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# Indecomposable $F_N$ -trees and minimal laminations

Thierry Coulbois, Arnaud Hilion and Patrick Reynolds

October 18, 2011

## Abstract

We extend the techniques of [CH] to build an inductive procedure for studying actions in the boundary of the Culler-Vogtmann Outer Space, the main novelty being an adaptation of the classical Rauzy-Veech induction for studying actions of *surface type*. As an application, we prove that a tree in the boundary of Outer space is free and indecomposable if and only if its dual lamination is *minimal up to diagonal leaves*. Our main result generalizes [BFH97, Proposition 1.8] as well as the main result of [KL11].

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# 1 Introduction

We consider  $\mathbb{R}$ -trees  $T$  equipped with a minimal very small action of the free group  $F_N$  of rank  $N$  by isometries: these are points in the closure of the celebrated unprojectivized Culler-Vogtmann Outer Space  $\overline{cv}_N$  [CV86]. A dual lamination  $L(T)$  is associated to such trees [CHL08a]. We recall that a lamination for the free group is a closed  $F_N$ -invariant, flip invariant, subset of the double Gromov boundary  $\partial^2 F_N = (\partial F_N)^2 \setminus \Delta$ , where  $\Delta$  is the diagonal. The main result of the paper relates *minimality* properties of the tree to *minimality* properties of its dual lamination.

For an action  $F_N \curvearrowright T$  there are several notions of minimality for the dynamics of the action; see Section 2.8. Mixing trees were considered by Morgan [Mor88], and Guirardel introduced the stronger notion of indecomposability in [Gui08]. Mixing and indecomposable trees are the elementary objects from which any tree in the closure of Outer Space can be built [Rey10].

**Definition 1.1.** *An  $\mathbb{R}$ -tree  $T \in \overline{cv}_N$  is **indecomposable** if for any non-degenerate segments  $I$  and  $J$  in  $T$ , there exist finitely many elements  $u_1, \dots, u_n$  in  $F_N$  such that*

1.  $I \subseteq u_1 J \cup u_2 J \cup \dots \cup u_n J$
2.  $u_i J \cap u_{i+1} J$  is a non degenerate segment for any  $i = 1, \dots, n - 1$ .

A lamination  $L$  is minimal if it does not contain a proper sublamination, or, equivalently, if the orbit of any leaf of  $L$  is dense in  $L$ . Laminations dual to trees are always diagonally closed: if  $L(T)$  contains leaves  $l_1 = (X_1, X_2)$  and  $l_2 = (X_2, X_3)$  then  $L(T)$  also contains the leaf  $l = (X_1, X_3)$ . Oftentimes the diagonal leaves are isolated in the dual lamination. Thus we need a slightly modified notion of minimality for laminations dual to  $\mathbb{R}$ -trees.

**Definition 1.2.** A lamination  $L$  is *minimal up to diagonal leaves* if :

- (i) there is a unique minimal sublamination  $L_0 \subseteq L$ , and
- (ii)  $L \setminus L_0$  consists of finitely many  $F_N$ -orbits of leaves, each of which is diagonal over  $L_0$ .

In the definition, a leaf  $l \in L$  is **diagonal** over a sublamination  $L_0 \subseteq L$  if there are leaves  $(X_1, X_2), (X_2, X_3), \dots, (X_{n-1}, X_n) \in L_0$  such that  $l = (X_1, X_n)$ . Note that the above notion of minimality coincides with the correct notion of minimality for foliations on a closed surface. Our main result is:

**Theorem A.** *Let  $T$  be an  $\mathbb{R}$ -tree with a free, minimal action of  $F_N$  by isometries with dense orbits. The tree  $T$  is indecomposable if and only if  $L(T)$  is minimal up to diagonal leaves.*

*In this case the unique minimal sublamination is the derived sublamination  $L(T)'$ : the subset of non-isolated leaves.*

The third author [Rey10] proved that under the same hypotheses,  $T$  is indecomposable if and only if no leaf of the dual lamination  $L(T)$  is carried by a finitely generated subgroup of infinite index. We use this characterization in the proof of Theorem A.

Theorem A generalizes the main result of [KL11]. Indeed, it is shown in [CH] that the repelling tree  $T_{\Phi^{-1}}$  of a fully irreducible (iwip) outer automorphism  $\Phi$  of  $F_N$  is indecomposable, and  $T_{\Phi^{-1}}$  is free exactly when  $\Phi$  is non-geometric. Moreover, the attracting lamination  $L_{\Phi}$  of  $\Phi$  is minimal [BFH97] and contained in the dual lamination of the repelling tree (see [CH]).

**Corollary 1.3** ([KL11]). *Let  $\Phi$  be a non-geometric iwip outer automorphism of  $F_N$ . The dual lamination of the repelling tree  $L(T_{\Phi^{-1}})$  is the diagonal closure of the attracting lamination  $L_{\Phi}$ .  $\square$*

Recall that a current on  $F_N$  is a positive  $F_N$ -invariant, flip invariant, Radon measure  $\mu$  on  $\partial^2 F_N$  [Kap06]. The support  $Supp(\mu)$  of  $\mu$  is a lamination (see [CHL08c]). We consider the intersection form  $i_{KL}$  between currents and trees:  $i_{KL} : \overline{cv}_N \times Curr(F_N) \rightarrow \mathbb{R}_{\geq 0}$  [KL09]. A current  $\mu$  is orthogonal to a tree  $T \in \overline{cv}_N$  (i.e  $i_{KL}(T, \mu) = 0$ ) if and only if the support of  $\mu$  is a sublamination of the dual lamination  $L(T)$  [KL10]. Theorem A implies:

**Corollary 1.4.** *Let  $T \in \overline{cv}_N$  be a free, indecomposable  $F_N$ -tree, and let  $\mu$  be a geodesic current. The following are equivalent:*

- (i)  $\text{Supp}(\mu) \subseteq L(T)$ ,
- (ii)  $i_{KL}(T, \mu) = 0$ ,
- (iii)  $\text{Supp}(\mu) = L(T)'$ . □

Our strategy for proving Theorem A is summarized as follows. We encode a tree  $T \in \overline{\text{cv}}_N$  in a system of isometries on a compact  $\mathbb{R}$ -tree as in [CHL09], and then develop an inductive procedure for studying such systems of isometries. There are two cases to consider: if the tree  $T$  is of Levitt type we use the Rips induction [CH]; if the tree  $T$  is of surface type we use a splitting procedure inspired by the Rauzy-Veech induction for flat surfaces. The result is a sequence of systems of isometries encoding the tree  $T$ ; there is a train track associated to each of these systems of isometries, and the dual lamination  $L(T)$  is the inverse limit of this sequence of train tracks. We are able to read off from the sequence of train tracks the desired minimality properties of the lamination. The motivation for our procedure comes from surface theory.

## 1.1 The Case of a Surface Lamination

The proof of Theorem A relies on an adaptation to the free group  $F_N$  of *train track expansions* of a measured lamination on a hyperbolic surface, which we now casually recall. First, we mention the correspondence between measured laminations, interval exchanges, and train tracks, as this correspondence suggests the procedure presented in this paper. Let  $\Sigma$  be a non-exceptional hyperbolic surface equipped with a measured geodesic lamination  $\mathcal{L} = (L, \mu)$ . We suppose that  $L$  is minimal and filling. Choosing an arc  $I \subseteq \Sigma$  that is transverse to  $L$ , one can consider the first return map  $f : I \rightarrow I$  induced by the holonomy of  $L$ . The map  $f$  is a classical *interval exchange transformation*, whose dynamics coincide with those of a finitely generated pseudogroup  $S$  of partial isometries on  $I$ : the generators of  $S$  are the maximal continuous restrictions of  $f$ . Suspending  $S$ , see Section 2.3, gives a foliated space  $\mathcal{S}$ , which is homeomorphic to  $\Sigma$  with finitely many discs removed; after gluing in these discs and extending the foliation to them in the obvious way, one obtains a surface homeomorphic to  $\Sigma$ , carrying a measured foliation which corresponds to measured lamination  $\mathcal{L}$ ; see [Lev83].

The suspension  $\mathcal{S}$  of the interval exchange  $f$  has the homotopy type of a finite graph—collapsing each band of  $\mathcal{S}$  onto one of its leaves gives the desired homotopy equivalence  $h : \mathcal{S} \rightarrow \Gamma$ . The graph  $\Gamma$  can be equipped with a *train track structure* as follows: let  $e$  and  $e'$  be (oriented) edges of  $\Gamma$

such that the endpoint  $v$  of  $e$  coincides with the initial point of  $e'$ ; make  $e$  and  $e'$  tangent at the vertex  $v$  if and only if there is a non-singular leaf of  $\mathcal{S}$  that crosses the (ordered) pair of bands corresponding to  $(e, e')$ . The widths of the bands of  $\mathcal{S}$  give an assignment of weights to the edges of  $\Gamma$ , which clearly satisfy a *switch condition*. In fact, one sees that  $\Gamma$  can be embedded in  $\Sigma$  as a measured train track *carrying*  $L$ .

We now consider the effect of the *Rauzy-Veech induction*. Recall that given an interval exchange transformation  $f : I \rightarrow I$  with corresponding partition  $I = I_1 \cup \dots \cup I_r$ , one step of the Rauzy-Veech induction consists of: first removing the right most interval  $I_r$ , then replacing  $f$  with the first return map  $f'$  on the interval  $I \setminus I_r$ . Equivalently, consider the suspension  $\mathcal{S}$ , and split apart the right-most singularity of the foliation on  $\mathcal{S}$  (all singularities lie on  $I$ ) until  $I$  is again reached. Rauzy-Veech induction is defined in the generic case, and extra information is available when it is undefined; see Proposition 4.7. This process gives a foliated space  $\mathcal{S}'$ , which is the suspension of the interval exchange transformation  $f'$ . In the present note, we consider this “splitting” point of view of Rauzy-Veech induction.

