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MEASURE CONTRACTION PROPERTIES OF CARNOT GROUPS

RIZZI LUCA

ABSTRACT. We prove that any corank 1 Carnot group of dimension $k + 1$ equipped with a left-invariant measure satisfies the MCP(K, N) iff $K \leq 0$ and $N \geq k + 3$. This generalizes the well known result by Juillet for the Heisenberg group \mathbb{H}_{2d+1} to a larger class of structures, which admit non-trivial abnormal minimizing curves.

The number $k + 3$ coincides with the geodesic dimension of the Carnot group, which we define here for a general metric space. We discuss some of its properties, and its relation with the curvature exponent (the least N such that the MCP($0, N$) is satisfied). We prove that, on a metric measure space, the curvature exponent is always larger than the geodesic dimension which, in turn, is larger than the Hausdorff one. When applied to Carnot groups, this result improves a previous lower bound due to Rifford.

As a byproduct, we prove that a Carnot group is ideal if and only if it is fat.

1. SUMMARY OF THE RESULTS

Let (X, d, μ) be a metric measure space which is assumed to be geodesic with negligible cut locus. In particular, for any $x \in X$ there exists a negligible set $\mathcal{C}(x)$ and a measurable map $\Phi^x : X \setminus \mathcal{C}(x) \times [0, 1] \rightarrow X$, such that the curve $\gamma(t) = \Phi^x(y, t)$ is the unique minimizing geodesic joining x with y . Moreover μ is a Borel measure such that $0 < \mu(\mathcal{B}(x, r)) < +\infty$ for any $r > 0$, where $\mathcal{B}(x, r)$ is the metric ball of radius r centered in x . Any complete Riemannian manifold, equipped with the Riemannian measure, provides an example.

For any set Ω , we consider its geodesic homothety of center $x \in X$ and ratio $t \in [0, 1]$:

$$(1) \quad \Omega_t := \{\Phi^x(y, t) \mid y \in X \setminus \mathcal{C}(x)\}.$$

For any $K \in \mathbb{R}$, define the function

$$(2) \quad s_K(t) := \begin{cases} (1/\sqrt{K}) \sin(\sqrt{K}t) & \text{if } K > 0, \\ t & \text{if } K = 0, \\ (1/\sqrt{-K}) \sinh(\sqrt{-K}t) & \text{if } K < 0. \end{cases}$$

Definition 1 (Ohta [15]). Let $K \in \mathbb{R}$ and $N > 1$, or $K \leq 0$ and $N = 1$. We say that (X, d, μ) satisfies the *measure contraction property* MCP(K, N) if for any $x \in M$ and any measurable set Ω with $0 < \mu(\Omega) < +\infty$ (and with $\Omega \subset \mathcal{B}(x, \pi\sqrt{N-1}/K)$ if $K > 0$)

$$(3) \quad \mu(\Omega_t) \geq \int_{\Omega} t \left[\frac{s_K(td(x, z)/\sqrt{N-1})}{s_K(d(x, z)/\sqrt{N-1})} \right]^{N-1} d\mu(z), \quad \forall t \in [0, 1],$$

where we set $0/0 = 1$ and the term in square bracket is 1 if $K \leq 0$ and $N = 1$.

In this setting, the measure contraction property is a global control on the evolution of the measure of Ω_t . The function s_K comes from the exact behavior of the Jacobian determinant of the exponential map on Riemannian space forms of constant curvature K and dimension N , where (3) is an equality. On a complete Riemannian manifold M equipped with the Riemannian measure, the MCP(K, N) is equivalent to $\text{Ric} \geq K$ and

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$\dim M \leq N$ (see [15]). Thus, the measure contraction property is a synthetic replacement for Ricci curvature bounds on more general metric measure spaces, and is actually one of the weakest. It has been introduced independently by Ohta [15] and Sturm [20]. See also [19, 20, 12] for other (stronger) synthetic curvature conditions, including the popular geometric curvature dimension condition $\text{CD}(K, N)$. An important property, shared by all these synthetic conditions, is their stability under (pointed) Gromov-Hausdorff limits.

It is interesting to investigate whether the synthetic theory of curvature bounds can be applied to the study of sub-Riemannian manifolds. These are an interesting class of metric spaces, that generalize Riemannian geometry with non-holonomic constraints. Even though some sub-Riemannian structures can be seen as Gromov-Hausdorff limits of sequences of Riemannian ones with the same dimension, these turn out to have Ricci curvature unbounded from below [16, 3]. For this reason a direct analysis is demanded.

In this paper we focus on Carnot groups. In the following, any Carnot group G is considered as a metric measure space (G, d, μ) equipped with the Carnot-Carathéodory distance d and a left-invariant measure μ . The latter coincides with the Popp [14, 7] and with the Hausdorff one [1], up to a constant rescaling. All of them coincide with the Lebesgue measure when we identify $G \simeq \mathbb{R}^n$ in a set of adapted coordinates.

1.1. The Heisenberg group. In [9], Juillet proved that the $2d + 1$ dimensional Heisenberg group \mathbb{H}_{2d+1} does not satisfy the $\text{CD}(K, N)$ condition, for any value of K and N . On the other hand, it satisfies the $\text{MCP}(K, N)$ if and only if $K \leq 0$ and $N \geq 2d + 3$.

The number $\mathcal{N} = 2d + 3$, which is the lowest possible dimension for the synthetic condition $\text{MCP}(0, N)$ in \mathbb{H}_{2d+1} , is surprisingly larger than its topological dimension ($2d + 1$) or the Hausdorff one ($2d + 2$). This is essentially due to the fact that, letting $\Omega = \mathcal{B}(x, 1)$, we have $\Omega_t \subset \mathcal{B}(x, t)$ strictly, and

$$(4) \quad \mu(\Omega_t) \sim t^{2d+3}, \quad \text{while} \quad \mu(\mathcal{B}(x, t)) \sim t^{2d+2},$$

for $t \rightarrow 0^+$ (see [9, Remark 2.7]).

1.2. Corank 1 Carnot groups. Our first result is an extension of the MCP results of [9] to any corank 1 Carnot group.

Theorem 2. *Let (G, d, μ) be a corank 1 Carnot group of rank k . Then it satisfies the $\text{MCP}(K, N)$ if and only if $K \leq 0$ and $N \geq k + 3$.*

Remark 1. We stress that, in general, corank 1 Carnot groups admit abnormal minimizing curves (albeit not strictly abnormal ones).

1.3. The geodesic dimension. The geodesic dimension has been introduced in [3] for sub-Riemannian structures. We present it here in a different and more general setting.

Definition 3. Let (X, d, μ) be a metric measure space. For any $x \in X$ and $s > 0$, define

$$(5) \quad C_s(x) := \sup \left\{ \limsup_{t \rightarrow 0^+} \frac{1}{t^s} \frac{\mu(\Omega_t)}{\mu(\Omega)} \mid \Omega \text{ measurable, bounded, } 0 < \mu(\Omega) < +\infty \right\},$$

where Ω_t is the homothety of Ω with center x and ratio t as in (1). We define the *geodesic dimension* of (X, d, μ) at $x \in X$ as the non-negative real number

$$(6) \quad \mathcal{N}(x) := \inf\{s > 0 \mid C_s(x) = +\infty\} = \sup\{s > 0 \mid C_s(x) = 0\},$$

with the conventions $\inf \emptyset = +\infty$ and $\sup \emptyset = 0$.

Roughly speaking, the measure of $\mu(\Omega_t)$ vanishes at least as $t^{\mathcal{N}(x)}$ or more rapidly, for $t \rightarrow 0$. The two definitions in (6) are equivalent since $s \geq s'$ implies $C_s(x) \geq C_{s'}(x)$.

Remark 2. Indeed $\mathcal{N}(x)$ does not change if we replace μ with any commensurable measure (two measures μ and μ' are commensurable if $\mu \ll \mu'$ and $\mu' \ll \mu$).

The geodesic dimension $\mathcal{N}(x)$ is a local property. In fact, for sufficiently small $t > 0$, the set Ω_t lies in an arbitrarily small neighborhood of x . The next theorem puts it in relation with Hausdorff dimension $\dim_H(B)$ of a subset $B \subseteq X$ (see [5] for reference).

Theorem 4. *Let (X, d, μ) be a metric measure space. Then, for any Borel subset B*

$$(7) \quad \sup\{\mathcal{N}(x) \mid x \in B\} \geq \dim_H(B).$$

The next result appears in [3, Proposition 5.49], and we give here a self-contained proof.

