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Solving Absolute Value Equation using Complementarity and Smoothing Functions

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Abstract

In this paper, we consider the NP-hard problem of solving absolute value equation (AVE). We reformulate this problem as an horizontal linear complementarity problem and then use a smoothing technique to solve the problem as a sequence of concave minimization problem. This approach leads to a new method valid for a general equation without additional hypothesis on the set of solutions furthermore it remains valid as a method to solve the linear complementarity problem. We prove convergence to the original problem and an error estimate for the sequence of solutions. We also provide remarks about the algorithm and its implementation. The concave minimization problem are solved by considering a sequence of linear programs. Finally numerical results on randomly generated problems and applications are used to validate our approach and show its interest as in the general case it manages to reduce the number of unsolved problems compare to existing methods in the literature.

Keywords : smoothing function ; concave minimization ; complementarity ; absolute value equation

AMS Subject Classification : 90C59 ; 90C30 ; 90C33 ; 65K05 ; 49M20

1 Introduction

In this paper we consider the absolute value equation, which is to find $x \in \mathbb{R}^N$ such that

$$Ax - |x| = b, \quad (\text{AVE})$$

where $A \in \mathbb{R}^{N \times N}$ and $b \in \mathbb{R}^N$. This problem has been introduced by [1] in a more general form as

$$Ax + B|x| = b,$$

where $A, B \in \mathbb{R}^{M \times N}$, $b \in \mathbb{R}^M$ and unknown $x \in \mathbb{R}^N$. We focus here on (AVE), which has been more popular in the literature. The recent interest for these

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problems can be explained since frequently occurring optimization problems such as linear complementarity problem and mixed integer programming problem can be reformulated as an (AVE), see [2, 3]. The general NP-hard linear complementarity problem can be formulated as an (AVE), which implies that it is NP-hard in general. Besides in [2] it has been proved that checking if (AVE) has one or an infinite number of solutions is NP-complete.

Theoretical criteria regarding existence of solutions and unique solvability of (AVE) have been studied in [2, 4, 5, 6, 7]. An important criterion among others is that (AVE) has a unique solution if all of the singular values of the matrix A exceed 1. In the special case where the problem is uniquely solvable a family of Newton methods has been proposed first in [8], then completed with global and quadratic convergence in [9], an inexact version in [10] and other related methods [11, 12, 13]. Also Picard-HSS iteration methods and nonlinear HSS-like methods have been considered for instance in [14, 15, 16]. It is of a great interest to consider methods that remains valid in the general case. Most of such method valid in the general case are due to Mangasarian in [17, 18, 19] by considering a concave or a bilinear reformulation of (AVE) solved by a sequence of linear program. An hybrid method mixing Newton approach of [8] and [19] can be found in [20]. A method based on interval matrix has been studied by Rohn in [21, 22].

The special case where (AVE) is not solvable also received some interests in the literature. Prokopyev shows numerical results using a mixed integer programming solver in [2]. Theoretical study in order to correct b and A to make (AVE) feasible can be found in [23, 24].

Our aim in this paper is to pursue the study of (AVE) without additional hypothesis and propose a new method, which solves a sequence of linear program. The motivation is to diminish the number of instances where usual methods can not solve the problem. We propose a new reformulation of (AVE) as a sequence of concave minimization problem using complementarity and a smoothing technique. We provide analysis of the algorithm with convergence study, error estimate as well as numerical results in order to validate our approach.

This paper is organised as follow. Section 2 presents the new formulation of (AVE) as a sequence of concave minimization problem. Section 3 gives convergence to the original problem and Section 4 shows error estimate. Finally, Section 5 provides numerical results with simple examples and random generated problems.

2 AVE as a Sequence of Concave Minimization Program

We present in this section a reformulation of (AVE) as a sequence of concave minimization problem. First, we use a classical decomposition of the absolute

value to reformulate (AVE) as an horizontal linear complementarity problem. Set $x = x^+ - x^-$, where $x^+ \geq 0$, $x^- \geq 0$ and $x^+ \perp x^-$, so that $x^+ = \max(x, 0)$ and $x^- = \max(-x, 0)$. This decomposition guarantees that $|x| = x^+ + x^-$. So (AVE) is equivalent to the following complementarity problem

$$\begin{aligned} A(x^+ - x^-) - (x^+ + x^-) &= b \\ x^+ \geq 0, x^- \geq 0 & \\ x^+ \perp x^- & \end{aligned} .$$

Now, we reformulate this problem as a sequence of concave optimization problem using a smoothing technique. This technique has been first studied in [25, 26] and uses a family of non-decreasing continuous smooth concave functions $\theta : \mathbb{R} \rightarrow]-\infty, 1[$ that verifies

$$\theta(t) < 0 \text{ if } t < 0, \theta(0) = 0 \text{ and } \lim_{t \rightarrow +\infty} \theta(t) = 1 .$$

One possible way to build such function is to consider non-increasing probability density functions $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and then take the corresponding cumulative distribution function

$$\theta(t) = \int_0^t f(x) dx .$$

By definition of f we can verify that

$$\lim_{t \rightarrow +\infty} \theta(t) = \int_0^{+\infty} f(x) dx = 1 \text{ and } \theta(0) = \int_0^0 f(x) dx = 0 .$$

The non-decreasing hypothesis gives the concavity of θ . We then extend this functions for negative values in a smooth way. Examples of this family are $\theta^1(t) = t/(t+1)$ if $t \geq 0$ and $\theta^1(t) = t/r$ if $t < 0$, $\theta^2(t) = 1 - e^{-t}$ with $t \in \mathbb{R}$.

We introduce $\theta_r(t) := \theta\left(\frac{t}{r}\right)$ for $r > 0$. This definition is similar to the perspective functions in convex analysis. This functions satisfy

$$\theta_r(0) = 0 \forall r > 0 \text{ and } \lim_{r \searrow 0} \theta_r(t) = 1 \forall t > 0 .$$

Previous examples now become $\theta_r^1(t) = t/(t+r)$ if $t \geq 0$ and $\theta_r^1(t) = t$ if $t < 0$, $\theta_r^2(t) = 1 - e^{-t/r}$ $t \in \mathbb{R}$.

