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Solving Absolute Value Equation using Complementarity and Smoothing Functions

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In this paper, we consider the NP-hard problem of solving absolute value equation (AVE). It is sometimes difficult to solve general (AVE) and our concern is to propose a new method, which reduces the number of unsolved problems. This approach leads to a new method valid for general (AVE). We transform (AVE) as an horizontal linear complementarity problem, then we reformulate it in a sequence of concave optimization problems. We show convergence to the original problem, an error estimate for the sequence of solutions. Moreover we give remarks about the algorithm, where we solve sequences of linear programs, and numerical results, which shows that this new method manages to improve the number of unsolved problems compare to usual one.

Keywords: : smoothing function ; concave minimization ; complementarity ; absolute value equation

AMS Subject Classification: 90C59 ; 90C30 ; 90C33 ; 65K05 ; 49M20

1. Introduction

We consider the absolute value equation

$$\text{(AVE) } Ax - |x| = b \ , \tag{1}$$

where $A \in \mathrm{R}^{N\times N}$, $b \in \mathrm{R}^N$ and unknown $x \in \mathrm{R}^N$. This problem has been introduced by [1] in a more general form

$$\text{(GAVE) } Ax + B|x| = b \ , \tag{2}$$

where $A, B \in \mathrm{R}^{M\times N}$, $b \in \mathrm{R}^M$ and unknown $x \in \mathrm{R}^N$. We focus here on (AVE), which has received more interest in the literature. The recent interest for this problem can be explained since frequently occurring optimization problems such as linear complementarity problem or mixed integer programming can be reformulated as an (AVE), see [2, 3]. The general NP-hard linear complementarity problem can be formulated as an (AVE), which implies that (AVE) is NP-hard in general. In [2] it has been proved that checking if (AVE) has one or an infinite number of solutions is NP-complete.

Theoretical criteria concerning existence of solutions and unique solvability of (1) have been studied in [2, 4–7]. An important criterion among others is that

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(AVE) has a unique solution if singular values of matrix A exceed 1. In [8] Rohn proposed an exponential time algorithm to find every solution of (AVE), also a major interest is to find methods to find a solution in general. In the special case where the problem is uniquely solvable a family of Newton methods has been proposed first in [9], then completed with global and quadratic convergence in [10] and other related methods [11, 12]. Also Picard-HSS iteration method and nonlinear HSS-like method have been considered for instance in [13, 14]. Most of the methods valid in the general case are due to Mangasarian [15–17] mainly considering a concave or bilinear programming reformulation of (1) solved by a sequence of linear program. An hybrid method mixing Newton approach of [9] and [17] can be found in [18]. Finally another interesting method in the general case based on interval matrix is the one from Rohn [19, 20]. The case where (AVE) is not solvable has some motivations. Prokopyev gives numerical results using a mixed integer programming solver in [2], theoretical study in order to correct b and A to make (AVE) feasible can be found in [21, 22].

Our aim in this paper is to pursue the study of the general (AVE) and propose a new method, which solves a sequence of linear program. The motivation is to diminish the number of instances where usual methods can not solve the problem. We propose a new reformulation of (1) as a sequence of concave minimization problems using complementarity and smoothing functions based on an idea from [23, 24]. We provide analysis of the algorithm with convergence study, error estimate as well as numerical results in order to validate the interest in our approach.

This paper is organised as follow. Section 2 presents the new formulation of (AVE) as a sequence of concave minimization programs. Section 3 gives convergence to the original problem and Section 4 shows error estimate. Finally, Section 5 provides numerical results with simple examples and random generated problems.

2. AVE as a sequence of concave minimization programs

Using a classical decomposition of the absolute value we reformulate (1) as an horizontal linear complementarity problem. Set $x = x^+ - x^-$, where $x^+ \geq 0$, $x^- \geq 0$ and $x^+ \perp x^-$, in other words $x^+ = \max(x, 0)$ and $x^- = \min(x, 0)$. This decomposition guarantees that $|x| = x^+ + x^-$. So that (AVE) is equivalent to the following complementarity problem

$$\begin{cases} A(x^+ - x^-) - (x^+ + x^-) = b \\ x^+ \geq 0,\ x^- \geq 0 \\ x^+ \perp x^- \end{cases} . \tag{3}$$

We reformulate this problem as a sequence of concave optimization problems using a smoothing technique. This technique has already been studied in [23, 24] and uses a family of non-decreasing continuous smooth concave functions $\theta : \mathrm{R} \to]-\infty, 1[$, such that

$$\theta(t) < 0 \text{ if } t < 0,\ \theta(0) = 0 \text{ and } \lim_{t \to +\infty} \theta(t) = 1 \ . \tag{4}$$

One possible way to build such function is to consider non-increasing probability density functions $f : \mathrm{R}_+ \to \mathrm{R}_+$ and then take the corresponding cumulative

distribution function

$$\theta(t) = \int_0^t f(x)dx \ . \tag{5}$$

By definition of f we can verify that

$$\lim_{t\to+\infty} \theta(t) = \int_0^{+\infty} f(x)dx = 1 \text{ and } \theta(0) = \int_0^0 f(x)dx = 0 \ . \tag{6}$$

The non-decreasing hypothesis gives the concavity of θ. We then extend this functions for negative values in a smooth way.

Examples of this family are $\theta^1(t) = t/(t+1)$ if $t \geq 0$ and $\theta^1(t) = t$ if $t < 0$, $\theta^2(t) = 1 - e^{-t}$ with $t \in \mathrm{R}$.

