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# MEASURE CONTRACTION PROPERTIES OF CARNOT GROUPS

RIZZI LUCA

ABSTRACT. We prove that any corank 1 Carnot group of dimension  $k + 1$  equipped with a left-invariant measure satisfies the MCP( $K, N$ ) iff  $K \leq 0$  and  $N \geq k + 3$ . This generalizes the well known result by Juillet for the Heisenberg group  $\mathbb{H}_{2d+1}$  to a larger class of structures, which admit non-trivial abnormal minimizing curves.

The number  $k + 3$  coincides with the geodesic dimension of the Carnot group, which we define here for a general metric space. We discuss some of its properties, and its relation with the curvature exponent (the least  $N$  such that the MCP( $0, N$ ) is satisfied). We prove that, on a metric measure space, the curvature exponent is always larger than the geodesic dimension which, in turn, is larger than the Hausdorff one. When applied to Carnot groups, this result improves a previous lower bound due to Rifford.

As a byproduct, we prove that a Carnot group is ideal if and only if it is fat.

## 1. SUMMARY OF THE RESULTS

Let  $(X, d, \mu)$  be a metric measure space which is assumed to be geodesic with negligible cut locus. In particular, for any  $x \in X$  there exists a negligible set  $\mathcal{C}(x)$  and a measurable map  $\Phi^x : X \setminus \mathcal{C}(x) \times [0, 1] \rightarrow X$ , such that the curve  $\gamma(t) = \Phi^x(y, t)$  is the unique minimizing geodesic joining  $x$  with  $y$ . Moreover  $\mu$  is a Borel measure such that  $0 < \mu(\mathcal{B}(x, r)) < +\infty$  for any  $r > 0$ , where  $\mathcal{B}(x, r)$  is the metric ball of radius  $r$  centered in  $x$ . Any complete Riemannian manifold, equipped with the Riemannian measure, provides an example.

For any set  $\Omega$ , we consider its geodesic homothety of center  $x \in X$  and ratio  $t \in [0, 1]$ :

$$(1) \quad \Omega_t := \{\Phi^x(y, t) \mid y \in X \setminus \mathcal{C}(x)\}.$$

For any  $K \in \mathbb{R}$ , define the function

$$(2) \quad s_K(t) := \begin{cases} (1/\sqrt{K}) \sin(\sqrt{K}t) & \text{if } K > 0, \\ t & \text{if } K = 0, \\ (1/\sqrt{-K}) \sinh(\sqrt{-K}t) & \text{if } K < 0. \end{cases}$$

**Definition 1** (Ohta [15]). Let  $K \in \mathbb{R}$  and  $N > 1$ , or  $K \leq 0$  and  $N = 1$ . We say that  $(X, d, \mu)$  satisfies the *measure contraction property* MCP( $K, N$ ) if for any  $x \in M$  and any measurable set  $\Omega$  with  $0 < \mu(\Omega) < +\infty$  (and with  $\Omega \subset \mathcal{B}(x, \pi\sqrt{N-1}/K)$  if  $K > 0$ )

$$(3) \quad \mu(\Omega_t) \geq \int_{\Omega} t \left[ \frac{s_K(td(x, z)/\sqrt{N-1})}{s_K(d(x, z)/\sqrt{N-1})} \right]^{N-1} d\mu(z), \quad \forall t \in [0, 1],$$

where we set  $0/0 = 1$  and the term in square bracket is 1 if  $K \leq 0$  and  $N = 1$ .

In this setting, the measure contraction property is a global control on the evolution of the measure of  $\Omega_t$ . The function  $s_K$  comes from the exact behavior of the Jacobian determinant of the exponential map on Riemannian space forms of constant curvature  $K$  and dimension  $N$ , where (3) is an equality. On a complete Riemannian manifold  $M$  equipped with the Riemannian measure, the MCP( $K, N$ ) is equivalent to  $\text{Ric} \geq K$  and

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$\dim M \leq N$  (see [15]). Thus, the measure contraction property is a synthetic replacement for Ricci curvature bounds on more general metric measure spaces, and is actually one of the weakest. It has been introduced independently by Ohta [15] and Sturm [20]. See also [19, 20, 12] for other (stronger) synthetic curvature conditions, including the popular geometric curvature dimension condition  $\text{CD}(K, N)$ . An important property, shared by all these synthetic conditions, is their stability under (pointed) Gromov-Hausdorff limits.

It is interesting to investigate whether the synthetic theory of curvature bounds can be applied to the study of sub-Riemannian manifolds. These are an interesting class of metric spaces, that generalize Riemannian geometry with non-holonomic constraints. Even though some sub-Riemannian structures can be seen as Gromov-Hausdorff limits of sequences of Riemannian ones with the same dimension, these turn out to have Ricci curvature unbounded from below [16, 3]. For this reason a direct analysis is demanded.

In this paper we focus on Carnot groups. In the following, any Carnot group  $G$  is considered as a metric measure space  $(G, d, \mu)$  equipped with the Carnot-Carathéodory distance  $d$  and a left-invariant measure  $\mu$ . The latter coincides with the Popp [14, 7] and with the Hausdorff one [1], up to a constant rescaling. All of them coincide with the Lebesgue measure when we identify  $G \simeq \mathbb{R}^n$  in a set of adapted coordinates.

**1.1. The Heisenberg group.** In [9], Juillet proved that the  $2d + 1$  dimensional Heisenberg group  $\mathbb{H}_{2d+1}$  does not satisfy the  $\text{CD}(K, N)$  condition, for any value of  $K$  and  $N$ . On the other hand, it satisfies the  $\text{MCP}(K, N)$  if and only if  $K \leq 0$  and  $N \geq 2d + 3$ .

The number  $\mathcal{N} = 2d + 3$ , which is the lowest possible dimension for the synthetic condition  $\text{MCP}(0, N)$  in  $\mathbb{H}_{2d+1}$ , is surprisingly larger than its topological dimension ( $2d + 1$ ) or the Hausdorff one ( $2d + 2$ ). This is essentially due to the fact that, letting  $\Omega = \mathcal{B}(x, 1)$ , we have  $\Omega_t \subset \mathcal{B}(x, t)$  strictly, and

$$(4) \quad \mu(\Omega_t) \sim t^{2d+3}, \quad \text{while} \quad \mu(\mathcal{B}(x, t)) \sim t^{2d+2},$$

for  $t \rightarrow 0^+$  (see [9, Remark 2.7]).

**1.2. Corank 1 Carnot groups.** Our first result is an extension of the MCP results of [9] to any corank 1 Carnot group.

**Theorem 2.** *Let  $(G, d, \mu)$  be a corank 1 Carnot group of rank  $k$ . Then it satisfies the  $\text{MCP}(K, N)$  if and only if  $K \leq 0$  and  $N \geq k + 3$ .*

*Remark 1.* We stress that, in general, corank 1 Carnot groups admit abnormal minimizing curves (albeit not strictly abnormal ones).

**1.3. The geodesic dimension.** The geodesic dimension has been introduced in [3] for sub-Riemannian structures. We present it here in a different and more general setting.

**Definition 3.** Let  $(X, d, \mu)$  be a metric measure space. For any  $x \in X$  and  $s > 0$ , define

$$(5) \quad C_s(x) := \sup \left\{ \limsup_{t \rightarrow 0^+} \frac{1}{t^s} \frac{\mu(\Omega_t)}{\mu(\Omega)} \mid \Omega \text{ measurable, bounded, } 0 < \mu(\Omega) < +\infty \right\},$$

where  $\Omega_t$  is the homothety of  $\Omega$  with center  $x$  and ratio  $t$  as in (1). We define the *geodesic dimension* of  $(X, d, \mu)$  at  $x \in X$  as the non-negative real number

$$(6) \quad \mathcal{N}(x) := \inf\{s > 0 \mid C_s(x) = +\infty\} = \sup\{s > 0 \mid C_s(x) = 0\},$$

with the conventions  $\inf \emptyset = +\infty$  and  $\sup \emptyset = 0$ .

Roughly speaking, the measure of  $\mu(\Omega_t)$  vanishes at least as  $t^{\mathcal{N}(x)}$  or more rapidly, for  $t \rightarrow 0$ . The two definitions in (6) are equivalent since  $s \geq s'$  implies  $C_s(x) \geq C_{s'}(x)$ .

*Remark 2.* Indeed  $\mathcal{N}(x)$  does not change if we replace  $\mu$  with any commensurable measure (two measures  $\mu$  and  $\mu'$  are commensurable if  $\mu \ll \mu'$  and  $\mu' \ll \mu$ ).

The geodesic dimension  $\mathcal{N}(x)$  is a local property. In fact, for sufficiently small  $t > 0$ , the set  $\Omega_t$  lies in an arbitrarily small neighborhood of  $x$ . The next theorem puts it in relation with Hausdorff dimension  $\dim_H(B)$  of a subset  $B \subseteq X$  (see [5] for reference).

**Theorem 4.** *Let  $(X, d, \mu)$  be a metric measure space. Then, for any Borel subset  $B$*

$$(7) \quad \sup\{\mathcal{N}(x) \mid x \in B\} \geq \dim_H(B).$$

The next result appears in [3, Proposition 5.49], and we give here a self-contained proof.

**Theorem 5.** *Let  $(X, d, \mu)$  be the metric measure space defined by an equiregular sub-Riemannian or Riemannian structure, equipped with a smooth measure  $\mu$ , then*

$$(8) \quad \mathcal{N}(x) \geq \dim_H(X) \geq \dim(X), \quad \forall x \in X,$$

and the equality holds if and only if  $(X, d, \mu)$  is Riemannian.