The following lemma shows the link between this family of functions and the complementarity in one dimension.

Lemma 2.1. *Given $s, t \in \mathbb{R}_+$ and the parameter $r > 0$, then*

$$s \perp t \iff \lim_{r \searrow 0} \theta_r(s) + \theta_r(t) \leq 1 .$$

Proof. Prove by contradiction that

$$\lim_{r \searrow 0} \theta_r(s) + \theta_r(t) \leq 1 \implies s \perp t .$$

Suppose $s, t > 0$, then

$$\lim_{r \searrow 0} (\theta_r(s) + \theta_r(t)) = \lim_{r \searrow 0} \theta_r(s) + \lim_{r \searrow 0} \theta_r(t) = 2 .$$

This leads to a contradiction and therefore $s \perp t$. Conversely it is clear that $s \perp t$ implies $s = 0$ or $t = 0$ and the result follows. \square

In the case of the function θ_r^1 we even have the equality in (2.1) and by definition of this function it holds that

$$\theta_r^1(s) + \theta_r^1(t) = 1 \iff st = r^2 .$$

Using the previous lemma, the complementarity constraint is replaced by a sequence of concave optimization problem for $r > 0$:

$$\begin{aligned} \min_{x^+, x^- \in \mathbb{R}^N} \quad & \sum_{i=1}^N \theta_r(x_i^+) + \theta_r(x_i^-) - 1 \\ A(x^+ - x^-) - (x^+ + x^-) &= b \\ x^+ \geq 0, x^- \geq 0 \end{aligned} \quad . \quad (1)$$

In order to avoid compensation phenomenon and generate strictly feasible iterate we consider a relaxed version defined as

$$\begin{aligned} \min_{x^+, x^- \in \mathbb{R}^N} \quad & \sum_{i=1}^N \theta_r(x_i^+) + \theta_r(x_i^-) - 1 \\ b - g(r)|A|e - g(r)e \leq A(x^+ - x^-) - (x^+ + x^-) &\leq b + g(r)|A|e + g(r)e, (P_r) \\ x^+ + x^- \geq g(r)e \\ 0 \leq x^+ \leq M, 0 \leq x^- \leq M \end{aligned}$$

where e is the unit vector, $|A|$ denotes the matrix where each element is the absolute value of the corresponding element in A and $g : \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$ is a function which goes to 0 slower than r , that is

$$\lim_{r \searrow 0} \frac{r}{g(r)} = 0 \text{ and } \lim_{r \searrow 0} g(r) = 0$$

for instance let $g(r) = r^\alpha$ with $0 < \alpha < 1$.

3 Convergence

From now on, we suppose that the set of solution of (AVE) is non-empty and bounded. We will denote it by $S_{(AVE)}^*$. Denote $S_{(P_r)}^*$ the optimal set of (P_r) . In order to simplify the notation, we denote $x \in S_{(P_r)}^*$ when $(x^+, x^-) \in S_{(P_r)}^*$

with $x = x^+ - x^-$ and $x^+ = \max(x, 0)$, $x^- = \max(-x, 0)$. Let M be a positive constant such that

$$M \geq \max_{x \in S_{(AVE)}^*} \|x\|_\infty .$$

The following theorem shows that for $r \geq 0$, the set of solutions $S_{(P_r)}^*$ is non-empty.

Theorem 3.1. *(P_r) has at least one solution for any $r \geq 0$.*

Proof. Since $S_{(AVE)}^* \neq \emptyset$ then there exists a point $\bar{x} \in S_{(AVE)}^*$ and $\bar{x} = \bar{x}^+ - \bar{x}^-$ with $\bar{x}^+ \perp \bar{x}^-$. It follows that $(y^{r+} := \bar{x}^+ + g(r), y^{r-} := \bar{x}^-)$ is a feasible point of (P_r) . Furthermore, we minimize a continuous function over a non-empty compact set so the objective function attains its minimum. \square

We present now two lemmas that are used to prove the main convergence Theorem 3.5.

Lemma 3.2. *Given the functions θ_r and g defined above and $x^+, x^- \in \mathbb{R}_+^N$, $r \in \mathbb{R}_+^*$ such that $x^+ + x^- \geq g(r)e$. It holds that for all $i \in \{1, \dots, N\}$*

$$\theta_r(x_i^+) + \theta_r(x_i^-) - 1 \geq \theta_r(g(r)) - 1 .$$

Proof. θ_r is concave and $\theta_r(0) = 0$ so θ_r is subadditive. So, for all $i \in \{1, \dots, N\}$ it follows that

$$\theta_r(x_i^+) + \theta_r(x_i^-) - 1 \geq \theta_r(x_i^+ + x_i^-) - 1 .$$

Since θ_r is non-decreasing and $x^+ + x^- \geq g(r)e$ we can conclude that

$$\theta_r(x_i^+) + \theta_r(x_i^-) - 1 \geq \theta_r(g(r)) - 1 .$$

\square

Lemma 3.3. *Given functions θ_r and g defined above we have*

$$\lim_{r \searrow 0} \theta_r(g(r)) - 1 = 0 .$$

Proof. By definition $\theta_r(g(r)) = \theta_{\frac{r}{g(r)}}(1)$ and since $\lim_{r \searrow 0} r/g(r) = 0$, it holds

$$\lim_{r \searrow 0} \theta_r(g(r)) = \lim_{r \searrow 0} \theta_{\frac{r}{g(r)}}(1) = 1 . \quad (2)$$

\square

In the special case, where every solution of (AVE) has at least a zero component it can be difficult to find a feasible point that satisfy the constraint $x^+ + x^- \geq g(r)e$. The following lemma explains how to build such point in this case.

Lemma 3.4. Let \bar{x} be a solution of (AVE) and $r > 0$ be such that $g(r) < \min_{\bar{x}_i \neq 0} |\bar{x}_i|$. Then $y^r := \bar{x} + g(r)$ is a solution of $(AVE)_r$ defined as

$$Ax - |x| = b + g(r)Ae - g(r)\delta(x) , \quad ((AVE)_r)$$

where $\delta(x) \in \mathbb{R}^N$ is such that $\delta_i(x) := \begin{cases} 1 & \text{if } x_i \geq 0 \\ -1 & \text{if } x_i < 0 \end{cases}$.