Theorem 5. *Let (X, d, μ) be the metric measure space defined by an equiregular sub-Riemannian or Riemannian structure, equipped with a smooth measure μ , then*

$$(8) \quad \mathcal{N}(x) \geq \dim_H(X) \geq \dim(X), \quad \forall x \in X,$$

and the equality holds if and only if (X, d, μ) is Riemannian.

Remark 3. For an equiregular (sub-)Riemannian structure, the Hausdorff measure is commensurable w.r.t. any smooth one [13]. This is no longer true in the non-equiregular case [8]. By choosing the Hausdorff measure instead of a smooth one, one obtains, a priori, a different geodesic dimension $\mathcal{N}(x)$.

1.4. A lower bound for the MCP dimension. If (X, d, μ) satisfies the $\text{MCP}(K, N)$, then $N \geq \mathcal{N}(x)$ at any point. We give here a general statement for metric measure spaces.

Theorem 6. *Let (X, d, μ) be a metric measure space, with geodesic dimension $\mathcal{N}(x)$, that satisfies the $\text{MCP}(K, N)$, for some $K \in \mathbb{R}$ and $N \in \mathbb{N}$ or $K \leq 0$ and $N = 1$. Then*

$$(9) \quad N \geq \sup\{\mathcal{N}(x) \mid x \in X\}.$$

The following definition was given originally in [16] for Carnot groups.

Definition 7. Let (X, d, μ) be a metric measure space that satisfies the $\text{MCP}(0, N)$ for some $N \geq 1$. Its *curvature exponent* is

$$(10) \quad N_0 := \inf\{N > 1 \mid \text{MCP}(0, N) \text{ is satisfied}\}.$$

When (X, d, μ) does not satisfy the $\text{MCP}(0, N)$ for all $N \geq 1$, we set $N_0 = +\infty$.

Indeed, if $N_0 < +\infty$, then the $\text{MCP}(0, N_0)$ is satisfied. Theorem 6 implies that $N_0 \geq \mathcal{N}$. It may happen that $N_0 > \mathcal{N}$ strictly, as in the following example.

Example 1 (Riemannian Heisenberg). Consider the Riemannian structure generated by the following global orthonormal vector fields, in coordinates $(x, y, z) \in \mathbb{R}^3$:

$$(11) \quad X = \partial_x - \frac{y}{2}\partial_z, \quad Y = \partial_y + \frac{x}{2}\partial_z, \quad Z = \partial_z.$$

Being a Riemannian structure, $\mathcal{N} = 3$. In [16] it is proved that, when equipped with the Riemannian volume, it satisfies the $\text{MCP}(0, 5)$. With the same computations it is easy to prove that the $\text{MCP}(0, 5 - \varepsilon)$ is violated for any $\varepsilon > 0$, so its curvature exponent is $N_0 = 5$.

1.5. Back to Carnot groups. More recently, in [16], Rifford studied the measure contraction properties of general Carnot groups. It may happen that $N_0 = +\infty$, that is the $\text{MCP}(0, N)$ is never satisfied. However, if the Carnot group is ideal (i.e. it does not admit non-trivial abnormal minimizing geodesics), we have the following result.

Theorem 8 (Rifford [16]). *Let (G, d, μ) be a Carnot group. Assume it is ideal. Then it satisfies the $\text{MCP}(0, N)$ for some $N > 1$. In particular its curvature exponent $N_0 < +\infty$.*

The proof of the above result is based on a semiconcavity property of the distance for ideal structures, which does not hold in general. Nevertheless, Theorem 2 shows that the above statement can hold even in presence of non-trivial abnormal minimizers. In general, nothing is known on the finiteness of N_0 , but we have the following lower bound.

Theorem 9 (Rifford [16]). *Let (G, d, μ) be a Carnot group. Assume it is geodesic with negligible cut locus. Then its curvature exponent N_0 satisfies*

$$(12) \quad N_0 \geq N_R := Q + n - k,$$

where Q is the Hausdorff dimension, n is the topological one, and k is the rank of the horizontal distribution.

For Carnot groups, the geodesic dimension $\mathcal{N}(x) = \mathcal{N}$ is clearly constant. In particular $N_0 \geq \mathcal{N}$, by Theorem 6. This lower bound improves (12), as a consequence of the following.

Theorem 10. *A Carnot group is ideal if and only if it is fat¹. In this case, $\mathcal{N} = N_R$. If a Carnot group is not fat, then*

$$(13) \quad \mathcal{N} > N_R.$$

Remark 4. Since fat Carnot groups do not admit non-trivial abnormal curves, the first part of Theorem 10 can be restated as follows: a Carnot group admits a non-trivial abnormal curve if and only if it admits a non-trivial abnormal minimizer (see Section 2).

Example 2 (Engel group). Consider the Carnot group in dimension 4, generated by the following global orthonormal left-invariant vector fields in coordinates $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$

$$(14) \quad X_1 = \partial_1, \quad X_2 = \partial_2 + x_1\partial_3 + x_1x_2\partial_4.$$

The Engel group is a metric space with negligible cut locus. It has rank 2, step 3, dimension 4 and growth vector $(2, 3, 4)$. Its Hausdorff dimension is $Q = 7$. The geodesic dimension is $\mathcal{N} = 10$, while $N_R = 9$. This is the lowest dimensional Carnot group where $N_R > \mathcal{N}$.

Checking whether the Engel group satisfies the MCP(0, \mathcal{N}) should be possible, at least in principle, as expressions for the Jacobian determinant are known [6].

1.6. Open problems. As a consequence of the formula for $\mathcal{N}(x)$ in the (sub-)Riemannian setting (see Section 7), for any corank 1 Carnot group we have

$$(15) \quad \mathcal{N} = k + 3.$$

Thus, Theorem 2 can be restated saying that *for any corank 1 Carnot group, the curvature exponent is equal to the geodesic dimension*. Moreover, for \mathbb{H}_{2d+1} , this gives $\mathcal{N} = 2d + 3$, and coincides with the ‘‘mysterious’’ integer originally found by Juillet.

We do not know whether non-ideal Carnot groups enjoy some MCP(0, N). It is not even known whether general Carnot groups have negligible cut locus (this is related with the Sard conjecture in sub-Riemannian geometry [18, 10]). However, if they do, it is natural to expect the curvature exponent to be equal to the curvature dimension.

Conjecture. *Let (X, d, μ) be a Carnot group. Assume that it has negligible cut locus. Then the geodesic dimension coincides with the curvature exponent.*

Preliminary results (using sub-Riemannian curvature techniques, in collaboration with D. Barilari) seem to provide evidence to the above claim for some step 2 Carnot group (these are Lipschitz and, in particular, have negligible cut locus [17]).

Structure of the paper. In Section 2 we collect some preliminaries of sub-Riemannian geometry and Carnot groups. In Section 3 we characterize the minimizers of corank 1 Carnot groups. In Section 4, 5, 6 we prove Theorems 2, 4, 6 respectively. In Section 7 we recall the formula for the geodesic dimension on general sub-Riemannian structures, we prove Theorem 5 and we discuss the Engel example. In Section 8 we prove Theorem 10.

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¹A sub-Riemannian structure (M, \mathcal{D}, g) is *fat* (or *strong bracket-generating*) if for all $x \in M$ and $X \in \mathcal{D}$, $X(x) \neq 0$, then $\mathcal{D}_x + [X, \mathcal{D}]_x = T_x M$. It is *ideal* if it does not admit non-trivial abnormal minimizers.

2. SUB-RIEMANNIAN GEOMETRY

We present some basic results in sub-Riemannian geometry. See [2, 17, 14] for reference.