We introduce $\theta_r(t) := \theta\left(\frac{t}{r}\right)$ for $r > 0$. This definition is similar to the perspective functions in convex analysis. This functions satisfy

$$\theta_r(0) = 0 \ \forall r > 0 \text{ and } \lim_{r\searrow 0} \theta_r(t) = 1 \ \forall t > 0 \ . \tag{7}$$

Examples of this family are $\theta_r^1(t) = t/(t+r)$ if $t \geq 0$ and $\theta_r^1(t) = t$ if $t < 0$, $\theta_r^2(t) = 1 - e^{-t/r}$ $t \in \mathrm{R}$. The function θ_r^1 is quite useful as it can sometimes be used as a "minimum" of this family, when proving some interesting results.

Next lemma will show the link between these functions and complementarity in one dimension.

Lemma 2.1 *Given $s, t \in \mathrm{R}_+$ and the parameter $r > 0$, then*

$$s \perp t \iff \lim_{r\searrow 0} \theta_r(s) + \theta_r(t) \leq 1 \ . \tag{8}$$

Proof. We show by contradiction that

$$\lim_{r\searrow 0} \theta_r(s) + \theta_r(t) \leq 1 \Longrightarrow s \perp t \ . \tag{9}$$

Suppose $s, t > 0$, then

$$\lim_{r\searrow 0} (\theta_r(s) + \theta_r(t)) = \lim_{r\searrow 0} \ \theta_r(s) + \lim_{r\searrow 0} \ \theta_r(t) = 2 \ . \tag{10}$$

Contradiction, so $s \perp t$. Conversely it is clear that $s \perp t \Longrightarrow s = 0$ or $t = 0$. □

In the case of the function θ_r^1 we even have the equality in (8) and by definition of this function we have

$$\theta_r^1(s) + \theta_r^1(t) = 1 \iff st = r^2 \ . \tag{11}$$

Now, we use the previous lemma to replace the complementarity constraint by a sequence of concave optimization problems : for $r > 0$

$$(\tilde{P}_r) \ \begin{cases} \min\limits_{x^+, x^- \in \mathrm{R}^N} \ \sum_{i=1}^N \theta_r(x_i^+) + \theta_r(x_i^-) - 1 \\ A(x^+ - x^-) - (x^+ + x^-) = b \\ x^+ \geq 0, \ x^- \geq 0 \end{cases} \ . \tag{12}$$

We will consider through this paper a relaxed version of $(\tilde{P}_r)$ to avoid compensation phenomenon and generate strictly feasible iterate :

$$(P_r) \begin{cases} \min\limits_{x^+,x^- \in \mathrm{R}^N} \sum_{i=1}^N \theta_r(x_i^+) + \theta_r(x_i^-) - 1 \\ b - g(r)|A|e - g(r)e \leq A(x^+ - x^-) - (x^+ + x^-) \leq b + g(r)|A|e + g(r)e \\ x^+ + x^- \geq g(r)e \\ 0 \leq x^+ \leq M,\ 0 \leq\ x^- \leq M \end{cases}, \tag{13}$$

where e is the unit vector, $|A|$ denotes the matrix where each element is the absolute value of the corresponding elements in A and $g : \mathrm{R}_+^* \to \mathrm{R}_+^*$ is a function which goes to 0 slower than r, that is

$$\lim_{r \searrow 0} \frac{r}{g(r)} = 0 \text{ and } \lim_{r \searrow 0} g(r) = 0 \tag{14}$$

for instance $g(r) = r^\alpha$ with $0 < \alpha < 1$.

3. Convergence

From now on, we will suppose that the solution set of (AVE) is non-empty and bounded. We will denote it by $S^*_{(AVE)}$. Denote $S^*_{(P_r)}$ the optimal set of (P_r). In order to simplify the notation, in the following we will denote $x \in S^*_{(P_r)}$ when $(x^+, x^-) \in S^*_{(P_r)}$ with $x = x^+ - x^-$ and $x^+ = \max(x, 0)$, $x^- = \min(x, 0)$. Let M be a positive constant such that

$$M \geq \max_{x \in S^*_{(AVE)}} ||x||_\infty \ . \tag{15}$$

First, we show in the following theorem that for $r \geq 0$, $S^*_{(P_r)}$ is non empty.

THEOREM 3.1 *The problem (P_r) has at least one solution for any $r \geq 0$.*

Proof. Since $S^*_{(AVE)} \neq \emptyset$ then there exists a point $\bar{x} \in S^*_{(AVE)}$ and $\bar{x} = \bar{x}^+ - \bar{x}^-$ with $\bar{x}^+ \perp \bar{x}^-$. It follows that $(y^{r+} = \bar{x}^+ + g(r), y^{r-} = \bar{x}^-)$ is a feasible point of (P_r). Moreover, we minimize a continuous function over a non-empty compact set so the objective function attains its minimum. □

We present now two lemmas that will be used to prove the convergence theorem.