Proof. \bar{x} is a solution of (AVE), then

$$A\bar{x} - |\bar{x}| = b .$$

Therefore it holds

$$\begin{aligned} \Rightarrow & A\bar{x} + g(r)Ae - |\bar{x}| - g(r)\delta(x) = b + g(r)Ae - g(r)\delta(x), \\ \Rightarrow & A(\bar{x} + g(r)e) - |\bar{x} + g(r)e| = b + g(r)Ae - g(r)\delta(x), \end{aligned}$$

and so $y^r = \bar{x} + g(r)$ is a solution of $((AVE)_r)$. \square

We now proceed to the convergence proof of the sequence of $\{x^r\}_{r>0}$ to an element of $S_{(AVE)}^*$, where $x^r := x^{r+} - x^{r-}$ with (x^{r+}, x^{r-}) optimal solution of (P_r) . It is to be noted that the set of solution $S_{(P_r)}^*$ is a singleton is not necessarily a singleton.

Theorem 3.5. Every limit point of the sequence $\{x^r\}$ such that $x^r \in S_{(P_r)}$ for $r > 0$ is a solution of (AVE).

Proof. By Theorem 3.1 there exists at least one solution of (AVE). According to Lemma 3.4 we can build a sequence $\{y^r\}_{r>0}$ where $y^r = y^{r+} - y^{r-}$ with $y^{r+} \perp y^{r-}$ that are solution of $(AVE)_r$. Furthermore, for r sufficiently small (y^{r+}, y^{r-}) is a feasible point of (P_r) . Let $x^r = (x^{r+}, x^{r-})$ with $\{x^r\}_{r>0}$ be a sequence of optimal solution of (P_r) , then

$$\sum_{i=1}^N (\theta_r(x_i^{r+}) + \theta_r(x_i^{r-}) - 1) \leq \sum_{i=1}^N (\theta_r(y_i^{r+}) + \theta_r(y_i^{r-}) - 1) \leq 0 .$$

For all $i \in \{1, \dots, N\}$, it holds

$$\theta_r(x_i^{r+}) + \theta_r(x_i^{r-}) - 1 \leq - \sum_{j=1; j \neq i}^N (\theta_r(x_j^{r+}) + \theta_r(x_j^{r-}) - 1) .$$

By Lemma 3.2, we have for all $i \in \{1, \dots, N\}$ that

$$\theta_r(x_i^{r+}) + \theta_r(x_i^{r-}) \leq 1 + (N - 1)(1 - \theta_r(g(r))) .$$

For every limit point $\bar{x} = (\bar{x}^+, \bar{x}^-)$ of the sequence $\{x^r\}_r$, where $\bar{x}^+ = \lim_{r \searrow 0} x^{r+}$ and $\bar{x}^- = \lim_{r \searrow 0} x^{r-}$ and using lemma (3.3) ($\lim_{r \searrow 0} 1 - \theta_r(g(r)) = 0$) passing to the limit it follows

$$\lim_{r \searrow 0} \theta_r(\bar{x}_i^+) + \theta_r(\bar{x}_i^-) \leq 1 .$$

We have $\bar{x}^+ \perp \bar{x}^-$ by the previous inequality and Lemma 2.1.

Now, we verify that \bar{x} is a solution of (AVE). Let x^r be a solution of (P_r) for $r > 0$, we have

$$b - g(r)|A|e - g(r)e \leq A(x^{r+} - x^{r-}) - (x^{r+} + x^{r-}) \leq b + g(r)|A|e + g(r)e.$$

Passing to the limit $r \searrow 0$ leads to

$$A(\bar{x}^+ - \bar{x}^-) - (\bar{x}^+ + \bar{x}^-) = b.$$

So, (\bar{x}^+, \bar{x}^-) is a solution of (P_r) and $\bar{x} = \bar{x}^+ - \bar{x}^-$ is a solution of (AVE). \square

4 Error Estimate

In this section we study the behaviour of the sequence $\{x^r\}_{r>0}$ of optimal solution of (P_r) when r becomes small. We remind the definition of the Landau notation O often used in the context of asymptotic comparison. Given two functions f and h . We have

$$f(x) = O_{x \rightarrow a}(h(x)) \text{ if } \exists C > 0, \exists d > 0, \forall x, |x - a| \leq d \implies |f(x)| \leq C|h(x)|.$$

We denote $O(h(x))$ when a is 0.

We first show a useful lemma, which does not need the hypothesis of the existence of a solution without zero component.

Lemma 4.1. *Let θ_r be such that $\theta_r \geq \theta_r^1$. For $x^r \in S_{(P_r)}^*$ and r sufficiently small, we have*

$$x_i^{r+} x_i^{r-} \leq O(rg(r)) \quad \forall i \in \{1, \dots, N\}.$$

Proof. Set $i \in \{1, \dots, N\}$. Thanks to the convergence proof of the Theorem 3.5 for r sufficiently small the following holds

$$\theta_r(x_i^{r+}) + \theta_r(x_i^{r-}) - 1 \leq (N - 1)(1 - \theta_r(g(r))).$$

Using the definition of θ_r^1 function gives

$$\begin{aligned} \theta_r^1(x_i^{r+}) + \theta_r^1(x_i^{r-}) - 1 &\leq \theta_r(x_i^{r+}) + \theta_r(x_i^{r-}) - 1, \\ &\leq (N - 1)(1 - \theta_r(g(r))), \\ &\leq (N - 1)(1 - \theta_r^1(g(r))). \end{aligned}$$