As noted before, there is a measured train track  $\Gamma$  corresponding to the space  $\mathcal{S}$ , such that  $\Gamma$  carries  $L$ . If  $\mathcal{S}'$  arises from  $\mathcal{S}$  via one step of the Rauzy-Veech induction, then we say that the measured train track  $\Gamma'$  corresponding to  $\mathcal{S}'$  arises from  $\Gamma$  via *splitting*; there is a homotopy equivalence  $\tau : \Gamma' \rightarrow \Gamma - \tau$  is a *fold*. The sequence of measured train tracks  $(\Gamma_i)$  arising by iterating this procedure is usually called a *splitting sequence*, or a *train track expansion* of  $L$ . The key features for us are that each  $\Gamma_i$  carries  $L$  and that the branches of  $\Gamma_i$  approximate leaves of  $L$  for  $i \gg 0$ .

## 1.2 The Case of a Very Small $F_N$ -Tree

We now outline our translation of the above technology. Let  $T$  be an  $\mathbb{R}$ -tree equipped with a free, minimal action of  $F_N$  by isometries, and suppose that the action  $F_N \curvearrowright T$  is mixing (see Section 2.8). Let  $L(T)$  denote the dual lamination associated to  $T$  (see Section 2.5). Choose a basis  $A$  for  $F_N$ , let  $X_A$  denote the Cayley tree of  $F_N$  with respect to this basis, and identify  $\partial^2 F_N = \partial^2 X_A$  in the obvious way. Let  $C_A$  denote the subset of  $\partial^2 F_N$  consisting of points represented by lines in  $X_A$  passing through the identity element. The set  $C_A$  is compact, hence  $L_A := L(T) \cap C_A$  is compact as well. As  $L(T)$  is  $F_N$ -invariant, we may consider the restricted action of  $F_N$  on  $L_A$ ; this turns out to be a classical *symbolic flow* (see [CHL08a]). We “geometrize” this flow using the tree  $T$ .

It is shown in [CHL09] that there is an  $F_N$ -equivariant, continuous map

$Q^2 : L(T) \rightarrow \overline{T}$ , where  $\overline{T}$  denotes the metric completion of  $T$ ; consider the compact subspace  $\Omega_A := Q^2(L(T) \cap C_A)$ —this is the analogue of an arc transverse to a surface lamination. From the equivariance of  $Q^2$ , we get a partial action of  $F_N$  on  $\Omega_A$ , which is just the restriction of the action  $F_N \curvearrowright T$ . It is shown in [CH] that there are two possibilities for the structure of  $\Omega_A$ : either  $\Omega_A$  is a finite union of compact  $\mathbb{R}$ -trees, in which case  $T$  is called *surface type*; or  $\Omega_A$  is totally disconnected, in which case  $T$  is called *Levitt type*. In either case the suspension of the (partial) action of  $F_N$  on  $\Omega_A$  is a compact foliated space that is a “geometric realization” of  $L_A$ . However, in the Levitt type case, it is more convenient to work with a nicer space: put  $K_A$  to be the convex hull of  $\Omega_A$  in  $\overline{T}$  (so  $K_A$  is a compact  $\mathbb{R}$ -tree), and suspend the (partial) action of  $F_N$  on  $K_A$ . The result is a foliated space  $\mathcal{S}$  in which some leaves are 1-ended; deleting all leaves with strictly less than two ends gives the suspension of  $\Omega_A$  (see Sections 2.6, 3.2, 2.3, and 3.1).

We now describe our generalization of Rauzy-Veech induction; our induction has two distinct procedures. In the case that  $T$  is surface type, we show (Proposition 4.7) that there exist in the suspension  $\mathcal{S}$  singularities which look like singularities which arise in foliated surface, *i.e.* they are “splittable”; splitting apart these singularities as above gives a homotopy equivalence  $h_1 : \mathcal{S}' \rightarrow \mathcal{S}$  ( $h_1$  “zips up” the splitting). There is a homotopy equivalence  $h : \mathcal{S} \rightarrow \Gamma$  to a finite graph  $\Gamma$  got by collapsing each band onto a vertical fiber, and splitting induces a homotopy equivalence  $\Gamma' \rightarrow \Gamma$ , where  $\Gamma'$  is the graph associated to  $\mathcal{S}'$  (see Section 4).

In the case that  $T$  is Levitt type, we operate on  $\mathcal{S}$  using the *Rips machine*: erase from  $K_A$  all points  $x$  such that  $x$  is an endpoint of a leaf to get  $K'_A \subseteq \Omega_A$ , and suspend  $K'_A$  get  $\mathcal{S}'$ . Again there are homotopy equivalences  $\mathcal{S} \rightarrow \Gamma$ ,  $\mathcal{S}' \rightarrow \Gamma'$ , and  $\Gamma' \rightarrow \Gamma$ .

In either case, we get a sequence of foliated spaces  $(\mathcal{S}_i)$  and graphs  $\Gamma_i$  with homotopy equivalences  $\tau_i : \Gamma_i \rightarrow \Gamma_{i-1}$ . Any sublamination  $L_0 \subseteq L(T)$  defines a *train track structure* on  $\Gamma_i$  just as in Section 1.1, where leaves in  $L_0$  are treated as “non-singular.” As one might expect, choosing too large of a sublamination  $L_0$  gives too many legal turns; however, it follows from the results of this paper that for  $T$  *indecomposable* (see Section 2.8), the derived set  $L(T)'$  consists of leaves that are morally non-singular; see Sections 2.7 and 5.2. As before, the main features are that the train tracks  $\Gamma_i$  all carry  $L(T)$  and that for  $i \gg 0$ , the branches of  $\Gamma_i$  “approximate” leaves of  $L(T)'$ . Our train-tracks will have no transverse measure; however, the informed reader will realize that currents are the natural object for measuring these train tracks. This idea will be explored in future work.

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## 2 Background

In this section we briefly review the relevant definitions around  $\mathbb{R}$ -trees, Outer space, and laminations. In what follows  $F_N$  denotes the free group of rank  $N$ .

### 2.1 Basics About $\mathbb{R}$ -Trees

A metric space  $(T, d)$  is an  $\mathbb{R}$ -**tree** (or just a **tree**) if for any two points  $x, y \in T$ , there is a unique topological arc  $p_{x,y} : [0, 1] \rightarrow T$  connecting  $x$  to  $y$ , and the image of  $p_{x,y}$  is isometric to the segment  $[0, d(x, y)]$ . As is usual, we let  $[x, y]$  stand for  $\text{Im}(p_{x,y})$ , and we call  $[x, y]$  the **segment** (also called an **arc**) in  $T$  from  $x$  to  $y$ . A segment is **non-degenerate** if it contains strictly more than one point. We let  $\bar{T}$  stand for the metric completion of  $T$ . Unless otherwise stated, we regard  $T$  as a topological space with the metric topology. If  $T$  is a tree, and  $x \in T$ , then  $x$  is a **branch point** if the cardinality of  $\pi_0(T - \{x\})$  is strictly greater than two. For  $x \in T$ , the elements of  $\pi_0(T - \{x\})$  are the **directions** at  $x$ .

In this paper, all the trees we consider are equipped with an isometric (left) action of  $F_N$ , i.e. a group morphism  $\rho : F_N \rightarrow \text{Isom}(T)$ ; as usual, we suppress the morphism  $\rho$  and identify  $F_N$  with  $\rho(F_N)$ . A tree  $T$  equipped with an isometric action will be called an  $F_N$ -**tree**, and we denote this situation by  $F_N \curvearrowright T$ . Notice that an action  $F_N \curvearrowright T$  induces an action of  $F_N$  on the sets of directions and branch points of  $T$ . We identify two  $F_N$ -trees  $T, T'$  if there is an  $F_N$ -equivariant isometry between them.

There are two sorts of isometries of trees: an isometry  $g$  of  $T$  is **elliptic** if  $g$  fixes some point of  $T$ , while an isometry  $h$  of  $T$  is **hyperbolic** if it is not elliptic. Any hyperbolic isometry  $h$  of  $T$  leaves invariant a unique isometric copy of  $\mathbb{R}$  in  $T$  which is the **axis**,  $A(h)$ , of  $h$ . If  $g$  is an elliptic isometry, we let  $A(g)$  stand for the fixed point set of  $g$ , i.e.  $A(g) := \{x \in T \mid gx = x\}$ . The **translation length function** of a  $F_N$ -tree  $T$  is  $l_T : F_N \rightarrow \mathbb{R}$ , where

$$l_T(g) := \inf\{d(x, gx) \mid x \in T\}.$$

The number  $l_T(g)$  is the **translation length** of  $g$ , and for any  $g \in F_N$ , the infimum is always realized on  $A(g)$ , so that  $g$  acts on  $A(g)$  as a translation



of length  $l_T(g)$ . If  $H \leq F_N$  is a finitely generated subgroup containing a hyperbolic isometry, then  $H$  leaves invariant the set

$$T_H^{min} := \cup_{l_T(h) > 0} A(h).$$

which is a subtree of  $T$ , and is minimal in the set of  $H$ -invariant subtrees of  $T$ ;  $T_H^{min}$  is the **minimal invariant subtree for  $H$** . In the case that  $H = F_N$ , we omit  $H$  and write  $T^{min}$  for  $T_{F_N}^{min}$ . An action  $F_N \curvearrowright T$  is **minimal** if  $T = T^{min}$ ; a minimal action  $F_N \curvearrowright T$  is **non-trivial** if  $T$  contains strictly more than one point.

