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A NONCOMMUTATIVE VERSION OF THE JULIA-WOLFF-CARATHÉODORY THEOREM

SERBAN TEODOR BELINSCHI

ABSTRACT. The classical Julia-Wolff-Carathéodory Theorem characterizes the behaviour of the derivative of an analytic self-map of a unit disc or of a half-plane of the complex plane at certain boundary points. We prove a version of this result that applies to noncommutative self-maps of noncommutative half-planes in von Neumann algebras at points of the distinguished boundary of the domain. Our result, somehow surprisingly, relies almost entirely on simple geometric properties of noncommutative half-planes, which are quite similar to the geometric properties of classical hyperbolic spaces.

1. INTRODUCTION

The classical Julia-Wolff-Carathéodory Theorem describes the behaviour of the derivative of an analytic self-map of the unit disc \mathbb{D} or of the upper half-plane \mathbb{C}^+ of the complex plane \mathbb{C} at certain boundary points. Numerous generalizations, to self-maps of balls or polydisks in \mathbb{C}^n , analytic functions with values in spaces of linear operators, analytic self-maps on domains in spaces of operators or in more general Banach spaces etc. - see for example [22, 14, 16, 25, 1, 19, 18, 2] (the list is not exhaustive) - are known. This note gives a version of this theorem for noncommutative self-maps of the noncommutative upper half-plane of a von Neumann algebra \mathcal{A} . The result builds on the recent literature in the field - see [5, 7, 20], and falls under the programme aiming to find the noncommutative versions of classical complex analysis results - see for example [3, 4, 8, 9, 10].

In the second section we state our main result, and provide the required background. The third section is dedicated to proving a Schwarz lemma-type result for noncommutative functions. In this same section we give a simple (not necessarily original, though) proof of the classical Julia-Wolff-Carathéodory Theorem in order to make this article self-contained and some lemmas that make use of it. Finally, in the last section we prove our main result.

2. NONCOMMUTATIVE FUNCTIONS AND THE JULIA-CARATHÉODORY THEOREM

2.1. Noncommutative functions. Noncommutative sets and functions originate in [23, 24]. Here we largely follow [17] in their definition. We refer to this excellent monograph for details on, and proofs of, the statements below.

First a notation: if S is a nonempty set, we denote by $M_{m \times n}(S)$ the set of all matrices with m rows and n columns having entries from S . For simplicity, we let $M_n(S) := M_{n \times n}(S)$. Given C^* -algebra \mathcal{A} , a *noncommutative set* is a family $\Omega := (\Omega_n)_{n \in \mathbb{N}}$ such that

- (a) for each $n \in \mathbb{N}$, $\Omega_n \subseteq M_n(\mathcal{A})$;
- (b) for each $m, n \in \mathbb{N}$, we have $\Omega_m \oplus \Omega_n \subseteq \Omega_{m+n}$.

The noncommutative set Ω is called *right admissible* if in addition the condition (c) below is satisfied:

- (c) for each $m, n \in \mathbb{N}$ and $a \in \Omega_m, b \in \Omega_n, w \in M_{m \times n}(\mathcal{A})$, there is an $\epsilon > 0$ such that $\begin{pmatrix} a & zw \\ 0 & b \end{pmatrix} \in \Omega_{m+n}$ for all $z \in \mathbb{C}, |z| < \epsilon$.

Given C^* -algebras \mathcal{A}, \mathcal{C} and a noncommutative set $\Omega \subseteq \coprod_{n=1}^{\infty} M_n(\mathcal{A})$, a *noncommutative function* is a family $f := (f_n)_{n \in \mathbb{N}}$ such that $f_n: \Omega_n \rightarrow M_n(\mathcal{C})$ and

- (1) $f_m(a) \oplus f_n(b) = f_{m+n}(a \oplus b)$ for all $m, n \in \mathbb{N}, a \in \Omega_m, b \in \Omega_n$;
(2) for all $n \in \mathbb{N}, f_n(T^{-1}aT) = T^{-1}f_n(a)T$ whenever $a \in \Omega_n$ and $T \in GL_n(\mathbb{C})$ are such that $T^{-1}aT$ belongs to the domain of definition of f_n .