LEMMA 3.2 *Given the functions θ_r and g defined above and $x^+, x^- \in \mathrm{R}_+^N$, $r \in \mathrm{R}_+^*$ such that $x^+ + x^- \geq g(r)e$ we have*

$$\theta_r(x_i^+) + \theta_r(x_i^-) - 1 \geq \theta_r(g(r)) - 1,\ \forall i \in \{1, ..., N\}. \tag{16}$$

Proof. θ_r is concave and $\theta_r(0) = 0$ so θ_r is subadditive. So for all $i \in \{1, ..., N\}$ we have

$$\theta_r(x_i^+) + \theta_r(x_i^-) - 1 \geq \theta_r(x_i^+ + x_i^-) - 1 \ . \tag{17}$$

Thanks to $x^+ + x^- \geq g(r)e$ and since θ_r is non-decreasing we can conclude that

$$\theta_r(x_i^+) + \theta_r(x_i^-) - 1 \geq \theta_r(g(r)) - 1 \ . \tag{18}$$

□

LEMMA 3.3 *Given functions θ_r and g defined above we have*

$$\lim_{r \searrow 0} \ \theta_r(g(r)) - 1 = 0 \ . \tag{19}$$

Proof. $\theta_r(g(r)) = \theta_{\frac{r}{g(r)}}(1)$ and since $\lim_{r\searrow 0} r/g(r) = 0$, we have

$$\lim_{r \searrow 0} \ \theta_r(g(r)) = \lim_{r \searrow 0} \ \theta_{\frac{r}{g(r)}}(1) = 1 \ . \tag{20}$$

□

In the special case, where every solution of (AVE) has at least a zero component it can be difficult to find a feasible point that satisfy the constraint $x^+ + x^- \geq g(r)e$. The following lemma explains how to build such point in this case.

LEMMA 3.4 *Given $\bar{x}$ a solution of (AVE) and $r > 0$ such that $g(r) < r_0 = \min_{\bar{x}_i \neq 0} |\bar{x}_i|$, then $y^r = \bar{x} + g(r)$ is a solution of (AVE)$_r$, that is the equation*

$$\text{(AVE)}_r \ \ Ax - |x| = b + g(r)Ae - g(r)\delta(x) \ . \tag{21}$$

$\delta(x) \in \mathrm{R}^N$ *with* $\delta_i(x) = \begin{cases} 1 \ if \ x_i \geq 0 \\ -1 \ if \ x_i < 0 \end{cases}$.

Proof. Since $\bar{x}$ is a solution of (AVE)

$$A\bar{x} - |\bar{x}| = b \ , \tag{22}$$

$$\Rightarrow \ A\bar{x} + g(r)Ae - |\bar{x}| - g(r)\delta(x) = b + g(r)Ae - g(r)\delta(x), \tag{23}$$

$$\Rightarrow \ A(\bar{x} + g(r)e) - |\bar{x} + g(r)e| = b + g(r)Ae - g(r)\delta(x), \tag{24}$$

then $y^r = \bar{x} + g(r)$ is a solution of (AVE)$_r$ □

We now proceed to the convergence proof of the sequence of $\{x^r\}_{r>0}$ to an element of $S^*_{(AVE)}$, where $x^r = x^{r+} - x^{r-}$ with (x^{r+}, x^{r-}) optimal solution of (P_r). We never claim that for a given r the set of solution $S^*_{(P_r)}$ is a singleton.

THEOREM 3.5 *Every limit point of the sequence $\{x^r\}_{r>0}$ such that $x^r \in S_{(P_r)}$ is a solution of (AVE).*

Proof. By hypothesis there exists at least one solution of (AVE). Using lemma (3.4) we can build a sequence $\{y^r\}_{r>0}$ where $y^r = y^{r+} - y^{r-}$ with $y^{r+} \perp y^{r-}$ is a solution of (AVE)$_r$. Moreover, for r sufficiently small (y^{r+}, y^{r-}) is a feasible point of (P_r). Set $x^r = (x^{r+}, x^{r-})$ with $\{x^r\}_{r>0}$ is a sequence of optimal solution of (P_r),

then

$$\sum_{i=1}^{N}(\theta_r(x_i^{r+})+\theta_r(x_i^{r-})-1)\leq\sum_{i=1}^{N}(\theta_r(y_i^{r+})+\theta_r(y_i^{r-})-1)\leq 0\ . \tag{25}$$

So for all $i\in\{1,...,N\}$,

$$\theta_r(x_i^{r+})+\theta_r(x_i^{r+})-1\leq-\sum_{j=1;j\neq i}^{N}(\theta_r(x_j^{r+})+\theta_r(x_j^{r+})-1)\ . \tag{26}$$

And then by lemma (3.2), we have for all $i\in\{1,...,N\}$

$$\theta_r(x_i^{r+})+\theta_r(x_i^{r-})\leq 1+(N-1)(1-\theta_r(g(r)))\ . \tag{27}$$

So for every limit point $\bar{x}=(\bar{x}^+,\bar{x}^-)$ of the sequence $\{x^r\}_r$, where $\bar{x}^+=\lim_{r\searrow 0}x^{r+}$ and $\bar{x}^-=\lim_{r\searrow 0}x^{r-}$, by lemma (3.3) ($\lim_{r\searrow 0}1-\theta_r(g(r))=0$) when passing to the limit we obtain

$$\lim_{r\searrow 0}\ \theta_r(\bar{x}_i^+)+\theta_r(\bar{x}_i^-)\leq 1\ . \tag{28}$$

Thanks to the inequality (28) and the lemma (2.1) we have $\bar{x}^+\perp\bar{x}^-$.

Now, let verify that $\bar{x}$ is a solution of (AVE). Let x^r be a solution of (P_r) for $r>0$, we have

$$b-g(r)|A|e-g(r)e\leq A(x^{r+}-x^{r-})-(x^{r+}+x^{r-})\leq b+g(r)|A|e+g(r)e. \tag{29}$$

Passing to the limit $r\searrow 0$ we have

$$A(\bar{x}^+-\bar{x}^-)-(\bar{x}^++\bar{x}^-)=b. \tag{30}$$

So, $(\bar{x}^+,\bar{x}^-)$ is a solution of (13) and $\bar{x}=\bar{x}^+-\bar{x}^-$ is a solution of (AVE). □

4. Error estimate

In this section we study the behaviour of a sequence $\{x^r\}_{r>0}$ of optimal solution of (P_r) when r becomes small. We remind the definition of the Landau notation O often used in the context of asymptotic comparison. Given two functions f and h. We have

$$f(x)=O_{x\to a}(h(x))\text{ if }\exists C>0,\ \exists d>0,\ \forall x,\ |x-a|\leq d\Longrightarrow|f(x)|\leq C|h(x)|. \tag{31}$$

We denote $O(h(x))$ when a is 0. We first show a useful lemma, which doesn't need the hypothesis of the existence of a solution without zero component.