## 2.2 System of isometries

A metric space  $F$  is a **finite forest** if  $F$  has finitely many connected components, each of which is a compact  $\mathbb{R}$ -tree. A **partial isometry**  $a$  of  $F$  is an isometry between two closed (hence compact) subtrees of  $F$ .

A **system of isometries** is a pair  $S = (F, A)$  where  $F$  is a finite forest and  $A$  is a finite collection of non-empty partial isometries of  $F$ . By allowing inverses and composition we get a pseudo-group of partial isometries on  $F$  which we write on the right.

To a system of isometries  $S = (F, A)$  we associate a graph  $\Gamma$  whose vertices are the connected components of  $F$  and such that for each partial isometry  $a \in A$  there is an oriented edge starting at the connected component of  $F$  containing the domain,  $\text{dom}(a)$ , and ending at the connected component of  $F$  containing the image of  $a$ . Denote by  $V(\Gamma)$  and  $E(\Gamma)$  the sets of vertices and edges of  $\Gamma$ , respectively.

By a **path** in  $\Gamma$ , we mean a finite **edge path**, that is, a path starting and ending at vertices of  $\Gamma$ . A path  $\gamma$  in  $\Gamma$  is called **reduced** if  $\gamma$  is an immersion; any path is homotopic relative to endpoints to a reduced path. A reduced path  $\gamma$  in  $\Gamma$  defines a partial isometry of  $F$ , and we say that  $\gamma$  is **admissible** if the corresponding partial isometry has non-empty domain. We abuse notation, identifying an admissible path  $\gamma$  with the partial isometry corresponding to it. An **infinite reduced path**  $\gamma$  is an immersion  $\gamma : \mathbb{R}_{\geq 0} \rightarrow \Gamma$  such that  $\gamma^{-1}(V(\Gamma)) = \mathbb{N}$ . For an infinite reduced path  $\gamma$ , the  **$i$ -prefix** of  $\gamma$ , denoted  $\gamma_i$ , is the restriction  $\gamma|_{[0, i]}$ . For  $i \geq j$ , one has that  $\text{dom}(\gamma_i) \subseteq \text{dom}(\gamma_j)$ , and we put  $\text{dom}(\gamma) := \cap_i \text{dom}(\gamma_i)$ . We say that an infinite path  $\gamma$  is **admissible** if  $\text{dom}(\gamma) \neq \emptyset$ . A **bi-infinite reduced path**  $\gamma$  is an immersion  $\gamma : \mathbb{R} \rightarrow \Gamma$  such that  $\gamma^{-1}(V(\Gamma)) = \mathbb{Z}$ . Given a bi-infinite path  $\gamma$ , the **halves** of  $\gamma$  are the restrictions  $\gamma^- := \gamma|_{\mathbb{R}_{\leq 0}}$  and  $\gamma^+ := \gamma|_{\mathbb{R}_{\geq 0}}$ ; we reparametrize  $\gamma^-$  in order to regard it as an infinite path. A bi-infinite reduced path is **admissible** if the domains of its halves have non-empty

intersection. Equivalently a bi-infinite path  $\gamma$  is admissible if and only if for any  $i \leq j \in \mathbb{Z}$ , the restriction  $\gamma|_{[i,j]}$  is admissible.

The set of bi-infinite admissible paths is called the **admissible lamination** of the system of isometries  $S = (F, A)$  and is denoted  $L(S)$ .

### 2.3 The Suspension and the Dual Tree

Let  $S = (F, A)$  be a system of isometries, and let  $I = [0, 1]$  denote the unit interval. For each  $a_i \in A$ , let  $b_i := \text{dom}(a_i)$ ; one forms a **band**  $B_i := b_i \times I$ . Identify  $b_i$  with  $b_i \times \{0\} \subseteq B_i$ , and denote  $\tilde{b}_i := b_i \times \{1\}$ . Say that  $b_i$  and  $\tilde{b}_i$  are the **bases** of the band  $B_i$ .

**Definition 2.1.** *The **suspension**  $\mathcal{S}$  of  $S$  is the quotient of  $F \amalg \coprod_i B_i$ , where one identifies  $b_i$  with  $\text{dom}(a_i)$  and  $\tilde{b}_i$  with  $\text{im}(a_i) = \text{dom}(a_i) \cdot a_i$ .*

The suspension of a system of isometries is a compact, Hausdorff space that has the homotopy type of a finite graph  $\Gamma$ . The graph  $\Gamma$  is a deformation retract of  $\mathcal{S}$  obtained by contracting each connected component of the forest  $F$  to a point and each band  $b_i \times I$  to a core  $\{pt\} \times I$ . The graph  $\Gamma$  is the graph associated to the system of isometries in Section 2.2. In the sequel, we always suppose that  $\mathcal{S}$  is connected. Hence,  $\pi_1(\mathcal{S})$  is a free group  $F_N$ .

We think of each band as being **foliated** by leaves of the form  $\{pt\} \times [0, 1]$ , and the foliations of the bands of  $\mathcal{S}$  give rise to a foliation on  $\mathcal{S}$ : define a relation  $R$  on points of  $\mathcal{S}$  by declaring  $xRy$  if and only if  $x$  and  $y$  lie in the same leaf of the foliation on some band. The classes of the smallest equivalence relation containing  $R$  are the **leaves** of the foliation on  $\mathcal{S}$ . For a point  $x \in F$ , we let  $l(x)$  denote the leaf of the foliation on  $\mathcal{S}$  containing  $x$ .

We consider the path metric on leaves coming from the metric on  $[0, 1]$ . A finite, infinite, or bi-infinite path  $\gamma$  in  $\mathcal{S}$  is an **admissible leaf path** if  $\gamma : J \rightarrow \mathcal{S}$ ,  $J$  a closed subinterval of  $\mathbb{R}$  with extremities  $\partial J \subseteq \mathbb{Z} \cup \{\pm\infty\}$ , is a locally isometric, immersed leaf path, so  $\gamma^{-1}(F) = J \cap \mathbb{Z}$ . The **lamination** (which is rather a foliation in this setting)  $L(\mathcal{S})$  is the set of bi-infinite admissible leaf paths.

Any admissible leaf path  $\gamma$  defines an admissible path in  $\Gamma$  which we also denote by  $\gamma$ . Any bi-infinite admissible leaf path in  $L(\mathcal{S})$  defines a bi-infinite admissible path in the admissible lamination  $L(S)$ . Thus, the admissible lamination  $L(S)$  associated to a system of isometries  $S = (F, A)$  is a combinatorial version of the foliation on the suspension  $\mathcal{S}$ .

A **length measure**  $\mu$  on a tree  $T$  is a collection  $\{\mu_I\}_{I \subseteq T}$  of finite Borel measures on the compact arcs  $I$  of  $T$ , such that if  $J \subseteq I$ , then  $\mu_J = \mu_I|_J$ .

For any tree  $T$ , we have the **Lebesgue measure**  $\mu_L$  on  $T$  consisting of the Lebesgue measures on the compact intervals of  $T$ . The foliated space  $\mathcal{S}$  inherits a **transverse measure** from the Lebesgue measures on the bases.

Let  $S = (F, A)$  be a system of isometries, and let  $\tilde{\mathcal{S}}$  denote the universal cover of  $\mathcal{S}$ . Lift the foliation and transverse measure from  $\mathcal{S}$  to  $\tilde{\mathcal{S}}$ . Then  $F_N = \pi_1(\mathcal{S})$  acts on  $\tilde{\mathcal{S}}$  by deck transformations, and the foliation and transverse measure are preserved. Collapsing to a point each leaf of the foliation of  $\tilde{\mathcal{S}}$  gives an  $\mathbb{R}$ -tree  $T_{\mathcal{S}}$ . As the action  $F_N \curvearrowright \tilde{\mathcal{S}}$  preserves the foliation and transverse measure, one gets an isometric action  $F_N \curvearrowright T_{\mathcal{S}}$ ; see [CH] or [BF95]. The action  $F_N \curvearrowright T_{\mathcal{S}}$  is **dual** to the system of isometries  $S$ .

## 2.4 Outer Space and its Closure

Recall that an action  $F_N \curvearrowright T$  is **free** if for any  $1 \neq g \in F_N$  one has  $l_T(g) > 0$ . If  $X \subseteq T$ , then the **stabilizer** of  $X$  is  $\text{Stab}(X) := \{g \in F_N \mid gX = X\}$  — the setwise stabilizer of  $X$ . An action  $F_N \curvearrowright T$  is **very small** if:

- (i)  $F_N \curvearrowright T$  is minimal,
- (ii) for any non-degenerate arc  $I \subseteq T$ ,  $\text{Stab}(I) = \{1\}$  or  $\text{Stab}(I)$  is a maximal cyclic subgroup of  $F_N$ ,
- (iii) stabilizers of tripods are trivial.

A minimal action  $F_N \curvearrowright T$  is **discrete** (or **simplicial**) if the  $F_N$ -orbit of any point of  $T$  is a discrete subset of  $T$ ; in this case  $T$  is obtained by equivariantly assigning a metric to the edges of a (genuine) simplicial tree. Note that the metric topology is weaker than the simplicial topology if the tree is not locally compact.

