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# Solving Absolute Value Equation using Complementarity and Smoothing Functions

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## Abstract

In this paper, we consider the NP-hard problem of solving absolute value equation (AVE). We transform (AVE) as an horizontal linear complementarity problem, then we reformulate it in a sequence of concave optimization problems. We show convergence to the original problem, error estimate, remarks about the algorithm and numerical results.

**Keywords** : smoothing function ; concave minimization ; complementarity ; absolute value equation

**AMS Subject Classification** : 90C59 ; 90C30 ; 90C33 ; 65K05 ; 49M20

## 1 Introduction

In this document we consider the absolute value equation

$$(AVE) \quad Ax - |x| = b, \quad (1)$$

where  $A \in \mathbb{R}^{N \times N}$ ,  $b \in \mathbb{R}^N$  and unknown  $x \in \mathbb{R}^N$ . A more general form is the general absolute value equation

$$(GAVE) \quad Ax + B|x| = b \quad (2)$$

where  $A, B \in \mathbb{R}^{M \times N}$ ,  $b \in \mathbb{R}^M$  and unknown  $x \in \mathbb{R}^N$ . We are here focus on (AVE) which has received more interest in the literature. First, in [1] a reformulation of (AVE) to a linear complementarity problem (LCP) has been shown, which imply that (AVE) is NP-hard. In [2] it has been said that checking if (AVE) has one or an infinite number of solution is NP-complete. Finally, [1] and [3] give a series of existence conditions for the solutions of (AVE). Several methods has been investigated to solve (AVE), such as a reformulation to a concave minimization problem in [4], a smoothing Newton method in [5] or a particle swarm optimization technique [6]. We propose in this paper an iterative method which at each step solve a concave minimization problem.

We use a decomposition of the absolute value to reformulate (AVE) as an horizontal linear complementarity problem, then we reformulate this new problem as an optimization problem. Set  $x = x^+ - x^-$ , where  $x^+ \geq 0$  and  $x^- \geq 0$ , then  $|x| = x^+ + x^-$  with the complementarity constraint  $x^+ \perp x^-$ . (AVE) becomes

$$\begin{cases} A(x^+ - x^-) - (x^+ + x^-) = b \\ x^+ \geq 0, x^- \geq 0 \\ x^+ \perp x^- \end{cases} . \quad (3)$$

We reformulate this problem as a sequence of concave optimization problems using a smoothing technique. This technique has already been studied in [7, 8] and uses a family of non-decreasing continuous smooth concave functions  $\theta : \mathbb{R} \rightarrow ]-\infty, 1[$ , such that

$$\theta(t) < 0 \text{ if } t < 0, \theta(0) = 0 \text{ and } \lim_{t \rightarrow +\infty} \theta(t) = 1. \quad (4)$$

One possible way to build  $\theta$  functions is to consider non-increasing probability density functions  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  and then take the corresponding cumulative distribution function

$$\theta(t) = \int_0^t f(x)dx . \quad (5)$$

By definition of  $f$  we can verify that

$$\lim_{t \rightarrow +\infty} \theta(t) = \int_0^{+\infty} f(x)dx = 1 \text{ and } \theta(0) = \int_0^0 f(x)dx = 0 . \quad (6)$$

The non-decreasing hypothesis gives the concavity of  $\theta$ .

Examples of this family are  $\theta^1(t) = t/(t+1)$  if  $t \geq 0$  and  $\theta^1(t) = t$  if  $t < 0$ ,  $\theta^2(t) = 1 - e^{-t}$  with  $t \in \mathbb{R}$ .

We introduce  $\theta_r(t) := \theta(\frac{t}{r})$  for  $r > 0$ . This definition is similar to the perspective functions in convex analysis. This functions satisfy

$$\theta_r(0) = 0 \quad \forall r > 0 \text{ and } \lim_{r \searrow 0} \theta_r(t) = 1 \quad \forall t > 0. \quad (7)$$

Examples of this family are  $\theta_r^1(t) = t/(t+r)$  if  $t \geq 0$  and  $\theta_r^1(t) = t$  if  $t < 0$ ,  $\theta_r^2(t) = 1 - e^{-t/r}$   $t \in \mathbb{R}$ . The function  $\theta_r^1(t)$  is quite useful as it can sometimes be used as a "minimum" of this family.

Next lemma will show the link between this functions and the complementarity in one dimension.

**Lemma 1.1.** *Given  $s, t \in \mathbb{R}_+$  and the parameter  $r > 0$ , then*

$$s \perp t \iff \lim_{r \searrow 0} \theta_r(s) + \theta_r(t) \leq 1. \quad (8)$$

*Proof.* We show by contradiction that

$$\lim_{r \searrow 0} \theta_r(s) + \theta_r(t) \leq 1 \implies s \perp t. \quad (9)$$

Suppose  $s, t > 0$ , then

$$\lim_{r \searrow 0} (\theta_r(s) + \theta_r(t)) = \lim_{r \searrow 0} \theta_r(s) + \lim_{r \searrow 0} \theta_r(t) = 2. \quad (10)$$

Contradiction, so  $s \perp t$ . Conversely it is clear that  $s \perp t \implies s = 0$  or  $t = 0$ .  $\square$

In the case of the functions  $\theta_r^1(x)$  we even have the equality in (8) and by definition of this function we have

$$\theta_r^1(s) + \theta_r^1(t) = 1 \iff st = r^2. \quad (11)$$

Now, we will use the previous lemma to replace the complementarity constraint by a minimization of the functions  $\theta$ . We get the sequence of concave optimization problems for  $r > 0$

$$(\tilde{P}_r) \begin{cases} \min_{x^+, x^- \in \mathbb{R}^N} \sum_{i=1}^N \theta_r(x_i^+) + \theta_r(x_i^-) - 1 \\ A(x^+ - x^-) - (x^+ + x^-) = b \\ x^+ \geq 0, x^- \geq 0 \end{cases}. \quad (12)$$

This paper is organised as follow. In section 2 we show convergence results, then in section 3 we look at convergence rates, to continue in section 4 we present the algorithm and finally in section 5 we give numerical results.