2.1. Basic definitions. A sub-Riemannian manifold is a triple (M, \mathcal{D}, g) , where M is a smooth, connected manifold of dimension $n \geq 3$, \mathcal{D} is a vector distribution of constant rank $k \leq n$ and g is a smooth metric on \mathcal{D} . We always assume that the distribution is bracket-generating. A *horizontal curve* $\gamma : [0, 1] \rightarrow M$ is a Lipschitz continuous path such that $\dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)}$ for almost any t . Horizontal curves have a well defined *length*

$$(16) \quad \ell(\gamma) = \int_0^1 \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))} dt.$$

The *sub-Riemannian (or Carnot-Carathéodory) distance* is defined by:

$$(17) \quad d(x, y) = \inf\{\ell(\gamma) \mid \gamma(0) = x, \gamma(1) = y, \gamma \text{ horizontal}\}.$$

By the Chow-Rashevskii theorem, under the bracket-generating condition, $d : M \times M \rightarrow \mathbb{R}$ is finite and continuous. A sub-Riemannian manifold is complete if (M, d) is complete as a metric space. In this case, for any $x, y \in M$ there exists a minimizing geodesic joining the two points. In place of the length ℓ , one can consider the *energy functional* as

$$(18) \quad J(\gamma) = \frac{1}{2} \int_0^1 g(\dot{\gamma}(t), \dot{\gamma}(t)) dt.$$

It is well known that, on the space of horizontal curves with fixed endpoints, the minimizers of $J(\cdot)$ coincide with the minimizers of $\ell(\cdot)$ with constant speed. Since ℓ is invariant by reparametrization (and in particular we can always reparametrize horizontal curves in such a way that they have constant speed), we do not lose generality in defining *geodesics* as horizontal curves that are locally energy minimizers between their endpoints.

2.2. Hamiltonian. We define the *Hamiltonian function* $H : T^*M \rightarrow \mathbb{R}$ as

$$(19) \quad H(\lambda) = \frac{1}{2} \sum_{i=1}^k \langle \lambda, X_i \rangle, \quad \lambda \in T^*M,$$

for any local orthonormal frame X_1, \dots, X_k for \mathcal{D} . Here $\langle \lambda, \cdot \rangle$ denotes the dual action of covectors on vectors. The cotangent bundle $\pi : T^*M \rightarrow M$ is equipped with a natural symplectic form σ . The Hamiltonian vector field \vec{H} is the unique vector field such that $\sigma(\cdot, \vec{H}) = dH$. In particular, the *Hamilton equations* are

$$(20) \quad \dot{\lambda}(t) = \vec{H}(\lambda(t)), \quad \lambda(t) \in T^*M.$$

If (M, d) is complete, any solution of (20) can be extended to a smooth curve for all times.

2.3. End-point map. Let $\gamma_u : [0, 1] \rightarrow M$ be an horizontal curve joining x and y . Up to restriction and reparametrization, we assume that the curve has no self-intersections. Thus we can find a smooth orthonormal frame X_1, \dots, X_k of horizontal vector fields, defined in a neighborhood of γ_u . Moreover, there is a *control* $u \in L^\infty([0, 1], \mathbb{R}^k)$ such that

$$(21) \quad \dot{\gamma}_u(t) = \sum_{i=1}^k u_i(t) X_i(\gamma_u(t)), \quad \text{a.e. } t \in [0, 1].$$

Let $\mathcal{U} \subset L^\infty([0, 1], \mathbb{R}^k)$ be the open set such that, for $v \in \mathcal{U}$, the solution of

$$(22) \quad \dot{\gamma}_v(t) = \sum_{i=1}^k v_i(t) X_i(\gamma_v(t)), \quad \gamma_v(0) = x,$$

is well defined for a.e. $t \in [0, 1]$. Indeed $u \in \mathcal{U}$. We define the *end-point map* with base x as $E_x : \mathcal{U} \rightarrow M$ that sends $v \mapsto \gamma_v(1)$. The end-point map is smooth on \mathcal{U} .

2.4. Lagrange multipliers. We can see $J : \mathcal{U} \rightarrow \mathbb{R}$ as a smooth functional on \mathcal{U} (we are identifying \mathcal{U} with a neighborhood of γ_u in the space of horizontal curves starting from x). A minimizing geodesic γ_u is a solution of the constrained minimum problem

$$(23) \quad J(v) \rightarrow \min, \quad E_x(v) = y, \quad v \in \mathcal{U}.$$

By the Lagrange multipliers rule, there exists a non-trivial pair (λ_1, ν) , such that

$$(24) \quad \lambda_1 \circ D_u E_x = \nu D_u J, \quad \lambda_1 \in T_y^* M, \quad \nu \in \{0, 1\},$$

where \circ denotes the composition and D the (Fréchet) differential. If $\gamma_u : [0, 1] \rightarrow M$ with control $u \in \mathcal{U}$ is an horizontal curve (not necessarily minimizing), we say that a non-zero pair $(\lambda_1, \nu) \in T_y^* M \times \{0, 1\}$ is a *Lagrange multiplier* for γ_u if (24) is satisfied. The multiplier (λ_1, ν) is called *normal* if $\nu = 1$ and *abnormal* if $\nu = 0$. Observe that Lagrange multipliers are not unique, and a horizontal curve may be both normal *and* abnormal. Observe also that (λ_1, ν) is an abnormal multiplier if and only if u is a critical point of E_x . The following characterization is a consequence of the Pontryagin Maximum Principle [4].

Theorem 11. *Let $\gamma_u : [0, 1] \rightarrow M$ be an horizontal curve joining x with y . A non-zero pair $(\lambda_1, \nu) \in T_y^* M \times \{0, 1\}$ is a Lagrange multiplier for γ_u if and only if there exists a Lipschitz curve $\lambda(t) \in T_{\gamma_u(t)}^* M$ with $\lambda(1) = \lambda_1$, such that*

- if $\nu = 1$ then $\dot{\lambda}(t) = \vec{H}(\gamma(t))$, i.e. it is a solution of Hamilton equations,
- if $\nu = 0$ then $\sigma(\dot{\lambda}(t), T_{\lambda(t)} \mathcal{D}^\perp) = 0$,

where $\mathcal{D}^\perp \subset T^* M$ is the sub-bundle of covectors that annihilate the distribution.

In the first (resp. second) case, $\lambda(t)$ is called a *normal* (resp. *abnormal*) *extremal*. Normal extremals are precisely the integral curves $\lambda(t)$ of \vec{H} . As such, they are smooth, and are characterized by their *initial covector* $\lambda = \lambda(0)$. A geodesic is normal (resp. abnormal) if admits a normal (resp. abnormal) extremal. On the other hand, it is well known that the projection $\gamma_\lambda(t) = \pi(\lambda(t))$ of a normal extremal is locally minimizing, hence it is a normal geodesic. The *exponential map* with base $x \in M$ is the map

$$(25) \quad \exp_x : T_x^* M \rightarrow M,$$

which assigns to $\lambda \in T_x^* M$ the final point $\pi(\lambda(1))$ of the corresponding normal geodesic. The curve $\gamma_\lambda(t) := \exp_x(t\lambda)$, for $t \in [0, 1]$, is the normal geodesic corresponding to λ , which has constant speed $\|\dot{\gamma}_\lambda(t)\| = \sqrt{2H(\lambda)}$ and length $\ell(\gamma|_{[t_1, t_2]}) = \sqrt{2H(\lambda)}(t_2 - t_1)$.

2.5. Carnot groups. A *Carnot group* (G, \star) of step s is a connected, simply connected Lie group of dimension n , such that its Lie algebra $\mathfrak{g} = T_e G$ is stratified of step s , that is

$$(26) \quad \mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_s,$$

with

$$(27) \quad [\mathfrak{g}_1, \mathfrak{g}_j] = \mathfrak{g}_{1+j}, \quad \forall 1 \leq j \leq s, \quad \mathfrak{g}_s \neq \{0\}, \quad \mathfrak{g}_{s+1} = \{0\}.$$

The *group exponential map* $\exp_G : \mathfrak{g} \rightarrow G$ associates with $V \in \mathfrak{g}$ the element $\gamma_V(1)$, where $\gamma_V : [0, 1] \rightarrow G$ is the integral line, starting at $\gamma_V(0) = e$, of the left invariant vector field associated with V . Since G is simply connected and \mathfrak{g} is nilpotent, \exp_G is a smooth diffeomorphism. Thus, by choosing a basis of \mathfrak{g} , we identify $G \simeq \mathbb{R}^n$.

Let \mathcal{D} be the left-invariant distribution generated by \mathfrak{g}_1 , with a left-invariant scalar product g . This defines a sub-Riemannian structure (G, \mathcal{D}, g) on the Carnot group. For $x \in G$, we denote with $L_x(y) := x \star y$ the left translation. The map $L_x : G \rightarrow G$ is a smooth isometry. Any Carnot group is a metric measure space (X, d, μ) where d is the Carnot-Carathéodory distance and μ the Lebesgue measure of $G = \mathbb{R}^n$. Haar, Popp, Lebesgue and top-dimensional Hausdorff measures are all left-invariant and proportional.