*Remark 3.* For an equiregular (sub-)Riemannian structure, the Hausdorff measure is commensurable w.r.t. any smooth one [13]. This is no longer true in the non-equiregular case [8]. By choosing the Hausdorff measure instead of a smooth one, one obtains, a priori, a different geodesic dimension  $\mathcal{N}(x)$ .

**1.4. A lower bound for the MCP dimension.** If  $(X, d, \mu)$  satisfies the  $\text{MCP}(K, N)$ , then  $N \geq \mathcal{N}(x)$  at any point. We give here a general statement for metric measure spaces.

**Theorem 6.** *Let  $(X, d, \mu)$  be a metric measure space, with geodesic dimension  $\mathcal{N}(x)$ , that satisfies the  $\text{MCP}(K, N)$ , for some  $K \in \mathbb{R}$  and  $N \in \mathbb{N}$  or  $K \leq 0$  and  $N = 1$ . Then*

$$(9) \quad N \geq \sup\{\mathcal{N}(x) \mid x \in X\}.$$

The following definition was given originally in [16] for Carnot groups.

**Definition 7.** Let  $(X, d, \mu)$  be a metric measure space that satisfies the  $\text{MCP}(0, N)$  for some  $N \geq 1$ . Its *curvature exponent* is

$$(10) \quad N_0 := \inf\{N > 1 \mid \text{MCP}(0, N) \text{ is satisfied}\}.$$

When  $(X, d, \mu)$  does not satisfy the  $\text{MCP}(0, N)$  for all  $N \geq 1$ , we set  $N_0 = +\infty$ .

Indeed, if  $N_0 < +\infty$ , then the  $\text{MCP}(0, N_0)$  is satisfied. Theorem 6 implies that  $N_0 \geq \mathcal{N}$ . It may happen that  $N_0 > \mathcal{N}$  strictly, as in the following example.

*Example 1 (Riemannian Heisenberg).* Consider the Riemannian structure generated by the following global orthonormal vector fields, in coordinates  $(x, y, z) \in \mathbb{R}^3$ :

$$(11) \quad X = \partial_x - \frac{y}{2}\partial_z, \quad Y = \partial_y + \frac{x}{2}\partial_z, \quad Z = \partial_z.$$

Being a Riemannian structure,  $\mathcal{N} = 3$ . In [16] it is proved that, when equipped with the Riemannian volume, it satisfies the  $\text{MCP}(0, 5)$ . With the same computations it is easy to prove that the  $\text{MCP}(0, 5 - \varepsilon)$  is violated for any  $\varepsilon > 0$ , so its curvature exponent is  $N_0 = 5$ .

**1.5. Back to Carnot groups.** More recently, in [16], Rifford studied the measure contraction properties of general Carnot groups. It may happen that  $N_0 = +\infty$ , that is the  $\text{MCP}(0, N)$  is never satisfied. However, if the Carnot group is ideal (i.e. it does not admit non-trivial abnormal minimizing geodesics), we have the following result.

**Theorem 8 (Rifford [16]).** *Let  $(G, d, \mu)$  be a Carnot group. Assume it is ideal. Then it satisfies the  $\text{MCP}(0, N)$  for some  $N > 1$ . In particular its curvature exponent  $N_0 < +\infty$ .*

The proof of the above result is based on a semiconcavity property of the distance for ideal structures, which does not hold in general. Nevertheless, Theorem 2 shows that the above statement can hold even in presence of non-trivial abnormal minimizers. In general, nothing is known on the finiteness of  $N_0$ , but we have the following lower bound.

**Theorem 9** (Rifford [16]). *Let  $(G, d, \mu)$  be a Carnot group. Assume it is geodesic with negligible cut locus. Then its curvature exponent  $N_0$  satisfies*

$$(12) \quad N_0 \geq N_R := Q + n - k,$$

where  $Q$  is the Hausdorff dimension,  $n$  is the topological one, and  $k$  is the rank of the horizontal distribution.

For Carnot groups, the geodesic dimension  $\mathcal{N}(x) = \mathcal{N}$  is clearly constant. In particular  $N_0 \geq \mathcal{N}$ , by Theorem 6. This lower bound improves (12), as a consequence of the following.

**Theorem 10.** *A Carnot group is ideal if and only if it is fat<sup>1</sup>. In this case,  $\mathcal{N} = N_R$ . If a Carnot group is not fat, then*

$$(13) \quad \mathcal{N} > N_R.$$

*Remark 4.* Since fat Carnot groups do not admit non-trivial abnormal curves, the first part of Theorem 10 can be restated as follows: a Carnot group admits a non-trivial abnormal curve if and only if it admits a non-trivial abnormal minimizer (see Section 2).

*Example 2* (Engel group). Consider the Carnot group in dimension 4, generated by the following global orthonormal left-invariant vector fields in coordinates  $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$

$$(14) \quad X_1 = \partial_1, \quad X_2 = \partial_2 + x_1\partial_3 + x_1x_2\partial_4.$$

The Engel group is a metric space with negligible cut locus. It has rank 2, step 3, dimension 4 and growth vector  $(2, 3, 4)$ . Its Hausdorff dimension is  $Q = 7$ . The geodesic dimension is  $\mathcal{N} = 10$ , while  $N_R = 9$ . This is the lowest dimensional Carnot group where  $N_R > \mathcal{N}$ .

Checking whether the Engel group satisfies the MCP(0,  $\mathcal{N}$ ) should be possible, at least in principle, as expressions for the Jacobian determinant are known [6].

**1.6. Open problems.** As a consequence of the formula for  $\mathcal{N}(x)$  in the (sub-)Riemannian setting (see Section 7), for any corank 1 Carnot group we have

$$(15) \quad \mathcal{N} = k + 3.$$

Thus, Theorem 2 can be restated saying that *for any corank 1 Carnot group, the curvature exponent is equal to the geodesic dimension*. Moreover, for  $\mathbb{H}_{2d+1}$ , this gives  $\mathcal{N} = 2d + 3$ , and coincides with the ‘‘mysterious’’ integer originally found by Juillet.

We do not know whether non-ideal Carnot groups enjoy some MCP(0,  $N$ ). It is not even known whether general Carnot groups have negligible cut locus (this is related with the Sard conjecture in sub-Riemannian geometry [18, 10]). However, if they do, it is natural to expect the curvature exponent to be equal to the curvature dimension.

**Conjecture.** *Let  $(X, d, \mu)$  be a Carnot group. Assume that it has negligible cut locus. Then the geodesic dimension coincides with the curvature exponent.*

Preliminary results (using sub-Riemannian curvature techniques, in collaboration with D. Barilari) seem to provide evidence to the above claim for some step 2 Carnot group (these are Lipschitz and, in particular, have negligible cut locus [17]).

**Structure of the paper.** In Section 2 we collect some preliminaries of sub-Riemannian geometry and Carnot groups. In Section 3 we characterize the minimizers of corank 1 Carnot groups. In Section 4, 5, 6 we prove Theorems 2, 4, 6 respectively. In Section 7 we recall the formula for the geodesic dimension on general sub-Riemannian structures, we prove Theorem 5 and we discuss the Engel example. In Section 8 we prove Theorem 10.

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<sup>1</sup>A sub-Riemannian structure  $(M, \mathcal{D}, g)$  is *fat* (or *strong bracket-generating*) if for all  $x \in M$  and  $X \in \mathcal{D}$ ,  $X(x) \neq 0$ , then  $\mathcal{D}_x + [X, \mathcal{D}]_x = T_x M$ . It is *ideal* if it does not admit non-trivial abnormal minimizers.

## 2. SUB-RIEMANNIAN GEOMETRY

We present some basic results in sub-Riemannian geometry. See [2, 17, 14] for reference.

**2.1. Basic definitions.** A sub-Riemannian manifold is a triple  $(M, \mathcal{D}, g)$ , where  $M$  is a smooth, connected manifold of dimension  $n \geq 3$ ,  $\mathcal{D}$  is a vector distribution of constant rank  $k \leq n$  and  $g$  is a smooth metric on  $\mathcal{D}$ . We always assume that the distribution is bracket-generating. A *horizontal curve*  $\gamma : [0, 1] \rightarrow M$  is a Lipschitz continuous path such that  $\dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)}$  for almost any  $t$ . Horizontal curves have a well defined *length*

$$(16) \quad \ell(\gamma) = \int_0^1 \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))} dt.$$

The *sub-Riemannian (or Carnot-Carathéodory) distance* is defined by:

$$(17) \quad d(x, y) = \inf\{\ell(\gamma) \mid \gamma(0) = x, \gamma(1) = y, \gamma \text{ horizontal}\}.$$

By the Chow-Rashevskii theorem, under the bracket-generating condition,  $d : M \times M \rightarrow \mathbb{R}$  is finite and continuous. A sub-Riemannian manifold is complete if  $(M, d)$  is complete as a metric space. In this case, for any  $x, y \in M$  there exists a minimizing geodesic joining the two points. In place of the length  $\ell$ , one can consider the *energy functional* as

$$(18) \quad J(\gamma) = \frac{1}{2} \int_0^1 g(\dot{\gamma}(t), \dot{\gamma}(t)) dt.$$

It is well known that, on the space of horizontal curves with fixed endpoints, the minimizers of  $J(\cdot)$  coincide with the minimizers of  $\ell(\cdot)$  with constant speed. Since  $\ell$  is invariant by reparametrization (and in particular we can always reparametrize horizontal curves in such a way that they have constant speed), we do not lose generality in defining *geodesics* as horizontal curves that are locally energy minimizers between their endpoints.