The **unprojectivised Outer Space** of rank  $N$ , denoted  $\text{cv}_N$ , is the topological space whose underlying set consists of free, minimal, discrete, isometric actions of  $F_N$  on  $\mathbb{R}$ -trees. A minimal  $F_N$ -tree is completely determined by its translation length function (see, for example, [Chi01]): we can embed  $\text{cv}_N \subseteq \mathbb{R}^{F_N}$ . The closure  $\overline{\text{cv}}_N$  in  $\mathbb{R}^{F_N}$  consists of very small isometric actions of  $F_N$  on  $\mathbb{R}$ -trees [CL95, BF94]. For more background on  $\text{cv}_N$  and its closure, see [Vog02] and the references therein.

## 2.5 The Map $Q$ and the Dual Lamination

Here we recall dual algebraic laminations associated to  $F_N$ -trees; see [CHL08a] and [CHL08b] for a careful development of the general theory. Let  $\partial F_N$  denote the Gromov boundary of  $F_N$  — *i.e.* the Gromov boundary of any Cayley

graph of  $F_N$ ; let  $\partial^2(F_N) := \partial F_N \times \partial F_N \setminus \Delta$ , where  $\Delta$  is the diagonal. The left action of  $F_N$  on a Cayley graph induces actions by homeomorphisms of  $F_N$  on  $\partial F_N$  and  $\partial^2 F_N$ . Let  $i : \partial^2 F_N \rightarrow \partial^2 F_N$  denote the involution that exchanges the factors. A **lamination** is a non-empty, closed,  $F_N$ -invariant,  $i$ -invariant subset  $L \subseteq \partial^2 F_N$ .

**Remark 2.2.** *In the setting of Sections 2.2 and 2.3, if the graph  $\Gamma$  is connected, then its fundamental group is a free group  $F_N$ . Specifying a marking isomorphism  $\pi_1(\Gamma) \simeq F_N$  gives a homeomorphism  $\partial\tilde{\Gamma} \simeq \partial F_N$ , where  $\tilde{\Gamma}$  is the universal cover of  $\Gamma$ . A bi-infinite reduced path in  $\Gamma$  lifts to bi-infinite reduced paths in  $\tilde{\Gamma}$  that are completely described by their pairs of ends in  $\partial^2 F_N$ . We regard  $L(S)$  and  $L(\mathcal{S})$  as laminations.*

**Proposition 2.3** ([LL03]). *Let  $T \in \overline{cv}_N$  have dense orbits, and suppose that  $X \in \partial F_N$ . There is a unique point  $Q(X) \in \overline{T} \cup \partial T$  such that there exists a sequence  $u_n$  in  $F_N$  converging to  $X$  and a point  $P \in T$  such that  $u_n P$  converges to  $Q(X)$ .  $\square$*

More intuitively, the map  $Q$  is the continuous extension of the map  $Q_P : F_N \rightarrow T$ ,  $u \rightarrow uP$  to  $\partial F_N \rightarrow \hat{T}^{\text{obs}}$ , where  $\hat{T} = \overline{T} \cup \partial T$  is endowed with the weaker observers' topology—the set of directions in  $\hat{T}$  is a basis of open sets for the observers' topology. The space  $\hat{T}^{\text{obs}}$  is Hausdorff and compact.

**Proposition 2.4** ([LL03]). *Let  $T \in \overline{cv}_N$  have dense orbits. The map  $Q : \partial F_N \rightarrow \hat{T}$  is  $F_N$ -equivariant and surjective; further, points in  $\partial T$  have exactly one pre-image by  $Q$ .  $\square$*

The crucial property for us is that  $Q$  can be used to associate to  $T$  a lamination [CHL08b].

**Definition 2.5.** *Let  $T \in \overline{cv}_N$  have dense orbits. The **dual lamination** of  $T$  is*

$$L(T) := \{(X, Y) \in \partial^2(F_N) \mid Q(X) = Q(Y)\}.$$

## 2.6 The Map $Q^2$ and the Compact Heart

For a tree  $T \in \overline{cv}_N$  with dense orbits, the map  $Q : \partial F_N \rightarrow \hat{T} = \overline{T} \cup \partial T$  induces a map  $Q^2 : L(T) \rightarrow \overline{T} : (X, Y) \mapsto Q(X) = Q(Y)$ . The metric topology on  $T$  canonically extends to  $\overline{T}$ , and we have:

**Proposition 2.6** ([CHL09]). *The map  $Q^2 : L(T) \rightarrow \overline{T}$  is continuous.  $\square$*

The space  $\partial^2 F_N$  is not compact, but there are many “nice” coverings of  $\partial^2 F_N$  by compact sets. Fixing a basis  $A$  for  $F_N$  gives an identification of  $\partial F_N$  with the space of infinite reduced words in  $A^{\pm 1}$ , hence an identification of  $\partial^2 F_N$  with the space of pairs  $(X, Y)$  of distinct infinite reduced words  $X \neq Y$ . For an infinite word  $X$ , we let  $X_1$  stand for the first letter of  $X$ . The **unit cylinder** of  $\partial^2 F_N$  with respect to  $A$  is the subset  $C_A := \{(X, Y) \in \partial^2 F_N \mid X_1 \neq Y_1\}$ . For any basis  $A$  of  $F_N$ ,  $C_A$  is a compact subspace of  $\partial^2 F_N$ , and  $\partial^2 F_N = \cup_{g \in F_N} gC_A$ .

For a tree  $T \in \overline{cv}_N$  with dense orbits the **limit set** of  $T$  is  $\Omega = Q^2(L(T)) \subseteq \overline{T}$ . For any basis  $A$  of  $F_N$ , the **compact limit set** of  $T$  (with respect to  $A$ ) is  $\Omega_A := Q^2(L(T) \cap C_A) \subseteq \overline{T}$ . The **heart** of  $T$  (with respect to  $A$ ) is the convex hull  $K_A$  of  $\Omega_A$  in  $\overline{T}$ . It follows from Proposition 2.6 that  $\Omega_A$  is a compact subset of  $\overline{T}$ , so  $K_A$  is a compact subtree of  $\overline{T}$ .

Now, fix a tree  $T \in \overline{cv}_N$  with dense orbits and a basis  $A$  for  $F_N$ . For a compact subtree  $K \subseteq \overline{T}$ , let  $S = (K, A)$  denote the system of isometries with  $a \in A$  the partial isometry got by the (maximal) restriction of the action of  $a^{-1}$  to  $K$ . Let  $\mathcal{S}$  denote the suspension of  $S$ , and let  $T_{\mathcal{S}}$  denote the  $F_N$ -tree dual to  $\mathcal{S}$ . The following result is essential for the present note.

**Proposition 2.7** ([CHL09]). *Let  $T \in \overline{cv}_N$  have dense orbits, and fix a basis  $A$  for  $F_N$ . Let  $K_A$  denote the heart of  $T$  with respect to  $A$ ; let  $S = (K_A, A)$  denote the associated system of isometries, and let  $\mathcal{S}$  denote its suspension.*

- (i) *The tree  $T$  is dual to  $S$ :  $T_{\mathcal{S}}^{\min} = T$ ,*
- (ii)  *$L(T) \cap C_A = L(S) = L(\mathcal{S})$ , and*
- (iii) *for any infinite admissible leaf path  $X$  in  $\mathcal{S}$ ,  $\text{dom}(X) = \{Q(X)\}$ .  $\square$*

Proposition 2.7 allows us to fully transfer the problem of understanding the dual lamination of a tree in the boundary of Outer space to the problem of understanding the associated systems of isometries.

A tree  $T \in \overline{cv}_N$  is **geometric** if its compact heart  $K_A$  for some (hence any) basis  $A$  of  $F_N$  is a finite tree, that is to say a compact  $\mathbb{R}$ -tree which is the convex hull of finitely many points, see [CHL09].

## 2.7 Regular Leaves and the Derived Sublamination

In this Section, we collect some results regarding the derived space of  $L(T)$  and its relationship to regular leaves in systems of isometries associated to  $T$ .

**Definition 2.8.** Let  $T$  be an  $F_N$ -tree in  $\overline{cv}_N$  with dense orbits. A leaf  $l \in L(T)$  is **regular** if there exists a sequence  $l_n \in L(T)$  of leaves converging to  $l$  and such that the  $x_n = Q^2(l_n)$  are distinct. The set of regular leaves is the **regular sublamination**  $L_r(T)$ .

If the action of  $F_N$  on  $T$  is free, by [CH, Theorem 5.3] we have that for all  $x \in \overline{T}$ ,  $(Q^2)^{-1}(x)$  is a finite set of uniformly bounded cardinality. Recall that the set of non-isolated points of a topological space,  $X$ , is its **derived space**,  $X'$ .

**Lemma 2.9.** Let  $T \in \overline{cv}_N$  be free with dense orbits. The regular sublamination of  $T$  is equal to the derived lamination:

$$L_r(T) = L(T)'. \quad \square$$

## 2.8 Mixing Properties for $F_N$ -trees

From the work of V. Guirardel [Gui08] (after J. Morgan [Mor88]), there are several notions of “minimality” for the dynamics of an action of a group on an  $\mathbb{R}$ -tree. These notions are hierarchized as follows:

1. **dense orbits:** the  $F_N$ -orbit of some (hence any) point  $P$  of  $T$  is dense in  $T$ ;
2. **arc-dense:** every orbit meets every non-degenerate segment of  $T$ ;
3. **arc-dense directions:** for each  $x \in T$ , each direction  $d$  at  $x$ , and each non-degenerate arc  $I \subseteq T$ , there exists  $g \in F_N$  such that  $gx \in I$  and  $gd \cap I$  is non-degenerate;
4. **mixing:** for any non-degenerate segments  $I$  and  $J$  in  $T$ , there exists finitely many elements  $u_1, \dots, u_n$  in  $F_N$  such that  $I \subseteq u_1J \cup u_2J \cup \dots \cup u_nJ$ ;
5. **indecomposable:** for any non-degenerate segments  $I$  and  $J$  in  $T$ , there exists finitely many elements  $u_1, \dots, u_n$  in  $F_N$  such that
  - (a)  $I \subseteq u_1J \cup u_2J \cup \dots \cup u_nJ$
  - (b)  $u_iJ \cap u_{i+1}J$  is a non degenerate segment for any  $i = 1, \dots, n - 1$ .