## 2 Convergence

We denote  $S_{(AVE)}^*$  the set of solution of (AVE). From now on we suppose that  $S_{(AVE)}^*$  is non-empty and bounded, which means that there exists a constant  $M$  such that

$$M \geq \max_{x \in S_{(AVE)}^*} \|x\|_\infty. \quad (13)$$

We use a relaxed version of  $(\tilde{P}_r)$  to generate strictly feasible iterate, avoid local minima of the objective function and compensation phenomenon.

$$(P_r) \begin{cases} \min_{x^+, x^- \in \mathbb{R}^N} \sum_{i=1}^N \theta_r(x_i^+) + \theta_r(x_i^-) - 1 \\ b - g(r)|A|e - g(r)e \leq A(x^+ - x^-) - (x^+ + x^-) \leq b + g(r)|A|e + g(r)e \\ x^+ + x^- \geq g(r)e \\ 0 \leq x^+ \leq M, 0 \leq x^- \leq M \end{cases}, \quad (14)$$

where  $e$  is the unit vector,  $|A|$  denotes the matrix where each element is the absolute value of the elements of  $A$  and  $g : \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$  is a function which goes to 0 slower than  $r$ , that is

$$\lim_{r \searrow 0} \frac{r}{g(r)} = 0, \quad (15)$$

for instance  $g(r) = r^\alpha$  with  $0 < \alpha < 1$ .

We show in a first step that  $(P_r)$  has solutions.

**Theorem 2.1.** We denote  $S_{(P_r)}^*$  the set of solutions of  $(P_r)$  for a given  $r > 0$ . If  $S_{(AVE)}^* \neq \emptyset$  the problem  $(P_r)$  has at least one solution.

*Proof.* Suppose that  $S_{(AVE)}^* \neq \emptyset$  then there exists a point  $y_r = \bar{x} + g(r)$  with  $\bar{x} \in S_{(AVE)}^*$  which is a feasible point of  $(P_r)$ . Moreover, we minimize a continuous function over a non-empty compact set so the objective function attains its minimum.  $\square$

We present now two lemmas, which we will need for the convergence theorems.

**Lemma 2.2.** Given the functions  $\theta$  and  $g$  define above and  $x \in \mathbb{R}^N$ ,  $r \in \mathbb{R}_+^*$  such that  $x^+ + x^- \geq g(r)e$ , then

$$\theta_r(x_i^+) + \theta_r(x_i^-) - 1 \geq \theta_r(g(r)) - 1, \quad \forall i \in \{1, \dots, N\}. \quad (16)$$

*Proof.* Thanks to  $x^+ + x^- \geq g(r)e$  and that the functions  $\theta$  are non-decreasing and subadditive (concave and  $\theta_r(0) = 0$ ), for all  $i \in \{1, \dots, N\}$  we have

$$\theta_r(x_i^+) + \theta_r(x_i^-) - 1 \geq \theta_r(x_i^+ + x_i^-) - 1, \quad (17)$$

$$\geq \theta_r(g(r)) - 1. \quad (18)$$

$\square$

**Lemma 2.3.** Given functions  $\theta$  and  $g$  define above, then

$$\lim_{r \searrow 0} \theta_r(g(r)) = 1. \quad (19)$$

*Proof.* Using the definition of functions  $\theta$

$$\lim_{r \searrow 0} \theta_r(g(r)) = \lim_{r \searrow 0} \theta_{\frac{r}{g(r)}}(1) = 1. \quad (20)$$

$\square$

We now proceed to the convergence proof of the sequence of  $\{x_r\}_{r>0}$  to an element of  $S_{(AVE)}^*$ , where  $x_r$  is one solution of problem  $(P_r)$ . We never claim that for a given  $r$  the set of solution  $S_{(P_r)}^*$  is a singleton.

We will split this proof in two part : first, if there exists a solution of (AVE) with no zero component, theorem (2.4), and then if every solution of (AVE) contains at least one zero component, theorem (2.6).

**Theorem 2.4.** Suppose that there exists a solution of (AVE) such that this solution has no zero component, then every limit point of the sequence  $\{x^r\}_{r>0}$  of solution of  $(P_r)$  is a solution of (AVE).

*Proof.* We note  $z = (z^+, z^-)$  a solution of (AVE) with no zero component such that  $z^+ \perp z^-$  and for  $r$  sufficiently small  $z^+ + z^- \geq g(r)e$ . So  $z$  is a feasible point

of  $(P_r)$ . Set  $x^r = (x^{r+}, x^{r-})$  with  $\{x^r\}_{r>0}$  the sequence of optimal solution of  $(P_r)$ , then

$$\sum_{i=1}^N (\theta_r(x_i^{r+}) + \theta_r(x_i^{r-}) - 1) \leq \sum_{i=1}^N (\theta_r(z_i^+) + \theta_r(z_i^-) - 1) \leq 0, \quad (21)$$

$$\theta_r(x_i^{r+}) + \theta_r(x_i^{r-}) - 1 \leq - \sum_{j=1; j \neq i}^N (\theta_r(x_j^{r+}) + \theta_r(x_j^{r-}) - 1). \quad (22)$$

We remind that lemma (2.2) gives

$$\theta_r(x_j^+) + \theta_r(x_j^-) - 1 \geq \theta_r(g(r)) - 1, \quad \forall j \in \{1, \dots, N\}. \quad (23)$$

So, in (22) we get

$$\theta_r(x_i^{r+}) + \theta_r(x_i^{r-}) \leq 1 + (N-1)(1 - \theta_r(g(r))). \quad (24)$$

By lemma (2.3) we have  $\lim_{r \rightarrow 0} 1 - \theta_r(g(r)) = 0$ , then we consider the limit point  $\bar{x} = (\bar{x}^+, \bar{x}^-)$  of the sequence  $\{x^r\}_r$ , where  $\bar{x}^+ = \lim_{r \rightarrow 0} x^{r+}$  and  $\bar{x}^- = \lim_{r \rightarrow 0} x^{r-}$ . We obtain

$$\lim_{r \searrow 0} \theta_r(\bar{x}_i^+) + \theta_r(\bar{x}_i^-) \leq 1. \quad (25)$$

Thanks to the equation (25) and the lemma (1.1) we have  $\bar{x}^+ \perp \bar{x}^-$ .