**2.2. Hamiltonian.** We define the *Hamiltonian function*  $H : T^*M \rightarrow \mathbb{R}$  as

$$(19) \quad H(\lambda) = \frac{1}{2} \sum_{i=1}^k \langle \lambda, X_i \rangle, \quad \lambda \in T^*M,$$

for any local orthonormal frame  $X_1, \dots, X_k$  for  $\mathcal{D}$ . Here  $\langle \lambda, \cdot \rangle$  denotes the dual action of covectors on vectors. The cotangent bundle  $\pi : T^*M \rightarrow M$  is equipped with a natural symplectic form  $\sigma$ . The Hamiltonian vector field  $\vec{H}$  is the unique vector field such that  $\sigma(\cdot, \vec{H}) = dH$ . In particular, the *Hamilton equations* are

$$(20) \quad \dot{\lambda}(t) = \vec{H}(\lambda(t)), \quad \lambda(t) \in T^*M.$$

If  $(M, d)$  is complete, any solution of (20) can be extended to a smooth curve for all times.

**2.3. End-point map.** Let  $\gamma_u : [0, 1] \rightarrow M$  be an horizontal curve joining  $x$  and  $y$ . Up to restriction and reparametrization, we assume that the curve has no self-intersections. Thus we can find a smooth orthonormal frame  $X_1, \dots, X_k$  of horizontal vector fields, defined in a neighborhood of  $\gamma_u$ . Moreover, there is a *control*  $u \in L^\infty([0, 1], \mathbb{R}^k)$  such that

$$(21) \quad \dot{\gamma}_u(t) = \sum_{i=1}^k u_i(t) X_i(\gamma_u(t)), \quad \text{a.e. } t \in [0, 1].$$

Let  $\mathcal{U} \subset L^\infty([0, 1], \mathbb{R}^k)$  be the open set such that, for  $v \in \mathcal{U}$ , the solution of

$$(22) \quad \dot{\gamma}_v(t) = \sum_{i=1}^k v_i(t) X_i(\gamma_v(t)), \quad \gamma_v(0) = x,$$

is well defined for a.e.  $t \in [0, 1]$ . Indeed  $u \in \mathcal{U}$ . We define the *end-point map* with base  $x$  as  $E_x : \mathcal{U} \rightarrow M$  that sends  $v \mapsto \gamma_v(1)$ . The end-point map is smooth on  $\mathcal{U}$ .

**2.4. Lagrange multipliers.** We can see  $J : \mathcal{U} \rightarrow \mathbb{R}$  as a smooth functional on  $\mathcal{U}$  (we are identifying  $\mathcal{U}$  with a neighborhood of  $\gamma_u$  in the space of horizontal curves starting from  $x$ ). A minimizing geodesic  $\gamma_u$  is a solution of the constrained minimum problem

$$(23) \quad J(v) \rightarrow \min, \quad E_x(v) = y, \quad v \in \mathcal{U}.$$

By the Lagrange multipliers rule, there exists a non-trivial pair  $(\lambda_1, \nu)$ , such that

$$(24) \quad \lambda_1 \circ D_u E_x = \nu D_u J, \quad \lambda_1 \in T_y^* M, \quad \nu \in \{0, 1\},$$

where  $\circ$  denotes the composition and  $D$  the (Fréchet) differential. If  $\gamma_u : [0, 1] \rightarrow M$  with control  $u \in \mathcal{U}$  is an horizontal curve (not necessarily minimizing), we say that a non-zero pair  $(\lambda_1, \nu) \in T_y^* M \times \{0, 1\}$  is a *Lagrange multiplier* for  $\gamma_u$  if (24) is satisfied. The multiplier  $(\lambda_1, \nu)$  is called *normal* if  $\nu = 1$  and *abnormal* if  $\nu = 0$ . Observe that Lagrange multipliers are not unique, and a horizontal curve may be both normal *and* abnormal. Observe also that  $(\lambda_1, \nu)$  is an abnormal multiplier if and only if  $u$  is a critical point of  $E_x$ . The following characterization is a consequence of the Pontryagin Maximum Principle [4].

**Theorem 11.** *Let  $\gamma_u : [0, 1] \rightarrow M$  be an horizontal curve joining  $x$  with  $y$ . A non-zero pair  $(\lambda_1, \nu) \in T_y^* M \times \{0, 1\}$  is a Lagrange multiplier for  $\gamma_u$  if and only if there exists a Lipschitz curve  $\lambda(t) \in T_{\gamma_u(t)}^* M$  with  $\lambda(1) = \lambda_1$ , such that*

- if  $\nu = 1$  then  $\dot{\lambda}(t) = \vec{H}(\gamma(t))$ , i.e. it is a solution of Hamilton equations,
- if  $\nu = 0$  then  $\sigma(\dot{\lambda}(t), T_{\lambda(t)} \mathcal{D}^\perp) = 0$ ,

where  $\mathcal{D}^\perp \subset T^* M$  is the sub-bundle of covectors that annihilate the distribution.

In the first (resp. second) case,  $\lambda(t)$  is called a *normal* (resp. *abnormal*) *extremal*. Normal extremals are precisely the integral curves  $\lambda(t)$  of  $\vec{H}$ . As such, they are smooth, and are characterized by their *initial covector*  $\lambda = \lambda(0)$ . A geodesic is normal (resp. abnormal) if admits a normal (resp. abnormal) extremal. On the other hand, it is well known that the projection  $\gamma_\lambda(t) = \pi(\lambda(t))$  of a normal extremal is locally minimizing, hence it is a normal geodesic. The *exponential map* with base  $x \in M$  is the map

$$(25) \quad \exp_x : T_x^* M \rightarrow M,$$

which assigns to  $\lambda \in T_x^* M$  the final point  $\pi(\lambda(1))$  of the corresponding normal geodesic. The curve  $\gamma_\lambda(t) := \exp_x(t\lambda)$ , for  $t \in [0, 1]$ , is the normal geodesic corresponding to  $\lambda$ , which has constant speed  $\|\dot{\gamma}_\lambda(t)\| = \sqrt{2H(\lambda)}$  and length  $\ell(\gamma|_{[t_1, t_2]}) = \sqrt{2H(\lambda)}(t_2 - t_1)$ .

**2.5. Carnot groups.** A *Carnot group*  $(G, \star)$  of step  $s$  is a connected, simply connected Lie group of dimension  $n$ , such that its Lie algebra  $\mathfrak{g} = T_e G$  is stratified of step  $s$ , that is

$$(26) \quad \mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_s,$$

with

$$(27) \quad [\mathfrak{g}_1, \mathfrak{g}_j] = \mathfrak{g}_{1+j}, \quad \forall 1 \leq j \leq s, \quad \mathfrak{g}_s \neq \{0\}, \quad \mathfrak{g}_{s+1} = \{0\}.$$

The *group exponential map*  $\exp_G : \mathfrak{g} \rightarrow G$  associates with  $V \in \mathfrak{g}$  the element  $\gamma_V(1)$ , where  $\gamma_V : [0, 1] \rightarrow G$  is the integral line, starting at  $\gamma_V(0) = e$ , of the left invariant vector field associated with  $V$ . Since  $G$  is simply connected and  $\mathfrak{g}$  is nilpotent,  $\exp_G$  is a smooth diffeomorphism. Thus, by choosing a basis of  $\mathfrak{g}$ , we identify  $G \simeq \mathbb{R}^n$ .

Let  $\mathcal{D}$  be the left-invariant distribution generated by  $\mathfrak{g}_1$ , with a left-invariant scalar product  $g$ . This defines a sub-Riemannian structure  $(G, \mathcal{D}, g)$  on the Carnot group. For  $x \in G$ , we denote with  $L_x(y) := x \star y$  the left translation. The map  $L_x : G \rightarrow G$  is a smooth isometry. Any Carnot group is a metric measure space  $(X, d, \mu)$  where  $d$  is the Carnot-Carathéodory distance and  $\mu$  the Lebesgue measure of  $G = \mathbb{R}^n$ . Haar, Popp, Lebesgue and top-dimensional Hausdorff measures are all left-invariant and proportional.

3. CORANK 1 CARNOT GROUPS

A Corank 1 Carnot group is a Carnot groups of step  $s = 2$ , with  $\dim \mathfrak{g}_1 = k$  and  $\dim \mathfrak{g}_2 = 1$ . In exponential coordinates  $(x, z)$  on  $\mathbb{R}^k \times \mathbb{R}$ , they are generated by the following set of global orthonormal left-invariant frames

$$(28) \quad X_i = \partial_{x_i} - \frac{1}{2} \sum_{j=1}^k A_{ij} x_j \partial_z, \quad i = 1, \dots, k,$$

where  $A$  is a  $k \times k$  skew symmetric matrix. Observe that

$$(29) \quad [X_i, X_j] = A_{ij} \partial_z, \quad i, j = 1, \dots, k.$$

Let  $0 < \alpha_1 \leq \dots \leq \alpha_d$  be the non-zero singular values of  $A$ . In particular,  $\dim \ker A = k - 2d$ . Up to an orthogonal change of coordinates, we can assume that

$$(30) \quad A = \begin{pmatrix} \alpha_1 J & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \alpha_d J \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The first zero block has dimension  $k - 2d$ , while each other diagonal block is  $2 \times 2$ . We split the coordinate  $x = (x^0, x^1, \dots, x^d)$ , where  $x^0 \in \mathbb{R}^{k-2d}$  and  $x^i \in \mathbb{R}^2$ , for  $i = 1, \dots, d$ .