We remark that this hierarchy is not exactly strict as

**Lemma 2.10** ([Rey10, Lemma 12.6]). Let  $T \in \overline{cv}_N$ . The action  $F_N \curvearrowright T$  is mixing if and only if it has arc-dense directions.  $\square$

In this paper we will use two characterizations of indecomposable trees. A **transverse family** for an action  $F_N \curvearrowright T$  of  $F_N$  on an  $\mathbb{R}$ -tree  $T$  is an  $F_N$ -invariant family  $\{T_v\}_{v \in V}$  of non-degenerate, proper subtrees of  $T$  such that if  $T_v \neq T_{v'}$ , then  $T_v \cap T_{v'}$  contains at most one point.

**Proposition 2.11** ([Rey11, Lemma 4.1]). *Let  $F_N \curvearrowright T$  be an action of  $F_N$  on an  $\mathbb{R}$ -tree  $T$ . Then  $F_N \curvearrowright T$  is indecomposable if and only if there is no transverse family.  $\square$*

In the proof of Lemma 5.1, we need a refined understanding of transverse families that occur in free  $F_N$ -trees. We collect the following:

**Proposition 2.12** ([Rey10, Lemma 4.4] and [Lev94, Theorem 5]). *Let  $T \in \overline{cv}_N$  be free with dense orbits. If  $T$  is not indecomposable there exists a non-degenerate subtree  $T_0$  of  $T$ , such that*

1.  $\{gT_0 \mid g \in F_N\}$  is a transverse family,
2.  $H = \text{Stab}(T_0)$  is a free factor of  $F_N$ ,
3.  $H$  acts on  $T_0$  with dense orbits.  $\square$

A finitely generated subgroup  $H$  of  $F_N$  is quasiconvex, thus the boundaries  $\partial H \subseteq \partial F_N$  come with a natural inclusion. We say that a line  $(X, Y) \in \partial^2 F_N$  is **carried** by  $H$  if  $(X, Y) \in \partial^2 H$ .

**Proposition 2.13** ([Rey11, Corollary 4.8]). *Let  $T \in \overline{cv}_N$  be free and indecomposable, and let  $H \leq F_N$  be finitely generated. Then  $H$  carries a leaf of  $L(T)$  if and only if  $H$  has finite index in  $F_N$ .  $\square$*

## 3 The Rips Machine and Types of Actions

### 3.1 The Rips Machine

We recall the generalization of Process I of the Rips Machine [GLP94, BF95] that was first studied in the present context in [CH].

Let  $S = (F, A)$  be a system of isometries. The output of one step of the Rips Machine applied to  $S$  is a new system of isometries  $S' = (F', A')$  defined as follows:

$$F' := \{x \in F \mid \exists a \neq a' \in A^{\pm 1}, x \in \text{dom}(a) \cap \text{dom}(a')\}$$

Since  $A$  is finite and since intersections of domains of isometries are compact  $\mathbb{R}$ -trees, we have that  $F'$  is again a finite forest. We let  $A'$  consist of all

maximal restrictions of the elements of  $A$  to pairs of connected components of  $F'$ , so  $S' = (F', A')$  is indeed a system of isometries, as required.

The suspension  $\mathcal{S}'$  of  $S'$  is a subspace of the suspension  $\mathcal{S}$  of  $S$ . We can regard each leaf-path in  $\mathcal{S}'$  as a leaf-path in  $\mathcal{S}$ , in particular for bi-infinite admissible leaf paths  $L(\mathcal{S}') \subseteq L(\mathcal{S})$ . On the other hand, the Rips Machine does not modify bi-infinite admissible leaf paths, thus:

**Lemma 3.1.** *Let  $S = (F, A)$  be a system of isometries, and let  $S' = (F', A')$  denote the output of the Rips machine applied to  $S$ . The laminations are equal:  $L(\mathcal{S}) = L(\mathcal{S}')$ .  $\square$*

### 3.2 Types of Actions

We consider the output of iterating the Rips Machine on a system of isometries  $S_0 = (F_0, A_0)$ ; we denote by  $S_i$  the output of the  $i^{\text{th}}$  iteration of the Rips Machine. If for some  $i_0$ , one has that  $F_{i_0} = F_{i_0+1}$ , *i.e.* the Rips Machine **halts** on  $S_{i_0}$ , then we say that the Rips Machine **eventually halts** on  $S_0$ .

**Definition 3.2.** *Let  $S_0$  be a system of isometries. If the Rips Machine eventually halts on  $S_0$ , then  $S_0$  is called **surface type**.*

The **limit set** of  $S_0$  is  $\Omega = \bigcap_{i \in \mathbb{N}} F_i$ . If  $S_0$  is of surface type then  $\Omega = F_{i_0}$ . If  $S = (K_A, A)$  is a system of isometries associated to a tree  $T \in \overline{\text{cv}}_N$  with dense orbits, then the limit set  $\Omega$  of the system of isometries is equal to the compact limit set  $\Omega_A$  with respect to the basis  $A$  defined in Section 2.6.

**Definition 3.3.** *Let  $S_0$  be a system of isometries, and suppose that the Rips machine does not eventually halt on  $S_0$ . If the limit set  $\Omega$  associated to  $S_0$  is totally disconnected, then  $S_0$  is said to be **Levitt type**.*

In [CH] it is shown that for  $T \in \overline{\text{cv}}_N$  with dense orbits, if for some basis  $A$ , the system of isometries associated to  $K_A$  is of surface type (resp. Levitt type), then for every basis  $A'$ , the system of isometries associated to  $K_{A'}$  is of surface type (resp. Levitt type). In this case we say that  $T$  is of **surface type** (resp. **Levitt type**). It should be noted that there are trees in  $\overline{\text{cv}}_N$  that are neither of surface type nor Levitt type; however, we have the following:

**Proposition 3.4** ([CH, Proposition 5.14]). *Let  $T \in \overline{\text{cv}}_N$  have dense orbits. If the action  $F_N \curvearrowright T$  is mixing, then  $T$  is either of surface type or Levitt type.  $\square$*



### 3.3 Levitt type Actions

Let  $T \in \overline{cv}_N$  has dense orbits, and let  $S_0 = (K_A, A) = (F_0, A_0)$  be an associated system of isometries. Denote by  $S_i$  the output of the  $i^{\text{th}}$  iteration of the Rips machine.

Recall the definition of the graph  $\Gamma$  associated to a system of isometries  $S$  from Section 2.2. Let  $\Gamma_i$  denote the graph associated to  $S_i$ :  $\Gamma_i$  is got by contracting each band of  $\mathcal{S}_i$  onto one of its leaves. There are induced graph morphisms  $\tau_i : \Gamma_i \rightarrow \Gamma_{i-1}$ . The following Lemma follows from [CH, Propositions 3.12, 3.13, and 5.6].

**Lemma 3.5.** *Let  $T \in \overline{cv}_N$  have dense orbits; let  $A$  a basis for  $F_N$ ; and let  $S_0$  denote the associated system of isometries. Denote by  $S_i$  the output of the  $i^{\text{th}}$  iteration of the Rips machine applied to  $S_0$ , and let  $\Gamma_i$  be the associated graph.*

(i)  $\Gamma_i$  has no vertices of valence 0 or 1, and

(ii) the maps  $\tau_i : \Gamma_i \rightarrow \Gamma_{i-1}$  are homotopy equivalences.  $\square$

Note that, as  $F_0 = K_A$  is connected,  $\Gamma_0$  is a rose with  $N$  petals, so Lemma 3.5 gives a uniform bound  $2N - 2$  on the number of vertices of valence strictly greater than two in  $\Gamma_i$ .

**Lemma 3.6.** *Let  $T \in \overline{cv}_N$  be free with dense orbits, and suppose that  $T$  is of Levitt type. If  $L_0 \subsetneq L_r(T)$  is a proper sublamination, then every leaf of  $L_0$  is carried by a proper free factor of  $F_N$ .*

*Proof.* Fix a basis  $A$  for  $F_N$ , and let  $S_0 = (K_A, A)$  be the associated system of isometries; let  $S_i = (F_i, A_i)$  denote the output of the  $i^{\text{th}}$  iteration of the Rips machine, and let  $\Gamma_i$  denote the graph associated to  $S_i$ . Recall that since  $T$  is of Levitt type, the limit set  $\Omega$  is totally disconnected, hence the number of vertices of  $\Gamma_i$  goes to infinity with  $i$ . Let  $l$  be a bi-infinite admissible leaf path in  $L_r(\mathcal{S}_0) \setminus L_0$ . There exists a sequence  $l_n$  in  $L(\mathcal{S}_0)$  converging to  $l$  such that the  $x_n = Q^2(l_n)$  are distinct and distinct from  $x = Q^2(l)$ . Additionally, we can assume that  $l_n|_{[-n,n]} = l|_{[-n,n]}$  (viewed as admissible paths in  $\Gamma_0$ ). As  $\Omega$  is totally disconnected, for any  $m$ , there is  $i(m)$  such that  $x_n$  lie in separate components of  $F_i$  for  $n \leq m$  and  $i \geq i(m)$ . Also, since  $l \notin L_0$ , there is  $M$  such that

$$\{l' \in L_0 \mid l'|_{[-M,M]} = l|_{[-M,M]}\} = \emptyset.$$

We now apply Lemma 3.1 to view the leaves  $l_n$  as bi-infinite admissible leaf-paths in the suspensions  $\mathcal{S}_i$ .