A remarkable result (see [17, Theorem 7.2]) states that, under very mild conditions on Ω , local boundedness for f implies each f_n is analytic as a map between Banach spaces. Indeed, a hint towards the proof of this result is the following essential property of noncommutative functions: if Ω is admissible, $a \in \Omega_n, b \in \Omega_m, c \in M_{n \times m}(\mathcal{A})$, such that $\begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \in \Omega_{n+m}$, then there exists a linear map $\Delta f_{n,m}(a, b): M_{n \times m}(\mathcal{A}) \rightarrow M_{n \times m}(\mathcal{C})$ such that

$$(1) \quad f_{n+m} \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} = \begin{pmatrix} f_n(a) & \Delta f_{n,m}(a, b)(c) \\ 0 & f_m(b) \end{pmatrix}.$$

Obviously, this implies in particular that f_{n+m} extends to the set of all elements $\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}$ such that $a \in \Omega_n, b \in \Omega_m, c \in M_{n \times m}(\mathcal{A})$ (see [17, Section 2.2]). Two properties of this operator that are important for us are

$$(2) \quad \Delta f_{n,n}(a, b)(a - b) = f(a) - f(b) = \Delta f_{n,n}(b, a)(a - b), \quad \Delta f_{n,n}(a, a)(c) = f'_n(a)(c),$$

the classical Frechet derivative of f_n in a applied to the element $c \in M_n(\mathcal{A})$. Moreover, $\Delta f_{n,m}(a, b)$ as functions of a and b , respectively, satisfy properties similar to the ones described in items (1), (2) above (see [17, Sections 2.3–2.5] for details). For convenience, from now on we shall suppress the indices denoting the level for noncommutative functions, as it will almost always be obvious from the context.

We provide three examples of noncommutative sets:

- (i) The noncommutative upper half-plane $H^+(\mathcal{A}) = (H^+(M_n(\mathcal{A})))_{n \in \mathbb{N}}$, where $H^+(M_n(\mathcal{A})) = \{b \in M_n(\mathcal{A}) : \Im b > 0\}$,
(ii) The set of nilpotent matrices with entries from \mathcal{A} , and
(iii) The unit ball $(B(M_n(\mathcal{A})))_{n \in \mathbb{N}}$, where $B(M_n(\mathcal{A})) = \{b \in M_n(\mathcal{A}) : \|b\| < 1\}$.

Our paper will focus on the first example.

2.2. The Julia-Wolff-Carathéodory Theorem, classical and noncommutative. We state the classical Julia-Wolff-Carathéodory Theorem for analytic self-maps of the upper half-plane \mathbb{C}^+ at a point of the real line \mathbb{R} . In the following we denote by $\lim_{z \rightarrow \alpha}^{\triangleleft}$ the nontangential limit at a point $\alpha \in \mathbb{R}$ (see, for ex. [15]).

Theorem 2.1. *Let $f: \mathbb{C}^+ \rightarrow \mathbb{C}^+$ be analytic and $\alpha \in \mathbb{R}$ be fixed.*

- (1) *Assume that*

$$(3) \quad c := \liminf_{z \rightarrow \alpha} \frac{\Im f(z)}{\Im z} < \infty.$$

Then $\lim_{z \underset{\triangleleft}{\rightarrow} \alpha} f(z) := f(\alpha)$ exists and belongs to \mathbb{R} , and

$$(4) \quad \lim_{z \underset{\triangleleft}{\rightarrow} \alpha} f'(z) = \lim_{z \underset{\triangleleft}{\rightarrow} \alpha} \frac{f(z) - f(\alpha)}{z - \alpha} = \lim_{y \downarrow 0} \frac{\Im f(\alpha + iy)}{y} = c.$$

(2) Assume that $\lim_{y \downarrow 0} f(\alpha + iy) := f(\alpha)$ exists and belongs to \mathbb{R} . If

$$\lim_{z \underset{\triangleleft}{\rightarrow} \alpha} \frac{f(z) - f(\alpha)}{z - \alpha} = c \in \mathbb{C},$$

then $c \in (0, +\infty)$ and

$$c = \liminf_{z \rightarrow \alpha} \frac{\Im f(z)}{\Im z} = \lim_{z \underset{\triangleleft}{\rightarrow} \alpha} f'(z).$$

(3) Assume that $\lim_{z \underset{\triangleleft}{\rightarrow} \alpha} f'(z) = c \in \mathbb{C}$ and $\lim_{z \underset{\triangleleft}{\rightarrow} \alpha} f(z) = f(\alpha) \in \mathbb{R}$. Then

$$c = \liminf_{z \rightarrow \alpha} \frac{\Im f(z)}{\Im z} = \lim_{z \underset{\triangleleft}{\rightarrow} \alpha} \frac{f(z) - f(\alpha)}{z - \alpha} \in \mathbb{R}.$$

