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# On the Doubly Sparse Compressed Sensing Problem

Grigory Kabatiansky<sup>1</sup>, Serge Vlăduț<sup>1,2</sup>, Cedric Tavernier<sup>3</sup>

<sup>1</sup> Institute for Information Transmission Problems, Russian Academy of Sciences,  
Moscow, Russia,

<sup>2</sup> Institut de Mathématiques de Marseille, Aix-Marseille Université, IML, Marseille,  
France

<sup>3</sup> Assystem AEOS, France

kaba@iitp.ru, vl@iitp.ru, tavernier.cedric@gmail.com

**Abstract.** A new variant of the Compressed Sensing problem is investigated when the number of measurements corrupted by errors is upper bounded by some value  $l$  but there are no more restrictions on errors. We prove that in this case it is enough to make  $2(t+l)$  measurements, where  $t$  is the sparsity of original data. Moreover for this case a rather simple recovery algorithm is proposed. An analog of the Singleton bound from coding theory is derived what proves optimality of the corresponding measurement matrices.

## 1 Introduction and Definitions

A vector  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$  in  $n$ -dimensional vector space  $\mathbb{R}^n$  called  $t$ -sparse if its Hamming weight  $wt(x)$  or equivalently its  $l_0$  norm  $\|x\|_0$  is at most  $t$ , where by the definition  $wt(x) = \|x\|_0 = |\{i : x_i \neq 0\}|$ . Let us recall that the Compressed Sensing (CS) Problem [1,2] is a problem of reconstructing of an  $n$ -dimensional  $t$ -sparse vector  $x$  by a few ( $r$ ) linear measurements  $s_i = \langle h^{(i)}, x \rangle$  (i.e. inner product of vectors  $x$  and  $h^{(i)}$ ), assuming that measurements  $\langle h^{(i)}, x \rangle$  are known with some errors  $e_i$ , for  $i = 1, \dots, r$ . Saying in other words, one needs to construct an  $r \times n$  matrix  $H$  with minimal number of rows  $h^{(1)}, \dots, h^{(r)}$ , such that the following equation

$$\hat{s} = Hx^T + e, \quad (1)$$

has either a unique  $t$ -sparse solution or all such solutions are “almost equal”. The compressed sensing problem was mainly investigated under the assumption that the vector  $e = (e_1, \dots, e_r)$ , is called the error vector, has relatively small Euclidean norm (length)  $\|e\|_2$ . We consider another problem’s statement assuming that the error vector  $e$  is also sparse but its Euclidean norm can be arbitrary large. In other words, we consider the doubly sparse CS problem when  $\|x\|_0 \leq t$  and  $\|e\|_0 \leq l$ . The assumption  $\|e\|_0 \leq l$  was first time considered in [3] as a

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proper replacement for discrete version of CS-problem of usual assumption that an error vector  $e$  has relatively small Euclidean norm.

**Definition 1.** A real  $r \times n$  matrix  $H$  called a  $(t, l)$ -compressed sensing (CS) matrix if

$$\|Hx^T - Hy^T\|_0 \geq 2l + 1 \quad (2)$$

for any two distinct vectors  $x, y \in \mathbb{R}^n$  such that  $\|x\|_0 \leq t$  and  $\|y\|_0 \leq t$ .

This definition immediately leads (see [3]) to the following

**Proposition 1.** A real  $r \times n$  matrix  $H$  is a  $(t, l)$ -CS matrix iff

$$\|Hz^T\|_0 \geq 2l + 1 \quad (3)$$

for any nonzero vector  $z \in \mathbb{R}^n$  such that  $\|z\|_0 \leq 2t$ .

Our main result is an explicit and simple construction of  $(t, l)$ -CS matrices with  $r = 2(t + l)$  for any  $n$ . We show this value of  $r$  is the minimal possible for  $(t, l)$ -CS matrices by proving an analog of the well-known in coding theory Singleton bound for the compressed sensing problem. Besides that we propose an efficient recovery (decoding) algorithm for the considered double sparse CS-problem.

## 2 Optimal Matrices for Doubly Sparse Compressed Sensing Problem

We start with constructing of  $(t, l)$ -CS matrices. Let a real  $\tilde{r} \times n$  matrix  $\tilde{H}$  be a parity-check matrix of an  $(n, n - \tilde{r})$ -code over  $\mathbb{R}$ , correcting  $t$  errors, i.e. any  $2t$  columns  $\tilde{h}_{i_1}, \dots, \tilde{h}_{i_{2t}}$  of  $\tilde{H}$  are linearly independent. And let  $G$  be a generator matrix of an  $(r, \tilde{r})$ -code over  $\mathbb{R}$  of length  $r$ , correcting  $l$  errors. Let matrix  $H$  consists of the columns  $h_1, \dots, h_n$ , where

$$h_j^T = \tilde{h}_j^T G \quad (4)$$

and transposition  $^T$  means, that vectors  $h_j$  and  $\tilde{h}_j$  are considered in (4) as row vectors, i.e.

$$H = G^T \tilde{H} \quad (5)$$

In other words, we encode columns of parity-check matrix  $\tilde{H}$ , which is already capable to correct  $t$  errors, by a code, correcting  $l$  errors, in order to restore correctly the syndrom of  $\tilde{H}$ .

