



Simultaneous Empirical Interpolation and Reduced Basis method for non-linear problems

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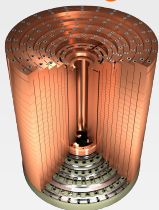
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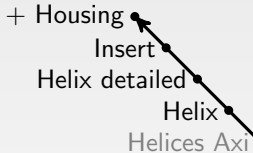




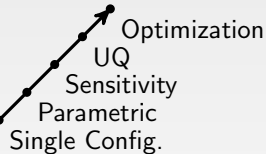
High Field magnet modeling



Geometries

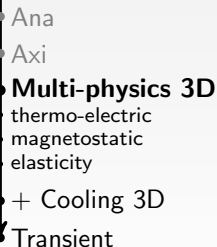


Studies



HifiMagnet project

- 3D Multi-physics model
- Generic (Geometries, materials, ...)
- Efficient
 - HPC
 - Reduced Order Modeling
- Reliable
 - Uncertainty quantification



Physics



Plan

- 1 Reduced Basis method
- 2 Non-affinely parametrized problems
- 3 SER method
- 4 Numerical results
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Improve model efficiency : Reduced Basis Method

Weak formulation of the model : $a(u(\mu), v; \mu) = f(v; \mu)$

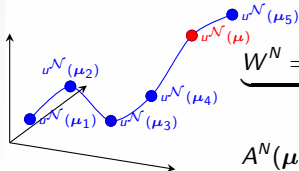
Quantity of interest : $s(\mu) = \ell(u(\mu)) \implies$ **Evaluation for several $\mu \in \mathcal{D} \subset \mathbb{R}^p$**

FEM approximation

$$\underbrace{X^{\mathcal{N}} = \text{span}\{\phi_1, \dots, \phi_{\mathcal{N}}\}}_{\text{FEM approximation space}} \longrightarrow \underbrace{u^{\mathcal{N}}(\mu) = \sum_{i=1}^{\mathcal{N}} u_i^{\mathcal{N}} \phi_i}_{\mathcal{N} \approx 10^6} \longrightarrow \underbrace{A^{\mathcal{N}}(\mu) u^{\mathcal{N}}(\mu) = F^{\mathcal{N}}(\mu)}_{\substack{\mathcal{N} \times \mathcal{N} \text{ system } \textbf{expensive to solve} \\ \approx 10 \text{ min on } 32 \text{ procs}}}$$

Objective : reduce the computational cost

$u^N(\mu) \approx u^{\mathcal{N}}(\mu)$: linear combination of FEM solution



$$\underbrace{W^N = \text{span}\{u^{\mathcal{N}}(\mu_1), \dots, u^{\mathcal{N}}(\mu_N)\}}_{\text{RB approximation space}} \longrightarrow \underbrace{u^N(\mu) = \sum_{i=1}^N u_i^N(\mu) \underline{u}^{\mathcal{N}}(\mu_i)}_{10 \leq N \leq 100}$$

$$A^N(\mu) u^N(\mu) = F^N(\mu) \longrightarrow \mathbf{N} \times \mathbf{N} \text{ system } \textbf{cheaper to solve}$$

$$\sum_{i=1}^N \left[\sum_{q=1}^{Q_a} \theta_a^q(\boldsymbol{\mu}) \underbrace{a^q(\underline{u}^N(\boldsymbol{\mu}_i), \underline{u}^N(\boldsymbol{\mu}_j))}_{\text{precomputed}} \right] \underline{u}_i^N(\boldsymbol{\mu}) = \sum_{q=1}^{Q_f} \theta_f^q(\boldsymbol{\mu}) \underbrace{f^q(\underline{u}^N(\boldsymbol{\mu}_j))}_{\text{precomputed}}$$

Offline : Sampling build

$$S_N \leftarrow \{\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_N\}$$

Offline : Basis build

for $\boldsymbol{\mu} \in S_N$ do

$$A_{i,j}^N = \sum_{q=1}^{Q_a} \theta_a^q(\boldsymbol{\mu}) a^q(\phi_j, \phi_i)$$

$$F_j^N = \sum_{q=1}^{Q_f} \theta_f^q(\boldsymbol{\mu}) f^q(\phi_j)$$

solve $A^N \underline{u}^N(\boldsymbol{\mu}) F^N$

$$W_N \leftarrow W_N \cup \underline{u}^N(\boldsymbol{\mu})$$

end for

Offline : Pre-computations

$$A_{i,j}^{N,q} = a^q(\underline{u}^N(\boldsymbol{\mu}_j), \underline{u}^N(\boldsymbol{\mu}_i))$$