If  $A$  has trivial kernel (in particular,  $k$  is even), we are in the case of a *contact Carnot group*, and there are no non-trivial minimizers. However, when  $A$  has a non-trivial kernel, then non-trivial abnormal minimizers appear. To prove Theorem 6, we need a complete characterization of the minimizing geodesics on a general corank 1 Carnot group. Our study extends the results of [1], where the case of a non-degenerate  $A$  is considered.

**3.1. Characterization of minimizers.** On any corank 1 sub-Riemannian distribution, all minimizing geodesics are normal (this is true for any step 2 distribution). In particular, they can be recovered by solving Hamilton equations. By left-invariance, it is sufficient to consider geodesics starting from the identity  $e = (0, 0)$ . Any covector  $\lambda \in T_e^*G$  has coordinates  $(p_x, p_z)$ , where we split  $p_x = (p_x^0, p_x^1, \dots, p_x^d)$ .

**Lemma 12.** *The exponential map  $\exp_e : T_e^*G \rightarrow G$  of a Corank 1 Carnot group is*

$$(31) \quad \exp_e(p_x^0, p_x^1, \dots, p_x^d, p_z) = (x^0, x^1, \dots, x^d, z),$$

where, for all  $i = 1, \dots, d$  we have

$$(32) \quad x^0 = p_x^0,$$

$$(33) \quad x^i = \left( \frac{\sin(\alpha_i p_z)}{\alpha_i p_z} I + \frac{\cos(\alpha_i p_z) - 1}{\alpha_i p_z} J \right) p_x^i,$$

$$(34) \quad z = \sum_{i=1}^d \|p_x^i\|^2 \frac{\alpha_i p_z - \sin(\alpha_i p_z)}{2\alpha_i p_z^2}.$$

If  $p_z = 0$ , one must consider the limit  $p_z \rightarrow 0$ , that is  $\exp_e(p_x, 0) = (p_x, 0)$ .

*Remark 5* (Abnormal geodesics). A non-zero covector  $\lambda = (p_x, p_z)$  such that  $A p_x = 0$ , that is of the form  $(p_x^0, 0, \dots, 0, p_z)$  corresponds to an abnormal geodesic. A way to see this is to observe that there is an infinite number of initial covectors giving the same geodesic

$$(35) \quad \exp_e(tp_x^0, 0, \dots, 0, tp_z) = (tp_x^0, 0, \dots, 0, 0), \quad \forall p_z \in \mathbb{R}.$$

A direct analysis of the end-point map shows that abnormal geodesic are all of this type.



*Proof.* Let  $h_x = (h_1, \dots, h_k) : T^*G \rightarrow \mathbb{R}^k$  and  $h_z : T^*G \rightarrow \mathbb{R}$ , where  $h_i(\lambda) := \langle \lambda, X_i \rangle$ , for  $i = 1, \dots, k$  and  $h_z(\lambda) := \langle \lambda, \partial_z \rangle$ . Indeed  $H = \frac{1}{2} \|h_x\|^2$ . Hamilton equations are

$$(36) \quad \dot{h}_z = 0, \quad \dot{h}_x = -h_z A h_x, \quad \dot{x} = h_x, \quad \dot{z} = -\frac{1}{2} h_x^* A x,$$

where, without risk of confusion, the dot denotes the derivative w.r.t.  $t$ . Indeed we have

$$(37) \quad h_z(t) = p_z, \quad h_x(t) = e^{-p_z A t} p_x.$$

The equations for  $(x, z)$  can be easily integrated, using the block-diagonal structure of  $A$ . Split  $h_x = (h_x^0, h_x^1, \dots, h_x^d)$ , with  $h_x^0 \in \mathbb{R}^{k-2d}$  and  $h_x^i \in \mathbb{R}^2$  for  $i = 1, \dots, d$ . We obtain

$$(38) \quad h_x^0(t) = p_x^0, \quad h_x^i(t) = [\cos(\alpha_i p_z t) I - \sin(\alpha_i p_z t) J] p_x^i,$$

where  $I$  is the  $2 \times 2$  identity matrix. Integrating the above on  $[0, t]$ , we obtain

$$(39) \quad x^0(t) = p_x^0 t, \quad x^i(t) = \left( \frac{\sin(\alpha_i p_z t)}{\alpha_i p_z} I + \frac{\cos(\alpha_i p_z t) - 1}{\alpha_i p_z} J \right) p_x^i.$$

Finally, for the coordinate  $z$  we obtain

$$(40) \quad \begin{aligned} z &= -\frac{1}{2} \int_0^1 h_x^*(s) A x(s) ds = -\frac{1}{2} \sum_{i=1}^d \int_0^1 h_x^i(s)^* \alpha_i J x^i(s) ds \\ &= \frac{1}{2 p_z} \sum_{i=1}^d \|p_x^i\|^2 \int_0^1 (1 - \cos(\alpha_i p_z s)) ds = \sum_{i=1}^d \|p_x^i\|^2 \left( \frac{\alpha_i p_z - \sin(\alpha_i p_z)}{2 \alpha_i p_z^2} \right). \quad \square \end{aligned}$$

**Lemma 13.** *The Jacobian determinant of the exponential map is*

$$(41) \quad J(p_x, p_z) = \frac{2^{2d}}{\alpha^2 p_z^{2d+2}} \sum_{i=1}^d \|p_x^i\|^2 \prod_{j \neq i} \sin\left(\frac{\alpha_j p_z}{2}\right) \sin\left(\frac{\alpha_i p_z}{2}\right) \left( \sin\left(\frac{\alpha_i p_z}{2}\right) - \frac{\alpha_i p_z}{2} \cos\left(\frac{\alpha_i p_z}{2}\right) \right),$$

where  $\alpha = \prod_{i=1}^d \alpha_i$  is the product of the non-zero singular values of  $A$ . If  $p_z = 0$ , the formula must be taken in the limit  $p_z \rightarrow 0$ . In particular  $J(p_x, 0) = \frac{1}{12} \sum_{i=1}^d \|p_x^i\|^2 \alpha_i^2$ .

*Proof.* For any matrix with the following block structure

$$(42) \quad M = \begin{pmatrix} B & v \\ w^* & \theta \end{pmatrix},$$

where the only constraint is that  $\theta \in \mathbb{R}$  is a one-dimensional block, we have

$$(43) \quad \det(M) = \theta \det(B) - v^* \operatorname{cof}(B) w,$$

where  $\operatorname{cof}$  denotes the matrix of cofactors. More in general, let

$$(44) \quad M = \begin{pmatrix} B_0 & & & v_0 \\ & B_1 & & v_1 \\ & & \ddots & \vdots \\ & & & B_d & v_d \\ w_0^* & w_1^* & \dots & w_d^* & \theta \end{pmatrix},$$

where  $B_0, \dots, B_d$  are square blocks of arbitrary (possibly different) dimension,  $\theta \in \mathbb{R}$  and  $v_i, w_j$  are column vectors of the appropriate dimension. In this case we have

$$(45) \quad \det(M) = \theta \prod_{i=0}^d \det B_i - \sum_{i=0}^d \left( \prod_{j \neq i} \det B_j \right) v_i^* \operatorname{cof}(B_i) w_i.$$

If  $B_i = a_i I + b_i J$ , then  $\operatorname{cof}(B_i) = B_i$ . If we also assume that  $B_0 = 1$ ,  $v_0 = w_0 = 0$ , we have

$$(46) \quad \det(M) = \theta \prod_{i=1}^d \det B_i - \sum_{i=1}^d \left( \prod_{j \neq i} \det B_j \right) v_i^* B_i w_i.$$

From Lemma 12, the differential of the exponential map has the above form, with

$$(47) \quad B_0 = \frac{\partial x^0}{\partial p_x^0} = 1, \quad B_i = \frac{\partial x^i}{\partial p_x^i} = \frac{\sin(\alpha_i p_z)}{\alpha_i p_z} I + \frac{\cos(\alpha_i p_z) - 1}{\alpha_i p_z} J,$$

$$(48) \quad v_i = \frac{\partial x^i}{\partial p_z} = \frac{\alpha_i p_z \cos(\alpha_i p_z) - \sin(\alpha_i p_z)}{\alpha_i p_z^2} I p_x^i + \frac{1 - \cos(\alpha_i p_z) - \alpha_i p_z \sin(\alpha_i p_z)}{\alpha_i p_z^2} J p_x^i,$$

$$(49) \quad w_i = \frac{\partial z}{\partial p_x^i} = \frac{\alpha_i p_z - \sin(\alpha_i p_z)}{\alpha_i p_z^2} p_x^i,$$

$$(50) \quad \theta = \frac{\partial z}{\partial p_z} = \sum_{i=1}^d \|p_x^i\|^2 \left( \frac{2 \sin(\alpha_i p_z) - \alpha_i p_z - \alpha_i p_z \cos(\alpha_i p_z)}{2 \alpha_i p_z^3} \right).$$

The result follows applying formula (46) and observing that, for  $i = 1, \dots, d$ , we have

$$(51) \quad \det(B_i) = \frac{4 \sin^2(\alpha_i p_z / 2)}{(\alpha_i p_z)^2},$$

$$v_i^* B_i w_i = \frac{\alpha_i^2 \|p_x^i\|^2}{(\alpha_i p_z)^5} (\alpha_i p_z - \sin(\alpha_i p_z)) (\alpha_i p_z \sin(\alpha_i p_z) + 2 \cos(\alpha_i p_z) - 2). \quad \square$$

**Lemma 14** (Characterization of the cotangent injectivity domain). *Consider the set*

$$(52) \quad D := \left\{ \lambda = (p_x, p_z) \in T_e^* G \text{ such that } |p_z| < \frac{2\pi}{\alpha_d} \text{ and } A p_x \neq 0 \right\} \subset T_e^* G.$$

Then  $\exp_e : D \rightarrow \exp_e(D)$  is a smooth diffeomorphism and  $\mathcal{C}(e) := G \setminus \exp_e(D)$  is a closed set with zero measure.