We now check that the limit point of the sequence of  $\{x^r\}_{r>0}$  is a solution of (AVE). Let  $x^r$  be a solution of  $(P_r)$  for  $r > 0$ , we have

$$b - g(r)|A|e - g(r)e \leq A(x^{r+} - x^{r-}) - (x^{r+} + x^{r-}) \leq b + g(r)|A|e + g(r)e. \quad (26)$$

Going through the limit  $r \searrow 0$  we have

$$A(\bar{x}^+ - \bar{x}^-) - (\bar{x}^+ + \bar{x}^-) = b. \quad (27)$$

So,  $\bar{x}$  is a solution of (AVE).  $\square$

Now we consider the second part of our convergence proof, where every solution of (AVE) contain at least one zero component. This proof is similar to the previous one to the difference that there exists some difficulties to satisfy the constraints  $x^{r+} + x^{r-} \geq g(r)e$ . We will first show a useful lemma, which defines a relaxed (AVE).

**Lemma 2.5.** *Given  $\bar{x}$  a solution of (AVE) and  $r > 0$  such that  $g(r) < r_0 = \min_{\bar{x}_i \neq 0} |\bar{x}_i|$ , then  $y_r = \bar{x} + g(r)$  is a solution of  $(AVE)_r$ , that is the equation*

$$(AVE)_r \quad Ax - |x| = b + g(r)Ae - g(r)\delta(x). \quad (28)$$

$$\delta(x) \in \mathbb{R}^N \quad \text{with} \quad \delta_i(x) = \begin{cases} 1 & \text{if } x_i \geq 0 \\ -1 & \text{if } x_i < 0 \end{cases}.$$

*Proof.* Since  $\bar{x}$  is a solution of (AVE)

$$A\bar{x} - |\bar{x}| = b, \quad (29)$$

$$\Rightarrow A\bar{x} + g(r)Ae - |\bar{x}| - g(r)\delta(x) = b + g(r)Ae - g(r)\delta(x), \quad (30)$$

$$\Rightarrow A(\bar{x} + g(r)e) - |\bar{x} + g(r)e| = b + g(r)Ae - g(r)\delta(x), \quad (31)$$

then  $y_r = \bar{x} + g(r)$  is a solution of  $(AVE)_r$ .  $\square$

In the next theorem we prove the convergence in the case where every solution have at least one zero component.

**Theorem 2.6.** *Suppose that  $S_{(AVE)}^* \neq \emptyset$  and that every solution of (AVE) has at least one zero component, then every limit point of the sequence  $\{x^r\}_{r>0}$  of solution of  $(P_r)$  is a solution of (AVE).*

*Proof.* By hypothesis there exists at least one solution of (AVE). Using lemma (2.5) we can build a sequence  $\{y^r\}_{r>0}$  where  $y^r$  is solution of  $(AVE)_r$ . For  $r$  sufficiently small  $y_r$  is a feasible point of  $(P_r)$ . We can always choose the unique decomposition of  $y^r = y^{r+} - y^{r-}$  such that  $y^{r+} \perp y^{r-}$ . Set  $x^r = (x^{r+}, x^{r-})$  with  $\{x_r\}_{r>0}$  the sequence of optimal solution of  $(P_r)$ , then

$$\sum_{i=1}^N (\theta_r(x_i^{r+}) + \theta_r(x_i^{r-}) - 1) \leq \sum_{i=1}^N (\theta_r(y_i^{r+}) + \theta_r(y_i^{r-}) - 1) \leq 0, \quad (32)$$

$$\theta_r(x_i^{r+}) + \theta_r(x_i^{r+}) - 1 \leq - \sum_{j=1; j \neq i}^N (\theta_r(x_j^{r+}) + \theta_r(x_j^{r+}) - 1). \quad (33)$$

Again lemma (2.2) yields to

$$\theta_r(x_i^{r+}) + \theta_r(x_i^{r-}) \leq 1 + (N-1)(1 - \theta_r(g(r))). \quad (34)$$

By lemma (2.3) we have  $\lim_{r \rightarrow 0} 1 - \theta_r(g(r)) = 0$ , then we consider the limit point  $\bar{x} = (\bar{x}^+, \bar{x}^-)$  of the sequence  $\{x^r\}_r$ , where  $\bar{x}^+ = \lim_{r \rightarrow 0} x^{r+}$  and  $\bar{x}^- = \lim_{r \rightarrow 0} x^{r-}$ . We obtain

$$\lim_{r \searrow 0} \theta_r(\bar{x}_i^+) + \theta_r(\bar{x}_i^-) \leq 1. \quad (35)$$

Thanks to the equation (35) and the lemma (1.1) we have  $\bar{x}^+ \perp \bar{x}^-$ .

Finally we verify that  $\bar{x}$  is a solution of (AVE). Let  $x^r$  be a solution of  $(P_r)$  for  $r > 0$ , we have

$$b - g(r)|A|e - g(r)e \leq A(x^{r+} - x^{r-}) - (x^{r+} + x^{r-}) \leq b + g(r)|A|e + g(r)e. \quad (36)$$

Going through the limit  $r \searrow 0$  we have

$$A(\bar{x}^+ - \bar{x}^-) - (\bar{x}^+ + \bar{x}^-) = b. \quad (37)$$

So,  $\bar{x}$  is a solution of (AVE).  $\square$

### 3 Error estimate

In this section we study the behaviour of the sequence of  $\{x^r\}_{r>0}$  optimal solution of  $(P_r)$  when  $r$  becomes small.