For a given  $m$  the leaves  $l_n$ ,  $n \leq m$ , define distinct admissible paths  $l_n|_{[-M;M]}$  in  $\Gamma_{i(m)}$  each of length  $2M$ . As the action of  $F_N$  on  $T$  is free there exists  $j(M)$  such that for  $i \geq j(M)$  the size of any reduced loop in  $\Gamma_i$  is strictly bigger than  $2M$ . Recall that there are at most  $2N - 2$  vertices of valence strictly bigger than two in  $\Gamma_i$ . Thus for  $m$  large enough, for any  $i$  bigger than  $i(m)$  and  $j(M)$ , there exists  $n \leq m$  such that the admissible reduced path  $l_n|_{[-M;M]}$  in  $\Gamma_i$  does not cross any vertex of valence strictly greater than two.

Note that no leaf of  $L_0$  could cross any edge in the image of  $l_n|_{[-M;M]}$  in  $\Gamma_i$ ; thus every leaf of  $L_0$  is contained in the subgraph  $G_0 := (\Gamma_i \setminus \text{Im}(l_n|_{[-M;M]}))$ . By Lemma 3.5, we have that  $G_0$  corresponds to a proper free factor  $H \leq F_N$  and every leaf of  $L_0$  is carried by  $H$ .  $\square$

We now have our first main result.

**Proposition 3.7.** *Let  $T \in \overline{\text{cv}}_N$ , and assume that  $T$  is free and indecomposable and of Levitt type. The regular sublamination  $L_r(T) \subseteq L(T)$  is minimal.*

*Proof.* According to Lemma 3.6, if there happened to be a proper sublamination  $L_0 \subsetneq L_r(T)$ , then every leaf of  $L_0$  would be carried by a proper free factor  $H$  of  $F_N$ . This is impossible by Proposition 2.13.  $\square$

## 4 Splitting

In this section we define an inductive procedure that allows us to study the dual lamination of a free, surface type tree. To define this procedure, which is a generalization of the classical Rauzy-Veech induction, we need to find “good” singularities in systems of isometries associated to trees of surface type. Toward that end we recall some results regarding indices of trees.

### 4.1 $Q$ -Index and Geometric Index

Let  $T \in \overline{\text{cv}}_N$  have dense orbits, and let  $Q : \partial F_N \rightarrow \widehat{T} = \overline{T} \cup \partial T$  be the map defined in Section 2.5. For  $x \in \overline{T}$ , let  $\text{Stab}(x) \leq F_N$  denote the stabilizer of  $x$ . It was shown in [GL95] that there are finitely many orbits of points in  $T$  with non-trivial stabilizer and that  $\text{Stab}(x)$  is finitely generated. Note that for  $x \in \widehat{T} \setminus T$ ,  $\text{Stab}(x)$  is always trivial. From the definition of  $Q$ , one sees that  $\partial \text{Stab}(x) \subseteq Q^{-1}(x)$ ; put  $Q_r^{-1}(x) := Q^{-1}(x) \setminus \partial \text{Stab}(x)$ . Evidently,  $\text{Stab}(x)$  acts on  $Q^{-1}(x)$ , leaving invariant  $Q_r^{-1}(x)$ . For  $x \in \widehat{T}$ , the  **$Q$ -index** of  $x$  is

$$\text{ind}_Q(x) := |Q_r^{-1}(x)/\text{Stab}(x)| + 2\text{Rank}(\text{Stab}(x)) - 2$$

The  $Q$ -index is constant on  $F_N$ -orbits in  $T$ , and the  $Q$ -index of  $T$  is

$$\text{ind}_Q(T) := \sum_{[x] \in \widehat{T}/F_N} \max\{0, \text{ind}_Q(x)\}$$

As  $Q$  is injective on  $Q^{-1}(\partial T)$ , only points of  $\overline{T}$  contribute to the  $Q$ -index of  $T$ . The following is established in [CH]:

**Theorem 4.1** ([CH, Theorem 5.3]). *Let  $T \in \overline{cv}_N$  have dense orbits. Then  $\text{ind}_Q(T) \leq 2N - 2$ . Moreover,  $T$  is surface type if and only if  $\text{ind}_Q(T) = 2N - 2$ .  $\square$*

Let  $T \in \overline{cv}_N$  have dense orbits, and let  $x \in T$ . Then  $\text{Stab}(x)$  acts on  $\pi_0(T \setminus \{x\})$ . Following [GL95], one defines the **geometric index** of  $x$  to be

$$\text{ind}_{\text{geom}}(x) := |\pi_0(T \setminus \{x\})/\text{Stab}(x)| + 2\text{Rank}(\text{Stab}(x)) - 2$$

The geometric index is constant on  $F_N$ -orbits in  $T$ , and one defines the **geometric index** of  $T$  to be

$$\text{ind}_{\text{geom}}(T) := \sum_{[x] \in T/F_N} \text{ind}_{\text{geom}}(x)$$

We have the following:

**Theorem 4.2** ([GL95]). *Let  $T \in \overline{cv}_N$ . Then  $\text{ind}_{\text{geom}}(T) \leq 2N - 2$ . Moreover,  $T$  is geometric if and only if  $\text{ind}_{\text{geom}}(T) = 2N - 2$ .  $\square$*

## 4.2 Finding Splitting Points

**Convention 4.3.** *If  $T \in \overline{cv}_N$  is of surface type and if  $B$  is a basis for  $F_N$ , we let  $S = (K_B, B)$  denote the associated system of isometries. In the sequel, we assume that any system of isometries  $S = (F, A)$  associated to a surface type action is obtained from some  $S = (K_B, B)$  by running the Rips machine until it halts.*

A point  $x$  is **extremal** in a tree  $K$  if it is not contained in the interior of an arc contained in  $K$ . In this section we are interested in points which are extremal in some bases of a system of isometries but which are non-extremal in the underlying forest.

Let  $S = (F, A)$  be a system of isometries. A partial isometry  $a \in A^{\pm 1}$  is **defined in direction**  $d$  at a point  $x \in F$  if  $d$  is a direction in  $\text{dom}(a)$ :  $x \in \text{dom}(a)$  and  $d \cap \text{dom}(a) \neq \emptyset$ .

**Proposition 4.4.** [CH, Proposition 4.3] *Let  $T \in \overline{cv}_N$  be a tree with dense orbits of surface type and  $S = (F, A)$  be a system of isometries associated to  $T$ . For each direction  $d$  at a point  $x$  in  $F$  there are exactly two partial isometries  $a, b \in A^{\pm 1}$  defined in  $d$ .  $\square$*

In the surface type case, we can locally compare the geometric and  $Q$ -indices.

**Lemma 4.5.** *Let  $T \in \overline{cv}_N$  be free with dense orbits of surface type, and let  $S = (F, A)$  be a system of isometries associated to  $T$ . Let  $x \in F$  lie in the intersection of at least three distinct bases. If every point of the  $S$ -orbit of  $x$  is non-extremal in all bases that contain it, then  $\text{ind}_Q(x) \leq \text{ind}_{\text{geom}}(x)$ .*

*Proof.* Let  $\Gamma_x$  the (infinite) graph with vertices  $V(\Gamma_x)$  the  $S$ -orbit of  $x$  and with an edge labeled by  $a \in A^{\pm 1}$  between each pair of vertices  $x.u$  and  $x.ua$  with  $u \in F_N$  a partial isometry defined at  $x$ . As  $T$  is free, the graph  $\Gamma_x$  is a tree and from Proposition 2.7, its space of ends can be identified with  $Q^{-1}(x) \subseteq \partial F_N$ . From Theorem 4.1,  $\Gamma_x$  has finitely many ends.

Let  $\Gamma_x^d$  be the (infinite) graph with vertices the directions in  $F$  at points in the  $S$ -orbit of  $x$  and with an edge labeled by  $a \in A^{\pm 1}$  between each pair of vertices  $d$  and  $d.a$  (in particular  $a$  is defined in  $d$ ). The number of connected components in  $\Gamma_x^d$  is  $\text{ind}_{\text{geom}}(x) + 2$ . From Theorem 4.2,  $\Gamma_x^d$  has finitely many connected components.

By Proposition 4.4,  $\Gamma_x^d$  is a disjoint union of bi-infinite lines. By our hypothesis on  $x$ , for each edge labeled by  $a$  from  $x.u$  to  $x.ua$  in  $\Gamma_x$  there are at least two edges labeled by  $a$  in  $\Gamma_x^d$  from  $d_1$  to  $d_1.a$  and from  $d_2$  to  $d_2.a$  where  $d_1, d_2 \in V(\Gamma_x^d)$  are directions at  $x.u$ .