The noncommutative version of this theorem becomes quite obvious in light of (2) and of the formulations of the corresponding main result from [25] as well as the recent work [20]. In the following, when we make a statement about a completely positive map, we usually write the statement for level one, and, unless the contrary is explicitly stated, we mean that the property in question holds for all levels n . Thus, for example, the statement

$$\lim_{z \underset{\triangleleft}{\rightarrow} 0} f'(\alpha + zv) := f'(\alpha)$$

exists and is completely positive for $\alpha = \alpha^*$ and $0 < v \in \mathcal{A}$ means that for any $n \in \mathbb{N}$ and any $v \in M_n(\mathcal{A})$,

$$\lim_{z \underset{\triangleleft}{\rightarrow} 0} f'(\alpha \otimes 1_n + zv) = f'(\alpha \otimes 1_n) = f'(\alpha) \otimes \text{Id}_n$$

is a positive map on $M_n(\mathcal{A})$.

Theorem 2.2. *Let \mathcal{A} be a von Neumann algebra and $f: H^+(\mathcal{A}) \rightarrow H^+(\mathcal{A})$ be a noncommutative map. Fix $\alpha = \alpha^* \in \mathcal{A}$.*

(1) *Assume that for any $v \in \mathcal{A}, v > 0$ and any state φ on \mathcal{A} ,*

$$(5) \quad \liminf_{z \rightarrow 0, z \in \mathbb{C}^+} \frac{\varphi(\Im f(\alpha + zv))}{\Im z} < \infty.$$

Then there exists $c = c(v) \in \mathcal{A}, c > 0$ such that

$$(6) \quad \text{wo-}\lim_{y \downarrow 0} \frac{\Im f(\alpha + iyv)}{y} = c.$$

Moreover, $\lim_{z \underset{\triangleleft}{\rightarrow} 0} f(\alpha + zv) = f(\alpha)$ exists, does not depend on v and is self-adjoint. The limits

$$(7) \quad \lim_{z \underset{\triangleleft}{\rightarrow} 0} \Delta f(\alpha + iyv_1, \alpha + iyv_2) \quad \text{and} \quad \lim_{z \underset{\triangleleft}{\rightarrow} 0} f'(\alpha + zv)$$

exist pointwise for any $v, v_1, v_2 > 0$, and $\lim_{z \underset{\triangleleft}{\rightarrow} 0} f'(\alpha + zv)(v) = c(v)$.

- (1') Assume in addition to the hypothesis (5) that for any $v, w > 0$ in \mathcal{A} and any state φ on \mathcal{A} , the gradient of the two-variable complex function $\mathbb{C}^+ \times \mathbb{C}^+ \ni (z, \zeta) \mapsto \varphi(f(\alpha + zv + \zeta w)) \in \mathbb{C}^+$ admits the limit

$$\lim_{y_1, y_2 \rightarrow 0} (\varphi(f'(\alpha + iy_1 v + iy_2 w)(v)), \varphi(f'(\alpha + iy_1 v + iy_2 w)(w))).$$

Then the limits (7) are equal to each other, completely positive and do not depend on v, v_1, v_2 .

- (2) Assume that the pointwise wo limit $\lim_{y \downarrow 0} f'(\alpha + iyv) := f'(\alpha)$ exists for any $v > 0$, does not depend on v and $f'(\alpha)$ is a completely bounded operator on \mathcal{A} . Then $f'(\alpha)$ is completely positive, $\text{wo-}\lim_{z \nearrow 0} f(\alpha + zv) := f(\alpha)$ exists, does not depend on v and is selfadjoint, and

$$f'(\alpha)(v) = \text{wo-}\lim_{z \nearrow 0} \frac{\Im f(\alpha + iyv)}{y} \quad \text{for any } v > 0.$$

Unfortunately, unlike in the classical case of Theorem 2.1, and similar to the case of functions of several complex variables [22, 1], item (1') above cannot be improved upon. Indeed, it was observed in [5] that for analytic functions of two complex variables on the bidisk with values in the unit disk, there exist examples that satisfy the commutative equivalent of (5) for the bidisk, and yet the gradient map does not have a nontangential limit. The equivalent of condition (5) implies the existence of all *directional* derivatives in permissible directions, but these directional derivatives do not necessarily “add up” to a linear map. This commutative example has a natural noncommutative extension, as shown in [20]. It is enough for our purposes to treat a simplified version of this extension. It is shown in [6] that any Loewner map from the n -dimensional upper half-plane $(\mathbb{C}^+)^n$ to \mathbb{C}^+ has a certain operatorial realization: for any such $h: (\mathbb{C}^+)^n \rightarrow \mathbb{C}^+$ there exist Hilbert spaces \mathcal{N}, \mathcal{M} , a selfadjoint densely defined operator A on \mathcal{M} , a real number s an orthogonal decomposition $P = \{P_1, \dots, P_n\}$ of $\mathcal{N} \oplus \mathcal{M}$ (i.e. $P_i P_j = P_j P_i = \delta_{ij} P_j = \delta_{ij} P_j^*$ and $P_1 + \dots + P_n = 1_{\mathcal{M} \oplus \mathcal{N}}$) and a vector state $\varphi_v(\cdot) = \langle \cdot v, v \rangle$ on the von Neumann algebra of bounded linear operators on $\mathcal{N} \oplus \mathcal{M}$ such that

