Entropy of polyhedral billiard

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ENTROPY OF POLYHEDRAL BILLIARD

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ABSTRACT. We consider the billiard map in a convex polyhedron of $\mathbb{R}^3$, and we prove that it is of zero topological entropy.

1. Introduction

A billiard ball, i.e. a point mass, moves inside a polyhedron P with unit speed along a straight line until it reaches the boundary ∂P, then it instantaneously changes direction according to the mirror law, and continues along the new line.

Label the faces of P by symbols from a finite alphabet $\mathcal{A}$ whose cardinality equals the number of faces of P. Consider the set of all billiard orbits. After coding, the set of all the words is a language. We define the complexity of the language, $p(n)$, by the number of words of length n that appears in this system. How complex is the game of billiard inside a polygon or a polyhedron? For the cube the computations have been done, see [Bed03, BHar], but there is no result for a general polyhedron. One way to answer this question is to compute the topological entropy of the billiard map.

There are three different proofs that polygonal billiard have zero topological entropy [Kat87, GKT95, GH97]. Here we consider the billiard map inside a polyhedron. We want to compute the topological entropy of the billiard map in a polyhedron. The idea is to improve the proof of Katok. Thus we must compute the metric entropy of each ergodic measure. When we follow this proof some difficulties appear. In particular a non atomic ergodic measure for the related shift can have its support included in the boundary of the definition set. Such examples were known for some piecewise isometries of $\mathbb{R}^2$ since the works of Adler, Kitchens and Tresser [AKT01]; Goetz and Poggiaspalla [Goe98, GP04]. Piecewise isometries and billiard are related since the first return map of the directional billiard flow inside a rational polyhedron is a piecewise isometry.

Our main result is the following

Theorem 1. *Let P be a convex polyhedron of $\mathbb{R}^3$ and let T be the billiard map, then*

$$h_{top}(T) = 0.$$

Corollary 2. *The complexity of the billiard map satisfies*

$$\lim_{n\to+\infty} \frac{\log p(n)}{n} = 0.$$

For the standard definitions and properties of entropy we refer to Katok and Hasselblatt [HK02].

1.1. **Overview of the proof.** We consider the shift map associated to the billiard map, see Section 2, and compute the metric entropy for each ergodic measure of this shift. We must treat several cases depending on the support of the measure. If the ergodic measure has its support included in the definition set, then the method of Katok can be used with minor changes, see Section 3. The other case can not appear in dimension two and represent the main problem in dimension three. We treat this case by looking at the billiard orbits which pass through singularities. By a geometric argument we prove in Section 4 that the support of a such measure is the union of two sets: a countable set and a set of words whose complexity can be bounded, see Proposition 28 and Lemma 29.

If we want to generalize this result to any dimension some problems appear. Im dimension three, we treat two cases by different methods depending on the dimension of the cells. In dimension d there would be at least $d-1$ different cases and actually we have no method for these cases. Moreover we must generalize Lemma 22 and the followings . Unfortunately this is much harder and cannot be made with computations.

2. Background and notations

2.1. **Definitions.** We consider the billiard map inside a convex polyhedron P. This map is defined on the set $E \subset \partial P \times \mathbb{PR}^3$, by the following method:

First we define the set $E' \subset \partial P \times \mathbb{PR}^3$. A point (m, θ) belongs to E' if and only if one of the two following points is true:

• The line $m + \mathbb{R}^*[\theta]$ intersects an edge of P, where $[\theta]$ is a vector of $\mathbb{R}^3$ which represents θ.

• The line $m + \mathbb{R}^*[\theta]$ is included inside the face of P which contains m.

Then we define E as the set

$$E = (\partial P \times \mathbb{PR}^3) \setminus E'.$$

Now we define the map T: Consider $(m, \theta) \in E$, then we have $T(m, \theta) = (m', \theta')$ if and only if mm' is colinear to $[\theta]$, and $[\theta'] = s[\theta]$, where s is the linear reflection over the face which contains m'.

$$T : E \to \partial P \times \mathbb{PR}^3$$

$$T : (m, \theta) \mapsto (m', \theta')$$

Remark 3. *In the following we identify $\mathbb{PR}^3$ with the unit vectors of $\mathbb{R}^3$ (i.e we identify θ and $[\theta]$).*

Definition 4. *The set E is called the phase space.*

2.2. **Combinatorics.**

Definition 5. *Let $\mathcal{A}$ be a finite set called the alphabet. By a language L over $\mathcal{A}$ we mean always a factorial extendable language: a language is a collection of sets $(L_n)_{n\geq 0}$ where the only element of L_0 is the empty word, and each L_n consists of words of the form $a_1 a_2 \dots a_n$ where $a_i \in \mathcal{A}$ and such that for each $v \in L_n$ there exist $a, b \in \mathcal{A}$ with $av, vb \in L_{n+1}$, and for all $v \in L_{n+1}$ if $v = au = u'b$ with $a, b \in \mathcal{A}$ then $u, u' \in L_n$.*

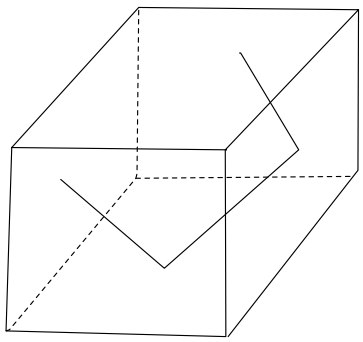

FIGURE 1. Billiard map inside the cube

The complexity function of the language L, $p : \mathbb{N} \to \mathbb{N}$ is defined by $p(n) = card(L_n)$.