Each end of  $\Gamma_x$  is reached by at least two bi-infinite lines in  $\Gamma_x^d$  and a bi-infinite line has two ends, thus the number of lines in  $\Gamma_x^d$  is bounded below by the number of ends in  $\Gamma_x$ :

$$\text{ind}_Q(x) \leq \text{ind}_{\text{geom}}(x). \quad \square$$

**Definition 4.6.** *A **splitting point** in a system of isometries  $S = (F, A)$  is a point  $x$  in the connected component  $K_x$  in  $F$  such that*

- (S1)  *$x$  is not extremal in  $K_x$ ;*
- (S2) *there exists a partial isometry  $a_0 \in A^{\pm 1}$  defined at  $x$  such that  $x$  is extremal in the base  $\text{dom}(a_0)$ , and  $\text{dom}(a_0)$  is not reduced to  $\{x\}$ . We denote by  $d_x$  the unique direction at  $x$  which meets  $\text{dom}(a_0)$ . We call  $d_x$  the **splitting direction***

(S3) *There exists exactly one other partial isometry  $a_1 \in A^{\pm 1} \setminus \{a_0\}$  defined at  $x$  and such that  $\text{dom}(a_1)$  meets  $d_x$ .*

(S4) *The point  $x$  is not extremal in  $\text{dom}(a_1)$ .*

From the previous Proposition we get that splitting points exists.

**Proposition 4.7.** *Let  $T \in \overline{cv}_N$  be free, indecomposable, and of surface type. Let  $S = (F, A)$  be a system of isometries dual to  $T$ . There exists a splitting point  $x$  in  $S$ .*

*Proof.* We first prove that there exist points satisfying conditions (S1) and (S2).

If there were no points satisfying conditions (S1) and (S2) then we can use Proposition 4.5 to get

$$\text{ind}_{\text{geom}}(T) = \sum_{[x] \in T/F_N} \text{ind}_{\text{geom}}(x) \geq \sum_{[x] \in T/F_N} \text{ind}_Q(x) = \text{ind}_Q(T).$$

As  $T$  is of surface type, from Theorem 4.1,  $\text{ind}_Q(T) = 2N - 2$  and by Theorem 4.2 we get that  $T$  is geometric. By definition of geometric trees the forest  $F$  has finitely many extremal points and as conditions (S1) and (S2) fail, partial isometries send extremal points to extremal points, and the action is not free, a contradiction.

Thus, there exists a point  $x$  in  $F$  and a partial isometry  $a_0 \in A^{\pm 1}$  satisfying conditions (S1) and (S2). As  $T$  is of surface type, according to Proposition 4.4 condition (S3) is satisfied. If condition (S4) does not hold then  $x$  locally separates the suspension  $\mathcal{S}$  of  $S$ . In this case there is a proper free factor  $F'$  of  $F_N$  that carries every leaf of  $L_r(\mathcal{S})$ , contradicting Proposition 2.13.  $\square$

### 4.3 Splitting

Let  $S = (F, A)$  be a system of isometries, and let  $\Gamma$  be its associated graph. Assume that  $x$  is a splitting point for  $S$ . We use the notations of Definition 4.6.

We split the connected component  $K_x$  of  $x$  into two new compact  $\mathbb{R}$ -trees:  $K' = d_x \cup \{x\}$  and  $K'' = K_x \setminus d_x$ . We denote by  $F'$  the finite forest obtained by replacing  $K_x$  by two disjoint compact  $\mathbb{R}$ -trees  $K'$  and  $K''$  (in particular there are two copies of  $x$  in  $F'$ ). Let  $A_0$  be the set of partial isometries which do not meet  $x$ :

$$A_0 = \{a \in A \mid x \notin \text{dom}(a) \cup \text{dom}(a^{-1})\}.$$

Let  $a'_0$  be the same partial isometry as  $a_0$  with domain in  $K'$ . Let  $a'_1$  be the restriction of  $a_1$  to  $K'$  and  $a''_1$  be the restriction of  $a_1$  to  $K''$ . For any other partial isometry  $a \in A^{\pm 1} \setminus \{a_0, a_1\}$  such that  $x \in \text{dom}(a)$  we let  $a''$  be the same partial isometry as  $a$  defined on  $K''$ . Put  $A' = A_0 \cup \{a'_0, a'_1, a''_1\} \cup \{a'' \mid a \in A^{\pm 1} \setminus \{a_0, a_1\}, x \in \text{dom}(a)\}$ . We say that the system of isometries  $S' = (F', A')$  is obtained from  $S = (F, A)$  by **splitting** at  $x$  in the splitting direction  $d_x$ .

The suspension  $\mathcal{S}'$  of  $S'$  can be “zipped-up” to recover the suspension  $\mathcal{S}$  of  $S$ : the map  $z : \mathcal{S}' \rightarrow \mathcal{S}$  which identify the leaves  $\{(x, t) \mid t \in [0; 1]\}$  in the band  $K' \times [0; 1]$  with  $\{(x, t) \mid t \in [0; 1]\}$  in the band  $K'' \times [0; 1]$  is a homotopy equivalence.

**Lemma 4.8.** *A regular bi-infinite admissible leaf  $l$  in  $\mathcal{S}$  can be lifted by  $z$  to a regular bi-infinite admissible leaf in  $\mathcal{S}'$ .*

*Proof.* There exists a sequence  $l_n$  of bi-infinite admissible leaves in  $\mathcal{S}$  converging to  $l$  such that the  $x_n = Q(l_n)$  are distinct and distinct from  $x$ . We can assume that for each  $n$  the finite admissible leaf path  $l_n|_{[-n, n]}$  does not cross  $x$ . From the definition of splitting,  $l_n|_{[-n, n]}$  can be lifted to a finite admissible leaf path  $\gamma_n$  in  $\mathcal{S}'$ . The paths  $\gamma_n$  converges to a bi-infinite admissible leaf path  $l'$  in  $\mathcal{S}'$  which is a lift of  $l$ .  $\square$

We can now iteratively split our system of isometries dual to a tree of surface type to get the analogue of Lemma 3.5. As there might be more than one splitting point (there are at most finitely many), we choose, as a convention, to split all splitting points simultaneously.

**Lemma 4.9.** *Let  $T \in \overline{cv}_N$  be free with dense orbits, and suppose that  $T$  is of surface type. Let  $S_0$  denote a system of isometries associated to  $T$ . Denote by  $S_i$  the output of splitting  $S_{i-1}$ , and let  $\Gamma_i$  be the graph associated to  $S_i$ .*

(i)  $\Gamma_i$  has no vertices of valence 0 or 1, and

(ii) the maps  $\tau_i : \Gamma_i \rightarrow \Gamma_{i-1}$  are homotopy equivalences.  $\square$

#### 4.4 Surface Type Actions

We now establish analogues of Lemma 3.6 and Proposition 3.7 for actions of surface type.

**Lemma 4.10.** *Let  $T \in \overline{cv}_N$  be free with dense orbits, and suppose that  $T$  is of surface type. If  $L_0 \subsetneq L_r(T)$  is a proper sublamination, then every leaf of  $L_0$  is carried by a proper free factor of  $F_N$ .*

*Proof.* Let  $S = (K, A)$  be a system of isometries associated to  $T$ . By definition of surface type after finitely many steps the Rips machine starting on  $S$  halts on a surface type system of isometries  $S_0$ . According to Proposition 4.7 and its proof and Corollary 4.9, either we can then perform splittings on  $S_0$  or every leaf of  $L_r(T)$  is carried by a proper free factor of  $F_N$ . Let  $S_i = (F_i, A_i)$  denote the result of the  $i^{\text{th}}$  iteration of splitting applied to  $S_0$ .

By Lemma 2.10, directions are arc dense in  $T$  and by Proposition 2.7, directions are arc dense in  $F_i$  under the action of the pseudogroup  $S_i$ . In particular if  $d_x$  is the first splitting direction, for any non-degenerate arc  $[y, y'] \subseteq F$ , there is a finite admissible path  $\gamma$  in the graph  $\Gamma_0$  associated to  $S_0$  such that  $x.\gamma_0 \in [y, y']$  and  $d_x.\gamma_0$  meets  $[y, y']$ . For all  $i \geq i(y, y') = |\gamma|$ , the images of  $y$  and  $y'$  in  $F_i$  lie in different components of  $F_i$ .

Using Lemma 4.8 and Proposition 4.9, we may now conclude exactly as in the proof of Lemma 3.6.  $\square$

We have:

**Proposition 4.11.** *Let  $T \in \overline{cv}_N$ , and assume that  $T$  is free, indecomposable and of surface type. The regular sublamination  $L_r(T) \subseteq L(T)$  is minimal.*

*Proof.* According to Lemma 4.10, if there happened to be a proper sublamination  $L_0 \subsetneq L_r(T)$ , then every leaf of  $L_0$  would be carried by a proper free factor  $H \leq F_N$ . This is impossible by Proposition 2.13.  $\square$

## 5 Diagonal Leaves

Recall from the introduction that a lamination  $L$  is minimal up to diagonal leaves if  $L$  contains a unique minimal sublamination  $L_0$ , such that  $L \setminus L_0$  consists of finitely many  $F_N$ -orbits of leaves that are diagonal over  $L_0$ .

In this section we use both the Rips machine and the splitting induction to reach our main result.

### 5.1 Decomposable trees

**Lemma 5.1.** *Let  $T \in \overline{cv}_N$  be free with dense orbits. If  $L(T)$  is minimal up to diagonal leaves, then  $T$  is indecomposable.*

*Proof.* We argue the contrapositive. Let  $T \in \overline{cv}_N$  have dense orbits, and suppose that  $T$  is not indecomposable. Following Proposition 2.12, there exists a non-degenerate transverse family  $\{gT_0 \mid g \in F_N\}$ , where  $T_0$  is a closed non-degenerate subtree of  $T$ . The stabilizer  $H = \text{Stab}(T_0)$  is a proper

free factor of  $F_N$  and acts on  $T_0$  with dense orbits. The dual lamination  $L_H(T_0) \subseteq \partial^2 H$  of  $T_0$  is non-empty, and  $L_H(T_0)$  is diagonally closed. Recall that  $H$  is quasi-convex in  $F_N$  and thus there is an embedding  $\partial^2 H \subseteq \partial^2 F_N$ . Moreover, as  $H$  is a free factor we have:

$$\forall g \in F_N, g\partial H \cap \partial H \neq \emptyset \iff g \in H.$$

The lamination generated by  $L(T_0)$ ,  $L_0 = F_N.L(T_0)$ , is a sublamination of  $L(T)$  closed by diagonal leaves.