We remind the definition of the Landau notation  $O$  often used in the context of asymptotic comparison. Given a real  $a$  and two functions  $f$  and  $h$ . We have

$$f(x) = O_{x \rightarrow a}(h(x)) \text{ if } \forall x, |x - a| \leq d \implies |f(x)| \leq C|h(x)|. \quad (38)$$

We denote  $O(h(x))$  when the real  $a$  is 0.

We first show a useful lemma, which doesn't need the hypothesis of the existence of a solution without zero component.

**Lemma 3.1.** *Let  $S_{(AVE)}^* \neq \emptyset$  and choose functions  $\theta$  such that  $\theta \geq \theta^1$ , with  $\theta_r^1(t) = t/(t+r)$  for  $t \geq 0$ . We have for  $x^r \in S_{(P_r)}^*$  that*

$$x_i^{r+} x_i^{r-} \leq O(g(r)r) \quad \forall i \in \{1, \dots, N\}, \quad (39)$$

with  $x^+ x^-$  is the componentwise product.

*Proof.* Set  $i \in \{1, \dots, N\}$ . We have thanks to the convergence proof of the theorem (2.4) that for  $r$  sufficiently small the following holds

$$\theta_r(x_i^{r+}) + \theta_r(x_i^{r-}) - 1 \leq (N-1)(1 - \theta_r(g(r))). \quad (40)$$

By using  $\theta^1$  function

$$\theta_r^1(x_i^{r+}) + \theta_r^1(x_i^{r-}) - 1 \leq \theta_r(x_i^{r+}) + \theta_r(x_i^{r-}) - 1, \quad (41)$$

$$\leq (N-1)(1 - \theta_r(g(r))), \quad (42)$$

$$\leq (N-1)(1 - \theta_r^1(g(r))), \quad (43)$$

then for  $r$  sufficiently small such that  $g(r) \geq r$  and  $(1 - (N-1)g(r)) > 0$

$$\frac{x_i^{r+}}{x_i^{r+} + r} + \frac{x_i^{r-}}{x_i^{r-} + r} - 1 \leq (N-1) \left(1 - \frac{g(r)}{g(r) + r}\right) \leq (N-1)g(r), \quad (44)$$

$$\frac{2x_i^{r+} x_i^{r-} + r x_i^{r+} + r x_i^{r-} - x_i^{r+} x_i^{r-} - r x_i^{r+} - r x_i^{r-} - r^2}{(x_i^{r+} + r)(x_i^{r-} + r)} \leq (N-1)g(r) \quad (45)$$

$$\frac{x_i^{r+} x_i^{r-} - r^2}{(x_i^{r+} + r)(x_i^{r-} + r)} \leq (N-1)g(r), \quad (46)$$

$$x_i^{r+} x_i^{r-} - r^2 \leq (N-1)g(r)(x_i^{r+} x_i^{r-} + r x_i^{r+} + r x_i^{r-} + r^2), \quad (47)$$

$$x_i^{r+} x_i^{r-} \leq r^2 g(r) \frac{1 + (N-1)}{1 - (N-1)g(r)} + r g(r) \frac{(N-1)(x_i^{r+} + x_i^{r-})}{1 - (N-1)g(r)}, \quad (48)$$

and the results.  $\square$

We first show an error estimate for the part of the iterate which is going to zero.



**Proposition 3.2.** *Let  $S_{(AVE)}^* \neq \emptyset$ . Given  $\bar{x}$  an accumulation point of the sequence  $\{x^r\}_r$  solution of  $(P_r)$ . We split  $\bar{x} = \bar{x}^+ + \bar{x}^-$  with one at least is always zero and  $x^r = x^{r+} + x^{r-}$ . The convergence of the variable  $x^{r+}$  or  $x^{r-}$  to the possibly zero part of the accumulation point is done in  $O(r)$ .*

*Proof.* Set  $i \in \{1, \dots, N\}$ . We will work with one component. Suppose that  $\bar{x}_i^+ = 0$ . The opposite case is completely similar. Lemma (3.1) and using that  $x^r$  is feasible ( $x^{r+} + x^{r-} \geq g(r)$ ) gives

$$x_i^{r+} x_i^{r-} \leq O(rg(r)), \quad (49)$$

$$x_i^{r+} \leq \frac{O(rg(r))}{x_i^{r-}}, \quad (50)$$

$$x_i^{r+} \leq \frac{O(rg(r))}{g(r)}, \quad (51)$$

$$|x_i^{r+} - (\bar{x}^+)_i| \leq O(r). \quad (52)$$

□

In the next theorem we are interested in the error estimate of the possibly non-zero part of the solution in the couple  $(x^+, x^-)$ .

We will present the Hoffman's lemma, which we need in the proof of the next theorem. This lemma gives an error bound in the case of a convex polyhedron depending on the matrix  $A$  and the residual vector  $(Ax - b)^+ = \Pi_{\mathbb{R}_+^n}(Ax - b)$ .

**Lemma 3.3** (Hoffman's lemma). [9] *Given a convex polyhedron  $P$  such that*

$$P = \{x \mid Ax \leq b\}. \quad (53)$$

*We set  $d_P(x)$  the distance between  $x$  and  $P$ , by choosing a norm  $\|\cdot\|$ , where  $d_P(x) = \inf_{y \in P} \|y - x\|$ . There exists a constant  $K$  which only depends on  $A$ , such that*

$$\forall b, \forall x \in \mathbb{R}^n : d_P(x) \leq K \|(Ax - b)^+\|. \quad (54)$$

Remark that if the constraints are given by  $Ax = b$  with  $A$  a full-rank matrix instead of  $Ax \leq b$  then the polyhedron contains only one point and we can estimate the constant as  $K = \|A^{-1}\|$ .