2.3. **Coding.** We label each face of the polyhedron with a letter from the alphabet $\{1 \dots N\}$. Let E be the phase space of the billiard map and $d = \{d_1 \dots d_N\}$ the cover of E related to the coding. The phase space is of dimension four : two coordinates for the point on the boundary of P and two coordinates for the direction.

Let E_0 be the points of E such that T^n is defined, continuous in a neighborhood for all $n \in \mathbb{Z}$. Denote by ϕ the coding map, it means the map

$$\phi : E_0 \to \{1, \dots, N\}^{\mathbb{Z}},$$
$$\phi(p) = (v_n)_{\mathbb{Z}},$$

where v_n is defined by $T^n(p) \in d_{v_n}$. Let S denote the shift map on $\{1 \dots N\}^{\mathbb{Z}}$. We have the diagram,

$$\begin{array}{ccc} E_0 & \xrightarrow{T} & E_0 \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ \phi(E_0) & \xrightarrow[S]{} & \phi(E_0) \end{array}$$

with the equation $\phi \circ T = S \circ \phi$.

We want to compute the topological entropy of the billiard map. We define the topological entropy of the billiard map as the topological entropy of the subshift, see Definition 7.

We remark that the proof of Theorem 1 given in [GKT95] as a corollary of their result is not complete: They do not consider the case, where the ergodic measure is supported on the boundary of $\phi(E_0)$.

2.4. **Notations.** Let Σ be the closure of $\phi(E_0)$, and consider the cover

$$d \vee T^{-1}d \vee \dots \vee T^{-n+1}d.$$

The cover d, when restricted to E_0, is a partition. The sets of this cover are called n-cells. If $v \in \Sigma$ we denote

$$\sigma_v = \bigcap_{n \in \mathbb{Z}} \overline{T^{-n}(d_{v_n} \cap E_0)} = \bigcap_{n \in \mathbb{Z}} T^{-n} d_{v_n}.$$

It is the closure of the set of points of E_0 such that the orbit is coded by v. If $v \in \phi(E_0)$ then σ_v is equal to $\phi^{-1}(v)$. We denote $d^- = \bigvee_{n=0}^{\infty} T^{-n}d$ and

$$\sigma_v^- = \bigcap_{n\geq 0} \overline{T^{-n}(d_{v_n} \cap \phi(E_0))} = \bigcap_{n\geq 0} T^{-n}d_{v_n}.$$

Definition 6. *Let $\xi = \{c_1, \dots, c_k\}$ be the partition of Σ given by*

$$c_k = \overline{\phi(d_k \cap E_0)}.$$

Finally we can define the topological entropy

Definition 7. *Consider a polyhedron of $\mathbb{R}^3$, and T the billiard map, then we define*

$$h_{top}(T) = \lim_{n\to+\infty} \frac{\log p(n)}{n},$$

where $p(n)$ is the number of n-cells.

This definition is made with the help of the following lemma which links it to the topological entropy of the shift.

Lemma 8. *With the same notation*

$$\lim_{n\to+\infty} \frac{\log p(n)}{n} = h_{top}(S|\Sigma).$$

Proof. The partition ξ, see Definition 6, is a topological generator of $(S|\Sigma)$ (see [Pet83] for a definition), thus

$$h(S|\Sigma) = \lim_{n\to+\infty} \frac{\log card\xi_n}{n},$$

and we have $\text{card}(\xi_n) = p(n)$. □

Remark 9. *The number of cells, $p(n)$, is equal to the complexity of the language Σ.*
There are several other possible definitions (Bowen definition ...) but we use this one since we are interested in the complexity function of the billiard map.

2.5. **Billiard.**

2.5.1. *Cell.* We denote by π the following map:

$$\pi : \partial P \times \mathbb{PR}^3 \mapsto \mathbb{PR}^3$$
$$\pi : (m, \theta) \to \theta.$$

Consider an infinite word $v \in \phi(E_0)$.

Definition 10. *We consider the elements (m, θ) of $\partial P \times \mathbb{PR}^3$ as vectors θ with base point m.*
We say that $X \subset \partial P \times \mathbb{PR}^3$ is a strip if all $x \in X$ are parallel vectors whose base points form an interval.
We say that $X \subset \partial P \times \mathbb{PR}^3$ is a tube if all $x \in X$ are parallel vectors whose base points form an open polygon or an open ellipse.

Now we recall the theorem of Galperin, Kruger and Troubetzkoy [GKT95], which describe the shape of σ_v^-:

Lemma 11. *Let $v \in \phi(E_0)$ be an infinite word, then there are three cases:*
The set σ_v^- consists of only one point.
The set σ_v^- is a strip.
The set σ_v^- is a tube.
Moreover if σ_v^- is a tube then v is a periodic word.