Lemma 4.1 *Let θ_r be such that $\theta_r\geq\theta_r^1$. For $x^r\in S^*_{(P_r)}$ and r sufficiently small, we have*

$$x_i^{r+}x_i^{r-}\leq O(rg(r))\ \forall i\in\{1,...,N\}\ . \tag{32}$$

Proof. Set $i \in \{1, ..., N\}$. Thanks to the convergence proof of the theorem (3.5) for r sufficiently small the following holds

$$\theta_r(x_i^{r+}) + \theta_r(x_i^{r-}) - 1 \leq (N-1)(1 - \theta_r(g(r))) \ . \tag{33}$$

By using θ_r^1 function

$$\begin{aligned} \theta_r^1(x_i^{r+}) + \theta_r^1(x_i^{r-}) - 1 &\leq \theta_r(x_i^{r+}) + \theta_r(x_i^{r-}) - 1, & (34)\\ &\leq (N-1)(1 - \theta_r(g(r))), & (35)\\ &\leq (N-1)(1 - \theta_r^1(g(r))), & (36) \end{aligned}$$

then for r sufficiently small such that $g(r) \geq r$ and $(1 - (N-1)g(r)) > 0$

$$\frac{x_i^{r+}}{x_i^{r+} + r} + \frac{x_i^{r-}}{x_i^{r-} + r} - 1 \leq (N-1)(1 - \frac{g(r)}{g(r) + r}) \leq (N-1)g(r), \tag{37}$$

$$\frac{2x_i^{r+}x_i^{r-} + rx_i^{r+} + rx_i^{r-} - x_i^{r+}x_i^{r-} - rx_i^{r+} - rx_i^{r-} - r^2}{(x_i^{r+} + r)(x_i^{r-} + r)} \leq (N-1)g(r), \tag{38}$$

$$\frac{x_i^{r+}x_i^{r-} - r^2}{(x_i^{r+} + r)(x_i^{r-} + r)} \leq (N-1)g(r), \tag{39}$$

$$x_i^{r+}x_i^{r-} - r^2 \leq (N-1)g(r)(x_i^{r+}x_i^{r-} + rx_i^{r+} + rx_i^{r-} + r^2), \tag{40}$$

$$x_i^{r+}x_i^{r-} \leq r^2 g(r)\frac{1 + (N-1)}{1 - (N-1)g(r)} + rg(r)\frac{(N-1)(x_i^{r+} + x_i^{r-})}{1 - (N-1)g(r)}, \tag{41}$$

and the results. □

The following proposition gives an error estimate for the components of (x^{r+}, x^{r-}) that go to zero.

PROPOSITION 4.2 *Let θ_r be such that $\theta_r \geq \theta_r^1$. Given $(\bar{x}^+, \bar{x}^-)$ a limit point of the sequence $\{x^{r+}, x^{r-}\}_r$ of optimal solutions of (P_r). The convergence of the components of the variable x^{r+} or x^{r-} to the possibly zero part of the accumulation point is done in $O(r)$.*

Proof. Set $i \in \{1, ..., N\}$. We will work with one component. Suppose that $\bar{x}_i^+ = 0$. The opposite case is completely similar. By lemma (4.1) and using that (x^{r+}, x^{r-}) is feasible $(x^{r+} + x^{r-} \geq g(r))$ we have

$$\begin{aligned} x_i^{r+}x_i^{r-} &\leq O(rg(r)), & (42)\\ x_i^{r+} &\leq \frac{O(rg(r))}{x_i^{r-}}, & (43)\\ x_i^{r+} &\leq \frac{O(rg(r))}{g(r)}, & (44)\\ |x_i^{r+} - \bar{x}_i^+| &\leq O(r). & (45) \end{aligned}$$

□

In the next theorem we are interested in the error estimate of the possibly non-zero part of the solution in the couple (x^+, x^-).

To establish this result, we will use the following Hoffman's lemma.

LEMMA 4.3 (Hoffman's lemma) *[25] Given a convex polyhedron P such that*

$$P = \{x \mid Ax \leq b\} \ . \tag{46}$$

We set $d_P(x)$ the distance from x to P, by choosing a norm $||.||$, where $d_P(x) = inf_{y \in P} \ ||y - x||$. There exists a constant K which only depends on A, such that

$$\forall b, \ \forall x \in \mathrm{R}^n \ : \ d_P(x) \leq K||(Ax - b)^+|| \ . \tag{47}$$

Remark that if the constraints are given by $Ax = b$ with A a square full-rank matrix instead of $Ax \leq b$ then the polyhedron is reduced to a singleton and we can estimate the constant as $K = ||A^{-1}||$.

THEOREM 4.4 *Given $(\bar{x}^+, \bar{x}^-)$ an accumulation point of the sequence $\{x^{r+}, x^{r-}\}_{r>0}$ of optimal solutions of (P_r), where we denote $\bar{x} = \bar{x}^+ - \bar{x}^-$ and $x^r = x^{r+} - x^{r-}$. For r sufficiently small*

$$d_{s^*}(x^r) = O(g(r)) \ , \tag{48}$$

where S^* denote the intersection of $S^*_{(AVE)}$ and a neighbourhood V of $\bar{x}$, such that any point in V has the same sign than $\bar{x}$.