**Theorem 3.4.** *Let  $S_{(AVE)}^* \neq \emptyset$ . Given  $\bar{x}$  an accumulation point of the sequence  $\{x^r\}_{r>0}$  solutions of  $(P_r)$ . For  $r$  sufficiently small*

$$d_{S^*}(x^r) = O(g(r)) \quad (55)$$

*where  $S^*$  denote the intersection of  $S_{(AVE)}^*$  and a neighbourhood  $V$  of  $\bar{x}$ , such that any point in  $V$  has the same sign than  $\bar{x}$ .*

*Proof.* We split the proof in two cases, either  $\min_{i \in \{1, \dots, N\}} |\bar{x}_i| \neq 0$ , either  $\exists i \in \{1, \dots, N\}$ ,  $\bar{x}_i = 0$ .

a) First, suppose there is no zero component in  $\bar{x}$ . We set  $\alpha = \min_{i \in \{1, \dots, N\}} |\bar{x}_i|/2$  and a neighbourhood  $V$  of  $\bar{x}$  defined as

$$V = B_\infty(\bar{x}, \alpha) = \{x \mid \max_{1 \leq i \leq N} |x_i - \bar{x}_i| \leq \alpha\}. \quad (56)$$

So  $\forall x \in V$ ,  $\bar{x}$  and  $x$  have the same sign. We set  $D = \text{diag}(\delta(\bar{x}))$ , where

$$\delta(x) \in \mathbb{R}^N \text{ with } \delta_i(x) = \begin{cases} 1 & \text{if } x_i \geq 0 \\ -1 & \text{if } x_i < 0 \end{cases}.$$

By taking  $S^* = \{x \in \mathbb{R}^n \mid Ax - Dx = b\} \cap V$  we obtain a convex polyhedron. This set is non-empty because  $\bar{x} \in S^*$ . In the neighbourhood  $V$  the resolution of  $Ax - Dx = b$  gives a solution of (AVE). Now we can use Hoffman lemma.

For  $r$  sufficiently small  $x_r \in V$  and then

$$d_{S^*}(x^r) \leq K \left\| \begin{array}{l} (A - D)x^r - b \\ (x^r - \alpha - \bar{x})^+ \\ (-x^r - \alpha + \bar{x})^+ \end{array} \right\| \leq K(\|(A - D)x^r - b\| + \|(x^r - \alpha - \bar{x})^+\|) + \|(-x^r - \alpha + \bar{x})^+\|, \quad (58)$$

$$= K\|(A - D)x^r - b\|, \quad (59)$$

$$= K\|Ax^r - |x^r| - b\|. \quad (60)$$

As  $x^r$  is feasible for  $(P_r)$ , we have

$$\|Ax^r - |x^r| - b\| = \|g(r)Ae - g(r)\delta(x^r)\|, \quad (61)$$

$$= \|(Ae - \delta(x^r))g(r)\|, \quad (62)$$

$$\leq \|Ae - \delta(x^r)\| |g(r)|, \quad (63)$$

$$= \|Ae - \delta(x^r)\| |g(r)| = O(g(r)). \quad (64)$$

We add this in (57)

$$d_{S^*}(x^r) \leq K\|Ae - \delta(x^r)\| |g(r)| = O(g(r)). \quad (65)$$

b) Now we move to the case where  $\exists i \in \{1, \dots, N\}$ ,  $\bar{x}_i = 0$ . We denote  $\sigma(t) = \{i \mid t_i \neq 0\}$ . We set  $\alpha = \min_{i \in \sigma(\bar{x})} |\bar{x}_i|/2$  and a neighbourhood  $V$  of  $\bar{x}$  defined as

$$V = B_\infty(\bar{x}, \alpha) = \{x \mid \max_{i \in \sigma(\bar{x})} |x_i - \bar{x}_i| \leq \alpha\}. \quad (66)$$

$V$  is non-empty because  $\bar{x} \in V$ . For all  $x \in V$ ,  $\bar{x}$  and  $x$  have the same sign only for the components  $\bar{x}_i$  with  $i \in \sigma(\bar{x})$ . For  $r$  sufficiently small we have  $x^r \in V$ .

By taking  $S^* = \{x \in \mathbb{R}^n \mid Ax - Dx = b, Dx \geq 0\} \cap V$  with  $D = \text{diag}(\delta(x^r))$  we obtain a convex polyhedron. The choice of  $D$  depending on  $x_r$  is not restrictive as we can always take a subsequence of the sequence  $\{x_r\}_{r>0}$ , which converge to  $\bar{x}$ , with constant signs near  $\bar{x}$ . This set is non-empty because  $\bar{x} \in S^*$ . In the neighbourhood  $V$  the resolution of  $Ax - Dx = b$  with the constraints

$Dx \geq 0$  gives a solution of (AVE). We can use Hoffman lemma

$$d_{S^*}(x^r) \leq K \left\| \begin{array}{c} (A-D)x^r - b \\ (x^r - \alpha - \bar{x})^+ \\ (-x^r - \alpha + \bar{x})^+ \\ (-Dx)^+ \end{array} \right\| \leq K(\|(A-D)x^r - b\| + \|(x^r - \alpha - \bar{x})^+\|) \quad (67)$$

$$+ \|(-x^r - \alpha + \bar{x})^+\| + \|(-Dx)^+\|, \quad (68)$$

$$= K\|(A-D)x^r - b\|, \quad (69)$$

$$= K\|Ax^r - |x^r| - b\|. \quad (70)$$

As  $x^r$  is feasible for  $(P_r)$ , we have

$$\|Ax^r - |x^r| - b\| \leq \|g(r)Ae - g(r)\delta(x^r)\|, \quad (71)$$

$$= \|(Ae - \delta(x^r))g(r)\|, \quad (72)$$

$$\leq \|Ae - \delta(x^r)\| |g(r)|, \quad (73)$$

$$= \|Ae - \delta(x^r)\|g(r) = O(g(r)). \quad (74)$$

We add this in (67)

$$d_{S^*}(x^r) \leq K\|Ae - \delta(x^r)\|g(r) = O(g(r)), \quad (75)$$

□

*Remark 1.* We can be a bit more specific in the case where  $(A-D)$  is invertible. In this case  $S^* = \{\bar{x}\}$ , so (55) becomes

$$\|x^r - \bar{x}\| \leq \|(A-D)^{-1}\| \|Ae - \delta(x)\|g(r) = O(g(r)). \quad (76)$$

This case correspond to the special cases where (AVE) has some isolated solutions.