Remark 12. *The preceding lemma shows that ϕ is not bijective on E_0.*

By the preceding lemma for each infinite word v the set $\pi(\sigma_v^-)$ is unique. If the base points form an interval we say that σ_v^- is of dimension one, and of dimension two if the base points form a polygon or an ellipse.

Definition 13. *As in the preceding lemma, if v is an infinite word we say that $\pi(\sigma_v^-)$ is the direction of the word.*
Moreover if v is an infinite word, we identify σ_v^- with the set of base points a which fulfills $\sigma_v^- = a \times \pi(\sigma_v^-)$.

2.5.2. *Geometry.* First we define the rational polyhedron. Let P be a polyhedron of $\mathbb{R}^3$, consider the linear reflections s_i over the faces of P.

Definition 14. *We denote by $G(P)$ the group generated by the s_i, and we say that P is rational if $G(P)$ is finite.*

In $\mathbb{R}^2$ a polygon is rational if and only if all the angles are rational multiples of π. Thus the rational polygons with k edges are dense in the set of polygons with k edges. In higher dimension, there is no simple characterization of rational polyhedrons, moreover their set is not dense in the set of polyhedrons with fixed combinatorial type (number of edges, vertices, faces).

An useful tool in the billiard study is the unfolding. When a trajectory passes through a face, there is reflection of the line. The unfolding consists in following the same line and in reflecting the polyhedron over the face. For example for the billiard in the square/cube, we obtain the usual square/cube tiling. In the following we will use this tool, and an edge means an edge of an unfolded polyhedron.

2.6. **Related results.** If P is a rational polyhedron, then we can define the first return map of the directional flow in a fixed direction ω. This map T_ω is a polygon exchange (generalization of interval exchange). Gutkin and Haydn have shown :

Theorem 15. [GH97] *Let P be a rational polyhedron and $w \in \mathbb{S}^2$ then*

$$h_{top}(T_\omega) = 0.$$

Moreover if μ is any invariant measure then

$$h_\mu(T) = 0.$$

Buzzi [Buz01], has generalized this result. He proves that each piecewise isometrie of $\mathbb{R}^n$ have zero topological entropy. Remark that a polygonal exchange is a piecewise isometry.

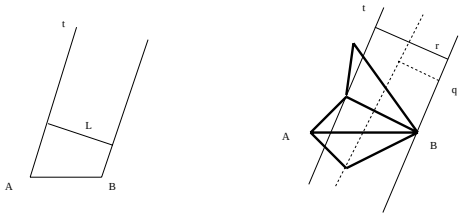

FIGURE 2. Billiard invariant

3. Variational principle

We use the variational principle to compute the entropy

$$h_{top}(S|\Sigma) = \sup_{\substack{\mu \\ ergo}} h_\mu(S|\Sigma).$$

Remark that we cannot apply it to the map T since it is not continuous on a compact metric space. The knowledge of $h_\mu(T)$ does not allow to compute $h_{top}(T)$. We are not interested in the atomic measures because the associated system is periodic, thus their entropy is equal to zero. We split into two cases $supp(\mu) \subset \phi(E_0)$ or not. We begin by treating the first case which is in the same spirit as the argument in Katok [Kat87].

Lemma 16. *Let μ be an ergodic measure with support in $\phi(E_0)$. We denote $\xi^- = \bigvee_{n=0}^{\infty} S^{-n}\xi$, where ξ is defined in Definition 6. Up to a set of μ measure zero we have*

$$S\xi^- = \xi^-.$$

Proof. As $\mu(\phi(E_0)) = 1$, the cover ξ can be thought as a partition of $\phi(E_0)$. Let $v \in \phi(E_0)$, then the set σ_v^- can be thought as an element of d^-. The set $\overline{\phi(\sigma_v^- \cap E_0)}$ coincides with the set of ξ^- which contains v.

By Lemma 11 the dimension of σ_v^- can take three values.

We have $\sigma^-_{S^{-1}v} \subset T^{-1}\sigma_v^-$, thus the set of v such that σ_v^- is a point is invariant by S. The ergodicity of μ implies that this set either has zero measure or full measure.

Assume it is of full measure, then d^- is a partition of points, and same thing for ξ^-. Then ξ^- is a refinement of $S\xi^-$, this implies that those two sets are equal.

Assume it is of zero measure. Then by ergodicity there are two cases : σ_v^- is an interval or of dimension two for a set of full measure.

• Assume σ_v^- is an interval for a full measure set of v.

If θ is the direction of v, then consider the strip $\sigma_v^- + \mathbb{R}\theta$. Consider a line included in the plane of the strip and orthogonal to the axis $\mathbb{R}\theta$, and denote $L(\sigma_v^-)$ the length of the set at the intersection of the line and the strip, see Figure 2.

Clearly we have $T(\sigma_v^-) \subset \sigma^-_{Sv}$, thus we have $L(T\sigma_v^-) \leq L(\sigma^-_{Sv})$. Since $L(T\sigma_v^-) = L(\sigma_v^-)$ we conclude that the function L is a sub-invariant of S.