Proof. Let $h_x = (h_1, \dots, h_k) : T^*G \rightarrow \mathbb{R}^k$ and $h_z : T^*G \rightarrow \mathbb{R}$, where $h_i(\lambda) := \langle \lambda, X_i \rangle$, for $i = 1, \dots, k$ and $h_z(\lambda) := \langle \lambda, \partial_z \rangle$. Indeed $H = \frac{1}{2} \|h_x\|^2$. Hamilton equations are

$$(36) \quad \dot{h}_z = 0, \quad \dot{h}_x = -h_z A h_x, \quad \dot{x} = h_x, \quad \dot{z} = -\frac{1}{2} h_x^* A x,$$

where, without risk of confusion, the dot denotes the derivative w.r.t. t . Indeed we have

$$(37) \quad h_z(t) = p_z, \quad h_x(t) = e^{-p_z A t} p_x.$$

The equations for (x, z) can be easily integrated, using the block-diagonal structure of A . Split $h_x = (h_x^0, h_x^1, \dots, h_x^d)$, with $h_x^0 \in \mathbb{R}^{k-2d}$ and $h_x^i \in \mathbb{R}^2$ for $i = 1, \dots, d$. We obtain

$$(38) \quad h_x^0(t) = p_x^0, \quad h_x^i(t) = [\cos(\alpha_i p_z t) I - \sin(\alpha_i p_z t) J] p_x^i,$$

where I is the 2×2 identity matrix. Integrating the above on $[0, t]$, we obtain

$$(39) \quad x^0(t) = p_x^0 t, \quad x^i(t) = \left(\frac{\sin(\alpha_i p_z t)}{\alpha_i p_z} I + \frac{\cos(\alpha_i p_z t) - 1}{\alpha_i p_z} J \right) p_x^i.$$

Finally, for the coordinate z we obtain

$$(40) \quad \begin{aligned} z &= -\frac{1}{2} \int_0^1 h_x^*(s) A x(s) ds = -\frac{1}{2} \sum_{i=1}^d \int_0^1 h_x^i(s)^* \alpha_i J x^i(s) ds \\ &= \frac{1}{2 p_z} \sum_{i=1}^d \|p_x^i\|^2 \int_0^1 (1 - \cos(\alpha_i p_z s)) ds = \sum_{i=1}^d \|p_x^i\|^2 \left(\frac{\alpha_i p_z - \sin(\alpha_i p_z)}{2 \alpha_i p_z^2} \right). \quad \square \end{aligned}$$

Lemma 13. *The Jacobian determinant of the exponential map is*

$$(41) \quad J(p_x, p_z) = \frac{2^{2d}}{\alpha^2 p_z^{2d+2}} \sum_{i=1}^d \|p_x^i\|^2 \prod_{j \neq i} \sin\left(\frac{\alpha_j p_z}{2}\right) \sin\left(\frac{\alpha_i p_z}{2}\right) \left(\sin\left(\frac{\alpha_i p_z}{2}\right) - \frac{\alpha_i p_z}{2} \cos\left(\frac{\alpha_i p_z}{2}\right) \right),$$

where $\alpha = \prod_{i=1}^d \alpha_i$ is the product of the non-zero singular values of A . If $p_z = 0$, the formula must be taken in the limit $p_z \rightarrow 0$. In particular $J(p_x, 0) = \frac{1}{12} \sum_{i=1}^d \|p_x^i\|^2 \alpha_i^2$.

Proof. For any matrix with the following block structure

$$(42) \quad M = \begin{pmatrix} B & v \\ w^* & \theta \end{pmatrix},$$

where the only constraint is that $\theta \in \mathbb{R}$ is a one-dimensional block, we have

$$(43) \quad \det(M) = \theta \det(B) - v^* \operatorname{cof}(B) w,$$

where cof denotes the matrix of cofactors. More in general, let

$$(44) \quad M = \begin{pmatrix} B_0 & & & v_0 \\ & B_1 & & v_1 \\ & & \ddots & \vdots \\ & & & B_d & v_d \\ w_0^* & w_1^* & \dots & w_d^* & \theta \end{pmatrix},$$

where B_0, \dots, B_d are square blocks of arbitrary (possibly different) dimension, $\theta \in \mathbb{R}$ and v_i, w_j are column vectors of the appropriate dimension. In this case we have

$$(45) \quad \det(M) = \theta \prod_{i=0}^d \det B_i - \sum_{i=0}^d \left(\prod_{j \neq i} \det B_j \right) v_i^* \operatorname{cof}(B_i) w_i.$$

If $B_i = a_i I + b_i J$, then $\operatorname{cof}(B_i) = B_i$. If we also assume that $B_0 = 1$, $v_0 = w_0 = 0$, we have

$$(46) \quad \det(M) = \theta \prod_{i=1}^d \det B_i - \sum_{i=1}^d \left(\prod_{j \neq i} \det B_j \right) v_i^* B_i w_i.$$

From Lemma 12, the differential of the exponential map has the above form, with

$$(47) \quad B_0 = \frac{\partial x^0}{\partial p_x^0} = 1, \quad B_i = \frac{\partial x^i}{\partial p_x^i} = \frac{\sin(\alpha_i p_z)}{\alpha_i p_z} I + \frac{\cos(\alpha_i p_z) - 1}{\alpha_i p_z} J,$$

$$(48) \quad v_i = \frac{\partial x^i}{\partial p_z} = \frac{\alpha_i p_z \cos(\alpha_i p_z) - \sin(\alpha_i p_z)}{\alpha_i p_z^2} I p_x^i + \frac{1 - \cos(\alpha_i p_z) - \alpha_i p_z \sin(\alpha_i p_z)}{\alpha_i p_z^2} J p_x^i,$$

$$(49) \quad w_i = \frac{\partial z}{\partial p_x^i} = \frac{\alpha_i p_z - \sin(\alpha_i p_z)}{\alpha_i p_z^2} p_x^i,$$

$$(50) \quad \theta = \frac{\partial z}{\partial p_z} = \sum_{i=1}^d \|p_x^i\|^2 \left(\frac{2 \sin(\alpha_i p_z) - \alpha_i p_z - \alpha_i p_z \cos(\alpha_i p_z)}{2 \alpha_i p_z^3} \right).$$

The result follows applying formula (46) and observing that, for $i = 1, \dots, d$, we have

$$(51) \quad \det(B_i) = \frac{4 \sin^2(\alpha_i p_z / 2)}{(\alpha_i p_z)^2},$$

$$v_i^* B_i w_i = \frac{\alpha_i^2 \|p_x^i\|^2}{(\alpha_i p_z)^5} (\alpha_i p_z - \sin(\alpha_i p_z)) (\alpha_i p_z \sin(\alpha_i p_z) + 2 \cos(\alpha_i p_z) - 2). \quad \square$$

Lemma 14 (Characterization of the cotangent injectivity domain). *Consider the set*

$$(52) \quad D := \left\{ \lambda = (p_x, p_z) \in T_e^* G \text{ such that } |p_z| < \frac{2\pi}{\alpha_d} \text{ and } A p_x \neq 0 \right\} \subset T_e^* G.$$

Then $\exp_e : D \rightarrow \exp_e(D)$ is a smooth diffeomorphism and $\mathcal{C}(e) := G \setminus \exp_e(D)$ is a closed set with zero measure.

Proof. Since all geodesic are normal and (G, d) is complete, each point of G is reached by at least one minimizing normal geodesic $\gamma_\lambda : [0, 1] \rightarrow G$, with $\lambda = (p_x, p_z) \in T_e^* G$. If $|p_z| > 2\pi/\alpha_d$ and $A p_x \neq 0$, then γ_λ is a strictly normal geodesic (i.e. not abnormal) with a conjugate time at $t_* = 2\pi/\alpha_d |p_z| < 1$. Strictly normal geodesics lose optimality after their first conjugate time (see [2]), hence $\gamma_\lambda(t)$ it is not minimizing on $[0, 1]$. On the other hand, if $A p_x = 0$, for any value of p_z we obtain the same abnormal geodesic (see Remark 5). It follows that $\exp_e : \bar{D} \rightarrow G$ is onto (the bar denotes the closure). Thus, $\exp_e : D \rightarrow \exp_e(D)$ is onto and $\mathcal{C}(e) = G \setminus \exp_e(D) = \exp_e(\partial D)$ has zero measure.