Therefore for r sufficiently small such that $g(r) \geq r$ and $(1 - (N - 1)g(r)) > 0$

it holds that

$$\begin{aligned}
\frac{x_i^{r+}}{x_i^{r+} + r} + \frac{x_i^{r-}}{x_i^{r-} + r} - 1 &\leq (N-1)\left(1 - \frac{g(r)}{g(r) + r}\right) \leq (N-1)g(r), \\
\frac{2x_i^{r+}x_i^{r-} + rx_i^{r+} + rx_i^{r-} - x_i^{r+}x_i^{r-} - rx_i^{r+} - rx_i^{r-} - r^2}{(x_i^{r+} + r)(x_i^{r-} + r)} &\leq (N-1)g(r), \\
\frac{x_i^{r+}x_i^{r-} - r^2}{(x_i^{r+} + r)(x_i^{r-} + r)} &\leq (N-1)g(r), \\
x_i^{r+}x_i^{r-} - r^2 &\leq (N-1)g(r)(x_i^{r+}x_i^{r-} + rx_i^{r+} + rx_i^{r-} + r^2), \\
x_i^{r+}x_i^{r-} &\leq r^2g(r)\frac{1 + (N-1)}{1 - (N-1)g(r)} + rg(r)\frac{(N-1)(x_i^{r+} + x_i^{r-})}{1 - (N-1)g(r)},
\end{aligned}$$

and the results follows. \square

The following proposition gives an error estimate of the components of (x^{r+}, x^{r-}) that go to zero.

Proposition 4.2. *Let θ_r be such that $\theta_r \geq \theta_r^1$. Let (\bar{x}^+, \bar{x}^-) be a limit point of the sequence $\{x^{r+}, x^{r-}\}_r$ of optimal solutions of (P_r) . The convergence of the components of the variable x^{r+} or x^{r-} to the possibly zero part of the accumulation point is done in $O(r)$.*

Proof. Set $i \in \{1, \dots, N\}$. We work with one component. Assume that $\bar{x}_i^+ = 0$. The opposite case is completely similar. By Lemma 4.1 and using that (x^{r+}, x^{r-}) is feasible so $x^{r+} + x^{r-} \geq g(r)$, we have

$$\begin{aligned}
x_i^{r+}x_i^{r-} &\leq O(rg(r)), \\
x_i^{r+} &\leq \frac{O(rg(r))}{x_i^{r-}}, \\
x_i^{r+} &\leq \frac{O(rg(r))}{g(r)}, \\
|x_i^{r+} - \bar{x}_i^+| &\leq O(r).
\end{aligned}$$

\square

In the next theorem we provide an error estimate of the possibly non-zero part of the solution in the couple (x^+, x^-) .

To establish this result, we use the classical Hoffman's lemma.

Lemma 4.3 (Hoffman's lemma). *[27] Given a convex polyhedron P such that*

$$P = \{x \mid Ax \leq b\}.$$

We set $d_P(x)$ the distance from x to P , by choosing a norm $\|\cdot\|$, where $d_P(x) = \inf_{y \in P} \|y - x\|$. There exists a constant K which only depends on A , such that

$$\forall b, \forall x \in \mathbb{R}^n : d_P(x) \leq K\|(Ax - b)^+\|.$$

It is to be noted that if the constraints are given by $Ax = b$ with A a square full-rank matrix instead of $Ax \leq b$ then the polyhedron is reduced to a singleton and we can estimate the constant as $K = \|A^{-1}\|$.

Theorem 4.4. *Given (\bar{x}^+, \bar{x}^-) an accumulation point of the sequence $\{x^{r+}, x^{r-}\}_{r>0}$ of optimal solutions of (P_r) , where we denote $\bar{x} = \bar{x}^+ - \bar{x}^-$ and $x^r = x^{r+} - x^{r-}$. Then, for r sufficiently small*

$$d_{S^*}(x^r) = O(g(r)) , \quad (3)$$

where S^* denote the intersection of $S^*_{(AVE)}$ and a neighbourhood V of \bar{x} , such that any point in V has the same sign than \bar{x} .

Proof. We split the proof in two cases, either $\min_{i \in \{1, \dots, N\}} |\bar{x}_i| \neq 0$, either $\exists i \in \{1, \dots, N\}$, $\bar{x}_i = 0$ respectively denoted as a) and b).

a) First, assume that there is no zero component in \bar{x} . Set $\alpha = \min_{i \in \{1, \dots, N\}} |\bar{x}_i|/2$ and a neighbourhood V of \bar{x} defined as

$$V = B_\infty(\bar{x}, \alpha) = \{x \mid \max_{1 \leq i \leq N} |x_i - \bar{x}_i| \leq \alpha\} .$$

For all $x \in V$, \bar{x} and x have the same sign. We set $D = \text{diag}(\delta(\bar{x}))$, where

$$\delta(x) \in \mathbb{R}^N \text{ with } \delta_i(x) = \begin{cases} 1 & \text{if } x_i \geq 0 \\ -1 & \text{if } x_i < 0 \end{cases} .$$

By taking $S^* = \{x \in \mathbb{R}^n \mid Ax - Dx = b\} \cap V$ we obtain a convex polyhedron. This set is non-empty because $\bar{x} \in S^*$. In the neighbourhood V solving $Ax - Dx = b$ gives a solution of (AVE). Use Hoffman lemma for r sufficiently small such that $x_r \in V$ then

$$\begin{aligned} d_{S^*}(x^r) &\leq K \left\| \begin{pmatrix} (A - D)x^r - b \\ (x^r - \alpha - \bar{x})^+ \\ (-x^r - \alpha + \bar{x})^+ \end{pmatrix} \right\| \leq K(\|(A - D)x^r - b\| + \|(x^r - \alpha - \bar{x})^+\| \\ &\quad + \|(-x^r - \alpha + \bar{x})^+\|), \\ &= K\|(A - D)x^r - b\|, \\ &= K\|Ax^r - |x^r| - b\|. \end{aligned}$$

Since x^r is feasible for (P_r) , it holds that

$$\begin{aligned} \|Ax^r - |x^r| - b\| &= \|g(r)Ae - g(r)\delta(x^r)\|, \\ &= \|(Ae - \delta(x^r))g(r)\|, \\ &\leq \|Ae - \delta(x^r)\| |g(r)|, \\ &= \|Ae - \delta(x^r)\|g(r) = O(g(r)). \end{aligned}$$

Combining both previous inequality gives

$$d_{S^*}(x^r) \leq K\|Ae - \delta(x^r)\|g(r) = O(g(r)) ,$$

and the result follows for the case a).

b) Now we move to the case where $\exists i \in \{1, \dots, N\}$, $\bar{x}_i = 0$. We denote $\sigma(t) = \{i | t_i \neq 0\}$. Set $\alpha = \min_{i \in \sigma(\bar{x})} |\bar{x}_i|/2$ and a neighbourhood V of \bar{x} defined as

$$V = B_\infty(\bar{x}, \alpha) = \{x \mid \max_{i \in \sigma(\bar{x})} |x_i - \bar{x}_i| \leq \alpha\}.$$

V is non-empty because $\bar{x} \in V$. For all $x \in V$, \bar{x} and x have the same sign only for the components \bar{x}_i with $i \in \sigma(\bar{x})$. Furthermore, for r sufficiently small we have $x^r \in V$.