$$F_j^{N,q} = f^q(\underline{u}^N(\boldsymbol{\mu}_j))$$

Online

$$A_{i,j}^N = \sum_{q=1}^{Q_a} \theta_a^q(\boldsymbol{\mu}) \underbrace{a^q(\underline{u}^N(\boldsymbol{\mu}_j), \underline{u}^N(\boldsymbol{\mu}_i))}_{\text{precomputed}}$$

$$F_j^N = \sum_{q=1}^{Q_f} \theta_f^q(\boldsymbol{\mu}) \underbrace{f^q(\underline{u}^N(\boldsymbol{\mu}_j))}_{\text{precomputed}}$$

solve $A^N \underline{u}^N(\boldsymbol{\mu}) F^N$



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EIM : Empirical Interpolation Method

$$\text{Build } w_M(u, \mathbf{x}; \boldsymbol{\mu}) = \sum_{m=1}^M \beta_m^M(u; \boldsymbol{\mu}) q_m(\mathbf{x}) \approx w(u, \mathbf{x}; \boldsymbol{\mu})$$

Training set $\Xi_{train}^\mu \subset \mathcal{D}^\mu$

- Offline step : Build basis functions $\mathbf{q}_m(\mathbf{x})$

→ Sample $\bar{S}_M = \{\bar{\boldsymbol{\mu}}_1 \in \Xi_{train}^\mu, \dots, \bar{\boldsymbol{\mu}}_m, \dots, \bar{\boldsymbol{\mu}}_M \in \Xi_{train}^\mu\}$

Interpolation points $\mathbf{t}_1, \dots, \mathbf{t}_m, \dots, \mathbf{t}_M \in \Omega$

→ Greedy : $\bar{\boldsymbol{\mu}}_m = \arg \max_{\boldsymbol{\mu} \in \Xi} \inf_{z \in W_{m-1}} \|w(\cdot, \cdot; \boldsymbol{\mu}) - z\|_{L^\infty(\Omega)}$

→ Approximation space $\bar{W}_M = \text{span}\{\mathbf{q}_1(\mathbf{x}), \dots, \mathbf{q}_m(\mathbf{x}), \dots, \mathbf{q}_M(\mathbf{x})\}$

Residual $\mathbf{r}_m(\mathbf{x}) = w(u, \mathbf{x}; \bar{\boldsymbol{\mu}}_m) - w_{m-1}(u, \mathbf{x}; \bar{\boldsymbol{\mu}}_m)$

$$\mathbf{t}_m = \arg \sup_{\mathbf{x} \in \Omega} |\mathbf{r}_m(\mathbf{x})|, \quad \mathbf{q}_m(\mathbf{x}) = \frac{\mathbf{r}_m(\mathbf{x})}{\mathbf{r}_m(\mathbf{t}_m)}$$

- Online step : Compute approximation coefficients $\beta_m(u; \boldsymbol{\mu})$

$$w_M(u, \mathbf{t}_i; \boldsymbol{\mu}) = \sum_{m=1}^M \beta_m^M(u; \boldsymbol{\mu}) q_m(\mathbf{t}_i) = w(u, \mathbf{t}_i; \boldsymbol{\mu}) \quad 1 \leq i \leq M$$

Non-linear and non-affine problems

Non-linearity : Iterative method

→ Picard

→ Newton

Offline : Basis build

for $\mu \in S_N$ **do**

${}^0 u^{\mathcal{N}} \leftarrow \text{initialGuess}(\mu)$

while increment > tol **do**

$$A_{i,j}^{\mathcal{N}} = \sum_{q=1}^{Q_a^{eim}} \sum_{m=1}^{M_q^a} \gamma_{a,m}^q({}^k u^{\mathcal{N}}, \mu) a_m^q(\phi_j, \phi_i) + \dots$$

$$F_j^{\mathcal{N}} = \sum_{q=1}^{Q_f^{eim}} \sum_{m=1}^{M_q^f} \gamma_{f,m}^q({}^k u^{\mathcal{N}}, \mu) f_m^q(\phi_j) + \dots$$