We now prove that, if $\lambda \in D$, then $\gamma_\lambda : [0, 1] \rightarrow G$ is the unique geodesic joining its endpoints. In fact, assume that there are two covectors $\lambda = (p_x, p_z)$ and $\bar{\lambda} = (\bar{p}_x, \bar{p}_z) \in D$, such that $\exp_e(\lambda) = \exp_e(\bar{\lambda})$. Since the two geodesics have the same length, $\|\lambda\| = \ell(\gamma_\lambda) = \ell(\gamma_{\bar{\lambda}}) = \|\bar{\lambda}\|$. Using Lemma 12, we have $p_x^0 = \bar{p}_x^0$ and

$$(53) \quad \|x^i\|^2 = 4 \|p_x^i\|^2 \operatorname{sinc}(\alpha_i p_z)^2 = 4 \|\bar{p}_x^i\|^2 \operatorname{sinc}(\alpha_i \bar{p}_z)^2, \quad \forall i = 1, \dots, d,$$

where $\operatorname{sinc}(w) = \sin(w)/w$ is positive and strictly decreasing on $[0, \pi)$. Since $A p_x, A \bar{p}_x \neq 0$, there exist two non-empty set of indices $I, \bar{I} \subset \{1, \dots, d\}$ such that, for $i \in I$ (resp. \bar{I}) we have $\|p_x^i\|^2 \neq 0$ (resp. $\|\bar{p}_x^i\|^2 \neq 0$). Since $\alpha_i p_z, \alpha_i \bar{p}_z < \pi$, by (53), we have $I = \bar{I}$.

Assume now that $\bar{p}_z > p_z$. Then by (53) $\|\bar{p}_x^i\|^2 > \|p_x^i\|^2$ for all $i \in I$. In particular

$$(54) \quad \|\bar{p}_x\|^2 = \|\bar{p}_x^0\|^2 + \sum_{i \in I} \|\bar{p}_x^i\|^2 > \|p_x^0\|^2 + \sum_{i \in I} \|p_x^i\|^2 = \|p_x\|^2,$$

which is absurd. Analogously if $\bar{p}_z < p_z$, with reversed inequalities. Thus $p_z = \bar{p}_z$. Using now the equations for the coordinate x^i of Lemma 12 we observe that

$$(55) \quad \left[\frac{\sin(\alpha_i p_z)}{\alpha_i p_z} I + \frac{\cos(\alpha_i p_z) - 1}{\alpha_i p_z} J \right] (\bar{p}_x^i - p_x^i) = 0, \quad \forall i = 1, \dots, d.$$

The 2×2 matrix on the l.h.s. is invertible (since if $\alpha_i p_z < \pi$), hence also $\bar{p}_x = p_x$. Thus $\exp_e : D \rightarrow \exp_e(D)$ is invertible.

Finally, no point $\lambda \in D$ can be critical for \exp_e . In fact, from Lemma 13 we have that $J(p_x, p_z) = \sum_{i=1}^d \|p_x^i\|^2 f_i(p_z)$, where each $f_i(p_z) > 0$ for $p_z < 2\pi/\alpha_d$. In particular $J(p_x, p_z) = 0$ if and only if $Ap_x = 0$. But this closed set was excluded from D . \square

Corollary 15. *For any $x \in G$, let $\mathcal{C}(x) := L_x \mathcal{C}(e)$, where $L_x : G \rightarrow G$ is the left-translation. There exists a measurable map $\Phi^x : G \setminus \mathcal{C}(x) \times [0, 1] \rightarrow G$, given by*

$$(56) \quad \Phi^x(y, t) = L_x \exp_e(t \exp_e^{-1}(L_x^{-1}y)),$$

such that $\Phi^x(y, t)$ is the unique minimizing geodesic joining x with y .

The next key lemma and its proof are a simplified version of the original concavity argument of Juillet for the Heisenberg group [9, Lemma 2.6].

Lemma 16. *Let $g(x) := \sin(x) - x \cos(x)$. Then, for all $x \in (0, \pi)$ and $t \in [0, 1]$,*

$$(57) \quad g(tx) \geq t^N g(x), \quad \forall N \geq 3.$$

Proof. The condition $N \geq 3$ is necessary, as $g(x) = x^3/3 + O(x^4)$. It is sufficient to prove the statement for $N = 3$. The cases $t = 0$ and $t = 1$ are trivial, hence we assume $t \in (0, 1)$. By Gronwall's Lemma the above statement is equivalent to the differential inequality

$$(58) \quad g'(s) \leq 3g(s)/s, \quad s \in (tx, x).$$

We need to prove the above inequality for all $s \in (0, \pi)$, which is equivalent to

$$(59) \quad f(s) := (3 - s^2) \sin(s) - 3s \cos(s) \geq 0, \quad s \in (0, \pi).$$

To prove it, we observe that $f(0) = 0$ and $f'(s) = s(\sin(s) - s \cos(s)) \geq 0$ on $(0, \pi)$. \square

Corollary 17. *For all $(p_x, p_z) \in D$, we have the following inequality*

$$(60) \quad \frac{J(tp_x, tp_z)}{J(p_x, p_z)} \geq t^2, \quad \forall t \in [0, 1].$$

Proof. Apply Lemma 16 to the explicit expression of J from Lemma 13, and the standard inequality $\sin(tx) \geq t \sin(x)$ for all $x \in [0, \pi]$ and $t \in [0, 1]$. \square

4. PROOF OF THEOREM 2

The proof combines the arguments of [9] and the computation of the Jacobian determinant of [1] for contact Carnot groups, extended here to the general corank 1 case.

4.1. Step 1. We first prove that the MCP(0, N) holds for $N \geq k + 3$. By left-translation, it is sufficient to prove the inequality (3) for the homothety with center equal to the identity $e = (0, 0)$. Let Ω be a measurable set with $0 < \mu(\Omega) < +\infty$.

By Lemma 14, up to removing a set of zero measure, $\Omega = \exp_e(A)$ for some $A \subset D \subset T_e^*G$. On the other hand, by Corollary 15, we have

$$(61) \quad \Omega_t = \exp_e(tA), \quad \forall t \in [0, 1],$$

where tA denotes the set obtained by multiplying by t any point of the set $A \subset T_e^*G$ (an Euclidean homothety). Thus, for all $t \in [0, 1]$ we have

$$(62) \quad \mu(\Omega_t) = \int_{\Omega_t} d\mu = \int_{tA} J(p_x, p_z) dp_x dp_z$$

$$(63) \quad = t^{k+1} \int_A J(tp_x, tp_z) dp_x dp_z \geq t^{k+3} \int_A J(p_x, p_z) dp_x dp_z = t^{k+3} \mu(\Omega),$$

where we used Corollary 17. In particular $\mu(\Omega_t) \geq t^N \mu(\Omega)$ for all $N \geq k + 3$.

4.2. **Step 2.** Fix $\varepsilon > 0$. We prove that the MCP(0, $k + 3 - \varepsilon$) does not hold. Let $\lambda = (p_x, 0) \in D$. By Lemma 13, and recalling that $J > 0$ on D , we have

$$(64) \quad J(tp_x, 0) = t^2 J(p_x, 0) < t^{2-\varepsilon} J(p_x, 0), \quad \forall t \in [0, 1].$$

By continuity of J and compactness of $[0, 1]$, we find an open neighborhood $A \subset D$ of λ such that $J(t\lambda) < t^{2-\varepsilon} J(\lambda)$, for all $t \in [0, 1]$. In particular, for $\Omega = \exp_e(A)$, we obtain

$$(65) \quad \mu(\Omega_t) < t^{k+3-\varepsilon} \mu(\Omega), \quad t \in [0, 1].$$