*Proof.* Since all geodesic are normal and  $(G, d)$  is complete, each point of  $G$  is reached by at least one minimizing normal geodesic  $\gamma_\lambda : [0, 1] \rightarrow G$ , with  $\lambda = (p_x, p_z) \in T_e^* G$ . If  $|p_z| > 2\pi/\alpha_d$  and  $A p_x \neq 0$ , then  $\gamma_\lambda$  is a strictly normal geodesic (i.e. not abnormal) with a conjugate time at  $t_* = 2\pi/\alpha_d |p_z| < 1$ . Strictly normal geodesics lose optimality after their first conjugate time (see [2]), hence  $\gamma_\lambda(t)$  it is not minimizing on  $[0, 1]$ . On the other hand, if  $A p_x = 0$ , for any value of  $p_z$  we obtain the same abnormal geodesic (see Remark 5). It follows that  $\exp_e : \bar{D} \rightarrow G$  is onto (the bar denotes the closure). Thus,  $\exp_e : D \rightarrow \exp_e(D)$  is onto and  $\mathcal{C}(e) = G \setminus \exp_e(D) = \exp_e(\partial D)$  has zero measure.

We now prove that, if  $\lambda \in D$ , then  $\gamma_\lambda : [0, 1] \rightarrow G$  is the unique geodesic joining its endpoints. In fact, assume that there are two covectors  $\lambda = (p_x, p_z)$  and  $\bar{\lambda} = (\bar{p}_x, \bar{p}_z) \in D$ , such that  $\exp_e(\lambda) = \exp_e(\bar{\lambda})$ . Since the two geodesics have the same length,  $\|\lambda\| = \ell(\gamma_\lambda) = \ell(\gamma_{\bar{\lambda}}) = \|\bar{\lambda}\|$ . Using Lemma 12, we have  $p_x^0 = \bar{p}_x^0$  and

$$(53) \quad \|x^i\|^2 = 4 \|p_x^i\|^2 \operatorname{sinc}(\alpha_i p_z)^2 = 4 \|\bar{p}_x^i\|^2 \operatorname{sinc}(\alpha_i \bar{p}_z)^2, \quad \forall i = 1, \dots, d,$$

where  $\operatorname{sinc}(w) = \sin(w)/w$  is positive and strictly decreasing on  $[0, \pi)$ . Since  $A p_x, A \bar{p}_x \neq 0$ , there exist two non-empty set of indices  $I, \bar{I} \subset \{1, \dots, d\}$  such that, for  $i \in I$  (resp.  $\bar{I}$ ) we have  $\|p_x^i\|^2 \neq 0$  (resp.  $\|\bar{p}_x^i\|^2 \neq 0$ ). Since  $\alpha_i p_z, \alpha_i \bar{p}_z < \pi$ , by (53), we have  $I = \bar{I}$ .

Assume now that  $\bar{p}_z > p_z$ . Then by (53)  $\|\bar{p}_x^i\|^2 > \|p_x^i\|^2$  for all  $i \in I$ . In particular

$$(54) \quad \|\bar{p}_x\|^2 = \|\bar{p}_x^0\|^2 + \sum_{i \in I} \|\bar{p}_x^i\|^2 > \|p_x^0\|^2 + \sum_{i \in I} \|p_x^i\|^2 = \|p_x\|^2,$$

which is absurd. Analogously if  $\bar{p}_z < p_z$ , with reversed inequalities. Thus  $p_z = \bar{p}_z$ . Using now the equations for the coordinate  $x^i$  of Lemma 12 we observe that

$$(55) \quad \left[ \frac{\sin(\alpha_i p_z)}{\alpha_i p_z} I + \frac{\cos(\alpha_i p_z) - 1}{\alpha_i p_z} J \right] (\bar{p}_x^i - p_x^i) = 0, \quad \forall i = 1, \dots, d.$$

The  $2 \times 2$  matrix on the l.h.s. is invertible (since if  $\alpha_i p_z < \pi$ ), hence also  $\bar{p}_x = p_x$ . Thus  $\exp_e : D \rightarrow \exp_e(D)$  is invertible.

Finally, no point  $\lambda \in D$  can be critical for  $\exp_e$ . In fact, from Lemma 13 we have that  $J(p_x, p_z) = \sum_{i=1}^d \|p_x^i\|^2 f_i(p_z)$ , where each  $f_i(p_z) > 0$  for  $p_z < 2\pi/\alpha_d$ . In particular  $J(p_x, p_z) = 0$  if and only if  $A p_x = 0$ . But this closed set was excluded from  $D$ .  $\square$

**Corollary 15.** *For any  $x \in G$ , let  $\mathcal{C}(x) := L_x \mathcal{C}(e)$ , where  $L_x : G \rightarrow G$  is the left-translation. There exists a measurable map  $\Phi^x : G \setminus \mathcal{C}(x) \times [0, 1] \rightarrow G$ , given by*

$$(56) \quad \Phi^x(y, t) = L_x \exp_e(t \exp_e^{-1}(L_x^{-1}y)),$$

such that  $\Phi^x(y, t)$  is the unique minimizing geodesic joining  $x$  with  $y$ .

The next key lemma and its proof are a simplified version of the original concavity argument of Juillet for the Heisenberg group [9, Lemma 2.6].

**Lemma 16.** *Let  $g(x) := \sin(x) - x \cos(x)$ . Then, for all  $x \in (0, \pi)$  and  $t \in [0, 1]$ ,*

$$(57) \quad g(tx) \geq t^N g(x), \quad \forall N \geq 3.$$

*Proof.* The condition  $N \geq 3$  is necessary, as  $g(x) = x^3/3 + O(x^4)$ . It is sufficient to prove the statement for  $N = 3$ . The cases  $t = 0$  and  $t = 1$  are trivial, hence we assume  $t \in (0, 1)$ . By Gronwall's Lemma the above statement is equivalent to the differential inequality

$$(58) \quad g'(s) \leq 3g(s)/s, \quad s \in (tx, x).$$

We need to prove the above inequality for all  $s \in (0, \pi)$ , which is equivalent to

$$(59) \quad f(s) := (3 - s^2) \sin(s) - 3s \cos(s) \geq 0, \quad s \in (0, \pi).$$

To prove it, we observe that  $f(0) = 0$  and  $f'(s) = s(\sin(s) - s \cos(s)) \geq 0$  on  $(0, \pi)$ .  $\square$

**Corollary 17.** *For all  $(p_x, p_z) \in D$ , we have the following inequality*

$$(60) \quad \frac{J(tp_x, tp_z)}{J(p_x, p_z)} \geq t^2, \quad \forall t \in [0, 1].$$

*Proof.* Apply Lemma 16 to the explicit expression of  $J$  from Lemma 13, and the standard inequality  $\sin(tx) \geq t \sin(x)$  for all  $x \in [0, \pi]$  and  $t \in [0, 1]$ .  $\square$

#### 4. PROOF OF THEOREM 2

The proof combines the arguments of [9] and the computation of the Jacobian determinant of [1] for contact Carnot groups, extended here to the general corank 1 case.

**4.1. Step 1.** We first prove that the MCP(0,  $N$ ) holds for  $N \geq k + 3$ . By left-translation, it is sufficient to prove the inequality (3) for the homothety with center equal to the identity  $e = (0, 0)$ . Let  $\Omega$  be a measurable set with  $0 < \mu(\Omega) < +\infty$ .

By Lemma 14, up to removing a set of zero measure,  $\Omega = \exp_e(A)$  for some  $A \subset D \subset T_e^*G$ . On the other hand, by Corollary 15, we have

$$(61) \quad \Omega_t = \exp_e(tA), \quad \forall t \in [0, 1],$$

where  $tA$  denotes the set obtained by multiplying by  $t$  any point of the set  $A \subset T_e^*G$  (an Euclidean homothety). Thus, for all  $t \in [0, 1]$  we have

$$(62) \quad \mu(\Omega_t) = \int_{\Omega_t} d\mu = \int_{tA} J(p_x, p_z) dp_x dp_z$$

$$(63) \quad = t^{k+1} \int_A J(tp_x, tp_z) dp_x dp_z \geq t^{k+3} \int_A J(p_x, p_z) dp_x dp_z = t^{k+3} \mu(\Omega),$$

where we used Corollary 17. In particular  $\mu(\Omega_t) \geq t^N \mu(\Omega)$  for all  $N \geq k + 3$ .