Proof. We split the proof in two cases, either $\min_{i \in \{1,...,N\}} |\bar{x}_i| \neq 0$, either $\exists i \in \{1, ..., N\}, \ \bar{x}_i = 0$.

a) First, suppose there is no zero component in $\bar{x}$. We set $\alpha = \min_{i \in \{1,...,N\}} |\bar{x}_i|/2$ and a neighbourhood V of $\bar{x}$ defined as

$$V = B_\infty(\bar{x}, \alpha) = \{x \mid \max_{1 \leq i \leq N} \ |x_i - \bar{x}_i| \leq \alpha\} \ . \tag{49}$$

So $\forall x \in V$, $\bar{x}$ and x have the same sign. We set $D = \ diag(\delta(\bar{x}))$, where $\delta(x) \in \mathrm{R}^N$ with $\delta_i(x) = \begin{cases} 1 \ if \ x_i \geq 0 \\ -1 \ if \ x_i < 0 \end{cases}$.

By taking $S^* = \{x \in \mathrm{R}^n \mid Ax - Dx = b\} \cap V$ we obtain a convex polyhedron. This set is non-empty because $\bar{x} \in S^*$. In the neighbourhood V the resolution of $Ax - Dx = b$ gives a solution of (AVE). Now we can use Hoffman lemma.

For r sufficiently small $x_r \in V$ and then

$$d_{S^*}(x^r) \leq K \left\| \begin{matrix} (A - D)x^r - b \\ (x^r - \alpha - \bar{x})^+ \\ (-x^r - \alpha + \bar{x})^+ \end{matrix} \right\| \leq K(\|(A - D)x^r - b\| + \|(x^r - \alpha - \bar{x})^+\| \tag{50}$$

$$+ \|(-x^r - \alpha + \bar{x})^+\|), \tag{51}$$

$$= K\|(A - D)x^r - b\|, \tag{52}$$

$$= K\|Ax^r - |x^r| - b\|. \tag{53}$$

As x^r is feasible for (P_r), we have

$$||Ax^r - |x^r| - b|| = ||g(r)Ae - g(r)\delta(x^r)||, \tag{54}$$
$$= ||(Ae - \delta(x^r))g(r)||, \tag{55}$$
$$\leq ||Ae - \delta(x^r)||\ |g(r)|, \tag{56}$$
$$= ||Ae - \delta(x^r)||g(r) = O(g(r)). \tag{57}$$

We add this in (50)

$$d_{S^*}(x^r) \leq K||Ae - \delta(x^r)||g(r) = O(g(r))\ . \tag{58}$$

b) Now we move to the case where $\exists i \in \{1, ..., N\},\ \bar{x}_i = 0$. We denote $\sigma(t) = \{i|t_i \neq 0\}$. We set $\alpha = \min_{i\in\sigma(\bar{x})} |\bar{x}_i|/2$ and a neighbourhood V of $\bar{x}$ defined as

$$V = B_\infty(\bar{x}, \alpha) = \{x \mid \max_{i\in\sigma(\bar{x})} |x_i - \bar{x}_i| \leq \alpha\}\ . \tag{59}$$

V is non-empty because $\bar{x} \in V$. For all $x \in V$, $\bar{x}$ and x have the same sign only for the components $\bar{x}_i$ with $i \in \sigma(\bar{x})$. For r sufficiently small we have $x^r \in V$.

By taking $S^* = \{x \in \mathrm{R}^n \mid Ax - Dx = b\ ,\ Dx \geq 0\} \cap V$ with $D = \ diag(\delta(x^r))$ we obtain a convex polyhedron. The choice of D depending on x^r is not restrictive as we can always take a subsequence of the sequence $\{x^r\}_{r>0}$, which converge to $\bar{x}$, with constant signs near $\bar{x}$. This set is non-empty because $\bar{x} \in S^*$. In the neighbourhood V the resolution of $Ax - Dx = b$ with the constraints $Dx \geq 0$ gives a solution of (AVE). We can use Hoffman lemma

$$d_{S^*}(x^r) \leq K \left\| \begin{matrix} (A-D)x^r - b \\ (x^r - \alpha - \bar{x})^+ \\ (-x^r - \alpha + \bar{x})^+ \\ (-Dx)^+ \end{matrix} \right\| \leq K(\|(A-D)x^r - b\| + \|(x^r - \alpha - \bar{x})^+\| \tag{60}$$
$$+ \|(-x^r - \alpha + \bar{x})^+\| + \|(-Dx)^+\|), \tag{61}$$
$$= K\|(A-D)x^r - b\|, \tag{62}$$
$$= K\|Ax^r - |x^r| - b\|. \tag{63}$$

As x^r is feasible for (P_r), we have

$$||Ax^r - |x^r| - b|| \leq ||g(r)Ae - g(r)\delta(x^r)||, \tag{64}$$
$$= ||(Ae - \delta(x^r))g(r)||, \tag{65}$$
$$\leq ||Ae - \delta(x^r)||\ |g(r)|, \tag{66}$$
$$= ||Ae - \delta(x^r)||g(r) = O(g(r)). \tag{67}$$

We add this in (60)

$$d_{S^*}(x^r) \leq K||Ae - \delta(x^r)||g(r) = O(g(r)), \tag{68}$$

□

Remark 1 We can be a bit more specific in the case where $(A-D)$ is invertible. In this case $S^* = \{\bar{x}\}$, so (48) becomes

$$||x^r - \bar{x}|| \leq ||(A-D)^{-1}||\ ||Ae - \delta(x)||g(r) = O(g(r))\ . \tag{69}$$

This case corresponds to the special cases where (AVE) has isolated solutions.