Assume first that there exists  $g \in F_N \setminus H$  such that  $g\overline{T_0} \cap \overline{T_0} \neq \emptyset$ , and let  $x, y \in \overline{T_0}$  such that  $x = gy$ . By Proposition 2.4 the map  $Q : \partial H \rightarrow \widehat{T_0}$  is onto, thus there exist,  $X, Y \in \partial H$  such that  $Q(X) = x$  and  $Q(Y) = y$ . By definition of the dual lamination,  $(X, gY)$  is a leaf of  $L(T)$ ; by construction  $(X, gY)$  is not diagonal over  $L_0$ .

We now assume, that  $\forall g \in F_N \setminus H, g\overline{T_0} \cap T_0 = \emptyset$ . The action of  $F_N$  on  $T$  has dense orbits, thus there exists a sequence  $g_n \in F_N \setminus H$  such that  $d(T_0, g_n T_0) < \frac{1}{n}$ . Fixing a basis  $B = \{a_1, \dots, a_r\}$  of  $H$  which is completed to a basis  $A = \{a_1, \dots, a_N\}$  of  $F_N$ , we can write  $g_n = h_n \cdot g'_n$  in reduced form with  $h_n \in H$  and  $g'_n$  starting with a letter in  $A \setminus B$ . Of course  $d(T_0, g'_n T_0) < \frac{1}{n}$ . The action of  $H$  on  $T_0$  has dense orbits thus for any point  $y \in T_0$  there exist  $h'_n \in H$  such that  $d(T_0, g'_n h'_n y) < \frac{1}{n}$ . By our assumption the sequence  $|g'_n h'_n|$  goes to infinity and there is a subsequence converging to  $Y \in \partial F_N$ . The first letter of  $Y$  written as an infinite reduced word is in  $A \setminus B$  thus  $Y \notin \partial H$ . Now we use the weaker observers' topology so that  $\widehat{T}^{\text{obs}}$  is compact and, we extract again a subsequence to have  $g'_n h'_n y$  converging to a point  $x \in \widehat{T_0}^{\text{obs}}$ . We get that  $Q(Y) = x$ , but as  $x \in \widehat{T_0}$  there exists  $X \in \partial H$  such that  $Q(X) = x$ . By definition of the dual lamination,  $(X, Y)$  is a leaf in  $L(T)$ ; by construction  $(X, Y)$  is not diagonal over  $L_0$ .  $\square$

Considering Propositions 3.7 and 4.11 and Lemma 5.1, to establish Theorem A, we need understand diagonal leaves in  $L(T)$  for  $T$  free and indecomposable.

## 5.2 Train Tracks and the Main Result

Let  $T$  be a free, indecomposable tree in  $\overline{\text{cv}}_N$ . Let  $A$  be a basis for  $F_N$ , and let  $S = (K_A, A)$  be the associated system of isometries. By Proposition 3.4,  $T$  is either surface or Levitt type. If  $T$  is surface, we run the Rips Machine on  $S$  until it halts. In either case, we get a system of isometries  $S_0 = (F_0, A_0)$  (which is equal to  $S$  if  $T$  is Levitt type), and we denote by  $S_i = (F_i, A_i)$



the result of running either the Rips machine or splitting on  $S_0$  for  $i$  steps. There are homotopy equivalences  $S_i \rightarrow \Gamma_i$  and  $\tau_i : \Gamma_i \rightarrow \Gamma_{i-1}$ .

A **turn** in  $\Gamma_i$  is a pair  $\{e, e'\}$  of directed edges with the same initial vertex. We give the graph  $\Gamma_i$  a **train track structure** by declaring a turn **legal** if it is crossed by a regular leaf, *i.e.* a regular leaf contains the subpath  $\bar{e}e'$ . Train track structures on graphs were introduced in [BH92].

**Remark 5.2.** *From Propostions 3.5 and 4.8, our inductive procedure (either the Rips machine or splitting) applied to  $\mathcal{S}_i$  has the effect of “splitting” an illegal turn in  $\Gamma_i$ : in other words, the graph morphisms  $\tau_i$  only fold at illegal turns.*

For a vertex  $v$  of  $\Gamma_i$ , let  $Leg(v)$  denote the set of legal turns in  $\Gamma_i$  at  $v$ , and let  $I(v)$  denote the set of edges of  $\Gamma_i$  with initial vertex  $v$ . Following [BH92] again, we define the **Whitehead graph**,  $Wh(v, \Gamma_i)$ , associated to the vertex  $v$  of  $\Gamma_i$ . The vertex set of  $Wh(v, \Gamma_i)$  is  $I(v)$  and there is an edge from  $e$  to  $e'$  if the turn  $\{e, e'\}$  is legal.

**Lemma 5.3.** *For every  $v \in V(\Gamma_i)$ , the Whitehead graph  $Wh(v, \Gamma_i)$  is connected.*

*Proof.* Toward contradiction suppose that there is  $i$  and  $v \in V(\Gamma_i)$  such that  $Wh(v, \Gamma_i)$  is not connected. Following the proof of [BH92, Proposition 4.5] this proves that every regular leaf of  $L(T)$  is carried by a proper free factor of  $F_N$ , contradicting Proposition 2.13.  $\square$

**Lemma 5.4.** *For any  $x$  in the limit set  $\Omega$ , there exists a regular leaf  $l \in L(T)$  such that  $Q^2(l) = x$ . In particular, if  $l \in L(T)$  is such that  $(Q^2)^{-1}(Q^2(l)) = \{l\}$  then  $l$  is regular.*

*Proof.* We first translate  $x$  in the compact limit set  $\Omega_A$ . For each  $i$  let  $v_i$  be the connected component of  $F_i$  containing  $x$ . By Lemma 5.3, there is a bi-infinite regular admissible leaf-path  $l_i$  passing through  $v_i$ . Up to passing to a subsequence,  $l_i$  converge to a bi-infinite regular admissible leaf path  $l$ . By the continuity of  $Q^2$  and arguing as in the proof of Lemma 4.10 and 3.6 we get that  $Q^2(l) = x$ .  $\square$

**Proposition 5.5.** *Let  $T \in \bar{c}v_N$  be free and indecomposable. Every leaf in  $L(T) \setminus L_r(T)$  is diagonal over  $L_r(T)$ , and there are finitely many  $F_N$ -orbits of such leaves.*

*Proof.* Let  $l$  be a leaf in  $L(T)$ , and let  $Q^2(l) = x$ . If  $(Q^2)^{-1}(x) = \{l\}$ , then by Lemma 5.4,  $l$  is regular. Assume now that  $|(Q^2)^{-1}(x)| > 1$ . From [CH],

there are finitely many orbits of such points  $x$  in  $\overline{T}$  and as the action is free,  $(Q^2)^{-1}(x)$  is finite and there are finitely many orbits of such leaves  $l$ . By Lemma 5.4, there are regular leaves in  $(Q^2)^{-1}(x)$ , and we now proceed to prove that  $l$  is in the diagonal closure of the regular leaves in  $(Q^2)^{-1}(x)$ .

Let  $A$  be a basis of  $F_N$  and let  $S = (K_A, A) = (F_0, A_0)$  be the system of isometries associated to  $T$ , and let  $S_i = (F_i, A_i)$  denote the output of  $i$  iterations of the appropriate inductive procedure (either the Rips Machine or splitting, depending on the type of  $T$ ). Let  $\Gamma_i$  denote the graph associated to  $S_i$ .

Let  $\Gamma_x$  be the (infinite graph) with vertex set the pseudo-orbit of  $x$  under the pseudo-group  $S$  (equivalently it is the intersection of the orbit of  $x$  in  $\overline{T}$  with  $F_0 = K_A$ ), and such that there is an edge labeled by  $a \in A^{\pm 1}$  between  $x.u$  and  $x.ua$ , where  $u \in F_N$  is a partial isometry defined at  $x$ .

A turn  $\{e, e'\}$  at the vertex  $y = x.u$  in  $\Gamma_x$  is legal if there exists a regular bi-infinite leaf-path  $l' \in (Q^2)^{-1}(y)$  such that  $l'([0, 1]) = e$  and  $l'([0, -1]) = e'$ . As we may have performed splittings, the point  $y$  may lie in more than one connected component of  $F_i$ . The Whitehead graph  $\text{Wh}(y, \Gamma_x)$  at  $y$  is the image of the union of the Whitehead graphs  $\text{Wh}(v, \Gamma_i)$  for all components  $v$  of  $F_i$  which contain a copy of  $y$ . From Lemma 5.3, we get that  $\text{Wh}(y, \Gamma_x)$  is connected. It follows that  $l$  is in the diagonal closure of the set of regular leaves in  $(Q^2)^{-1}(x)$ .  $\square$

Combining Propositions 3.7, 4.11, and 5.5 and Lemma 5.1 and 2.9, we get our main result:

**Theorem A.** *Let  $T$  be an  $\mathbb{R}$ -tree with a free, minimal action of  $F_N$  by isometries with dense orbits. The tree  $T$  is indecomposable if and only if  $L(T)$  is minimal up to diagonal leaves. In this case the unique minimal sublamination of  $L(T)$  is the regular sublamination, which is equal to derived sublamination of  $L(T)$ .*  $\square$

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