ∂P , then instantaneously changes direction according to the mirror law, and continues along the new line.

Label the sides of P by symbols from a finite alphabet Σ whose cardinality equal the number of faces of P . Fix a direction ω and code the orbit of a point by the sequence of sides it hits. Consider the set $\mathcal{L}(n, \omega)$ of all words of length n which arise via this coding, and let $p(n, \omega) = \text{card}(\mathcal{L}(n, \omega))$. This is the complexity function in the direction ω , it does not depend on the initial point if the billiard map is minimal. We want to compute the complexity.

There are no result on the complexity for general polyhedron P , the only known case is the cube if we use a coding with three letters using the same letter for parallel faces. It was first found by Arnoux, Mauduit, Shiokawa, and Tamura [?, ?]. They use the fact that the billiard map in the cube is a rotation on the torus \mathbb{T}^2 and compute the numbers of cells of the n iterate of the rotation. Their result generalizes the computation of the directional complexity in the square where the obtained sequences are Sturmian. This result was generalized to a cube in \mathbb{R}^s , $2 \leq s$ by Baryshnikov [?]. For the rational polygons Hubert has given an exact formula for the directional complexity, it is a linear polynomial in n and it does not depend on the direction [?].

To give a new proof of the complexity in the cube we consider the notion of a generalized diagonal, that is an orbit segment in the fixed direction which starts and ends in an edge of the cube. The combinatorial length of a generalized diagonal is the number of links. We note $N(\omega, n)$ the cardinal of the set of generalized diagonals of length n . It was considered in [?] for the global complexity of a polygonal billiard. The notion of generalized diagonals can be defined for any map with a partition, it is enough to exchange the words edge and discontinuity of the partition, in the definition.

In the case of the rational polygons the billiard map, in one direction, is an interval exchange map, for the polyhedra there is a similar result for which we need some definitions. We call a polyhedron rational if the group G generated by the linear reflections on the faces is finite. Furthermore we call a map an affine polygon exchange if there is an finite partition of a polygon

X in polygons, a map of X on itself which is defined on each polygon by an isometry and such that the image of the partition is a partition.

Let P be a rational polyhedron and G his group of reflections. The phase space of the biliard map in P has a decomposition into subspaces $\partial P \times G\omega$. This subspace can be viewed as a polyhedron Q where we identify the parallel copies of the same face. The biliard map in P becomes a geodesic flow in Q , and the geodesic flow in a given direction yields an affine polygonal exchange which each partition element is coded by a single letter.

In general each face is represented several times in the polygonal exchange, thus the complexity of the polygonal exchange yields bounds of the complexity of the biliard. However in the case of the cube, when coding the parallel faces by the same letters, the symbolic sequence of a given point produced by the biliard is the same as the one produced by the geodesic flow of the given point in the cube with parallel faces identified. Thus the complexity of the biliard in the cube and its associated polygonal exchange are the same.

Our main result relates the complexity of an IDOC2 polygon exchange to the number of its generalized diagonals.

Definition 1. *We consider a affine polygon exchange, the discontinuities are intervals, we consider two of them $a \neq b$. If $T^n a \cap T^m b$ is a point or is empty and if $T^n \partial a \cap T^m \partial b$ is empty for every $n, m \in \mathbb{Z}$ then the polygonal exchange is called IDOC2.*

Proposition 2. *Consider an affine polygon exchange which is IDOC2 and minimal. We call $p(n)$ the complexity of the polygon exchange, with the natural coding and $N(n)$ the number of generalized diagonals of combinatorial length n for this map.*

$$p(n) = (2 - n)p(1) + (n - 1)p(2) + \sum_{i=2}^{n-1} \sum_{j=1}^{i-1} N(j) \quad \forall n > 2.$$

In fact we can apply Proposition ?? to some rational polyhedra. The simplest ones are the right prisms with polygonal basis. In this case if we are able to bound the number of generalized diagonals in one direction we obtain bounds for the complexity.