Taking $S^* = \{x \in \mathbb{R}^n \mid Ax - Dx = b, Dx \geq 0\} \cap V$ with $D = \text{diag}(\delta(x^r))$ we obtain a convex polyhedron. The choice of D depending on x^r is not restrictive as we can always take a subsequence of the sequence $\{x^r\}_{r>0}$, which converge to \bar{x} , with constant signs near \bar{x} . This set is non-empty because $\bar{x} \in S^*$. In the neighbourhood V the solving $Ax - Dx = b$ with the constraints $Dx \geq 0$ gives a solution of (AVE). We can use Hoffman lemma to get

$$\begin{aligned} d_{S^*}(x^r) &\leq K \left\| \begin{array}{c} (A-D)x^r - b \\ (x^r - \alpha - \bar{x})^+ \\ (-x^r - \alpha + \bar{x})^+ \\ (-Dx)^+ \end{array} \right\| &\leq K(\|(A-D)x^r - b\| + \|(x^r - \alpha - \bar{x})^+\| \\ &+ \|(-x^r - \alpha + \bar{x})^+\| + \|(-Dx)^+\|), \\ &= K\|(A-D)x^r - b\|, \\ &= K\|Ax^r - |x^r| - b\|. \end{aligned}$$

As x^r is feasible for (P_r) , we have

$$\begin{aligned} \|Ax^r - |x^r| - b\| &\leq \|g(r)Ae - g(r)\delta(x^r)\|, \\ &= \|(Ae - \delta(x^r))g(r)\|, \\ &\leq \|Ae - \delta(x^r)\| |g(r)|, \\ &= \|Ae - \delta(x^r)\|g(r) = O(g(r)). \end{aligned}$$

Combining both previous inequalities gives

$$d_{S^*}(x^r) \leq K\|Ae - \delta(x^r)\|g(r) = O(g(r)),$$

and the result for the case b) follows. This completes the proof. \square

Remark 1. We can be a bit more specific in the case where $(A-D)$ is invertible. In this case $S^* = \{\bar{x}\}$, so (3) becomes

$$\|x^r - \bar{x}\| \leq \|(A-D)^{-1}\| \|Ae - \delta(x)\|g(r) = O(g(r)).$$

This case corresponds to the special cases where (AVE) has isolated solutions.

5 Algorithm

In the previous sections we have presented theoretical results about convergence and error estimate of an algorithm to compute a solution of (AVE). In this section we focus on the algorithm and we make some remarks about the parameters and its implementation. Consider the generic algorithm with C the feasible set of (P_{r^k}) :

$$\begin{cases} \{r^k\}_{k \in \mathbb{N}}, r^0 > 0 \text{ and } \lim_{k \rightarrow +\infty} r^k = 0 \\ \text{find } x^k : x^k \in \arg \min_{x \in C} \sum_{i=1}^n \theta_{r^k}(x_i^+) + \theta_{r^k}(x_i^-) - 1 \end{cases} \quad (\text{TAVE})$$

Considering a practical implementation of (TAVE) one should probably more likely use the initial problem (1) with the constraint $x^+ + x^- \geq g(r)e$. The sequence of computed points will probably be infeasible but we believe that it leads to improved numerical behaviour. The constraint $x^+ + x^- \geq g(r)e$ prevents the sequence to possibly go to a local minimum with a zero component.

Algorithm TAVE requires an initial point. In a same way as in [19, 20] we solve the following linear program

$$\begin{aligned} \min_{x^+, x^- \in \mathbb{R}^N} & (x^+ + x^-)^T e \\ & A(x^+ - x^-) - (x^+ + x^-) = b \cdot \\ & 0 \leq x^+, 0 \leq x^- \end{aligned}$$

This program find an initial feasible point of (P_r) and the objective function may encourage this point to satisfy the complementarity condition.

In this study we put the variables in a compact set. Indeed, the functions θ_r are more efficient when their arguments lives in $[0, 1]$. Besides, we use one way to express complementarity with Lemma 2.1 another way, which will be used in the numerical study, is to consider the following

$$\theta_r(s) + \theta_r(t) - \theta_r(s+t) = 0. \quad (4)$$

In this case we don't necessarily need the constraint $x^+ + x^- \geq r^\alpha$, since it is a reformulation of the complementarity and no longer a relaxation.

Regarding the choice of the parameters α , r_0 and the update parameter of r , it is to be noted that they are all used in the constraint $x^+ + x^- \geq g(r)e$ with $g(r) = r^\alpha$ and $0 < \alpha < 1$. Theorem 4.4 shows that the convergence to the zero part of the solution is a $O(g(r))$. So it is clear that α needs to be taken as big as possible, for instance $\alpha = 0.99$. Also there is a link between the value of α and the update in r . About the latter we choose to select a constant sequence of value with an update constant T , so that $r^{k+1} = \frac{r^k}{T}$. The initial parameter r_0 can be chosen according the relation in one dimension mentioned in the introduction of functions θ^1 , which we remind here

$$\theta_r^1(s) + \theta_r^1(t) = 1 \iff st = r^2 .$$

T	initial $r : r_0$	function θ_r	α
1.8	1	θ_r^2	0.99

Table 1: Parameters for the simulations

At each step in r we solve a concave optimization problem to get the current point. The following heuristic can be rather useful to accelerate convergence ensure a good precision when we are close to the solution and. After finding the current point x^k we solve if possible the linear system

$$(A - \text{diag}(\delta(x^k)))z = b. \quad (5)$$

If x solves (AVE), then the algorithm is finished and we solved (AVE) with the same precision as we solve the linear system. However, if x does not solve (AVE), then we continue the iteration in r with x^k . This idea is similar to compute a Newton iteration.