Since μ is ergodic the function L is constant μ *a.e.* Thus for μ *a.e* v we obtain two intervals of same length, one included in the other. They are

equal. We deduce $\sigma^-_{Sv} = T\sigma^-_v$. This implies that $v_1, v_2, \dots$ determines v_0 almost surely. It follows that

$$S\xi^- = \xi^- \mu a.e.$$

• If σ^-_v is of dimension 2 for a positive measure set of v, by ergodicity it is of the same dimension for μ *a.e* v. It implies that v is a periodic word μ *a.e*, thus $S\xi^- = \xi^- \mu$ *a.e.* □

Since $h(S, \xi) = H(S\xi^-|\xi^-) = 0$ we have :

Corollary 17. *If* $supp(\mu) \subset \phi(E_0)$ *then* $h_\mu(S|\Sigma) = 0$.

4. Measures on the boundary

We will treat the cases of ergodic measures, satisfying

$$X = supp(\mu) \subset \Sigma \setminus \phi(E_0).$$

First we generalize Lemma 11:

Lemma 18. *For a convex polyhedron, for any word* $v \in \Sigma \setminus \phi(E_0)$ *the set* σ^-_v *is connected and is a strip.*

We remark that Lemma 18 is the only place where we use the convexity of P.

Proof. First the word v is a limit of words v^n in $\phi(E_0)$. Each of these words v^n have a unique direction θ_n by Lemma 11. The directions θ_n converge to θ, this shows that the direction of σ^-_v is unique. Now by convexity of P the set σ^-_v is convex as intersection of convex sets. By definition the projection of σ^-_v on ∂P is included inside an edge, thus it is of dimension less than or equal to one. This implies that the set is an interval or a point. □

A priori there are several cases as $dim\sigma^-_v$ can be equal to 0 or 1. We see here a difference with the polygonal case. In this case the dimension was always equal to zero.

4.1. Orbits passing through several edges.

In this paragraph an edge means the edge which appears in the unfolding of P corresponding to v. We represent an edge by a point and a vector. The point is a vertex of a copy of P in the unfolding and the vector is the direction of the edge. We consider two edges A, B in the unfolding. Consider $m \in A$ and a direction θ such that the orbit of (m, θ) passes through an edge. We identify the point m with the distance $d(m, a)$ if a is one endpoint of the edge A. Moreover we denote by u an unit vector colinear to the edge A.

Lemma 19. *The set of* $(m, \theta), m \in A_0$ *such that the orbit of* (m, θ) *passes through an edge* A_1 *satisfies either*

(i) (m, θ) *is in the line or plane which contains* A_0, A_1. *Then there exists an affine map* f *such that* $f(\theta) = 0$.

or

(ii) there exists a map $F : \mathbb{R}^3 \to \mathbb{R}$ *such that* $m = F(\theta)$ *(it is the quotient of two linear polynomials). Moreover the map* $(A_0, A_1) \mapsto F$ *is injective.*

Remark 20. *The case where A_0, A_1 are colinear is included in the first case. In this case there are two equations of the form $f(\theta) = 0$ but we only use one of them.*

Proof. Consider the affine subspace generated by the edge A_0 and the line $m + \mathbb{R}\theta$. There are two cases :

- $A_1 \in Aff(A_0, m + \mathbb{R}\theta)$. Assume A_0, A_1 are not colinear, then the affine space generated by A_0, A_1 is of dimension two (or one), and several points m can be associated to the same direction θ. In the case it is of dimension 2, θ is in the plane which contains A_0, A_1. Then there exists an affine map f which gives the equation of the plane and we obtain $f(\theta) = 0$.
- $A_1 \notin Aff(A_0, m + \mathbb{R}\theta)$, then the space $Aff(A_0, A_1)$ is of dimension three. If the direction is not associated to a single point then the edges A_0, A_1 are coplanar. Thus in our case the direction is associated to a single point m. There exists a real number λ such that $m + \lambda\theta \in A_1$. Since A_1 is an edge, it is the intersection of two planes (we take the planes of the two faces of the polyhedron). We denote the two planes by the equations $h = 0; g = 0$ where $h, g \quad \mathbb{R}^3 \to \mathbb{R}$. We obtain the system

$$h(m + \lambda\theta) = 0,$$

$$g(m + \lambda\theta) = 0.$$

Here $h(x) =< v_h, x > +b_h$ where v_h is a vector and $< \cdot, \cdot >$ is the scalar product and similarly for g. Then we write $h(m) =< v_h, mu > +b_h = m < v_h, u > +b_h$, we do the same thing for g. Since A_0, A_1 are not coplanar the terms $< v_g, \theta >, < v_h, \theta >$ are non null, thus we obtain the expression for λ :

$$\lambda = \frac{-b_h - m < v_h, u >}{< v_h, \theta >} = \frac{-b_g - m < v_g, u >}{< v_g, \theta >}.$$

For a fixed θ, there can be only one point $m \in A_0$ which solves this equation, otherwise we would be in case (i). Thus we find $m = F(\theta)$ where F is the quotient of two linear polynomials :

$$m = \frac{b_g < v_h, \theta > -b_h < v_g, \theta >}{< v_h, u >< v_g, \theta > - < v_g, u >< v_h, \theta >} \quad (*).$$

Note that F does not depend on the concrete choices of the planes h, g, but only on the edges A_0, A_1.