**Theorem 1.** Matrix  $H = G^T \tilde{H}$  is a  $(t, l)$ -CS matrix.

*Proof.* According to Proposition 2 it is enough to prove that  $\|Hz^T\|_0 \geq 2l + 1$  for any nonzero vector  $z \in \mathbb{R}^n$  such that  $\|z\|_0 \leq 2t$ . Indeed,  $u = \tilde{H}z^T \neq 0$  since any  $2t$  columns of  $\tilde{H}$  are linear independent. Then  $Hz^T = G^T \tilde{H}z^T = G^T u = (u^T G)^T$  and  $u^T G$  is a nonzero vector of a code over  $\mathbb{R}$ , correcting  $l$  errors. Hence  $\|Hz^T\|_0 = \|u^T G\|_0 \geq 2l + 1$ .  $\square$

Now let us choose the well known Reed-Solomon (RS) codes (which are a particular case of evaluation codes construction) as both constituent codes. The length of the RS-code is restricted by the number of elements in the field so in the case of  $\mathbb{R}$  the length of evaluation code can be arbitrary large. Indeed, consider the corresponding evaluation code  $\mathbb{RS}_{(n,k)} = \{(f(a_1), \dots, f(a_n)) : \deg f(x) < k\}$ , where  $a_1, \dots, a_n \in \mathbb{R}$  are  $n$  different real numbers. The distance of the  $\mathbb{RS}_{(n,k)}$  code  $d = n - k + 1$  since the number of roots of a polynomial cannot exceed its degree and hence  $d \geq n - k + 1$ , but, on the other hand, the Singleton bound states that  $d \leq n - k + 1$  for any code, see [4]. Therefore the resulting matrix  $H$  is a  $(t, l)$ -CS matrix with  $r = 2(t + l)$ . The next result, which is a generalization of the Singleton bound for the doubly sparse CS problem, shows these matrices are optimal in the sense having the minimal possible number  $r$  of linear measurements.

**Theorem 2.** *For any  $(t, l)$ -CS  $r \times n$ -matrix*

$$r \geq 2(t + l). \quad (6)$$

*Proof.* Let  $H$  be any  $(t, l)$ -CS matrix of size  $r \times n$ , i.e.,  $\|Hz^T\|_0 \geq 2l + 1$  for any nonzero vector  $z \in \mathbb{R}^n$  such that  $\|z\|_0 \leq 2t$ . And let  $H_{2t-1}$  be the  $(2t - 1) \times n$  matrix consisting of the first  $2t - 1$  rows of  $H$ . There exists a nonzero vector  $\hat{z} = (\hat{z}_1, \dots, \hat{z}_{2t}, 0, 0, \dots, 0) \in \mathbb{R}^n$  such that  $H\hat{z}^T = 0$  (a system of linear homogenous equations with the number of unknown variables larger than the number of equations has a nontrivial solution). Then  $\|H\hat{z}^T\|_0 \leq r - (2t - 1)$  and finally  $r \geq 2t + 2l$  since  $\|H\hat{z}^T\|_0 \geq 2l + 1$ .  $\square$

### 3 Recovery Algorithm for Doubly Sparse Compressed Sensing Problem

Let us start from a simple remark that for  $e = 0$  recovering of the original sparse vector  $x$ , i.e., solving the equation (1), is the same as syndrome decoding of some code (over  $\mathbb{R}$ ) defined by matrix  $H$  as a parity-check matrix. In general, syndrome  $s = Hx^T$  is known with some error, namely, as  $\hat{s} = s + e$  and therefore we additionally encoded columns of  $H$  by some error-correcting code in order to recover the original syndrome  $s$  and then apply usual syndrome decoding algorithm. Therefore recovering, i.e., decoding algorithm for constructed in previous chapter optimal matrices is in some sense a “concatenation” of decoding algorithms of constituent codes.

Namely, first we decode vector  $\hat{s} = s + e$  by a decoding algorithm of the code with generator matrix  $G$ . Since  $\|e\|_0 \leq l$  this algorithm outputs the correct syndrome  $s$ . After that we form the syndrome  $\tilde{s}$  by selecting first  $\tilde{r}$  coordinates of  $s$  and then apply syndrome decoding algorithm (of the first code with parity-check matrix  $\tilde{H}$ ) for the following syndrom equation

$$\tilde{s} = \tilde{H}x^T. \quad (7)$$

Now let us discuss a right choice of constituent codes. It is very convenient to use the class of Reed-Solomon codes over  $\mathbb{R}$ . There are well known algorithms of their decoding up to half of the code distance (bounded distance decoding, see [4]), for instance, Berlekamp-Massey algorithm, which in our case (codes over  $\mathbb{R}$ ) is known also as Trench algorithm, see [5]. Hence the total decoding complexity does not exceed  $O(n^2)$  operations over real numbers. Moreover we can even decode these codes over their half distances by application of Guruswami-Sudan list decoding algorithm [6].