4.3. **Step 3.** To prove that MCP(K, N) does not hold for $K > 0$ and any $N > 1$, we observe that spaces verifying this condition are bounded, while $G \simeq \mathbb{R}^n$ clearly is not. Finally, assume that (G, d, μ) satisfies MCP(K, N) for some $K < 0$ and $N < k + 3$. Then the scaled space $(G, \varepsilon^{-1}d, \varepsilon^{-Q}\mu)$ (where $\varepsilon > 0$ and $Q = k + 2$ is the Hausdorff dimension of (G, d)) verifies MCP($\varepsilon^2 K, N$) for [15, Lemma 2.4]. But the two spaces (G, d, μ) and $(G, \varepsilon^{-1}d, \varepsilon^{-Q}\mu)$ are isometric through the dilation $\delta_\varepsilon(x, z) := (\varepsilon x, \varepsilon^2 z)$. In particular (G, d, μ) satisfies the MCP($\varepsilon K, N$) for all $\varepsilon > 0$, that is

$$(66) \quad \mu(\Omega_t) \geq \int_\Omega t \left[\frac{s_{\varepsilon K}(td(x, z)/\sqrt{N-1})}{s_{\varepsilon K}(d(x, z)/\sqrt{N-1})} \right]^{N-1} d\mu(z), \quad \forall t \in [0, 1].$$

Taking the limit for $\varepsilon \rightarrow 0^+$, we obtain that (G, d, μ) satisfies the MCP(0, N) with $N < k + 3$, but this is false by the previous step (these are the same arguments of [9]). \square

5. PROOF OF THEOREM 4

Assume that B is bounded. In particular, $\mu(B) < +\infty$. For any $k > 0$ let \mathcal{H}^k denote the k -dimensional Hausdorff measure on (X, d) . Let $k < \dim_H(B)$, then $\mathcal{H}^k(B) = +\infty$. By [5, Theorem 2.4.3], there exists an $x \in B$ such that

$$(67) \quad \limsup_{t \rightarrow 0^+} \frac{\mu(\mathcal{B}(x, t))}{t^k} < +\infty.$$

Let Ω be a bounded measurable set with $0 < \mu(\Omega) < +\infty$, and let Ω_t be its homothety with center x . Indeed $\Omega \subset \mathcal{B}(x, R)$ for some $R > 0$, and $\Omega_t \subseteq \mathcal{B}(x, tR)$. In particular

$$(68) \quad \limsup_{t \rightarrow 0^+} \frac{1}{t^{k-\varepsilon}} \frac{\mu(\Omega_t)}{\mu(\Omega)} \leq \limsup_{t \rightarrow 0^+} \frac{1}{t^{k-\varepsilon}} \frac{\mu(\mathcal{B}(x, tR))}{\mu(\Omega)} = 0, \quad \forall \varepsilon > 0.$$

Since this holds for any Ω , we have $C_{k-\varepsilon}(x) = 0$ for all $k < \dim_H(B)$ and $\varepsilon > 0$. By definition of geodesic dimension $\mathcal{N}(x) = \sup\{s > 0 \mid C_s(x) = 0\} \geq k$. In other words, for any $k < \dim_H(B)$ we have found $x \in B$ such that $\mathcal{N}(x) \geq k$, that implies the statement.

If B is not bounded, consider the increasing sequence of bounded sets $B_j := B \cap \mathcal{B}(x, j)$, with $j \in \mathbb{N}$, and observe that $\dim_H(B_j)$ is a non-decreasing sequence for $j \rightarrow \infty$. \square

6. PROOF OF THEOREM 6

By contradiction, assume that $N < \sup\{\mathcal{N}(x) \mid x \in M\}$. In particular there exists $x \in X$ such that $\mathcal{N}(x) > N$. Let $\Omega \subset X$ be a bounded, measurable set such that $0 < \mu(\Omega) < +\infty$, and with $\Omega \subset \mathcal{B}(x, \pi\sqrt{N-1}/K)$ if $K > 0$. By the MCP(K, N) we have

$$(69) \quad \frac{\mu(\Omega_t)}{\mu(\Omega)} \geq \frac{1}{\mu(\Omega)} \int_\Omega t \left[\frac{s_K(td(x, z)/\sqrt{N-1})}{s_K(d(x, z)/\sqrt{N-1})} \right]^{N-1} d\mu(z), \quad \forall t \in [0, 1].$$

Indeed $\Omega \subset \mathcal{B}(x, R\sqrt{N-1})$ for some sufficiently large R (with $R < \pi/\sqrt{K}$ if $K > 0$). Consider the functions $s_K(t\delta)/s_K(\delta)$, for $\delta \in (0, R)$. By explicit inspection using (53) we find a constant $A_{K,R} > 0$ (independent on δ) such that

$$(70) \quad s_K(t\delta)/s_K(\delta) \geq A_{K,R}t, \quad \forall t \in [0, 1], \quad \forall \delta \in (0, R).$$

Thus we have

$$(71) \quad \frac{\mu(\Omega_t)}{\mu(\Omega)} \geq \frac{1}{\mu(\Omega)} \int_{\Omega} A_{K,R}^{N-1} t^N d\mu(z) = A_{K,R}^{N-1} t^N, \quad \forall t \in [0, 1].$$

Let $\mathcal{N}(x) - N = 2\varepsilon > 0$. We have

$$(72) \quad \limsup_{t \rightarrow 0^+} \frac{1}{t^{\mathcal{N}(x)-\varepsilon}} \frac{\mu(\Omega_t)}{\mu(\Omega)} \geq \lim_{t \rightarrow 0^+} \frac{A_{K,R}^{N-1}}{t^{\mathcal{N}(x)-\varepsilon-N}} = \lim_{t \rightarrow 0^+} \frac{A_{K,R}^{N-1}}{t^\varepsilon} = +\infty.$$

In particular $C_{\mathcal{N}(x)-\varepsilon}(x) = +\infty$, where $C_s(x)$ is defined in (5). This is absurd since $\mathcal{N}(x) = \inf\{s > 0 \mid C_s(x) = +\infty\}$. \square

7. FORMULA FOR THE GEODESIC DIMENSION

We recall some of the results of [3] and we prove that the definition of geodesic dimension given in this paper coincides with the one of [3] for sub-Riemannian structures.

7.1. Flag of the distribution and Hausdorff dimension. Let (M, \mathcal{D}, g) be a fixed (sub-)Riemannian structure. The *flag of the distribution* at $x \in M$ is the filtration of vector subspaces $\mathcal{D}_x^1 \subseteq \mathcal{D}_x^2 \subseteq \dots \subseteq T_x M$ defined as

$$(73) \quad \mathcal{D}_x^1 := \mathcal{D}_x, \quad \mathcal{D}_x^{i+1} := \mathcal{D}_x^i + [\mathcal{D}, \mathcal{D}^i]_x,$$

where $[\mathcal{D}, \mathcal{D}^i]_x$ is the vector space generated by the iterated Lie brackets, up to length $i+1$, of local sections of \mathcal{D} , evaluated at x . We denote with s_x the *step of the distribution* at x , that is the smallest (finite) integer such that $\mathcal{D}_x^{s_x} = T_x M$.

We say that \mathcal{D} is *equiregular* if $\dim \mathcal{D}_x^i$ are constant for all $i \geq 0$. In this case the step is constant and equal to s . The *growth vector of the distribution* is

$$(74) \quad (d_1, \dots, d_s), \quad d_i := \dim \mathcal{D}^i.$$

Theorem 18 (Mitchell [13]). *Let (M, \mathcal{D}, g) an equiregular (sub-)Riemannian structure. Then its Hausdorff dimension is given by the following formula:*

$$(75) \quad \dim_H(M) = \sum_{i=1}^s i(d_i - d_{i-1}), \quad d_0 := 0.$$

7.2. Flag of the geodesic and geodesic dimension. Let $\gamma_\lambda : [0, \varepsilon) \rightarrow M$ be a normal geodesic, with initial covector λ , and $x = \gamma(0)$. Let $T \in \Gamma(\mathcal{D})$ any horizontal extension of γ , that is $T(\gamma(t)) = \dot{\gamma}(t)$ for all $t \in [0, \varepsilon)$. The *flag of the geodesic* is the filtration of vector subspaces $\mathcal{F}_\lambda^1 \subseteq \mathcal{F}_\lambda^2 \subseteq \dots \subseteq T_x M$ defined by

$$(76) \quad \mathcal{F}_\lambda^i := \text{span}\{\mathcal{L}_T^j(X)|_x \mid X \in \Gamma(\mathcal{D}), \quad j \leq i-1\} \subseteq T_x M, \quad i \geq 1,$$

where \mathcal{L} denotes the Lie derivative. By [3, Section 3.4], this definition does not depend on the choice of the extension T , but only on the germ of $\gamma(t)$ at $t=0$. In particular, it depends only on the initial covector $\lambda \in T_x^* M$. We define the *geodesic growth vector* as

$$(77) \quad \mathcal{G}_\lambda := (k_1, \dots, k_i, \dots), \quad k_i := \dim \mathcal{F}_\lambda^i.$$