We prove the last point by contradiction. If we have the same equation for two edges, it means that all the lines which pass through two edges pass through the third. We claim it implies that the three edges A_0, A_1, A_2 are coplanar : the first case is when A_1, A_2 are coplanar. Then the assumption implies that the third is coplanar, contradiction. Now assume that the three edges are pairwise not coplanar. Indeed consider a first line which passes through the three edges. Call m the point on A_0, and u the direction. Now consider a line which contains m and passes through A_1 with a different direction. Those two lines intersect A_1, thus m and the two lines are coplanar. Since A_2 is not coplanar with A_0, both lines can not intersect A_2, contradiction. To finish consider the case when two edges are colinear but the third one is not colinear with either of the other two. This case can be reduced to the first case by looking at the first and third edges. □

Lemma 21. *Consider two edges A_0, A_i which give the equation $m = F_i(\theta)$. Denote by p_i a point on A_i and x_i the direction of the line A_i. Then we have*

$$F_i(\theta) = \frac{< p_i \wedge x_i, \theta >}{< u \wedge x_i, \theta >},$$

where u is an unit vector colinear to the edge A_0.

Proof. By Lemma 19 each F_i is the quotient of two polynomials. Consider the denominator of F_i as function of θ (we use the notations of the preceding proof). By equation $(*)$ we obtain:

$$F_i(\theta) = \frac{N(\theta)}{D(\theta)},$$

$$D(\theta) = - < v_{h_i}, u >< v_{g_i}, \theta > + < v_{g_i}, u >< v_{h_i}, \theta > .$$

We remark for the map F_i that

$$- < v_{h_i}, u > v_{g_i} + < v_{g_i}, u > v_{h_i},$$

is orthogonal to u and to x_i. Thus this vector is colinear to $u \wedge x_i$:

$$- < v_{h_i}, u > v_{g_i} + < v_{g_i}, u > v_{h_i} = C_i u \wedge x_i.$$

Consider the numerator $(b_{h_i} v_{g_i} - b_{g_i} v_{h_i}, \theta)$ of F_i. The scalar product of $b_{h_i} v_{g_i} - b_{g_i} v_{h_i}$ with x_i is null, moreover the scalar product with p_i equals again zero by definition of $v_{g_i}, b_{g_i}, v_{h_i}, b_{h_i}$. Thus we obtain :

$$b_{h_i} v_{g_i} - b_{g_i} v_{h_i} = C_i' p_i \wedge x_i, \tag{1}$$

and :

$$F(\theta) = \frac{C_i'}{C_i} \frac{< p_i \wedge x_i, \theta >}{< u \wedge x_i, \theta >}.$$

We claim that $C_i = C_i' = 1$. We can choose the vectors v_{g_i}, v_{h_i} such that they are orthogonal and of norm 1. Then x_i is colinear to $v_{g_i} \wedge v_{h_i}$ and is of norm one, thus if we choose the proper orientation of x_i they are equal. Then we can have

$$- < v_{h_i}, u > v_{g_i} + < v_{g_i}, u > v_{h_i} = u \wedge (v_{g_i} \wedge v_{h_i}) = u \wedge x_i.$$

Thus we deduce $K_i' = 1$.

Now we compute the norm of the vector of the numerator $|b_{h_i} v_{g_i} - b_{g_i} v_{h_i}|^2 = b_{h_i}^2 + b_{g_i}^2$. By definition of b_{g_i}, b_{h_i}, p_i we obtain

$$b_{g_i} = - < v_{g_i}, p_i >; b_{h_i} = - < v_{h_i}, p_i > .$$

Thus we have $|b_{h_i} v_{g_i} - b_{g_i} v_{h_i}|^2 =< v_{g_i} | p_i >^2 + < v_{h_i} | p_i >^2$. Moreover by definition we have that $x_i = v_{g_i} \wedge v_{h_i}$ this implies that $|p_i \wedge x_i|^2 =< v_{g_i}, p_i >^2 + < v_{h_i}, p_i >^2$. Finally we deduce

$$|p_i \wedge x_i|^2 (C_i')^2 = |p_i \wedge x_i|^2.$$

□

Lemma 22. *Consider three edges A_0, A_1, A_2 such that $dim Aff(A_i, A_j) = 3$ for all i, j. Then the sets of lines d which pass through A_0, A_1, A_2 is contained in a surface which we call $S(A_0, A_1, A_2)$. Consider an orthonormal basis*

such that the direction u of A_0 satisfies $u = \begin{pmatrix}1\\0\\0\end{pmatrix}$. If we call (P_1, P_2, P_3) the coordinates of a point on this surface, then

(i) the equation of the surface can be written as $P_1 = f(P_2, P_3)$, where f is a polynomial.

(ii) there exists $N \leq 4$ such that any line which is not contained in S intersects S at most N times.

Proof. Consider a line $d = m + \mathbb{R}\theta, m \in A_0$ which passes through A_1, A_2. By Lemma 19 we obtain two equations $m = F_i(\theta)$. Then Lemma 21 implies that $F_i(\theta) = \frac{\sum a_{j,i}\theta_j}{\sum_{j=2}^{3} b_{i,j}\theta_j}$. Now call P_i the coordinates of a point P on d. We have $P = m + \lambda\theta$, thus we obtain

$$\begin{cases} P_1 = \frac{a_1\theta_1+a_2\theta_2+a_3\theta_3}{b_2\theta_2+b_3\theta_3} + \lambda\theta_1 \\ P_2 = \lambda\theta_2 \\ P_3 = \lambda\theta_3 \\ (F_1 - F_2)(\theta) = 0 \end{cases}$$

where $a_j = a_{j,1}$ and $b_j = b_{1,j}$.