6 Numerical Simulations

We present numerical results on two examples and randomly generated problems. These simulations have been done using MATLAB, [28], with the linear programming solver GUROBI, [29]. We use the Successive Linearisation Algorithm (SLA) of [30] to solve our concave minimization problem at each iteration in r .

Proposition 6.1 (SLA for concave minimization). *Given ϵ sufficiently small and r^k . Denote C the feasible set of (P_{r^k}) . Given $x^k = x^{k+} - x^{k-}$, x^{k+1} is designed as a solution of the linear problem*

$$\min_{y^+, y^- \in C} (y^+)^T \nabla \theta_{r^k}(x^{k+}) + (y^-)^T \nabla \theta_{r^k}(x^{k-}),$$

with $x^0 = x^{0+} - x^{0-}$ a random point. We stop when

$$x^{k+1} \in C \text{ and } (x^{k+1} - x^k)^T \nabla \theta_{r^k}(x^k) \leq \epsilon.$$

This algorithm generates a finite sequence with strictly decreasing objective function values.

Proof. see [[30], Theorem 4.2]. □

In SLA we add solving (AVE) as a supplementary stopping criterion. Along these simulations we used the parameters detailed in Table 1 for TAVE. The maximum number of iterations in r for one instance is 20 and the maximum number of linear program for one SLA is 10. We measure the time in seconds, the number of linear program solved and the number of linear system solved respectively denoted by nb-LP-method and nb-lin-syst-method.

In order to confirm the validity of our method we consider two concrete examples and then two kind of random generated problems. The first one is a second order ordinary differential equation with initial conditions and the second example is an obstacle problem. We remind that the principal motivation of this algorithm is to consider general kind of (AVE) and this case has been treated in the last simulation.

6.1 An Ordinary Differential Equation

We consider the ordinary differential equation

$$\ddot{x}(t) - |x(t)| = 0, \quad x(0) = x_0, \quad \dot{x}(0) = \gamma .$$

We get an (AVE) by using a finite difference scheme in order to discretize this equation. We use the following second-order backward difference to approximate the derivative

$$\frac{x_{i-2} - 2x_{i-1} + x_i}{h^2} - |x_i| = 0 .$$

This equation was derived with an equispace gridpoints $x_i = ih$, $i = 1, \dots, N$. In order Neumann boundary conditions were approximated using a center difference

$$\frac{x_{-1} - x_1}{2h} = \gamma .$$

We compare the obtained solution by TAVE to the one of the predefined Runge-Kutta ode45 function in MATLAB, [28]. The domain is $t \in [0, 4]$, initial conditions $x_0 = -1$, $\gamma = 1$ and $N = 100$. Results are presented in Figure 1. TAVE solves the problem and gives consistent results.

6.2 Obstacle Problem

The second example is a simple obstacle problem. We try to find a trajectory joining the bounds of a domain with an obstacle, g , and a minimal curvature, f . This can be formulated as the following equation

$$(\ddot{u}(x) - f(x))^T(u(x) - g(x)) = 0, \quad \ddot{u}(x) - f(x) \geq 0, \quad u(x) - g(x) \geq 0 .$$

We approximate the second order derivative with a second-order central difference, then the previous equation is similar to some discrete version on an equispace gridpoints $x_i = ih$, $i = 1, \dots, N$.

$$(Du - f)^T(u - g) = 0, \quad Du - f \geq 0, \quad u - g \geq 0 ,$$

where $g_i = g(x_i)$, $f_i = f(x_i)$. This can be reformulated as a linear complementarity problem with by setting $z = u - g$, $M = D$ and $q = Dg - f$, that is

$$(Mz + q)^T z = 0, \quad Mz + q \geq 0, \quad z \geq 0 .$$

This equation is equivalent to (AVE) whenever 1 is not an eigenvalue of M by proposition 2 of [17]. So, consider equation

$$(M - I)^{-1}(M + I)x - |x| = (M - I)^{-1}q .$$

We give results for our method and LPM method from [19], with $g(x) = \max(0.8 - 20(x - 0.2)^2, \max(1 - 20(x - 0.75)^2, 1.2 - 30(x - 0.41)^2))$, $f(x) = 1$, $N = 50$ in Figure 2. Both methods give 20 points on the curve g and none below g over 50 points. Once again TAVE method gives consistent results.

6.3 Random Uniquely Solvable Generated Problem

We now consider the special case where (AVE) is uniquely solvable. One way to generate such problem is to generate a matrix A with singular values exceeding 1. Following [31] the data (A, b) are generated by the following Matlab code for $n = 100, 200, 400, 800$

```
n=input('dimension of matrix A =');
rand('state',0);
R=rand(n,n);
b=rand(n,1);
A=R'*R+n*eye(n);
```

The required precision for solving (AVE) is 10^{-6} and thanks to the heuristic from Section 5 in equation (5) we get in the worst case 10^{-10} . For each n we consider 100 instances of the problem and compare TAVE to a Newton method from [8], which we denote GN. Results are sum up in Table 2, which give for TAVE the number of linear program solved, the time required to solve all the instances and give for GN the number of linear systems and the time required. Note that other Newton methods like [9, 11, 12] should give similar conclusions so we do not include them in our comparisons.

In every cases our method solves the problem, which once again valid our approach. It is to be noted that the number of solved linear program is very low. Indeed in every case the initialization step has been sufficient to solve the problem. Also we notice that the time required to solve problem is increasing significantly when the dimension grows. It is not a surprise that in this simulations the method GN outperform in time TAVE since it only requires to solve a few linear systems compare to few linear programs. Since some other methods to solve (AVE) have no problem solving this instances they are not our main focus here.