4.2. **Step 2.** Fix  $\varepsilon > 0$ . We prove that the MCP(0,  $k + 3 - \varepsilon$ ) does not hold. Let  $\lambda = (p_x, 0) \in D$ . By Lemma 13, and recalling that  $J > 0$  on  $D$ , we have

$$(64) \quad J(tp_x, 0) = t^2 J(p_x, 0) < t^{2-\varepsilon} J(p_x, 0), \quad \forall t \in [0, 1].$$

By continuity of  $J$  and compactness of  $[0, 1]$ , we find an open neighborhood  $A \subset D$  of  $\lambda$  such that  $J(t\lambda) < t^{2-\varepsilon} J(\lambda)$ , for all  $t \in [0, 1]$ . In particular, for  $\Omega = \exp_e(A)$ , we obtain

$$(65) \quad \mu(\Omega_t) < t^{k+3-\varepsilon} \mu(\Omega), \quad t \in [0, 1].$$

4.3. **Step 3.** To prove that MCP( $K, N$ ) does not hold for  $K > 0$  and any  $N > 1$ , we observe that spaces verifying this condition are bounded, while  $G \simeq \mathbb{R}^n$  clearly is not. Finally, assume that  $(G, d, \mu)$  satisfies MCP( $K, N$ ) for some  $K < 0$  and  $N < k + 3$ . Then the scaled space  $(G, \varepsilon^{-1}d, \varepsilon^{-Q}\mu)$  (where  $\varepsilon > 0$  and  $Q = k + 2$  is the Hausdorff dimension of  $(G, d)$ ) verifies MCP( $\varepsilon^2 K, N$ ) for [15, Lemma 2.4]. But the two spaces  $(G, d, \mu)$  and  $(G, \varepsilon^{-1}d, \varepsilon^{-Q}\mu)$  are isometric through the dilation  $\delta_\varepsilon(x, z) := (\varepsilon x, \varepsilon^2 z)$ . In particular  $(G, d, \mu)$  satisfies the MCP( $\varepsilon K, N$ ) for all  $\varepsilon > 0$ , that is

$$(66) \quad \mu(\Omega_t) \geq \int_\Omega t \left[ \frac{s_{\varepsilon K}(td(x, z)/\sqrt{N-1})}{s_{\varepsilon K}(d(x, z)/\sqrt{N-1})} \right]^{N-1} d\mu(z), \quad \forall t \in [0, 1].$$

Taking the limit for  $\varepsilon \rightarrow 0^+$ , we obtain that  $(G, d, \mu)$  satisfies the MCP(0,  $N$ ) with  $N < k + 3$ , but this is false by the previous step (these are the same arguments of [9]).  $\square$

## 5. PROOF OF THEOREM 4

Assume that  $B$  is bounded. In particular,  $\mu(B) < +\infty$ . For any  $k > 0$  let  $\mathcal{H}^k$  denote the  $k$ -dimensional Hausdorff measure on  $(X, d)$ . Let  $k < \dim_H(B)$ , then  $\mathcal{H}^k(B) = +\infty$ . By [5, Theorem 2.4.3], there exists an  $x \in B$  such that

$$(67) \quad \limsup_{t \rightarrow 0^+} \frac{\mu(\mathcal{B}(x, t))}{t^k} < +\infty.$$

Let  $\Omega$  be a bounded measurable set with  $0 < \mu(\Omega) < +\infty$ , and let  $\Omega_t$  be its homothety with center  $x$ . Indeed  $\Omega \subset \mathcal{B}(x, R)$  for some  $R > 0$ , and  $\Omega_t \subseteq \mathcal{B}(x, tR)$ . In particular

$$(68) \quad \limsup_{t \rightarrow 0^+} \frac{1}{t^{k-\varepsilon}} \frac{\mu(\Omega_t)}{\mu(\Omega)} \leq \limsup_{t \rightarrow 0^+} \frac{1}{t^{k-\varepsilon}} \frac{\mu(\mathcal{B}(x, tR))}{\mu(\Omega)} = 0, \quad \forall \varepsilon > 0.$$

Since this holds for any  $\Omega$ , we have  $C_{k-\varepsilon}(x) = 0$  for all  $k < \dim_H(B)$  and  $\varepsilon > 0$ . By definition of geodesic dimension  $\mathcal{N}(x) = \sup\{s > 0 \mid C_s(x) = 0\} \geq k$ . In other words, for any  $k < \dim_H(B)$  we have found  $x \in B$  such that  $\mathcal{N}(x) \geq k$ , that implies the statement.

If  $B$  is not bounded, consider the increasing sequence of bounded sets  $B_j := B \cap \mathcal{B}(x, j)$ , with  $j \in \mathbb{N}$ , and observe that  $\dim_H(B_j)$  is a non-decreasing sequence for  $j \rightarrow \infty$ .  $\square$

## 6. PROOF OF THEOREM 6

By contradiction, assume that  $N < \sup\{\mathcal{N}(x) \mid x \in M\}$ . In particular there exists  $x \in X$  such that  $\mathcal{N}(x) > N$ . Let  $\Omega \subset X$  be a bounded, measurable set such that  $0 < \mu(\Omega) < +\infty$ , and with  $\Omega \subset \mathcal{B}(x, \pi\sqrt{N-1}/K)$  if  $K > 0$ . By the MCP( $K, N$ ) we have

$$(69) \quad \frac{\mu(\Omega_t)}{\mu(\Omega)} \geq \frac{1}{\mu(\Omega)} \int_\Omega t \left[ \frac{s_K(td(x, z)/\sqrt{N-1})}{s_K(d(x, z)/\sqrt{N-1})} \right]^{N-1} d\mu(z), \quad \forall t \in [0, 1].$$

Indeed  $\Omega \subset \mathcal{B}(x, R\sqrt{N-1})$  for some sufficiently large  $R$  (with  $R < \pi/\sqrt{K}$  if  $K > 0$ ). Consider the functions  $s_K(t\delta)/s_K(\delta)$ , for  $\delta \in (0, R)$ . By explicit inspection using (53) we find a constant  $A_{K,R} > 0$  (independent on  $\delta$ ) such that

$$(70) \quad s_K(t\delta)/s_K(\delta) \geq A_{K,R}t, \quad \forall t \in [0, 1], \quad \forall \delta \in (0, R).$$

Thus we have

$$(71) \quad \frac{\mu(\Omega_t)}{\mu(\Omega)} \geq \frac{1}{\mu(\Omega)} \int_{\Omega} A_{K,R}^{N-1} t^N d\mu(z) = A_{K,R}^{N-1} t^N, \quad \forall t \in [0, 1].$$

Let  $\mathcal{N}(x) - N = 2\varepsilon > 0$ . We have

$$(72) \quad \limsup_{t \rightarrow 0^+} \frac{1}{t^{\mathcal{N}(x)-\varepsilon}} \frac{\mu(\Omega_t)}{\mu(\Omega)} \geq \lim_{t \rightarrow 0^+} \frac{A_{K,R}^{N-1}}{t^{\mathcal{N}(x)-\varepsilon-N}} = \lim_{t \rightarrow 0^+} \frac{A_{K,R}^{N-1}}{t^\varepsilon} = +\infty.$$

In particular  $C_{\mathcal{N}(x)-\varepsilon}(x) = +\infty$ , where  $C_s(x)$  is defined in (5). This is absurd since  $\mathcal{N}(x) = \inf\{s > 0 \mid C_s(x) = +\infty\}$ .  $\square$

## 7. FORMULA FOR THE GEODESIC DIMENSION

We recall some of the results of [3] and we prove that the definition of geodesic dimension given in this paper coincides with the one of [3] for sub-Riemannian structures.

**7.1. Flag of the distribution and Hausdorff dimension.** Let  $(M, \mathcal{D}, g)$  be a fixed (sub-)Riemannian structure. The *flag of the distribution* at  $x \in M$  is the filtration of vector subspaces  $\mathcal{D}_x^1 \subseteq \mathcal{D}_x^2 \subseteq \dots \subseteq T_x M$  defined as

$$(73) \quad \mathcal{D}_x^1 := \mathcal{D}_x, \quad \mathcal{D}_x^{i+1} := \mathcal{D}_x^i + [\mathcal{D}, \mathcal{D}_x^i],$$

where  $[\mathcal{D}, \mathcal{D}_x^i]$  is the vector space generated by the iterated Lie brackets, up to length  $i+1$ , of local sections of  $\mathcal{D}$ , evaluated at  $x$ . We denote with  $s_x$  the *step of the distribution* at  $x$ , that is the smallest (finite) integer such that  $\mathcal{D}_x^{s_x} = T_x M$ .