• First case $P_2 \neq 0$. This is equivalent to $\theta_2 \neq 0$.

$$\begin{cases} P_1 = \frac{a_1\theta_1+a_2\theta_2+a_3\theta_3}{b_2\theta_2+b_3\theta_3} + \lambda\theta_1 \\ P_2 = \lambda\theta_2 \\ \theta_3 = \frac{P_3}{P_2}\theta_2 \\ (F_1 - F_2)(\theta) = 0 \end{cases}$$

$$\begin{cases} P_1 = \frac{a_1\theta_1+\theta_2(a_2+a_3\frac{P_3}{P_2})}{\theta_2(b_2+\frac{P_3}{P_2})} + P_2\frac{\theta_1}{\theta_2} \\ P_2 = \lambda\theta_2 \\ \theta_3 = \frac{P_3}{P_2}\theta_2 \\ (F_1 - F_2)(\theta) = 0 \end{cases}$$

$$\begin{cases} P_1 = \frac{a_1}{(b_2+\frac{P_3}{P_2})}\frac{\theta_1}{\theta_2} + \frac{a_2+a_3\frac{P_3}{P_2}}{b_2+\frac{P_3}{P_2}} + P_2\frac{\theta_1}{\theta_2} \\ P_2 = \lambda\theta_2 \\ \theta_3 = \frac{P_3}{P_2}\theta_2 \\ (F_1 - F_2)(\theta) = 0 \end{cases}$$

$$\begin{cases} P_1 = (\frac{a_1}{b_2P_2+P_3} + 1)P_2\frac{\theta_1}{\theta_2} + \frac{a_2P_2+a_3P_3}{b_2P_2+P_3} \\ P_2 = \lambda\theta_2 \\ \theta_3 = \frac{P_3}{P_2}\theta_2 \\ (F_1 - F_2)(\theta) = 0 \end{cases}$$

Now the equation $(F_1 - F_2)(\theta) = 0$ can be written as

$$(\sum_{j=1}^{3} a_j\theta_j)(\sum_{j=2}^{3} b'_j\theta_j) = (\sum_{j=1}^{3} a'_j\theta_j)(\sum_{j=2}^{3} b_j\theta_j),$$

where $a'_j = a_{j,2}$ and $b'_j = b_{2,j}$.

$$(a_1\theta_1 + a_2\theta_2 + a_3\theta_3)(b'_2\theta_2 + b'_3\theta_3) = (a'_1\theta_1 + a'_2\theta_2 + a'_3\theta_3)(b_2\theta_2 + b_2\theta_3).$$

With the equation $\theta_3 = \frac{P_3}{P_2}\theta_2$ we obtain an equation of the following form.

$$\begin{gathered}(a_1\theta_1 P_2 + (a_2P_2 + a_3P_3)\theta_2)(b'_2P_2 + b'_3P_3) = \\ (a'_1\theta_1 P_2 + (a'_2P_2 + a'_3P_3)\theta_2)(b_2P_2 + b_3P_3). \\ (a_1\theta_1/\theta_2 P_2 + (a_2P_2 + a_3P_3))(b'_2P_2 + b'_3P_3) = \\ (a'_1\theta_1/\theta_2 P_2 + (a'_2P_2 + a'_3P_3))(b_2P_2 + b_3P_3).\end{gathered}$$

Thus we obtain the value of $\frac{\theta_1}{\theta_2}$.

$$\begin{gathered}\theta_1/\theta_2[a_1(b'_2P_2 + b'_3P_3) - a'_1(b_2P_2 + b_3P_3)]P_2 = \\ (a'_2P_2 + a'_3P_3)(b_2P_2 + b_3P_3) - (a_2P_2 + a_3P_3)(b'_2P_2 + b'_3P_3).\end{gathered}$$

If the coefficient of $\frac{\theta_1}{\theta_2}$ is null we obtain an equation of the form $P_2 = KP_3$. This implies that P is on a plane. It is impossible since the lines A_i are non coplanar. Thus we can obtain the value of $\frac{\theta_1}{\theta_2}$. Then the first line of the system gives an equation of the form

$$f(P_2, P_3) = P_1,$$

where f is a homogeneous rational map of twp variables.

• Second case $P_2 = 0$. We obtain

$$\begin{cases} P_1 = \frac{a_1\theta_1 + a_3\theta_3}{b_3\theta_3} + \lambda\theta_1 \\ P_3 = \lambda\theta_3 \\ (F_1 - F_2)(\theta) = 0 \end{cases}$$

Remark that $P_3 \neq 0$. Indeed if not the direction is included in A_0. Thus the system becomes

$$\begin{cases} P_1 = \frac{a_1\theta_1 + a_3\theta_3}{b_3\theta_3} + \lambda\theta_1 \\ \lambda = P_3/\theta_3 \\ (F_1 - F_2)(\theta) = 0 \end{cases}$$

$$\begin{cases} P_1 = \frac{a_1\theta_1 + a_3\theta_3}{b_3\theta_3} + P_3/\theta_3\theta_1 \\ P_3/\theta_3 = \lambda \\ (F_1 - F_2)(\theta) = 0 \end{cases}$$

And the equation $(F_1 - F_2)(\theta) = 0$ gives as in the first case the values of $\frac{\theta_1}{\theta_3}$.