6.4 Random Generated Problem

We present results for general (AVE), which is the main interest of our approach. The data are generated like in [17] for several n and for several values of the parameters, in each case we solve one hundred instances of the problem.

"Choose a random A from a uniform distribution on $[-10, 10]$, then choose a random x from a uniform distribution on $[-1, 1]$ and set $b = Ax - |x|$."

We compare 4 methods valid for general (AVE) : TAVE method from Algorithm TAVE, TAVE2 which is the same algorithm with the different objective (4), concave minimization method CMM from [17] and successive linear programming method LPM from [19].

We give "nnztot" the number of violated expression for all problems, "nnzx" the maximum violated expression for one problem, the number of iteration in r "out-iter" and the number of linear program solve for all the problems "in-iter". We also provide the time in seconds and the number of problems, which we did not solve. The results are presented in Table 3, 4, 5 and 6.

In every cases our methods manage to reduce the number of unsolved problem, which was our principal aim. This confirm the interest of the relaxation method presented here. Also one should note that an improved number of solve problem comes with a price, since it requires more time.

Table 4 shows promising results for TAVE2. It is a slightly different method, since it is not a relaxation but a reformulation of the complementarity. In every case it gives the smallest number of unsolved problem in a very reasonable time.

Conclusion and Perspectives

In this paper, a class of heuristics schemes to solve the NP-hard problem of solving (AVE) has been proposed. A complete analysis has been provided including convergence, error estimate and comments on the implementation. Furthermore, a numerical study shows that our method is full of interest. Indeed, our methods prove to be consistent on real examples and problems with unique solution. We do not compare the performance of our method with the methods designed specifically for these problems since they do not belong to same class. Finally, the last set of generated problems consider general (AVE). We compare our methods to existing methods and in each case it manages to improve the number of failure, which was our principle aim.

It is of interest to note that the methods presented here could also be used to solve the linear complementarity problem using the same technique as in the obstacle problem example.

Further studies could improve the choice of parameters in order to reduce the computational time to solve the problems especially for large instances. Promising results were shown by the modified algorithm TAVE2, which consider a slightly differ way to express the complementarity. So we may wonder if it is possible to improve our algorithms in this case and if there exists other similar reformulation of the complementarity which can give even better results. Finally in [20] they proposed an hybrid algorithm with the benefits of both the minimization methods and Newton methods with encouraging numerical results. This philosophy is fully applicable to the methods proposed here and could lead to further improvements.

References

- [1] Rohn J. A theorem of the alternatives for the equation $ax + b|x| = b$. *Linear and Multilinear Algebra*. 2004;52(6):421–426.
- [2] Prokopyev O. On equivalent reformulations for absolute value equations. *Computational Optimization and Applications*. 2009;44(3):363–372.
- [3] Mangasarian O. Linear complementarity as absolute value equation solution. *Optimization Letters*. 2014;8(4):1529–1534.
- [4] Mangasarian O, Meyer R. Absolute value equations. *Linear Algebra and Its Applications*. 2006;419(2):359–367.
- [5] Rohn J, Hooshyarbakhsh V, Farhadsefat R. An iterative method for solving absolute value equations and sufficient conditions for unique solvability. *Optimization Letters*. 2014;8(1):35–44.
- [6] Rohn J. On unique solvability of the absolute value equation. *Optimization Letters*. 2009;3(4):603–606.
- [7] Lotfi T, Veisheh H. A note on unique solvability of the absolute value equation. 2013;.
- [8] Mangasarian O. A generalized newton method for absolute value equations. *Optimization Letters*. 2009;3(1):101–108.
- [9] Caccetta L, Qu B, Zhou G. A globally and quadratically convergent method for absolute value equations. *Computational Optimization and Applications*. 2011;48(1):45–58.
- [10] Bello Cruz JY, Ferreira OP, Prudente LF. On the global convergence of the inexact semi-smooth Newton method for absolute value equation. *Computational Optimization and Applications*. 2016 sep;65(1):93–108.
- [11] Haghani FK. On generalized traub’s method for absolute value equations. *Journal of Optimization Theory and Applications*. 2015;:1–7.
- [12] Moosaei H, Ketabchi S, Noor M, Iqbal J, Hooshyarbakhsh V. Some techniques for solving absolute value equations. *Applied Mathematics and Computation*. 2015;268:696–705.
- [13] Yong L. A Smoothing Newton Method for Absolute Value Equation. *International Journal of Control and Automation*. 2016 feb;9(2):119–132.
- [14] Salkuyeh DK. The picard–hss iteration method for absolute value equations. *Optimization Letters*. 2014;8(8):2191–2202.
- [15] Qi YE, Zhu MZ. The nonlinear HSS-like iteration method for absolute value equations. *Applied Mathematics and Computation*. 2014 mar; 265(11271174):266–274.

- [16] Zhang JJ. The relaxed nonlinear phss-like iteration method for absolute value equations. *Applied Mathematics and Computation*. 2015;265:266–274.
- [17] Mangasarian O. Absolute value equation solution via concave minimization. *Optimization Letters*. 2007;1(1):3–8.
- [18] Mangasarian OL. Absolute value equation solution via dual complementarity. *Optimization Letters*. 2013;7(4):625–630.
- [19] Mangasarian OL. Absolute value equation solution via linear programming. *Journal of Optimization Theory and Applications*. 2014;161(3):870–876.
- [20] Mangasarian OL. A hybrid algorithm for solving the absolute value equation. *Optimization Letters*. 2015;;1–6.
- [21] Rohn J. An algorithm for solving the absolute value equation. *Electronic Journal of Linear Algebra*. 2009;18(589-599):5.
- [22] Rohn J. An algorithm for solving the absolute value equation: An improvement; 2010. Report no.:
- [23] Ketabchi S, Moosaei H. An efficient method for optimal correcting of absolute value equations by minimal changes in the right hand side. *Computers & Mathematics with Applications*. 2012;64(6):1882–1885.
- [24] Ketabchi S, Moosaei H, Fallahi S. Optimal error correction of the absolute value equation using a genetic algorithm. *Mathematical and Computer Modelling*. 2013;57(9):2339–2342.
- [25] Migot T, Haddou M. A smoothing method for sparse optimization over polyhedral sets. In: *Modelling, computation and optimization in information systems and management sciences*. Springer; 2015. p. 369–379.
- [26] Haddou M, Maheux P. Smoothing methods for nonlinear complementarity problems. *Journal of Optimization Theory and Applications*. 2014; 160(3):711–729.
- [27] Hoffman AJ. On approximate solutions of systems of linear inequalities. *Journal of Research of the National Bureau of Standard*. 1952;49(2):pp 263–265.
- [28] MATLAB. version r2013. Natick, Massachusetts: The MathWorks Inc.; 2014.
- [29] Gurobi Optimization I. Gurobi optimizer reference manual. 2015; Available from: <http://www.gurobi.com>.
- [30] Mangasarian OL. Machine learning via polyhedral concave minimization. In: *Applied mathematics and parallel computing*. Springer; 1996. p. 175–188.