## 4 Algorithm

In the two previous sections we have seen theoretical results about convergence and error estimate of an algorithm to find a solution of (AVE). In this section we focus on the algorithm and we make some remarks about the parameters and the implementation. We have the generic algorithm with  $C$  the feasible set of  $(P_{r^k})$ .

$$[\text{Theta-AVE}] \begin{cases} \{r^k\}_{k \in \mathbb{N}}, r^0 > 0 \text{ and } \lim_{k \rightarrow +\infty} r^k = 0 \\ \text{find } x^k : x^k \in \arg \min_{x \in C} \sum_{i=1}^n \theta_{r^k}(x_i^+) + \theta_{r^k}(x_i^-) - 1 \end{cases} \quad (77)$$

*Remark 2* (About the optimization problem in practice). When implementing our algorithms one should probably more likely use the initial problem  $(\tilde{P}_r)$  with infeasible iterates, which will have improved numerical behaviour and also may add the constraint  $x^+ + x^- \geq g(r)e$  otherwise the sequence will possibly go to a local minimum with a zero component.

*Remark 3* (About the use of functions  $\theta$ ). In this study we put the variables in a compact set. One should note that the functions  $\theta$  are more efficient when their arguments lives in  $[0, 1]$ . Also, we use one way to express complementarity with lemma (1.1) another way, which we will use in the numerical study, is to use the following

$$\theta_r(s) + \theta_r(t) - \theta_r(s + t) = 0. \quad (78)$$

*Remark 4.* [Acceleration and precision heuristic] At each step in  $r$  we solve the concave optimization problem to get the current iterate. The following heuristic can be rather useful to accelerate convergence when we are close to the solution and ensure a good precision. After finding the current iterate  $z^k$  we solve if invertible the linear system

$$(A - \text{diag}(\delta(z^k)))z = b. \quad (79)$$

We then check if  $z$  solves (AVE). If it does, then the algorithm is finished and we solved (AVE) with the same precision as we solve the linear system. Otherwise we just continue the iteration in  $r$  with  $z^k$ .

*Remark 5* (About the parameters). More than the choice of objective expression and the choice of  $\theta$  functions stay some parameters :  $\alpha$ ,  $r_0$  and the update in  $r$ . All this parameters are used in the constraint  $x^+ + x^- \geq g(r)e$  with  $g(r) = r^\alpha$  and  $0 < \alpha < 1$ . It has been shown in the error estimate theorem that the convergence to zero part of the solution is in  $g(r)$  so we have all interest to take  $\alpha$  as big as possible, for instance  $\alpha = 0.99$ . Also we understand that there is a link between the value of  $\alpha$  and update of  $r$ . About this one we choose to select a constant sequence of value with an update constant  $T$ , so that  $r^{k+1} = \frac{r^k}{T}$ . When you decrease this update constant you are allowed to increase  $\alpha$ . Finally about the  $r_0$  we can use the relation in one dimension mentioned in the introduction of functions  $\theta^1$

$$\theta_r^1(s) + \theta_r^1(t) = 1 \iff st = r^2. \quad (80)$$

## 5 Numerical Simulations

Thanks to the previous sections we have keys for an algorithm, we will show now some numerical results. These simulations have been done using MATLAB, [10], with the free software package for convex optimization CVX, [11]. We will use SLA algorithm (successive linearisation algorithm) to solve our concave minimization problem at each iteration in  $r$ . This algorithm is a finitely timestep Franck & Wolf algorithm, [12].

**Proposition 5.1** (SLA algorithm for concave minimization). *Given  $\epsilon$  sufficiently small and  $r^k$ . Denote  $C$  the feasible set of  $(P_{r^k})$ . We know  $x^k = x^{k+} - x^{k-}$  and we find  $x^{k+1}$  as a solution of the linear problem*

$$\min_{y^+, y^- \in C} y^{+T} \nabla \theta_{r^k}(x^{+k}) + y^{-T} \nabla \theta_{r^k}(x^{-k}), \quad (81)$$

with  $y^0 = y^{0+} - y^{0-}$  a random point. We stop when

$$x^{k+1} \in C \text{ and } (x^{k+1} - x^k)^t \nabla \theta_{r,k}(x^k) \leq \epsilon. \quad (82)$$

This algorithm generates a finite sequence with strictly decreasing objective function values.

*Proof.* see [[12], Theorem 4.2]. □

Also in the linearization we add a supplementary stopping criterion if we find a solution of (AVE).

In (table 1) we initialize our data the same way as in [6], which means that we consider some randomly generated (AVE) problem with singular values of A exceeding 1 where the data (A, b) are generated by the Matlab scripts:

```
n=input('dimension of matrix A =');
rand('state',0);
R=rand(n,n);
b=rand(n,1);
A=R'*R+n*eye(n);
```

for  $n = 4$  to 1024 in power of two.

The required precision for solving (AVE) is  $10^{-6}$  and thanks to the heuristic from remark (4) we get in the worst case  $10^{-10}$ . We use for  $r$  update the constant  $T = 1.8$ , as an initial  $r$  value  $r_0 = \sqrt{s^T t}$ , the functions  $\theta^1$ ,  $\alpha = 0.99$ . For every  $n$  we solve 10 instances of the problem, results are sum up in (table 1). In every cases our method solves the problem, also we notice that the time required to solve problem is increasing significantly when the dimension grows.

Now we are interested in another example, initializing the data like in [4] for several  $n$  and for several values of the parameters, in each situation we solve one hundred instances of the problem.

"Choose a random  $A$  from a uniform distribution on  $[-10, 10]$ , then choose a random  $x$  from a uniform distribution on  $[-1, 1]$  and set  $b = Ax - |x|$ ."