Definition 3. *Let P be a rational polyhedron and ω a direction. The direction ω is called BP irrational if there is no diagonal in this direction which passes through three sorts of edges and there is no diagonal which passes through two parallel edges.*

Corollary 4. *Let P be a rational polyhedron, ω a minimal and BP irrational direction, and $N(n, \omega)$ the number of generalized diagonals of length n in the direction ω . There exists $C \in \mathbb{N}$ and $b > 0$ constants such that for all $n > 2$.*

$$p(n, \omega) \leq (2 - n)p(1, \omega) + (n - 1)p(2, \omega) + b \sum_{i=2}^{n-1} \sum_{j=1}^{i-1} N(j, \omega) \leq p(C + n, \omega).$$

To apply Proposition ?? to the cube we have to compute the number of generalized diagonals and to understand the IDOC2 condition in the case of the cube.

Definition 5. *The initial direction of the billiard map is a vector $w = (w_1, w_2, w_3) \in \mathbb{R}^3$. If the ω_i are rationally independent we say that the direction is totally irrational, and if we add the condition that the (ω_i^{-1}) are \mathbb{Q} independent we call ω B irrational.*

The condition of total irrationality is the translation to the cube of the BP condition, and it implies that the polygon exchange is IDOC2.

Theorem 6. *Let ω be a B irrational direction, then the directional billiard complexity in the cube satisfies*

$$p(n, \omega) = n^2 + n + 1 \quad \forall n > 0.$$

Remark 7. *In [?, ?] the theorem was proven with the condition of total irrationality, but there exists some directions ω which are totally irrational, not B irrational and such that there exists n with $p(n, \omega) < n^2 + n + 1$. The mistake in their proof is minor. In [?] an alternative proof was given with the condition of B irrationality.*

Proposition ?? is true under a condition weaker than the affinity assumed. Without modification of its proof, the relation between the global complexity and the number of generalized diagonals of combinatoric length n holds for polygonal billiards generalizing the result of [?] to the non convex case.

We obtain a similar result for some right prisms.

Theorem 8. *Let a right prism with a tiling polygon as base. Consider a natural coding on this billiard table and let ω be a minimal and BP irrational direction, then there exists positives constants A, B such that*

$$B \leq \frac{p(n, \omega)}{n^2} \leq A \quad \forall n > 0.$$

2. POLYGONAL EXCHANGES

Proof of Proposition ??. First of all we have to recall some results of words combinatorics [?].

For any $n \geq 1$ let $s(n) := p(n+1) - p(n)$. For $v \in \mathcal{L}(n)$ let

$$\begin{aligned} m_l(v) &= \text{card}\{a \in \Sigma, va \in \mathcal{L}(n+1)\}, \\ m_r(v) &= \text{card}\{b \in \Sigma, bv \in \mathcal{L}(n+1)\}, \\ m_b(v) &= \text{card}\{a \in \Sigma, b \in \Sigma, bva \in \mathcal{L}(n+2)\}. \end{aligned}$$

A word is call right special if $m_r(v) \geq 2$, left special if $m_l(v) \geq 2$ and bispecial if it is right and left special. Let $\mathcal{BL}(n)$ be the set of the bispecial words. Cassaigne [?] has shown:

Lemma 9.

$$s(n+1) - s(n) = \sum_{v \in \mathcal{BL}(n)} m_b(v) - m_r(v) - m_l(v) + 1.$$

For the proof of the lemma we refer to [?] or [?].

Since the map is minimal we just have to count the number of initial words of length n . We consider an affine polygon exchange T , we label the polygons of the partition \mathcal{P} with the letters from a finite alphabet. Let a polygon of the partition corresponding to a letter a , it has a image by T

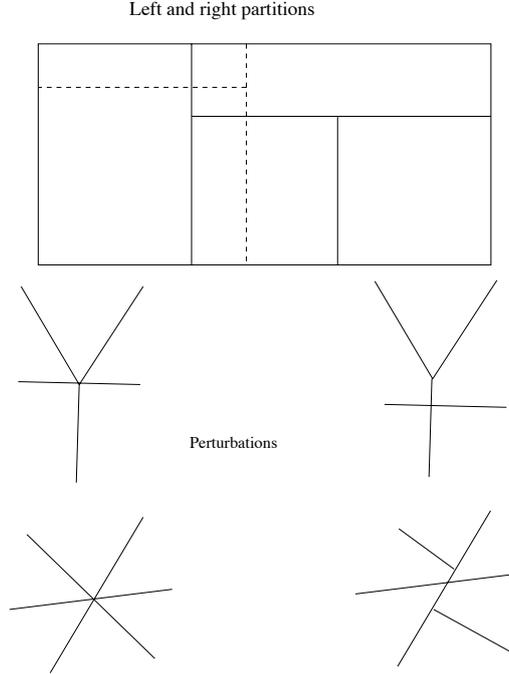


FIGURE 1.