We say that γ_λ is *ample* (at $t=0$) if there is a smallest integer $m \geq 1$ such that $\mathcal{F}_\lambda^m = T_x M$. In this case the growth vector is constant after its m -th entry, and m is called the *geodesic step*. Different initial covectors may give different growth vectors (possibly associated with non-ample geodesics when γ is abnormal). The *maximal geodesic growth vector* at x is

$$(78) \quad \mathcal{G}_x^{max} := (k_1^{max}, \dots, k_i^{max}, \dots), \quad k_i^{max} := \max\{\dim \mathcal{F}_\lambda^i \mid \lambda \in T_x^* M\}.$$

Theorem 19. *The set $\mathcal{A}_x \subset T_x^* M$ of initial covectors such that the corresponding geodesic is ample, and its growth vector is maximal is an open, non-empty Zariski subset.*

In particular, the generic normal geodesic starting at x has maximal growth vector and the minimal step $m(x)$. For a fixed $x \in M$, consider the following number:

$$(79) \quad \mathcal{N}(x) = \sum_{i=1}^{m(x)} (2i-1)(k_i^{max} - k_{i-1}^{max}), \quad k_0^{max} := 0.$$

Theorem 20. *Let (M, \mathcal{D}, g) be a sub-Riemannian manifold, equipped with a smooth measure. Assume that, as a metric measure space (M, d, μ) , has negligible cut locus. Let $x \in M$, and let Ω be any measurable, bounded subset with $0 < \mu(\Omega) < +\infty$. Then there exists a constant $C(\Omega) > 0$ such that*

$$(80) \quad \mu(\Omega_t) \sim C(\Omega)t^{\mathcal{N}(x)}, \quad t \rightarrow 0^+.$$

Equation (79) is the *definition* of geodesic dimension given in [3]. As a consequence of Theorem 20, it coincides with the one given in this paper, when specified to sub-Riemannian structures. In this case, to compute $\mathcal{N}(x)$, it is sufficient to compute the growth vector for the generic geodesic, and use (79). Theorems 19, 20 are also proved in [3], and are based on a deep relation between the geodesic growth vector and the asymptotics of the exponential map on a general sub-Riemannian manifold.

7.3. Proof of Theorem 5. If (M, \mathcal{D}, g) is Riemannian, for any point $x \in M$ we have $\mathcal{G}_x = (\dim(M))$ for any non-trivial initial covector and $\mathcal{N}(x) = \dim(M) = \dim_H(M)$.

If (M, \mathcal{D}, g) is sub-Riemannian (with $k = \text{rank } \mathcal{D} < n$), and equiregular of step s , let $q_i := d_i - d_{i-1}$, for $i = 1, \dots, s$ and $p_i := k_i^{max} - k_{i-1}^{max}$, for all $i = 1, \dots, m(x)$. Indeed $m(x) \geq s \geq 2$. By Mitchell's formula (75), and (79), we have

$$(81) \quad \dim_H(M) = \underbrace{1 + \dots + 1}_{q_1} + \underbrace{2 + \dots + 2}_{q_2} + \dots + \underbrace{s + \dots + s}_{q_s},$$

$$(82) \quad \mathcal{N}(x) = \underbrace{1 + \dots + 1}_{p_1} + \underbrace{3 + \dots + 3}_{p_2} + \dots + \underbrace{2m(x) - 1 + \dots + 2m(x) - 1}_{p_{m(x)}}.$$

Both sums have n terms and $q_1 = p_1 = k < n$. The terms of (82), after the k -th, are strictly greater than the ones of (81). Thus $\mathcal{N}(x) > \dim_H(M) > \dim(M)$. \square

7.4. The Engel group. We discuss more in detail the Engel group introduced in (Example 2). This is the Carnot group, in dimension $n = 4$, generated by the following global orthonormal left-invariant vector fields in coordinates $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$

$$(83) \quad X_1 = \partial_1, \quad X_2 = \partial_2 + x_1\partial_3 + x_1x_2\partial_4.$$

Indeed it is a rank 2 Carnot group of step 3, with $\mathfrak{g}_1 = \text{span}\{X_1, X_2\}$ and

$$(84) \quad \mathfrak{g}_2 = [X_1, X_2] = \partial_3 + x_2\partial_4, \quad \mathfrak{g}_3 = [X_2, [X_1, X_2]] = \partial_4,$$

where we omit the linear span. In particular, by left-invariance, $\mathcal{D}^1 = \mathfrak{g}_1$, $\mathcal{D}^2 = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and $\mathcal{D}^3 = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$. The growth vector of the distribution is $(2, 3, 4)$. By Mitchell's formula (75) for the Hausdorff dimension we have $Q = 2 + 2 + 3 = 7$.

Let us compute the geodesic growth vector. It is sufficient to choose the curve $\gamma(t) = e^{tX_2}(e)$ (this is a normal geodesic, by Lemma 21). Using the definition, we obtain $\mathcal{F}^1 = \text{span}\{X_1, X_2\}$ and, omitting the linear span,

$$(85) \quad \mathcal{F}^2 = [X_2, X_1] = \partial_3 + x_2\partial_4, \quad \mathcal{F}^3 = [X_2, [X_2, X_1]] = \partial_4.$$

Indeed this gives the maximal possible geodesic growth vector, hence

$$(86) \quad \mathcal{G}_x^{max} = (2, 3, 4), \quad \forall x \in G.$$

In particular, using (79) we obtain $\mathcal{N} = 2 + 3 + 5 = 10$.

8. PROOF OF THEOREM 10

Let (G, \mathcal{D}, g) be a Carnot group of step s and dimension n . We identify $\mathfrak{g} = T_e G$ (and its subspaces) with the vector space of left-invariant vector fields. In particular $\mathcal{D} = \mathfrak{g}_1$.

8.1. Step 1: Estimates in the fat case. We remind that a sub-Riemannian structure (M, \mathcal{D}, g) is *fat* (or *strong bracket-generating*) if for any $x \in M$ and $X \in \mathcal{D}$, $X(x) \neq 0$, then $\mathcal{D}_x + [X, \mathcal{D}]_x = T_x M$. It is well known that fat structures do not admit non-trivial abnormal minimizers [2, 14, 17]. If G is a fat Carnot group of rank k , then the geodesic growth vector of any geodesic is

$$(87) \quad \mathcal{G}_x^{max} = (k, n), \quad \forall x \in G.$$

By (79) we have $\mathcal{N} = k + 3(n - k) = 3n - 2k$. On the other hand, the Hausdorff dimension is $Q = k + 2(n - k) = 2n - k$. Moreover, from (12), we have $N_R = Q + n - k = 3n - 2k$. This proves that on a fat Carnot group $\mathcal{N} = N_R$.

To prove the inequality $\mathcal{N} > N_R$ when G is not fat, let $\mathcal{G}^{max} = (k_1, k_2, \dots, k_m)$ be the maximal geodesic growth vector, with geodesic step $m \geq s \geq 2$. Let $q_i := d_i - d_{i-1}$, for $i = 1, \dots, s$ and $p_i := k_i - k_{i-1}$, for $i = 1, \dots, m$. Since $d_0 = k_0 = 0$ by convention and $d_s = k_m = n$, we have $n = \sum_{i=1}^s q_i = \sum_{i=1}^m p_i$. Thus, from (79), we obtain

$$(88) \quad \mathcal{N} = \sum_{i=1}^m (2i - 1)p_i = \sum_{i=2}^m (2i - 2)p_i + n.$$

On the other hand, for $N_R = Q + n - k$ and using Mitchell's formula (75), we obtain

$$(89) \quad N_R = \sum_{i=1}^s i q_i + n - k = \sum_{i=2}^s i q_i + n.$$

Arranging the terms as we did in the proof of Theorem 5, we write

$$(90) \quad \mathcal{N} - n = \underbrace{2 + \dots + 2}_{p_2} + \underbrace{4 + \dots + 4}_{p_3} + \dots + \underbrace{2m - 2 + \dots + 2m - 2}_{p_m},$$

$$(91) \quad N_R - n = \underbrace{2 + \dots + 2}_{q_2} + \underbrace{3 + \dots + 3}_{q_3} + \dots + \underbrace{s + \dots + s}_{p_s}.$$

Both sums have $n - k = \sum_{i=2}^m p_i = \sum_{i=2}^s q_i$ entries. Moreover $m > 2$ (if $m = 2$, then $\mathcal{F}^2 = \mathfrak{g}_1 \oplus [X, \mathfrak{g}_1] = T_e M$ for some $X \in \mathfrak{g}_1$, which is absurd if G is not fat). Then the entries of (90), after the p_2 -th one, are strictly greater than the corresponding ones of (91), and $\mathcal{N} > N_R$.