• Now consider a transversal line d'. A point on this line depends on one parameter. If the point is on the surface, the parameter verifies a polynomial equation of degree four, thus there are a bounded number of solutions. □

Corollary 23. *Consider four edges A_0, A_1, A_2, A_3 two by two non coplanar such that $A_3 \notin S(A_0, A_1, A_2)$. Then the maps $F_1 - F_2, F_1 - F_3$ are linearly independent.*

Proof. We make the proof by contradiction. If the maps $F_1 - F_2, F_1 - F_3$ are linearly dependent, it means that F_3 is a linear combination of F_1, F_2. It implies that the system $\begin{cases} m = F_1(\theta) \\ m = F_2(\theta) \\ m = F_3(\theta) \end{cases}$ is equivalent to $\begin{cases} m = F_1(\theta) \\ m = F_2(\theta) \end{cases}$. Thus each line which passes through A_0, A_1, A_2 must passes through A_3. By preceding Lemma it implies that A_3 is in $S(A_0, A_1, A_2)$, contradiction. □

4.2. **Key point.**

Lemma 24. *Consider a point $(m, \theta) \in \overline{E_0}$; then the set of words v such that $(m, \theta) \in \sigma_v^-$ is at most countable.*

For the proof we refer to [Kat87]. This proof does not depend on the dimension.

4.2.1. *Definitions.* For a fixed word $v \in \Sigma \setminus \phi(E_0)$, the set σ_v^- is of dimension 0 or 1 and the direction θ is unique, see Lemma 18. Fix a word $v \in \Sigma \setminus \phi(E_0)$, we will consider several cases:

• First σ_v^- is an interval with endpoints a, b. For any $m \in]a, b[$ we consider the set of discontinuities met in the unfolding of (m, θ). This set is independent of $m \in]a, b[$ since σ_v^- is an interval. We denote it $Disc(v, int)$. If the endpoint a (resp. b) is included in the interval then the orbit of (m, θ) can meet other discontinuities. We call $Disc(v, a)$ (resp. $Disc(v, b)$) the set of those discontinuities.

• If σ_v^- is a point it is the same method as $Disc(v, int)$, we denote the set of discontinuities by $Disc(v, int)$.

Here there are two sorts of discontinuities. First the singularity is a point of the boundary of a face whose code contributes to v. Then the orbit is not transverse to the edge. Secondly they meet in the transversal sense. If the orbit is included in an edge, then the discontinuities met are the boundary points of that edge (and similarly if the orbit is in a face).

Definition 25. *Let $V = \Sigma \setminus \phi(E_0)$ and $X \subset V$ be the set of $v \in V$ such that the union of the elements A_i of $Disc(v, int), Disc(v, a), Disc(v, b)$ are contained in a finite union of hyperplanes and of surfaces $S(A_0, A_1, A_2)$.*

Suppose $v \in X$. Let $N(\sigma_v^-)$ be the number of planes containing $Disc(v)$ if σ_v^- is a point or $Disc(v, a)$ or $Disc(v, b)$ if σ_v^- is an interval.

In the following Lemma the function L refers to the width of the strip of singular orbits as it does in the proof of Lemma 16.

Lemma 26. *Suppose μ is an ergodic measure with support in $\Sigma \setminus \phi(E_0)$. Then*

(i) there exists a constant L such that $L(\sigma_v^-) = L$ for μ-a.e. $v \in \Sigma$ and thus for μ-a.e $v, w \in \Sigma$ if $w_i = v_i$ for $i \geq 0$ then $\sigma_w = \sigma_v$.

(ii) there exists a constant N such that $N(\sigma_v^-) = N$ for μ-a.e $v \in \Sigma$.

Proof. (i) If σ_v^- is a point then there is nothing to show. Let $L(\sigma_v^-)$ be as before. We have $L(\sigma_v^-) \leq L(\sigma_{S(v)}^-)$. Since S is ergodic, L is constant almost everywhere. Thus $L(\sigma_v) = L(\sigma_v^-)$ thus $\sigma_v = \sigma_v^-$. The same holds for w, thus since $\sigma_w^- = \sigma_v^-$ we have $\sigma_v = \sigma_w$.

(ii) We have $N(\sigma_v^-) \leq N(\sigma_{Sv}^-)$, thus the lemma follows since S is ergodic. □

Let D stand for $Disc(v, int), Disc(v, a), \text{or} \quad Disc(v, b)$.

Remark 27. *For two sets $A_i, A_j \in D$ the relation $dimAff(A_i, A_j) = 2$ is a transitive relation. Indeed consider three sets A_i, A_j, A_k such that $A_i \sim A_j$, and $A_j \sim A_k$. Since the line $m + \mathbb{R}\theta$ passes through A_i, A_j, A_k, we deduce $A_i \sim A_k$.*

Then we can show

Proposition 28. *The set $V \setminus X$ is at most countable.*

Proof. Let $v \in V$. Lemma 19 implies that we have for each pair of discontinuities an equation $m = F(\theta)$ or $f(\theta) = 0$. Denote the set D by $A_0, \ldots, A_n, \ldots$. Either there exist discontinuities $A_{i_0}, A_{i_1}, A_{i_2}, A_{i_3}$, such that the equations related to (A_{i_0}, A_{i_j}), for all $j \leq 3$, are of the form $m = F(\theta)$ or not. In the following we will assume, for simplicity, that these three discontinuities (if they exist) are denoted by A_0, A_1, A_2, A_3.