5. Algorithm

In the two previous sections we have seen theoretical results about convergence and error estimate of an algorithm to find a solution of (AVE). In this section we focus on the algorithm and we make some remarks about the parameters and the implementation. We have the generic algorithm with C the feasible set of (P_{r^k}) :

$$[\text{TAVE}] \begin{cases} \{r^k\}_{k\in\mathrm{N}},\ r^0 > 0 \text{ and } \lim\limits_{k\to+\infty} r^k = 0 \\ \text{find } x^k :\ x^k \in \arg\min\limits_{x\in C} \sum_{i=1}^n \theta_{r^k}(x_i^+) + \theta_{r^k}(x_i^-) - 1 \end{cases} . \tag{70}$$

Remark 2 (About the optimization problem in practice) When implementing the algorithm one should probably more likely use the initial problem $(\tilde{P}_r)$ with infeasible points, which will have improved numerical behaviour and also may add the constraint $x^+ + x^- \geq g(r)e$ otherwise the sequence will possibly go to a local minimum with a zero component.

Remark 3 (Initialization point) Numerical resolution of the problem may need an initial point. In a same way as in [17, 18] we can solve the following linear program

$$\begin{cases} \min\limits_{x^+,x^-\in\mathrm{R}^N} (x^+ + x^-)^T e \\ A(x^+ - x^-) - (x^+ + x^-) = b \\ 0 \leq x^+,\ 0 \leq\ x^- \end{cases} . \tag{71}$$

This program find an initial feasible point for (P_r) and the objective function may support this feasible point to satisfy the complementarity condition.

Remark 4 (About the use of functions θ_r) In this study we put the variables in a compact set. One should note that the functions θ_r are more efficient when their arguments lives in $[0,1]$. Moreover, we use one way to express complementarity with lemma (2.1) another way, which will be used in the numerical study, is to consider the following

$$\theta_r(s) + \theta_r(t) - \theta_r(s+t) = 0. \tag{72}$$

In this case we don't necessarily need the constraint $x^+ + x^- \geq r^\alpha$, since it is a reformulation of the complementarity and not a relaxation.

Remark 5 (About the parameters) Regarding the choice of the parameters α, r_0 and the update in r, we can note that they are all used in the constraint $x^+ + x^- \geq g(r)e$ with $g(r) = r^\alpha$ and $0 < \alpha < 1$. It has been shown in the error estimate theorem that the convergence to zero part of the solution is in $g(r)$ so it is natural to take α as big as possible, for instance $\alpha = 0.99$. Also we understand that there is

Table 1. Parameters for the simulations

T	initial $r : r_0$	function θ_r	α
1.8	1	θ_r^2	0.99

a link between the value of α and the update in r. About this one we choose to select a constant sequence of value with an update constant T, so that $r^{k+1} = \frac{r^k}{T}$. When you decrease this update constant you are allowed to increase α. Finally about the r_0 we can use the relation in one dimension mentioned in the introduction of functions θ^1

$$\theta_r^1(s) + \theta_r^1(t) = 1 \Longleftrightarrow st = r^2 \ . \tag{73}$$

Remark 6 [Acceleration and precision heuristic] At each step in r we solve a concave optimization problem to get the current point. The following heuristic can be rather useful to accelerate convergence when we are close to the solution and ensure a good precision. After finding the current point x^k we solve if possible the linear system

$$(A - \mathrm{diag}(\delta(x^k)))z = b. \tag{74}$$

We then check if x solves (AVE). If it does, then the algorithm is finished and we solved (AVE) with the same precision as we solve the linear system. Otherwise we just continue the iteration in r with x^k. This idea is similar to a Newton iteration.

6. Numerical Simulations

Thanks to the previous sections we have keys for an algorithm, we will present now some numerical results. These simulations have been done using MATLAB, [26], with the linear programming solver GUROBI, [27]. We used the SLA algorithm, [28], (successive linearisation algorithm) to solve our concave minimization problem at each iteration in r.

Proposition 6.1 (SLA algorithm for concave minimization) *Given ϵ sufficiently small and r^k. Denote C the feasible set of (P_{r^k}). Given $x^k = x^{k+} - x^{k-}$, x^{k+1} is designed as a solution of the linear problem*

$$\min_{y^+, y^- \in C} \ (y^+)^T \nabla\theta_{r^k}(x^{k+}) + (y^-)^T \nabla\theta_{r^k}(x^{k-}), \tag{75}$$

with $x^0 = x^{0+} - x^{0-}$ a random point. We stop when

$$x^{k+1} \in C \ \textit{and} \ (x^{k+1} - x^k)^T \nabla\theta_{r^k}(x^k) \leq \epsilon. \tag{76}$$

This algorithm generates a finite sequence with strictly decreasing objective function values.

Proof. see [[28], Theorem 4.2]. □

In the SLA we add a supplementary stopping criterion if we find a solution of (AVE). Through all this simulations we used the following parameters for TAVE, that are detailed in table 1. The maximum number of iterations in r for one instance is 20 and the maximum number of linear program for one SLA is 10. We measured

time in seconds, the number of linear program solved and the number of linear system solved respectively denoted by nb-LP-method and nb-lin-syst-method.