For each situation we compare our methods with the method presented in [4]. We give "nnztot" the number of violated expression for all problems, "nnzx" the maximum violated expression for one problem, the number of iteration in  $r$  "out-iter", the number of linear program solve for all the problems "in-iter". Finally we give the time needed in seconds, the minimum value of  $r$  " $r_{min}$ " where we manage to solve an instance, the number of problem where we didn't manage to solve (AVE), "Theta-AVE", which we will compare with the number of failure of SLA based method [4], "MM". We run large number of simulations with different configurations of the parameters and show here the best results.

In (table 2) and (table 3) we use the objective function as presented in this paper with different  $\theta$  functions, in the former with  $\theta_r^1(t) = t/(t+r)$  and in the latter with  $\theta_r^2(t) = 1 - e^{-t/r}$ . We can see that our methods slightly improves the number of failure compare to MM algorithm, except from the case  $n = 256$  in (table 3).

In (table 4) and (table 5) we only use function  $\theta^1$  but with the different objective function proposed in last section, equation (78). In this case we don't necessarily need to put the constraint  $x^+ + x^- \geq r^\alpha$ . We can see that we improved the number of failure from the two previous methods and we get improved results compare to the SLA based method. This can be explained as we noticed that for very low values of  $r$  we still manage to find new iterates, see value of  $r_{min}$ , and also explains the fact that we need more iterations and more time.

In order to confirm the validity of our method we consider two concrete examples. The first one is a second order ordinary differential equation with initial conditions and the second example is an obstacle problem.

We consider the ordinary differential equation

$$\ddot{x}(t) - |x(t)| = 0, \quad x(0) = x_0, \quad \dot{x}(0) = \gamma. \quad (83)$$

We get an (AVE) by using a finite difference scheme to linearize this equation. We use the following second-order backward difference to approximate the derivative

$$\frac{x_{i-2} - 2x_{i-1} + x_i}{2h} - |x_i| = 0. \quad (84)$$

Equation (84) was derived with equispace gridpoints  $x_i = ih$ ,  $i = 1, \dots, N$ . In order to approximate the Neumann boundary conditions we use a center difference

$$\frac{u_{-1} - u_1}{2h} = \gamma. \quad (85)$$

In (figure 1) we compare the resolution of equation (83) using MATLAB, [10], ode45 function with the resolution using finite difference scheme and ThetaAVE. The domain is  $t \in [0, 4]$ , initial conditions  $x_0 = -1$ ,  $\gamma = 1$  and  $N = 100$ .

We see that our method gives coherent results.

The second example is an obstacle problem. We try to find a trajectory joining the bounds of the domain with obstacle,  $g$ , and a minimal curvature,  $f$ . This can be formulated as the following equation

$$(\ddot{u}(x) - f(x))^T(u(x) - g(x)) = 0, \quad \ddot{u}(x) - f(x) \geq 0, \quad u(x) - g(x) \geq 0. \quad (86)$$

We approximate the second order derivative with a second-order central difference, then the problem (86) is similar to the discrete version on an equispace gridpoints  $x_i = ih$ ,  $i = 1, \dots, N$ .

$$(Du - f)^T(u - g) = 0, \quad Du - f \geq 0, \quad u - g \geq 0, \quad (87)$$

where  $g_i = g(x_i)$ ,  $f_i = f(x_i)$ . This can be reformulated as a linear complementarity problem with by setting  $z = u - g$ ,  $M = D$  and  $q = Dg - f$ , that is

$$(Mz + q)^T z = 0, \quad Mz + q \geq 0, \quad z \geq 0. \quad (88)$$

which if 1 is not an eigenvalue of  $M$  is equivalent to (AVE), ([1], Prop. 2),

$$(M - I)^{-1}(M + I)x - |x| = (M - I)^{-1}q . \quad (89)$$

We give results for our method and MM method from [4], with  $g(x) = \max(0.8 - 20(x - 0.2)^2, \max(1 - 20(x - 0.75)^2, 1.2 - 30(x - 0.41)^2))$ ,  $f(x) = 1$ ,  $N = 50$  in (figure 2).

Both methods gives 17 points on  $g$  and none under  $g$  over 50 points. Once again our ThetaAVE method gives coherent results.

## Conclusion and Perspectives

In this paper, we have proposed a class of heuristics schemes to solve the NP-hard problem of solving (AVE). We compared our methods with some existing methods, [6] and [4]. In each case our Theta-AVE algorithms manage to improve the number of failure, which was our principle aim.

Further studies can improve the choices of parameters in order to reduce the time needed to solve the problems, especially when the dimension grows. Also the last results involving another way to express the complementarity show promising results, so we can wonder if it is possible to improve our algorithms in this case and if there exists other similar reformulation of the complementarity which can give even better results.

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Table 1: Theta-algorithm for (AVE) in the case with singular values of A exceeding 1. Parameter :  $r = r/1.8$ ,  $\alpha = 0,99$ ,  $r_0 = \sqrt{s^T t}$ ,  $r_{max} = 10^{-4}$ . Function :  $\theta^1$ . Objective function :  $\theta + \theta - 1$ .

| $n$  | nb of instances | nb $r$ | nb LP | time     |
|------|-----------------|--------|-------|----------|
| 4    | 10              | 21     | 42    | 2.9      |
| 8    | 10              | 69     | 138   | 9.986    |
| 16   | 10              | 10     | 20    | 1.3614   |
| 32   | 10              | 31     | 62    | 4.7856   |
| 64   | 10              | 40     | 80    | 8.5690   |
| 128  | 10              | 30     | 60    | 11.4376  |
| 256  | 10              | 60     | 120   | 93.5494  |
| 512  | 10              | 70     | 140   | 702.1368 |
| 1024 | 10              | 70     | 140   | 4880.7   |

Table 2: Theta-algorithm on one hundred random cases of (AVE). Parameter :  $r = r/1.8$ ,  $\alpha = 0,99$ ,  $r_0 = \sqrt{s^T t}$ ,  $r_{fin} = 10^{-4}$ . Function :  $\theta^1$ . Objective function :  $\theta + \theta - 1$ .