which is an union of polygons. These polygons are associated to the letters b such that ab is a word of the language. So a word v of the language is represented by a polygon, it is partitionned into $m_r(v)$ polygons each of which corresponds to a different word. Since T is invertible we can repeat the same argument with T^{-1} and $m_l(v)$. If we intersect the two partitions of the polygon associated to v we obtain a partition into $m_b(v)$ polygons.

Now we use the Euler formula $F = E - V + 1$ for a partition of a polygon into F polygons with E edges and V vertices. There are three partitions so we obtain three equations.

$$m_r(v) = E_r(v) - V_r(v) + 1,$$

$$m_l(v) = E_l(v) - V_l(v) + 1,$$

$$m_b(v) = E_b(v) - V_b(v) + 1.$$

The vertices of the third partition are the vertices of the first one plus the vertices of the second one and the vertices created by the intersection of an edge of the first partition and an edge of the second one. Let us call this number $k(v)$, by the IDOC2 we obtain $V_b(v) - V_r(v) - V_l(v) = k(v)$.

For the edges we use the IDOC2 condition, and we assume, for the moment that an image of an edge does not intersect a vertex. The IDOC2 tells us that two edges can only intersect in one point, and we see that each point $k(v)$ create two edges, so $E_b(v) - E_r(v) - E_l(v) = 2k(v)$, thus $m_b(v) - m_r(v) - m_l(v) + 1 = k(v)$.

If we are in the case where an image of an edge intersect a vertex then we can make a small perturbation of some edge of one partition, see Figure

??, such that the number of polygons does not change, the formula is now true and we obtain

$$m_b(v) - m_r(v) - m_l(v) + 1 = k(v).$$

To finish the proof we just have to remark that $\sum_{v \in \mathcal{BL}(n)} k(v) = N(n)$. \square

3. GENERALIZED DIAGONALS FOR THE CUBIC BILLIARD

Proof of Theorem ??. When we play billiard in a polyhedron we can reflect the line or we can reflect the polyhedron and follow the same line, it is the unfolding. Let O be a vertex of the cube and consider the segment of

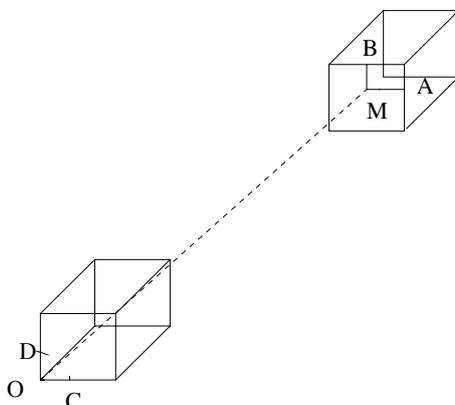


FIGURE 2.

direction ω who start from O and end at a point M after it pass through n cubes. M is a point of a face of an unfolding cube, if we translate M with a direction parallel to one of the two directions of the face we obtain a point A on an edge and if we call C the point such that $\vec{OC} = \vec{MA}$ then CA is a generalized diagonal, and we have another one, DB in the figure, arising from the second translation.

The symmetries of the cube implies that these diagonals are the only one. It remains to prove that the two generalized diagonals are of combinatorial length n .

The first thing to remark is that the condition of total irrationality implies that a generalized diagonal can not begin and end on two parallel edges. So the edges of begin and end are of different type.

To see that the combinatorial length is equal to n we can remark that the sum of the length of the projections is twice the length of the trajectory, so we just have to prove it for the projection, i.e. billiard in the square, where it follows from the symmetry. Lemma ?? shows us that the diagonals are non degenerate.

So we obtain that $s(n+1) - s(n) = 2$, since $p(1) = 3$ $p(2) = 7$, we obtain $p(n, \omega) = n^2 + n + 1$. \square

Next we show that the B condition of irrationality implies the impossibility to have a trajectory which pass through all three types of edges. It is the point that was wrong in the articles [?, ?].