In order to confirm the validity of our method we consider two concrete examples and then two kind of random generated problems. The first one is a second order ordinary differential equation with initial conditions and the second example is an obstacle problem. We remind that the principal motivation of this algorithm is to consider general kind of (AVE) and this case has been treated in the last simulation.

6.1. *An ordinary differential equation*

We consider the ordinary differential equation

$$\ddot{x}(t) - |x(t)| = 0,\ x(0) = x_0,\ \dot{x}(0) = \gamma\ . \tag{77}$$

We get an (AVE) by using a finite difference scheme to discretize this equation. We use the following second-order backward difference to approximate the derivative

$$\frac{x_{i-2} - 2x_{i-1} + x_i}{h^2} - |x_i| = 0\ . \tag{78}$$

Equation (78) was derived with equispace gridpoints $x_i = ih$, $i = 1, ...N$. In order to approximate the Neumann boundary conditions we use a center difference

$$\frac{x_{-1} - x_1}{2h} = \gamma\ . \tag{79}$$

We compare the obtained solution by TAVE to the one of the predefined Runge-Kutta ode45 function in MATLAB, [26]. The domain is $t \in [0, 4]$, initial conditions $x_0 = -1$, $\gamma = 1$ and $N = 100$. Results are presented in figure 1. TAVE solves the problem and results show that the method gives consistent results.

6.2. *Obstacle problem*

The second example is a simple obstacle problem. We try to find a trajectory joining the bounds of a domain with obstacle, g, and a minimal curvature, f. This can be formulated as the following equation

$$(\ddot{u}(x) - f(x))^T(u(x) - g(x)) = 0,\ \ddot{u}(x) - f(x) \geq 0,\ u(x) - g(x) \geq 0\ . \tag{80}$$

We approximate the second order derivative with a second-order central difference, then the problem (80) is similar to the discrete version on an equispace gridpoints $x_i = ih$, $i = 1, ...N$.

$$(Du - f)^T(u - g) = 0,\ Du - f \geq 0,\ u - g \geq 0\ , \tag{81}$$

where $g_i = g(x_i)$, $f_i = f(x_i)$. This can be reformulated as a linear complementarity problem with by setting $z = u - g$, $M = D$ and $q = Dg - f$, that is

$$(Mz + q)^T z = 0,\ Mz + q \geq 0,\ z \geq 0\ . \tag{82}$$

which if 1 is not an eigenvalue of M is equivalent to (AVE), ([15], Prop. 2),

$$(M-I)^{-1}(M+I)x-|x|=(M-I)^{-1}q\ . \tag{83}$$

We give results for our method and LPM method from [17], with $g(x)=\max(0.8-20(x-0.2)^2,\max(1-20(x-0.75)^2,1.2-30(x-0.41)^2))$, $f(x)=1$, $N=50$ in figure 2. Both methods give 20 points on the curve g and none below g over 50 points. Once again TAVE method gives consistent results.

6.3. *Random uniquely solvable generated problem*

We now consider the special case where (AVE) is uniquely solvable. One way to generate such (AVE) is to generate a matrix A with singular values exceeding 1. Following [29] the data (A, b) are generated by the Matlab script :

```
n=input('dimension of matrix A =');
rand('state',0);
R=rand(n,n);
b=rand(n,1);
A=R'*R+n*eye(n);
```

for $n=100,200,400,800$. The required precision for solving (AVE) is 10^{-6} and thanks to the heuristic from remark (6) we get in the worst case 10^{-10}. For each n we solve 100 instances of the problem and compare TAVE to a Newton method from [9], which we denote GN. Results are sum up in table 2 and give for TAVE the number of linear program solved, the time required to solve all the instances and the number of linear systems, time required for GN. Note that other Newton methods like [10–12] should give similar results so we do not include them in our comparisons.

In every cases our method solves the problem, which one again valid our approach. One should note that the number of solved linear program is very low. Indeed in every case the initialization program has been sufficient to solve the problem. Also we notice that the time required to solve problem is increasing significantly when the dimension grows. It is not a surprise that in this simulations the method GN outperform in time TAVE since it only requires to solve a few linear system compare to few linear program. Since classical methods to solve (AVE) have no problem solving this instances they are not our main focus.

6.4. *Random generated problem*

Now we present results for general (AVE), which is the main interest of our algorithm. The data are generated like in [15] for several n and for several values of the parameters, in each situation we solve one hundred instances of the problem.

"Choose a random A from a uniform distribution on $[-10,10]$, then choose a random x from a uniform distribution on $[-1,1]$ and set $b=Ax-|x|$."

We will compare 4 methods valid for general (AVE) : TAVE method from algorithm (70), TAVE2 which is the same algorithm with the slight modification from remark 6, concave minimization method CMM from [15] and successive linear programming method LPM from [17].

We give "nnztot" the number of violated expression for all problems, "nnzx" the maximum violated expression for one problem, the number of iteration in r "out-iter", the number of linear program solve for all the problems "in-iter". Finally we

give the time in seconds and the number of problem where we did not manage to solve (AVE).

In every cases our methods manage to reduce the number of unsolved problem, which was our principal aim. This confirm the interest in this relaxation method. Also one should note that an improved number of solve problem comes with a price, since it requires more time. In table 4 we see that TAVE2 gives promising results. It is a slightly different method, since it is not a relaxation but a reformulation of the complementarity. In every case it gives the smallest number of unsolved problem in a very reasonable time.

Conclusion and Perspectives

In this paper, we have proposed a class of heuristics schemes to solve the NP-hard problem of solving (AVE). We compared our methods to existing methods and in each case TAVE algorithm manages to improve the number of failure, which was our principle aim.