${}^{k+1} u^{\mathcal{N}}(\mu) \leftarrow \text{solve } A^{\mathcal{N}k} u^{\mathcal{N}}(\mu) F^{\mathcal{N}}$

end while

$W_N \leftarrow W_N \cup {}^k u^{\mathcal{N}}(\mu)$

end for

Non-linear and non-affine problems

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end while

$$W_N \leftarrow W_N \cup \underline{u}^{\mathcal{N}}(\mu)$$

end for

Offline : Pre-computations

$$A_{i,j}^{N,q,m} = a_m^q(\underline{u}^{\mathcal{N}}(\mu_j), \underline{u}^{\mathcal{N}}(\mu_i))$$

$$F_j^{N,q,m} = f_m^q(\underline{u}^{\mathcal{N}}(\mu_j))$$

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for $\mu \in S_N$ do

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end while

$$W_N \leftarrow W_N \cup \underline{u}^N(\mu)$$

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$$F_j^{N,q,m} = f_m^q(\underline{u}^N(\mu_j))$$

Online

${}^0 \underline{u}^N \leftarrow \text{initialGuess}(\mu)$, $k \leftarrow 0$

while increment > tol do

$$A_{i,j}^N = \sum_{q=1}^{Q_a^{eim}} \sum_{m=1}^{M_q^a} \gamma_{a,m}^q({}^k \underline{u}^N, \mu) \underbrace{A_{i,j}^{N,q,m}}_{\text{precomputed}}$$

+ ...

$$F_j^N = \sum_{q=1}^{Q_f^{eim}} \sum_{m=1}^{M_q^f} \gamma_{f,m}^q({}^k \underline{u}^N, \mu) \underbrace{F_j^{N,q,m}}_{\text{precomputed}}$$

+ ...

$${}^{k+1} \underline{u}^N \leftarrow \text{solve } A^N {}^k \underline{u}^N(\mu) F^N$$

end while



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How to reduce EIM cost ?

Greedy algorithm to select EIM parameters :

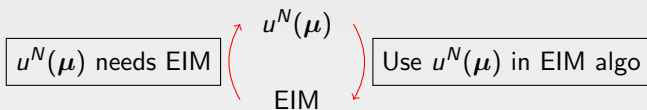
$$\bar{\mu}_m = \arg \max_{\mu \in \Xi} \inf_{z \in W_{m-1}} \|w(\cdot, \cdot; \mu) - z\|_{L^\infty(\Omega)}$$

```
for  $\mu_i \in \Xi_{train}^\mu$  do  
  Compute  $w(u^N(\mu_i), \mathbf{x}; \mu_i)$   
  ...  
end for
```

- Number of FE approximation required : proportional to Ξ_{train}^μ size
→ Ξ_{train}^μ can be large → **EIM parameter selection can be costly !**
- Proposed solution : use u^N in Greedy algorithm

$$\bar{\mu}_m = \arg \max_{\mu \in \Xi} \inf_{z \in W_{m-1}} \|w(u^{\mathcal{N} \rightarrow N}(\mu); \cdot; \mu) - z\|_{L^\infty(\Omega)}$$

Simultaneous EIM / RB : Initialization



Initialization : Build rough EIM approx $w_{M=1}(u(\mu), \mathbf{x}, \mu) \rightarrow$ Build $u^{N=1}(\mu)$

First EIM approximation

$$\bar{S}_1 = \{\bar{\mu}_1\}, \bar{\xi}_1 = w(u^N(\bar{\mu}_1), \mathbf{x}; \bar{\mu}_1), \mathbf{t}_1 = \arg \sup_{\mathbf{x} \in \Omega} |\bar{\xi}_1(\mathbf{x})|, \mathbf{q}_1 = \frac{\bar{\xi}_1(\mathbf{x})}{\bar{\xi}_1(\mathbf{t}_1)}$$

$$w_1(u(\mu), \mathbf{x}, \mu) = \beta_1^1(u, \mu) \mathbf{q}_1(\mathbf{x}) \leftarrow \text{Only one FE approximation}$$

First RB approximation

$$u^{N=1}(\mu) = u_1^1 u^N(\mu_1)$$



SER : Alternate build EIM / RB (Offline)

for $s=2 \rightarrow S$ do ($S = M = N$)

Build s^{th} EIM approximation $w_s \approx w$

$$\bar{\mu}_s = \arg \max_{\mu \in \Xi} \inf_{z \in W_{s-1}} \|w(u^{s-1}(\mu); \cdot; \mu) - z\|_{L^\infty(\Omega)}$$

[...]