8.2. Step 2: Ideal = Fat. To conclude the proof of Theorem 10, we prove that any ideal Carnot group is fat. Denote with $\text{ad}_X : \mathfrak{g} \rightarrow \mathfrak{g}$ the linear map:

$$(92) \quad \text{ad}_X(Y) := [X, Y] = \left. \frac{d}{dt} \right|_{t=0} e_*^{-tX} Y(\gamma_X(t)),$$

where $\gamma_X(t) = e^{tX}(x)$ is the integral curve of $X \in \mathfrak{g}_1$ starting from $x \in G$.

Lemma 21. *Let $\gamma_X(t) = e^{tX}(x)$ be the integral curve of the left-invariant vector field $X \in \mathfrak{g}_1$, starting from x . Then γ_X is a normal geodesic. It is also an abnormal geodesic if and only if there exists a non-zero $\lambda \in T_e^* G$ such that*

$$(93) \quad \langle \lambda, \text{ad}_X^i(\mathfrak{g}_1) \rangle = 0, \quad \forall i = 0, \dots, s - 1.$$

Proof. Let X_1, \dots, X_k be a basis of left-invariant vector fields. Indeed $X = \sum_{i=1}^k u_i X_i$ for a constant control $u \in L^\infty([0, 1], \mathbb{R}^k)$. By left-invariance we can set $x = e$. A well known formula for the differential of the end-point map [2, 17] $D_u E_e : T_u \mathcal{U} \simeq \mathcal{U} \rightarrow T_{\gamma(1)} G$, gives

$$(94) \quad D_u E_e(v) = \int_0^1 e_*^{(1-t)X} \sum_{i=1}^k v_i(t) X_i(\gamma(t)) dt, \quad \forall v \in L^\infty([0, 1], \mathbb{R}^k).$$

We first prove that $\gamma_X(t)$ is a normal geodesic. Consider the covector $\eta \in T_e G$ such that $\langle \eta, X_i \rangle = u_i$ and $\langle \eta, W \rangle = 0$ for all $W \in \mathfrak{g}_2 \oplus \dots \oplus \mathfrak{g}_s$. Then

$$(95) \quad \langle (e^{-X})^* \eta, D_u E_e(v) \rangle = \int_0^1 \sum_{i=1}^k v_i(t) \langle \eta, e_*^{-tX} X_i(\gamma(t)) \rangle dt = (u, v)_{L^2([0,1], \mathbb{R}^k)} = D_u J(v).$$

Thus $\gamma_X(t)$ with control u satisfies the normal Lagrange multiplier rule with covector $\eta_1 = (e^{-X})^* \eta \in T_{\gamma(1)}^* G$, and is a normal geodesic. By definition $\gamma_X(t)$ is also abnormal if and only if there exists a $\lambda_1 \in T_{\gamma(1)}^* G$ such that $\lambda_1 \circ D_u E_e = 0$. That is, if and only if there exists $\lambda = (e^X)^* \lambda_1 \in T_e^* G$ such that

$$(96) \quad 0 = \langle \lambda, e_*^{-X} D_u E_e(v) \rangle = \int_0^1 \sum_{i=1}^k v_i(t) \langle \lambda, e_*^{-tX} X_i(\gamma(t)) \rangle dt, \quad \forall v \in L^\infty([0,1], \mathbb{R}^k).$$

This is true if and only if $\langle \lambda, e_*^{-tX} Y \rangle = 0$ for any $Y \in \mathfrak{g}_1$ and $t \in [0,1]$. Since all the relevant data are analytic, $t \mapsto \langle \lambda, e_*^{-tX} Y \rangle$ is an analytic function of t . Hence it vanishes if and only if all its derivatives at $t=0$ are zero, and this condition coincides with 93. \square

Lemma 22. *Let G be a Carnot group of step $s \leq 2$. G is ideal if and only if it is fat.*

Proof. The implication fat \Rightarrow ideal is trivial. Then, assume that \mathfrak{g}_1 is not fat, i.e. there exists $X \neq 0 \in \mathfrak{g}_1$ such that $\mathfrak{g}_1 \oplus [X, \mathfrak{g}_1] \subsetneq T_e G$. Hence there exists $\lambda \neq 0 \in T_e^* G$ such that

$$(97) \quad 0 = \langle \lambda, \mathfrak{g}_1 \rangle = \langle \lambda, \text{ad}_X(\mathfrak{g}_1) \rangle.$$

By Lemma 21, $\gamma_X(t) = e^{tX}(e)$ is a normal and abnormal geodesic. In particular, a sufficiently short segment of it is a minimizing curve. \square

We learned the following fact by E. Le Donne. For the reader's convenience we provide a simple proof here, which is similar to the one in [11].

Lemma 23. *Let G be a Carnot group of step $s \geq 3$. Then there exists a non-zero $X \in \mathfrak{g}_1$ such that the integral curve $\gamma_X(t) = e^{tX}(e)$ is a normal and abnormal geodesic.*

Proof. If there exists a $X \neq 0 \in \mathfrak{g}_1$ such that $\text{ad}_X(\mathfrak{g}_1) \subsetneq \mathfrak{g}_2$, using the same argument of Lemma 22, we show that $\gamma_X(t)$ is abnormal and normal. Let $q_i := \dim \mathfrak{g}_i$, for $i = 1, \dots, s$. Then assume that, for any $X \neq 0 \in \mathfrak{g}_1$, we have $\text{ad}_X(\mathfrak{g}_1) = \mathfrak{g}_2$. This implies $q_2 \leq q_1 - 1$. Consider a basis Y_1, \dots, Y_{q_2} of \mathfrak{g}_2 and a basis Z_1, \dots, Z_{q_3} of \mathfrak{g}_3 . Let $\lambda \in T_e^* G$ such that $\langle \lambda, Z_i \rangle = 0$ if $i > 1$ and $\langle \lambda, Z_1 \rangle = 1$, while $\langle \lambda, \mathfrak{g}_i \rangle = 0$ for all $i \neq 3$. Consider the linear maps $A_i := \lambda \circ \text{ad}_{Y_i} : \mathfrak{g}_1 \rightarrow \mathbb{R}$, for $i = 1, \dots, q_2$. Indeed $\dim \ker A_i \geq q_1 - 1$. Moreover

$$(98) \quad \dim(\ker A_1 \cap \ker A_2) = \dim \ker A_1 + \dim \ker A_2 - \dim(\ker A_1 + \ker A_2)$$

$$(99) \quad \geq 2(q_1 - 1) - q_1 = q_1 - 2.$$

After a finite number of similar steps we arrive to

$$(100) \quad \dim(\ker A_1 \cap \dots \cap \ker A_{q_2}) \geq q_1 - q_2 \geq q_1 - (q_1 - 1) = 1.$$

Thus let $X \neq 0 \in \ker A_1 \cap \dots \cap \ker A_{q_2}$. We show that $\gamma_X(t) = e^{tX}(e)$, which is indeed a normal geodesic, verifies the abnormal characterization of Lemma 21 with covector λ . Since $\text{ad}_X^i(\mathfrak{g}_1) \subseteq \mathfrak{g}_{i+1}$, we have $\langle \lambda, \text{ad}_X^i(\mathfrak{g}_1) \rangle = 0$ for all $i \neq 2$ by construction of λ . Finally,

$$(101) \quad \langle \lambda, \text{ad}_X^2(\mathfrak{g}_1) \rangle = \langle \lambda, \text{ad}_X(\mathfrak{g}_2) \rangle = 0,$$

where in the last passage we used the fact that $\text{ad}_X(\mathfrak{g}_1) = \mathfrak{g}_2$, that the latter is generated by the Y_i , and the definition of X . Then $\gamma_X(t)$ is abnormal by Lemma 21. \square

To sum up, a Carnot group of step $s \geq 3$ admits non-trivial abnormal minimizing curves, and cannot be ideal. On the other hand if $s \leq 2$, then ideal \Leftrightarrow fat. \square

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