We say that  $\mathcal{D}$  is *equiregular* if  $\dim \mathcal{D}_x^i$  are constant for all  $i \geq 0$ . In this case the step is constant and equal to  $s$ . The *growth vector of the distribution* is

$$(74) \quad (d_1, \dots, d_s), \quad d_i := \dim \mathcal{D}_x^i.$$

**Theorem 18** (Mitchell [13]). *Let  $(M, \mathcal{D}, g)$  an equiregular (sub-)Riemannian structure. Then its Hausdorff dimension is given by the following formula:*

$$(75) \quad \dim_H(M) = \sum_{i=1}^s i(d_i - d_{i-1}), \quad d_0 := 0.$$

**7.2. Flag of the geodesic and geodesic dimension.** Let  $\gamma_\lambda : [0, \varepsilon) \rightarrow M$  be a normal geodesic, with initial covector  $\lambda$ , and  $x = \gamma(0)$ . Let  $T \in \Gamma(\mathcal{D})$  any horizontal extension of  $\gamma$ , that is  $T(\gamma(t)) = \dot{\gamma}(t)$  for all  $t \in [0, \varepsilon)$ . The *flag of the geodesic* is the filtration of vector subspaces  $\mathcal{F}_\lambda^1 \subseteq \mathcal{F}_\lambda^2 \subseteq \dots \subseteq T_x M$  defined by

$$(76) \quad \mathcal{F}_\lambda^i := \text{span}\{\mathcal{L}_T^j(X)|_x \mid X \in \Gamma(\mathcal{D}), \quad j \leq i-1\} \subseteq T_x M, \quad i \geq 1,$$

where  $\mathcal{L}$  denotes the Lie derivative. By [3, Section 3.4], this definition does not depend on the choice of the extension  $T$ , but only on the germ of  $\gamma(t)$  at  $t=0$ . In particular, it depends only on the initial covector  $\lambda \in T_x^* M$ . We define the *geodesic growth vector* as

$$(77) \quad \mathcal{G}_\lambda := (k_1, \dots, k_i, \dots), \quad k_i := \dim \mathcal{F}_\lambda^i.$$

We say that  $\gamma_\lambda$  is *ample* (at  $t=0$ ) if there is a smallest integer  $m \geq 1$  such that  $\mathcal{F}_\lambda^m = T_x M$ . In this case the growth vector is constant after its  $m$ -th entry, and  $m$  is called the *geodesic step*. Different initial covectors may give different growth vectors (possibly associated with non-ample geodesics when  $\gamma$  is abnormal). The *maximal geodesic growth vector* at  $x$  is

$$(78) \quad \mathcal{G}_x^{max} := (k_1^{max}, \dots, k_i^{max}, \dots), \quad k_i^{max} := \max\{\dim \mathcal{F}_\lambda^i \mid \lambda \in T_x^* M\}.$$

**Theorem 19.** *The set  $\mathcal{A}_x \subset T_x^* M$  of initial covectors such that the corresponding geodesic is ample, and its growth vector is maximal is an open, non-empty Zariski subset.*

In particular, the generic normal geodesic starting at  $x$  has maximal growth vector and the minimal step  $m(x)$ . For a fixed  $x \in M$ , consider the following number:

$$(79) \quad \mathcal{N}(x) = \sum_{i=1}^{m(x)} (2i-1)(k_i^{max} - k_{i-1}^{max}), \quad k_0^{max} := 0.$$

**Theorem 20.** *Let  $(M, \mathcal{D}, g)$  be a sub-Riemannian manifold, equipped with a smooth measure. Assume that, as a metric measure space  $(M, d, \mu)$ , has negligible cut locus. Let  $x \in M$ , and let  $\Omega$  be any measurable, bounded subset with  $0 < \mu(\Omega) < +\infty$ . Then there exists a constant  $C(\Omega) > 0$  such that*

$$(80) \quad \mu(\Omega_t) \sim C(\Omega)t^{\mathcal{N}(x)}, \quad t \rightarrow 0^+.$$

Equation (79) is the *definition* of geodesic dimension given in [3]. As a consequence of Theorem 20, it coincides with the one given in this paper, when specified to sub-Riemannian structures. In this case, to compute  $\mathcal{N}(x)$ , it is sufficient to compute the growth vector for the generic geodesic, and use (79). Theorems 19, 20 are also proved in [3], and are based on a deep relation between the geodesic growth vector and the asymptotics of the exponential map on a general sub-Riemannian manifold.

**7.3. Proof of Theorem 5.** If  $(M, \mathcal{D}, g)$  is Riemannian, for any point  $x \in M$  we have  $\mathcal{G}_x = (\dim(M))$  for any non-trivial initial covector and  $\mathcal{N}(x) = \dim(M) = \dim_H(M)$ .

If  $(M, \mathcal{D}, g)$  is sub-Riemannian (with  $k = \text{rank } \mathcal{D} < n$ ), and equiregular of step  $s$ , let  $q_i := d_i - d_{i-1}$ , for  $i = 1, \dots, s$  and  $p_i := k_i^{max} - k_{i-1}^{max}$ , for all  $i = 1, \dots, m(x)$ . Indeed  $m(x) \geq s \geq 2$ . By Mitchell's formula (75), and (79), we have

$$(81) \quad \dim_H(M) = \underbrace{1 + \dots + 1}_{q_1} + \underbrace{2 + \dots + 2}_{q_2} + \dots + \underbrace{s + \dots + s}_{q_s},$$

$$(82) \quad \mathcal{N}(x) = \underbrace{1 + \dots + 1}_{p_1} + \underbrace{3 + \dots + 3}_{p_2} + \dots + \underbrace{2m(x) - 1 + \dots + 2m(x) - 1}_{p_{m(x)}}.$$

Both sums have  $n$  terms and  $q_1 = p_1 = k < n$ . The terms of (82), after the  $k$ -th, are strictly greater than the ones of (81). Thus  $\mathcal{N}(x) > \dim_H(M) > \dim(M)$ .  $\square$

**7.4. The Engel group.** We discuss more in detail the Engel group introduced in (Example 2). This is the Carnot group, in dimension  $n = 4$ , generated by the following global orthonormal left-invariant vector fields in coordinates  $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$

$$(83) \quad X_1 = \partial_1, \quad X_2 = \partial_2 + x_1\partial_3 + x_1x_2\partial_4.$$

Indeed it is a rank 2 Carnot group of step 3, with  $\mathfrak{g}_1 = \text{span}\{X_1, X_2\}$  and

$$(84) \quad \mathfrak{g}_2 = [X_1, X_2] = \partial_3 + x_2\partial_4, \quad \mathfrak{g}_3 = [X_2, [X_1, X_2]] = \partial_4,$$

where we omit the linear span. In particular, by left-invariance,  $\mathcal{D}^1 = \mathfrak{g}_1$ ,  $\mathcal{D}^2 = \mathfrak{g}_1 \oplus \mathfrak{g}_2$  and  $\mathcal{D}^3 = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$ . The growth vector of the distribution is  $(2, 3, 4)$ . By Mitchell's formula (75) for the Hausdorff dimension we have  $Q = 2 + 2 + 3 = 7$ .

Let us compute the geodesic growth vector. It is sufficient to choose the curve  $\gamma(t) = e^{tX_2}(e)$  (this is a normal geodesic, by Lemma 21). Using the definition, we obtain  $\mathcal{F}^1 = \text{span}\{X_1, X_2\}$  and, omitting the linear span,

$$(85) \quad \mathcal{F}^2 = [X_2, X_1] = \partial_3 + x_2\partial_4, \quad \mathcal{F}^3 = [X_2, [X_2, X_1]] = \partial_4.$$

Indeed this gives the maximal possible geodesic growth vector, hence

$$(86) \quad \mathcal{G}_x^{max} = (2, 3, 4), \quad \forall x \in G.$$

In particular, using (79) we obtain  $\mathcal{N} = 2 + 3 + 5 = 10$ .

## 8. PROOF OF THEOREM 10

Let  $(G, \mathcal{D}, g)$  be a Carnot group of step  $s$  and dimension  $n$ . We identify  $\mathfrak{g} = T_e G$  (and its subspaces) with the vector space of left-invariant vector fields. In particular  $\mathcal{D} = \mathfrak{g}_1$ .

**8.1. Step 1: Estimates in the fat case.** We remind that a sub-Riemannian structure  $(M, \mathcal{D}, g)$  is *fat* (or *strong bracket-generating*) if for any  $x \in M$  and  $X \in \mathcal{D}$ ,  $X(x) \neq 0$ , then  $\mathcal{D}_x + [X, \mathcal{D}]_x = T_x M$ . It is well known that fat structures do not admit non-trivial abnormal minimizers [2, 14, 17]. If  $G$  is a fat Carnot group of rank  $k$ , then the geodesic growth vector of any geodesic is

$$(87) \quad \mathcal{G}_x^{max} = (k, n), \quad \forall x \in G.$$

By (79) we have  $\mathcal{N} = k + 3(n - k) = 3n - 2k$ . On the other hand, the Hausdorff dimension is  $Q = k + 2(n - k) = 2n - k$ . Moreover, from (12), we have  $N_R = Q + n - k = 3n - 2k$ . This proves that on a fat Carnot group  $\mathcal{N} = N_R$ .

To prove the inequality  $\mathcal{N} > N_R$  when  $G$  is not fat, let  $\mathcal{G}^{max} = (k_1, k_2, \dots, k_m)$  be the maximal geodesic growth vector, with geodesic step  $m \geq s \geq 2$ . Let  $q_i := d_i - d_{i-1}$ , for  $i = 1, \dots, s$  and  $p_i := k_i - k_{i-1}$ , for  $i = 1, \dots, m$ . Since  $d_0 = k_0 = 0$  by convention and  $d_s = k_m = n$ , we have  $n = \sum_{i=1}^s q_i = \sum_{i=1}^m p_i$ . Thus, from (79), we obtain

$$(88) \quad \mathcal{N} = \sum_{i=1}^m (2i - 1)p_i = \sum_{i=2}^m (2i - 2)p_i + n.$$

On the other hand, for  $N_R = Q + n - k$  and using Mitchell's formula (75), we obtain

$$(89) \quad N_R = \sum_{i=1}^s i q_i + n - k = \sum_{i=2}^s i q_i + n.$$

Arranging the terms as we did in the proof of Theorem 5, we write

$$(90) \quad \mathcal{N} - n = \underbrace{2 + \dots + 2}_{p_2} + \underbrace{4 + \dots + 4}_{p_3} + \dots + \underbrace{2m - 2 + \dots + 2m - 2}_{p_m},$$

$$(91) \quad N_R - n = \underbrace{2 + \dots + 2}_{q_2} + \underbrace{3 + \dots + 3}_{q_3} + \dots + \underbrace{s + \dots + s}_{p_s}.$$

Both sums have  $n - k = \sum_{i=2}^m p_i = \sum_{i=2}^s q_i$  entries. Moreover  $m > 2$  (if  $m = 2$ , then  $\mathcal{F}^2 = \mathfrak{g}_1 \oplus [X, \mathfrak{g}_1] = T_e M$  for some  $X \in \mathfrak{g}_1$ , which is absurd if  $G$  is not fat). Then the entries of (90), after the  $p_2$ -th one, are strictly greater than the corresponding ones of (91), and  $\mathcal{N} > N_R$ .