Lemma 10. *There exist a total irrational direction such that there exists n and a trajectory of length n which pass through three different edges.*

If ω is a B irrational direction this is impossible.

Proof. For the first point just consider the direction $(1 - \frac{\pi}{6}, 1, \frac{6-\pi}{12-\pi})$ and the points on the edges $(\pi/6, 0, 0), (1, 1, \frac{6-\pi}{12-\pi}), (2, \frac{12-\pi}{6-\pi}, 1)$.

The proof of the second point is by contradiction.

Let ω be a B irrational direction and three points on some edges $(x, 0, 0), (a, y, b), (c, d, z)$ with $a, b, c, d \in \mathbb{N}$.

The fact that the points are on a line with direction ω gives

$$\frac{c-x}{a-x} = \frac{d}{y} = \frac{z}{b}.$$

Furthermore ω is a multiple of $(a-x, y, b)$ and of $(c-x, d, z)$, thus

$$\frac{c-x}{\omega_1} = \frac{d}{\omega_2} \quad \frac{a-x}{\omega_1} = \frac{b}{\omega_3}.$$

So we have $\frac{a-c}{\omega_1} = \frac{b}{\omega_3} - \frac{d}{\omega_2}$ which is excluded by B irrationality. \square

4. RIGHT PRISMS.

We consider a right prism with a tiling polygon for base. To apply Corollary ??, we need to count the generalized diagonals. The same construction as for the cube works but there is less symmetry so we must consider all the vertices of the base polygon and we remark that for one vertex the number of generalized diagonal can be null.

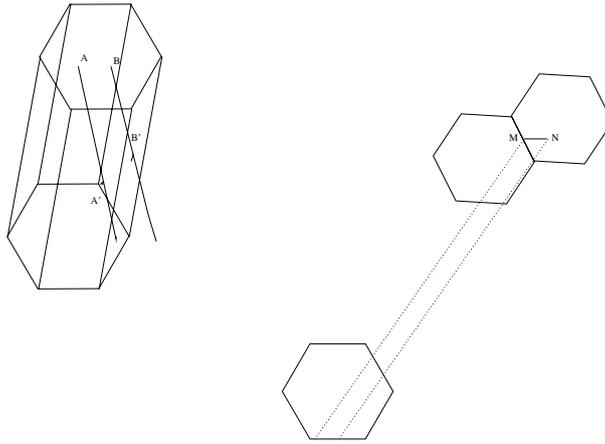


FIGURE 3.

Lemma 11. *Let P be a right prism and ω a BP irrational direction. Consider the two diagonals we have constructed, they are of combinatorial length n .*

Proof. Consider the following induction hypothesis for a right prism:

In a fixed direction all the trajectories who started from the same edge and stop on the same face of an unfolding polyhedron are of the same combinatorial length.

For $n = 1$ it is obvious, let us assume it is true for n and consider two diagonals of the rank $n + 1$ who started from the same edge. Call their ending points A and B and follow the trajectories in the inverse direction. The intersection with the first faces are A' , B' , see Figure ??, if they are on the same face we can apply the hypothesis, else there exists a finite sequence of faces who connect these faces, we consider the points on the common edges and we apply the same reasoning. \square

Lemma 12. *Let P be a right prism with regular hexagonal base and ω a BP irrational direction. It is impossible to have two diagonals of the same length which start from the same edge.*

If P is a right prism with tiling triangle base the number of diagonals which start from an edge is bounded.

Proof. The proof is by contradiction. Suppose the length of a side is one.

Assume that the initial edge is on the hexagon. Suppose that on this edge two diagonals start and are of length n . If the end points of the diagonals are on vertical edges we can translate one point to the other by a horizontal translation of length less than one, contradiction. Consider the plane parallel to the horizontal one and who pass on one of the end points M , see Figure ?. This plane intersects the diagonals on two points M and N . If M and N are in the same hexagon by the proof of the preceding lemma we have the same length for the two segments, so it is impossible. Since the hexagon tiles the plane the polygons do not overlap, thus we must have a point on an edge of an hexagon which by translation of length less than one parallel to one side go out of the hexagon, contradiction.

If the initial edge is vertical the final edges are of different hexagons, and a vertical translation of length less than one moves one to the other, contradiction.