Further studies can improve the choices of parameters in order to reduce the time needed to solve the problems, especially when the dimension grows.

Also the modify algorithm TAVE2 involving another way to express the complementarity show promising results, so we can wonder if it is possible to improve our algorithms in this case and if there exists other similar reformulation of the complementarity which can give even better results.

Finally in [18] they proposed an hybrid algorithm with the benefits of both general methods and Newton methods with encouraging numerical results. This philosophy is fully applicable to the method proposed in this paper and could lead to further improvements.

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Table 2. TAVE and Newton method from [9], GN, for (AVE) in the case with singular values of A exceeding 1.

n	nb-LP-TAVE	time TAVE	nb-lin-syst-GN	time GN
32	100	0.1841	217	0.0403
64	100	0.4702	224	0.0553
128	100	1.5880	219	0.1079
256	100	14.5161	226	0.3924
512	100	129.1686	214	2.2327

Table 3. TAVE

n	nnztot	nnzx	out-iter	in-iter	time	nb-failure
32	0	0	74	306	0.6234	0
64	3	1	156	491	2,8173	3
128	8	1	269	841	20,9447	8
256	8	1	324	1129	281,6190	8

Table 4. TAVE2

n	nnztot	nnzx	out-iter	in-iter	time	nb-failure
32	0	0	33	164	0.3137	0
64	2	1	81	221	1,1280	2
128	4	1	136	303	6,3953	4
256	4	1	131	292	56,4148	4

Table 5. CMM

n	nnztot	nnzx	out-iter	in-iter	time	nb-failure
32	9	1	-	485	1.0823	9
64	8	1	-	458	2,9234	8
128	10	1	-	568	18,4404	10
256	11	1	-	595	124,5728	11

Table 6. LPM

n	nnztot	nnzx	out-iter	in-iter	time	nb-failure
32	7	1	-	248	0.5546	7
64	19	4	-	342	2,5822	13
128	19	3	-	409	16,6830	13
256	29	5	-	439	143,0973	11

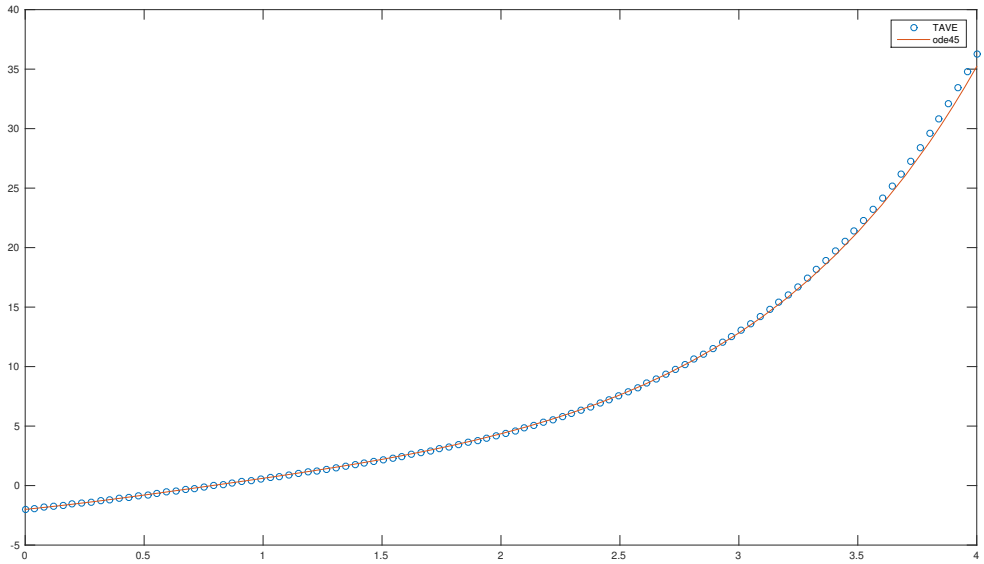

Figure 1. Numerical solution of equation (77) with edo45 and ThetaAVE.

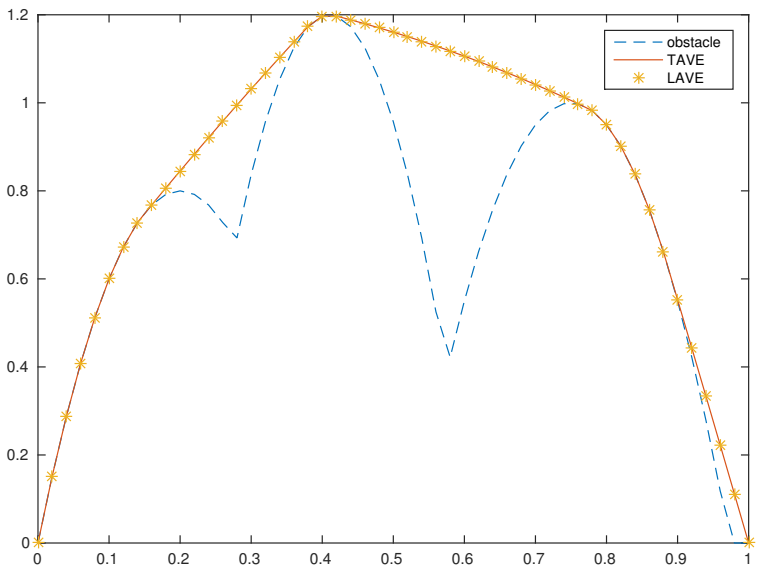

Figure 2. A solution of the obstacle problem (80) with ThetaAVE and method from [15]