$$r_s(\mathbf{x}) = w(u^{s-1}(\bar{\mu}_s), \mathbf{x}; \bar{\mu}_s) - w_{s-1}(u^{s-1}(\bar{\mu}_s), \mathbf{x}; \bar{\mu}_s)$$

$$t_s = \arg \sup_{\mathbf{x} \in \Omega} |r_s(\mathbf{x})|, \quad \mathbf{q}_s(\mathbf{x}) = \frac{r_s(\mathbf{x})}{r_s(t_s)}$$

▷ No FE approximation

end for

SER : Alternate build EIM / RB (Offline)

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Build s^{th} EIM approximation $w_s \approx w$

$$\bar{\mu}_s = \arg \max_{\mu \in \Xi} \inf_{z \in W_{s-1}} \|w(u^{s-1}(\mu); \cdot; \mu) - z\|_{L^\infty(\Omega)}$$

[...]

$$r_s(\mathbf{x}) = w(u^{s-1}(\bar{\mu}_s), \mathbf{x}; \bar{\mu}_s) - w_{s-1}(u^{s-1}(\bar{\mu}_s), \mathbf{x}; \bar{\mu}_s)$$

$$t_s = \arg \sup_{\mathbf{x} \in \Omega} |r_s(\mathbf{x})|, \quad \mathbf{q}_s(\mathbf{x}) = \frac{r_s(\mathbf{x})}{r_s(t_s)}$$

▷ No FE approximation

Build s^{th} RB approximation $\underline{u}^{N=s}(\mu)$

$${}^0 u^{\mathcal{N}} \leftarrow \text{initialGuess}(\mu_s)$$

while increment > tol do

$$A_{i,j}^{\mathcal{N}} = \sum_{q=1}^{Q_a^{\text{eim}}} \sum_{m=1}^s \gamma_{a,m}^q({}^k u^{\mathcal{N}}, \mu) a_m^q(\phi_j, \phi_i) + \dots, \quad F_j^{\mathcal{N}} = \sum_{q=1}^{Q_f^{\text{eim}}} \sum_{m=1}^s \gamma_{f,m}^q({}^k u^{\mathcal{N}}, \mu) f_m^q(\phi_j) + \dots$$

$${}^{k+1} u^{\mathcal{N}}(\mu) \leftarrow \text{solve } A^{\mathcal{N}k} u^{\mathcal{N}}(\mu) F^{\mathcal{N}}$$

end while

end for



SER : Characteristics of the method

Alternate build of EIM and RB basis functions in offline step

Reduction of computational cost

- Offline (standard basis build) : $N + (M \times \text{Card}(\Xi_{train}))$ FE approximations
- Offline (simultaneous basis build) : **$N + 1$ FE approximations**
→ EIM cost is reduced

SER customization :

- EIM / RB can be simultaneously built, per group of size r
→ $r = M$: standard build
→ $r = 1$: SER method
- W_N (RB space) can be recomputed at each step
→ EIM approximation changes at each step
→ Solve problem $A^N u^N = F^N$ changed due to EIM evolution



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Standard VS SER method - Benchmark [Daversin and Prud'Homme, 2015]

Non-linear and non-affine 2D problem *

$$\mu = (\mu_1, \mu_2) \in \mathcal{D} = [0.01, 10]^2$$

$$-\Delta u + \underbrace{\mu_1 \frac{e^{\mu_2 u} - 1}{\mu_2}}_{g(u, \mathbf{x}; \mu)} = 100 \sin(2\pi x) \sin(2\pi y) \text{ in } \Omega =]0, 1[^2 \text{ with } s = \int_{\Omega} u$$

$$\epsilon_{M,N}^{u,r} = \| u^{\mathcal{N}} - u_r^{\mathcal{N}} \|_{L_2}$$

$$\epsilon_{M,N}^{s,r} = | s^{\mathcal{N}} - s_r^{\mathcal{N}} |$$

N	M	$\max(\epsilon_{M,N}^{u,M})$	$\max(\epsilon_{M,N}^{s,M})$
4	5	7.38e-3	5.75e-3
8	10	1.01e-3	2.34e-4
12	15	1.49e-4	3.09e-5
16	20	2.21e-5	1.25e-5
20	25	5.88e-6	2.82e-6

(a) $r = M$

N	M	$\max(\epsilon_{M,N}^{u,1})$	$\max(\epsilon_{M,N}^{s,1})$
5	5	1.30e-2	1.02e-2
10	10	2.20e-3	1.50e-3
15	15	4.83e-4	4.05e-4
20	20	2.42e-4	1.98e-4
25	25	1.50e-5	1.24e-5