For the tiling triangles Figure ?? gives us that the number of diagonals starting from an edge is bounded. \square

Proof of Corollary ??. We introduce another coding. The polyhedron Q is made by gluing some copies of P . The new coding associates one letter to each polygon of the partition of Q . The orbit of a point has now three codings: one for the billiard map M , one for the new coding M' and one for the natural coding of the polygonal exchange M'' whose complexities are denoted by $p(M, n)$, $p(M', n)$, $p(M'', n)$ respectively. Since there any many copies of P tiling Q we have $p(M, n) \leq p(M', n)$.

The number of polygons of the exchange is related to the number of polygons of the partition of Q . One face gives one polygon of the exchange, except if the preimages of singularities intersect this face. In this case the

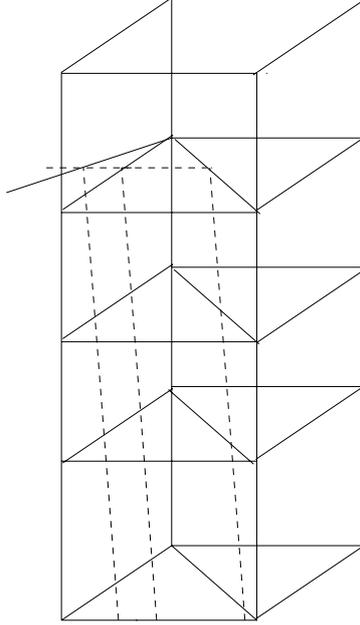


FIGURE 4.

face is cut into several polygons, thus $p(M', n) \leq p(M'', n)$. Since the number of singularities is bounded (by a constant C) we have that a word of length one in M'' is a word of length less than C in M' . We deduce that $p(M'', n) \leq p(M', C + n)$. The computation of $p(M'', n)$ can be completed using Proposition ?? when we remark that $N(n, \omega) = bN(n, \omega)$ where b is the number of polygons of the exchange. \square

Proof of Theorem ??. The coding is the natural one, with a letter for each face. The billiard map in the direction ω is a polygon exchange. Lemma ?? shows that the diagonals are of combinatorial length n . Lemma ?? implies that the number of generalized diagonals is bounded, the numbers of polygons of the exchange is also bounded, thus $bN(n, \omega)$ is bounded.

For a triangle a line which passes through an interior point of the triangle and which is parallel to an edge intersect the triangle. For a hexagon it is false but if we consider the lines parallel to two edges, one of them intersect the hexagon. So there is always at least one generalized diagonal of length n . So there exists D such that $1 \leq N(n, \omega) \leq D$ and Corollary 4 implies the existence of a, d with

$$a \leq p(M'', n)/n^2 \leq d.$$

Using $p(M', n) \leq p(M'', n) \leq p(M', C + n)$ we obtain the existence of A, B such that

$$B \leq p(M', n)/n^2 \leq A.$$

Like in the two dimensional case, [?], we prove.

Lemma 13. *There exists an integer m such that*

$$p(M, n) = p(M', n) \quad \forall n > m.$$

Proof. The proof is by contradiction. The language M' is bigger than M and a word of M' gives a word of M by a projection, π , letter by letter.

If it is false we can find for any n two different words v_n and w_n of M' such that their projections in the language M are equal. We can assume that the words v_n are successive and the same thing for w_n . Let us call v and w the two limit points of the sequences (v_n) and (w_n) .

There exists $m \in \partial P$ and ϕ a direction such that the orbit of (m, ϕ) has $\pi(v)$ as coding. The direction is unique, not necessary the point m , see [?].

If the orbit of (m, ϕ) is singular then there exists $e \in \partial P$ ϕ' a direction, k an integer such that the forward orbit of (e, ϕ') has $T^k(\pi(v))$ as coding and is non singular. Moreover $T^{-1}(e, \phi)$ belongs to an edge of P .

It remains to prove that the point on ∂P is unique.

If a point of a face has an orbit which passes through the same face with the same direction, the tiling property of P implies that those two faces are parallel. So the isometries of the polygonal exchange are translations, which implies that it is impossible to have two points with the same coding.

The point on ∂P is unique, and we have $v = w$.

□

So the last inequality is true for $p(n)$ for n large, if we change the constant B it is still true for all the integers.

□

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