**8.2. Step 2: Ideal = Fat.** To conclude the proof of Theorem 10, we prove that any ideal Carnot group is fat. Denote with  $\text{ad}_X : \mathfrak{g} \rightarrow \mathfrak{g}$  the linear map:

$$(92) \quad \text{ad}_X(Y) := [X, Y] = \left. \frac{d}{dt} \right|_{t=0} e_*^{-tX} Y(\gamma_X(t)),$$

where  $\gamma_X(t) = e^{tX}(x)$  is the integral curve of  $X \in \mathfrak{g}_1$  starting from  $x \in G$ .

**Lemma 21.** *Let  $\gamma_X(t) = e^{tX}(x)$  be the integral curve of the left-invariant vector field  $X \in \mathfrak{g}_1$ , starting from  $x$ . Then  $\gamma_X$  is a normal geodesic. It is also an abnormal geodesic if and only if there exists a non-zero  $\lambda \in T_e^* G$  such that*

$$(93) \quad \langle \lambda, \text{ad}_X^i(\mathfrak{g}_1) \rangle = 0, \quad \forall i = 0, \dots, s - 1.$$

*Proof.* Let  $X_1, \dots, X_k$  be a basis of left-invariant vector fields. Indeed  $X = \sum_{i=1}^k u_i X_i$  for a constant control  $u \in L^\infty([0, 1], \mathbb{R}^k)$ . By left-invariance we can set  $x = e$ . A well known formula for the differential of the end-point map [2, 17]  $D_u E_e : T_u \mathcal{U} \simeq \mathcal{U} \rightarrow T_{\gamma(1)} G$ , gives

$$(94) \quad D_u E_e(v) = \int_0^1 e_*^{(1-t)X} \sum_{i=1}^k v_i(t) X_i(\gamma(t)) dt, \quad \forall v \in L^\infty([0, 1], \mathbb{R}^k).$$

We first prove that  $\gamma_X(t)$  is a normal geodesic. Consider the covector  $\eta \in T_e G$  such that  $\langle \eta, X_i \rangle = u_i$  and  $\langle \eta, W \rangle = 0$  for all  $W \in \mathfrak{g}_2 \oplus \dots \oplus \mathfrak{g}_s$ . Then

$$(95) \quad \langle (e^{-X})^* \eta, D_u E_e(v) \rangle = \int_0^1 \sum_{i=1}^k v_i(t) \langle \eta, e_*^{-tX} X_i(\gamma(t)) \rangle dt = (u, v)_{L^2([0,1], \mathbb{R}^k)} = D_u J(v).$$

Thus  $\gamma_X(t)$  with control  $u$  satisfies the normal Lagrange multiplier rule with covector  $\eta_1 = (e^{-X})^* \eta \in T_{\gamma(1)}^* G$ , and is a normal geodesic. By definition  $\gamma_X(t)$  is also abnormal if and only if there exists a  $\lambda_1 \in T_{\gamma(1)}^* G$  such that  $\lambda_1 \circ D_u E_e = 0$ . That is, if and only if there exists  $\lambda = (e^X)^* \lambda_1 \in T_e^* G$  such that

$$(96) \quad 0 = \langle \lambda, e_*^{-X} D_u E_e(v) \rangle = \int_0^1 \sum_{i=1}^k v_i(t) \langle \lambda, e_*^{-tX} X_i(\gamma(t)) \rangle dt, \quad \forall v \in L^\infty([0,1], \mathbb{R}^k).$$

This is true if and only if  $\langle \lambda, e_*^{-tX} Y \rangle = 0$  for any  $Y \in \mathfrak{g}_1$  and  $t \in [0,1]$ . Since all the relevant data are analytic,  $t \mapsto \langle \lambda, e_*^{-tX} Y \rangle$  is an analytic function of  $t$ . Hence it vanishes if and only if all its derivatives at  $t=0$  are zero, and this condition coincides with 93.  $\square$

**Lemma 22.** *Let  $G$  be a Carnot group of step  $s \leq 2$ .  $G$  is ideal if and only if it is fat.*

*Proof.* The implication fat  $\Rightarrow$  ideal is trivial. Then, assume that  $\mathfrak{g}_1$  is not fat, i.e. there exists  $X \neq 0 \in \mathfrak{g}_1$  such that  $\mathfrak{g}_1 \oplus [X, \mathfrak{g}_1] \subsetneq T_e G$ . Hence there exists  $\lambda \neq 0 \in T_e^* G$  such that

$$(97) \quad 0 = \langle \lambda, \mathfrak{g}_1 \rangle = \langle \lambda, \text{ad}_X(\mathfrak{g}_1) \rangle.$$

By Lemma 21,  $\gamma_X(t) = e^{tX}(e)$  is a normal and abnormal geodesic. In particular, a sufficiently short segment of it is a minimizing curve.  $\square$

We learned the following fact by E. Le Donne. For the reader's convenience we provide a simple proof here, which is similar to the one in [11].

**Lemma 23.** *Let  $G$  be a Carnot group of step  $s \geq 3$ . Then there exists a non-zero  $X \in \mathfrak{g}_1$  such that the integral curve  $\gamma_X(t) = e^{tX}(e)$  is a normal and abnormal geodesic.*

*Proof.* If there exists a  $X \neq 0 \in \mathfrak{g}_1$  such that  $\text{ad}_X(\mathfrak{g}_1) \subsetneq \mathfrak{g}_2$ , using the same argument of Lemma 22, we show that  $\gamma_X(t)$  is abnormal and normal. Let  $q_i := \dim \mathfrak{g}_i$ , for  $i = 1, \dots, s$ . Then assume that, for any  $X \neq 0 \in \mathfrak{g}_1$ , we have  $\text{ad}_X(\mathfrak{g}_1) = \mathfrak{g}_2$ . This implies  $q_2 \leq q_1 - 1$ . Consider a basis  $Y_1, \dots, Y_{q_2}$  of  $\mathfrak{g}_2$  and a basis  $Z_1, \dots, Z_{q_3}$  of  $\mathfrak{g}_3$ . Let  $\lambda \in T_e^* G$  such that  $\langle \lambda, Z_i \rangle = 0$  if  $i > 1$  and  $\langle \lambda, Z_1 \rangle = 1$ , while  $\langle \lambda, \mathfrak{g}_i \rangle = 0$  for all  $i \neq 3$ . Consider the linear maps  $A_i := \lambda \circ \text{ad}_{Y_i} : \mathfrak{g}_1 \rightarrow \mathbb{R}$ , for  $i = 1, \dots, q_2$ . Indeed  $\dim \ker A_i \geq q_1 - 1$ . Moreover

$$(98) \quad \dim(\ker A_1 \cap \ker A_2) = \dim \ker A_1 + \dim \ker A_2 - \dim(\ker A_1 + \ker A_2)$$

$$(99) \quad \geq 2(q_1 - 1) - q_1 = q_1 - 2.$$

After a finite number of similar steps we arrive to

$$(100) \quad \dim(\ker A_1 \cap \dots \cap \ker A_{q_2}) \geq q_1 - q_2 \geq q_1 - (q_1 - 1) = 1.$$

Thus let  $X \neq 0 \in \ker A_1 \cap \dots \cap \ker A_{q_2}$ . We show that  $\gamma_X(t) = e^{tX}(e)$ , which is indeed a normal geodesic, verifies the abnormal characterization of Lemma 21 with covector  $\lambda$ . Since  $\text{ad}_X^i(\mathfrak{g}_1) \subseteq \mathfrak{g}_{i+1}$ , we have  $\langle \lambda, \text{ad}_X^i(\mathfrak{g}_1) \rangle = 0$  for all  $i \neq 2$  by construction of  $\lambda$ . Finally,

$$(101) \quad \langle \lambda, \text{ad}_X^2(\mathfrak{g}_1) \rangle = \langle \lambda, \text{ad}_X(\mathfrak{g}_2) \rangle = 0,$$

where in the last passage we used the fact that  $\text{ad}_X(\mathfrak{g}_1) = \mathfrak{g}_2$ , that the latter is generated by the  $Y_i$ , and the definition of  $X$ . Then  $\gamma_X(t)$  is abnormal by Lemma 21.  $\square$

To sum up, a Carnot group of step  $s \geq 3$  admits non-trivial abnormal minimizing curves, and cannot be ideal. On the other hand if  $s \leq 2$ , then ideal  $\Leftrightarrow$  fat.  $\square$



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