(b) $r = 1$ (W_N not recomputed)

Maximum absolute errors on solution u and on output s

SER method for high field magnet modeling

Equations

$$\begin{cases} -\nabla \cdot (\sigma(T)\nabla V) = 0 & \text{in } \Omega \\ -\nabla \cdot (k(T)\nabla T) = \sigma(T)\nabla V \cdot \nabla V & \text{in } \Omega \end{cases}$$

EIM approximations

$$\sigma_M \approx \sigma(T) = \frac{\sigma_0}{1 + \alpha(T - T_0)}$$

$$k_M \approx k(T) = LT\sigma(T)$$

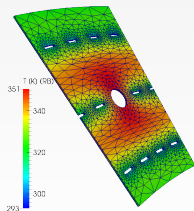
$$Q_M \approx \sigma(T)\nabla V \cdot \nabla V$$

Output : Mean temperature

$$\frac{1}{|\Omega|} \int_{\Omega} T$$

Inputs

- σ_0 : elec. conductivity
- α : temperature coeff
- L : Lorentz number
- j : current density
- h : Heat transfer coeff
- T_w : Water temperature





SER method for high field magnet modeling

N	M	$\max(\epsilon_{M,N}^u)$	$\max(\epsilon_{M,N}^s)$
5	5	3.44e-0	2.14e-2
7	7	1.74e-0	2.03e-2
10	10	3.06e-1	3.83e-3

$$r = M$$

N	M	$\max(\epsilon_{M,N}^u)$	$\max(\epsilon_{M,N}^s)$
5	5	1.67e+1	1.86e-1
7	7	9.02e-0	6.05e-2
10	10	3.04e-0	5.44e-3

(a) W_N built randomly

N	M	$\max(\epsilon_{M,N}^u)$	$\max(\epsilon_{M,N}^s)$
5	5	5.55e-0	2.88e-2
7	7	1.78e-0	6.27e-3
10	10	1.17e-0	2.35e-3

(b) W_N built from greedy

$$r = 1 \text{ (} W_N \text{ not recomputed)}$$



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Conclusion & Perspectives

SER : Simultaneous Empirical Interpolation method / Reduced Basis Method

- New approach for non-linear / non-affine problems
 - EIM uses reduced basis approximation
 - EIM and RB approximation spaces are built concurrently
 - Requires only $N + 1$ FE approximations

Ongoing work and Perspectives

- Apply SER method to multi-physics model
 - On real 3D magnet geometries
 - Adding physics : magnetostatic, elasticity
- Improve robustness
 - Identify limits of the method
 - Investigate error estimation possibilities



Reference



Daversin, C. and Prud'Homme, C. (2015).

Simultaneous empirical interpolation and reduced basis method for non-linear problems.

Accepted - C. R. Acad. Sci. Paris, Ser. I (2015).



Daversin, C., Veys, S., Trophime, C., and Prud'Homme, C. (2013).

A reduced basis framework: Application to large scale nonlinear multi-physics problems.

Esaim Proc., 43:225–254.



Grepl, M., Maday, Y., Nguyen, N. C., and Patera, A. (2007).

Efficient reduced-basis treatment of nonaffine and nonlinear partial differential equations.

ESAIM: Mathematical Modelling and Numerical Analysis, 41(03):575–605.

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(a) $r = M$

N	M	$\max(\epsilon_{M,N}^{u,1})$	$\max(\epsilon_{M,N}^{s,1})$
5	5	9.98e-3	7.77e-3
10	10	2.32e-3	1.86e-3
15	15	4.61e-4	3.75e-4
20	20	2.48e-4	2.02e-4
25	25	3.51e-5	2.33e-5

(b) $r = 1$
(W_N recomputed)

Maximum absolute errors on solution u and on output s



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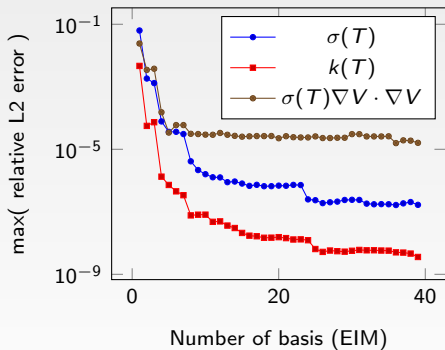
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(W_N not recomputed)

Maximum absolute errors on solution u and on output s



Electro-thermal RB model : Bitter Magnet

EIM - Convergence study



CRB - Convergence study