• First assume it is not the case. Then for any subset of $D\setminus\{A_0, A_1, A_2, A_3\}$ two elements give equations of the form $f(\theta) = 0$. By Remark 27 all the discontinuities in the set $D \setminus \{A_0, A_1, A_2, A_3\}$ are in a single hyperplane. Thus all the discontinuities of D are in a finite union of hyperplanes. We do the same thing for $Disc(v, a)$ and $Disc(v, b)$. We conclude $v \in X$.

• Now we treat the case where we obtain at least three equations of the form $m = F(\theta)$ for some choice of (m, θ).

Corollary 23 shows that two such equations are different since the discontinuities are not in the union of surfaces. Thus consider the three first equations $m = F(\theta) = G(\theta) = H(\theta)$. It gives two equations $(F - G)(\theta) = (F - H)(\theta) = 0$. Those two equations are different by Corollary 23, since F, G, H are different. We deduce that the direction θ is solution of a system of two independant equations, thus it is unique. We remark that the vertices which appear in unfolding have their coordinates in a countable set $\mathcal{C}$. Indeed we start from a finite number of points corresponding to the vertices and at each step of the unfolding we reflect them over some faces of P. Thus at each step there are a finite set of vertices. Moreover the coefficients of the edges are obtained by difference of coordinates of vertices. By the same argument the coefficients of cartesian equations of the hyperplanes which contains faces live in a countable set $\mathcal{C}$. There are only a countable collection of functions $m = F(\theta)$ which arise. Thus the solution θ corresponding to the equations $m = F(\theta) = G(\theta) = H(\theta)$ lives in a countable set. It determines (m, θ). The number of words associated to the orbit of (m, θ) is countable by Lemma 24. Thus the set of such words is countable. □

5. Proof of Theorem 1

Lemma 29. *Suppose that μ is an ergodic measure supported in $\Sigma\setminus\phi(E_0)$ such that $\mu(X) = 1$. Then $h_\mu(S) = 0$.*

Proof. By Lemma 26 we can assume there is a constant $L \geq 0$ such that $L(\sigma_v^-) = L$. Suppose first that $L > 0$. Suppose $v \in \text{support}(\mu)$. This

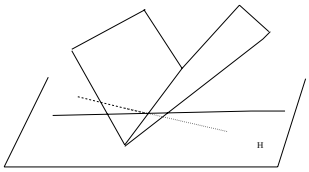

FIGURE 3. Coding of a word

implies that $Disc(v, int)$ is contained in a single plane. If $w \in \text{support}(\mu)$ satisfies $w_i = v_i$ for $i \geq 0$ then $Disc(w, int)$ is contained in the same plane. Each trajectory in $\phi(E_0)$ which approximates the future of v cuts this plane in a single point. Consider these sequence of approximating trajectories which converges to (m, θ). The limit of these trajectories cuts the surface at one (or zero) points. The point where it cuts the surface determines the backwards unfolding, and thus the backwards code. Thus if we ignore for the moment the boundary discontinuities the knowing the future $v_0, v_1, v_2, \dots$ determines $O(n)$ choices of the past $v_{-n}, \dots, v_{-1}$.

The boundary discontinuities and the case $L(\sigma_v^-) = 0$ are treated analogously. Let $(m, \theta) = \sigma_v^-$ (or one of the boundary points of σ_v^- in the case above). By Lemma 26 we can assume that $Disc(v, m)$ is contained in N planes, and that if $w \in \text{support}(\mu)$ satisfies $w_i = v_i$ for $i \geq 0$ then $Disc(w, int)$ is contained in the same planes. Arguing as above, the point where an approximating orbit cuts these planes determines the past. Thus the future $v_0, v_1, v_2, \dots$ determines $O(n^N)$ choices of the past $v_{-n}, \dots, v_{-1}$. Since $\lim_{n \to +\infty} \dfrac{\log n^N}{n} = 0$ we deduce the result. □

The preceding lemma and proposition allow to conclude

Corollary 30. *Let μ an ergodic measure with support in $\Sigma \setminus \phi(E_0)$, then*

$$h_\mu(S) = 0.$$

Proof. This follows immediately from Lemma 29 and Proposition 28. □

Lemma 8 reduces the problem to the computation of $h_{top}(S|\Sigma)$. Moreover we have

$$h_{top}(S|\Sigma) = \sup_{\substack{\mu \\ ergo, \\ supp(\mu) \subset \phi(E_0)}} h_\mu(S|\Sigma) + \sup_{\substack{\mu \\ ergo, \\ supp(\mu) \subset \Sigma \setminus \phi(E_0)}} h_\mu(S|\Sigma),$$

then Corollaries 17 and 30 imply:

$$h_{top}(S|\Sigma) = 0.$$

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