| $n$ | nnztot | nnzx | out-iter | in-iter | time | $r_{min}$ | Theta-AVE | MM |
|-----|--------|------|----------|---------|------|-----------|-----------|----|
| 32  | 9      | 1    | 407      | 963     | 107  | 0.0144    | 9         | 11 |
| 64  | 7      | 1    | 452      | 1297    | 159  | 0.0048    | 7         | 15 |
| 128 | 11     | 1    | 564      | 1890    | 427  | 0.0048    | 11        | 11 |
| 256 | 15     | 1    | 690      | 2381    | 2585 | 0.0014    | 15        | 16 |

Table 3: Theta-algorithm on one hundred random cases of (AVE). Parameter :  $r = r/2$ ,  $\alpha = 0,9999$ ,  $r_0 = \sqrt{s^T t}$ ,  $r_{fin} = 10^{-4}$ . Function :  $\theta^2$ . Objective function :  $\theta + \theta - 1$ .

| $n$ | nnztot | nnzx | out-iter | in-iter | time | $r_{min}$ | Theta-AVE | MM |
|-----|--------|------|----------|---------|------|-----------|-----------|----|
| 32  | 6      | 1    | 340      | 841     | 94   | 0.0081    | 6         | 11 |
| 64  | 10     | 1    | 432      | 1193    | 196  | 0.002     | 10        | 15 |
| 128 | 10     | 2    | 480      | 1461    | 481  | 0.0041    | 9         | 11 |
| 256 | 20     | 2    | 642      | 2130    | 2337 | 0.0018    | 19        | 16 |

Table 4: Theta-algorithm on one hundred random cases of (AVE). Parameter :  $r = r/2$ ,  $r_0 = \sqrt{\max(s_i t_i)}$ ,  $r_{fin} = 10^{-8}$ . Function :  $\theta^1$ . Objective function :  $\theta(s) + \theta(t) - \theta(s + t)$ .

| $n$ | nnztot | nnzx | out-iter | in-iter | time | $r_{min}$ | Theta-AVE | MM |
|-----|--------|------|----------|---------|------|-----------|-----------|----|
| 32  | 4      | 2    | 293      | 7378    | 602  | 3.36e-6   | 3         | 11 |
| 64  | 4      | 2    | 336      | 9495    | 1039 | 3e-6      | 3         | 15 |
| 128 | 8      | 2    | 338      | 9557    | 2004 | 3.6e-6    | 5         | 11 |

Table 5: Theta-algorithm on one hundred random cases of (AVE). Parameter :  $r = r/2$ ,  $r_0 = \sqrt{s^T t}$ ,  $r_{fin} = 10^{-8}$ . Function :  $\theta^1$ . Objective function :  $\theta(s) + \theta(t) - \theta(s+t)$ .

| $n$ | nnztot | nnzx | out-iter | in-iter | time | $r_{min}$ | Theta-AVE | MM |
|-----|--------|------|----------|---------|------|-----------|-----------|----|
| 32  | 5      | 2    | 312      | 8062    | 647  | 3.4e-6    | 3         | 11 |
| 64  | 7      | 2    | 354      | 10038   | 1118 | 3.8e-6    | 5         | 15 |
| 128 | 4      | 2    | 356      | 10400   | 2156 | 3.9e-6    | 2         | 11 |

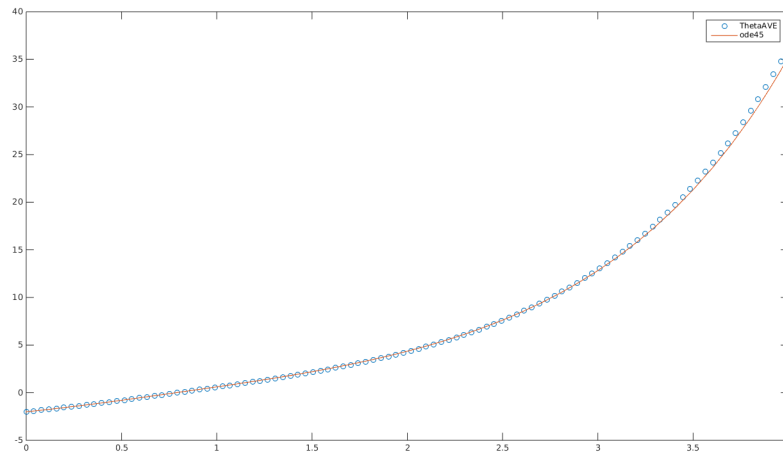


Figure 1: Numerical solution of equation (83) with edo45 and ThetaAVE.

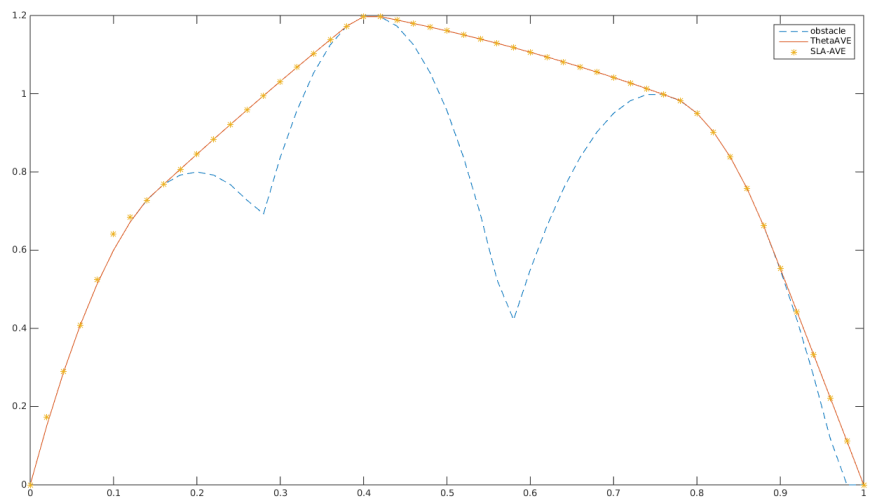


Figure 2: A solution of the obstacle problem (86) with ThetaAVE and method from [4]