It is well known that encoding-decoding procedures of Reed-Solomon codes become more simple in the case of cyclic codes, when the set  $a_1, \dots, a_n$  is a cyclic group under multiplication. In order to do it let us consider  $a_1, \dots, a_n$  as complex roots of degree  $n$  and define our codes through their “roots”, i.e. our codes consist of *polynomials*  $f(x)$  over  $\mathbb{R}$  such that  $f(e^{2\pi i \frac{m}{n}}) = 0$  for  $m \in \{-s, \dots, -1, 0, +1, \dots, +s\}$  with  $s = t$  for the first constituent code and  $s = l$  for the second. It easy to check that such codes achieve the Singleton bound with  $d = 2s + 2$ , so the corresponding doubly sparse code has redundancy  $r = 2(t + l + 1)$  what is slightly larger than the corresponding Singleton bound, but in return these codes can be decoded via FFT.

#### 4 Discussion - no small errors case and slightly beyond

Let us note that the initial papers on Compressed Sensing especially stated that this new technique (application of  $l_1$  minimization instead of  $l_0$ ) allows to recover information vector  $x \in \mathbb{R}^n$  in case when not many coordinates of  $x$  were affected by errors. For instance, “one can introduce errors of arbitrary large sizes and still recover the input vector exactly by solving a convenient linear program...”, see in [7]. To achieve such performance some special restriction on matrix  $H$  was placed, called Restricted Isometry Property (RIP), as follows

$$(1 - \delta_D) \|x\|_2 \leq \|Hx^T\|_2 \leq (1 + \delta_D) \|x\|_2, \quad (8)$$

for any vector  $x \in \mathbb{R}^n : \|x\|_0 \leq D$ , where  $0 < \delta_D < 1$ . The smallest possible  $\delta_D$  called the isometry constant.

Then typical result in [7] (Th. 1.1 ) is of the following form

*“if  $\delta_{3t} + 3\delta_{4t} < 2$  then the solution of linear programming problem is unique and equal to  $x$ ”*

Let us note that the condition  $\delta_{3t} + 3\delta_{4t} < 2$  implies  $\delta_{4t} < 2/3$  (of course, it implies that  $\delta_{4t} < 1/2$ , but for us enough to have  $\delta_{4t} < 1$ ). Hence  $Hx^T \neq \mathbf{0}$  for any nonzero  $x$  with  $wt(x) \leq 4t$ , or in other words, an error-correcting code (over reals) corresponding to such parity-check matrix  $H$  has the minimal distance at least  $4t + 1$  and can correct  $2t$  errors (instead of  $t$ ). So we lost twice in error-correction capability but maybe linear programming provides more easier way for decoding ? In fact, NOT, since it is well known in coding theory that such problem can be solved rather easily (in complexity) over *any infinite field* by usage of the corresponding Reed-Solomon codes and known decoding

algorithms. In case of real number or complex number fields one can use just an RS code with Fourier parity-check matrix, namely,  $h_{j,p} = \exp(2\pi i \frac{jp}{n})$ , where for complex numbers  $p \in \{1, 2, \dots, n\}$ ,  $j = a, a + d, a + 2d, \dots, a + (r - 1)d$ , and “reversible” RS-matrix  $H$  for real numbers, where  $p \in \{1, 2, \dots, n\}$ ,  $j \in \{-f, -f + 1, \dots, 0, 1, \dots, f\}$  and  $r = 2f + 1$ .

Fortunately, matrices with the RIP property allow to correct not only sparse errors but also additional errors with arbitrary support but relatively small (up to  $\varepsilon$ ) Euclidean norm. Again, the RIP property is good for linear programming decoding but is too strong in general. Namely, it is enough to have the following property

$$\lambda_{2t} \|x\|_2 \leq \|Hx^T\|_2, \quad (9)$$

for any vector  $x \in \mathbb{R}^n : \|x\|_0 \leq 2t$ , where  $\lambda_{2t} > 0$  and the largest such value we call *extension constant*. Indeed, then for any two solutions  $x$  and  $\tilde{x}$  of the Equation (1) we have that

$$\|x - \tilde{x}\|_2 \leq 2\lambda_{2t}^{-1} \varepsilon, \quad (10)$$

where  $\|e\|_2, \|\tilde{e}\|_2 \leq \varepsilon$ . Hence (10) shows that all solutions of the Equation (1) are “almost equal” if  $\lambda_{2t}$  is large enough. Let us note that for RS-matrices any  $r$  columns are linear independent and hence  $\lambda_{2t} > 0$ , but  $\lambda_{2t}$  tends to zero when  $n$  grows and code rate is fixed. To find better class of codes over the field of real (or complex) numbers is an open problem.

## 5 Conclusion

In this paper we extend technique, which was developed in [10] for error correction with errors in both the channel and syndrome, to the Compressed Sensing problem. We hope this approach will help to find limits of the unrestricted (i.e. without LP usage) compressed sensing.

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