- [31] Yong L. Particle swarm optimization for absolute value equations. *Journal of Computational Information Systems*. 2010;6(7):2359–2366.

n	nb-LP-TAVE	time TAVE	nb-lin-syst-GN	time GN
32	100	0.1841	217	0.0403
64	100	0.4702	224	0.0553
128	100	1.5880	219	0.1079
256	100	14.5161	226	0.3924
512	100	129.1686	214	2.2327

Table 2: TAVE and Newton method from [8], GN, for (AVE) in the case with singular values of A exceeding 1.

n	nnztot	nnzx	out-iter	in-iter	time	nb-failure
32	0	0	74	306	0.6234	0
64	3	1	156	491	2,8173	3
128	8	1	269	841	20,9447	8
256	8	1	324	1129	281,6190	8

Table 3: TAVE

n	nnztot	nnzx	out-iter	in-iter	time	nb-failure
32	0	0	33	164	0.3137	0
64	2	1	81	221	1,1280	2
128	4	1	136	303	6,3953	4
256	4	1	131	292	56,4148	4

Table 4: TAVE2

n	nnztot	nnzx	out-iter	in-iter	time	nb-failure
32	9	1	-	485	1.0823	9
64	8	1	-	458	2,9234	8
128	10	1	-	568	18,4404	10
256	11	1	-	595	124,5728	11

Table 5: CMM

n	nnztot	nnzx	out-iter	in-iter	time	nb-failure
32	7	1	-	248	0.5546	7
64	19	4	-	342	2,5822	13
128	19	3	-	409	16,6830	13
256	29	5	-	439	143,0973	11

Table 6: LPM

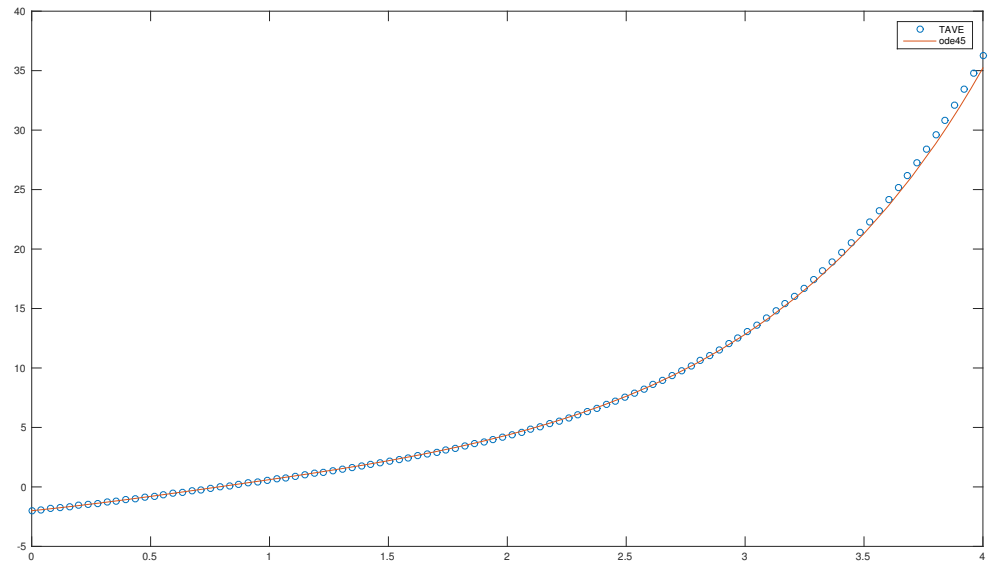


Figure 1: Numerical solution of equation (6.1) with edo45 and ThetaAVE.

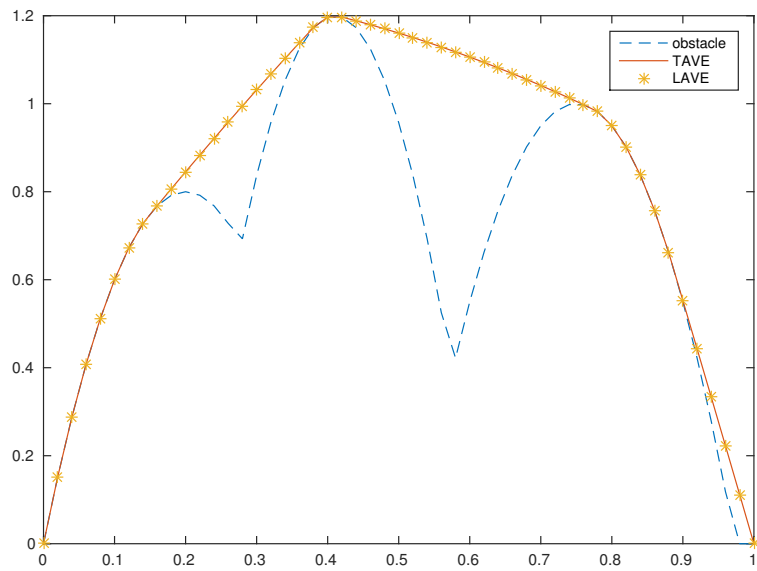


Figure 2: A solution of the obstacle problem (6.2) with ThetaAVE and method from [17]