$$h(z) = s + \varphi_v(M(z)), \quad z = (z_1, \dots, z_n) \in (\mathbb{C}^+)^n, 1 \leq j \leq n,$$

where

$$\begin{aligned} M(z) &= \begin{pmatrix} -i & 0 \\ 0 & 1 - iA \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} - (z_1 P_1 + \dots + z_n P_n) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)^{-1} \\ &\quad \times \left((z_1 P_1 + \dots + z_n P_n) \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} -i & 0 \\ 0 & 1 - iA \end{pmatrix}^{-1}. \end{aligned}$$

The 2×2 matrix decomposition is realized with respect to the canonical orthogonal decomposition of $\mathcal{N} \oplus \mathcal{M}$. We observe that such maps $M: (\mathbb{C}^+)^n \rightarrow B(\mathcal{N} \oplus \mathcal{M})$ have a natural noncommutative extension to $H^+(\mathbb{C}^n) := \coprod_{k \geq 1} \{a \in M_k(\mathbb{C}) : \Im a > 0\}^n$ given by replacing $(z_1 P_1 + \dots + z_n P_n)$ in the above formula of $M(z)$ by

$$\sum_{j=1}^n (P_j \otimes 1_k) a_j (P_j \otimes 1_k).$$

(While it is not obvious from its formula that $\Im M$ is positive when evaluated on $(\mathbb{C}^+)^n$, and even less when its amplification is evaluated on $\{a \in M_k(\mathbb{C}) : \Im a >$

$0\}^n$, a careful reading of the proofs of [6, Propositions 3.4 and 3.5] allows one to observe that they adapt without modification to show that $\Im M(a_1, \dots, a_n) > 0$ for $(a_1, \dots, a_n) \in \{a \in M_k(\mathbb{C}) : \Im a > 0\}^n$.) The extension of h becomes

$$h_k(a) = s \otimes 1_k + (\varphi_v \otimes \text{Id}_k)(M(a)),$$

for all $a = (a_1, \dots, a_n) \in \{a \in M_k(\mathbb{C}) : \Im a > 0\}^n$. For $n = 2$ any analytic function $h: \mathbb{C}^+ \times \mathbb{C}^+ \rightarrow \mathbb{C}^+$ admits such an operatorial realization, and hence it has a noncommutative extension as described above (see [5, 6, 7]). Considering the counterexample h provided in [5], the map $H: H^+(\mathbb{C}^2) \rightarrow H^+(\mathbb{C}^2)$ defined by $H(a) = (h(a), h(a))$ shows that we cannot dispense of item (1') in Theorem 2.2. However, observe that the noncommutative structure of the function f in Theorem 2.2 (1) allows for a slightly stronger conclusion than in classical case of [5]: the “directional derivative” becomes a bounded linear operator defined on all of \mathcal{A} .

As noted above, a classical analytic function is itself the first level of a noncommutative function, via the classical analytic functional calculus applied to matrices over \mathbb{C} . Relations (5), (6), (7) are the obvious consequences of relations (3) and (4) in this context. Thus the statements of Theorem 2.2 are anything but surprising. Indeed, if f has an analytic extension around α , then the proof of Theorem 2.2 is absolutely trivial.

3. A NORM ESTIMATE

Several slightly different proofs of Julia-Wolff-Carathéodory Theorem can be found in the literature. An essential element in one of them is the Schwarz-Pick Lemma: an analytic self-map of the upper half-plane is a contraction with respect to a “good” metric on \mathbb{C}^+ . In the next proposition, we obtain a similar result for noncommutative functions. We think that there is a rather striking resemblance between our result below and [19, Corollary 3.3], but it is not clear to us yet whether the two results can be obtained from each other, or even to what extent they are related. We intend to pursue this question later.

Proposition 3.1. *Let $f: H^+(\mathcal{A}) \rightarrow H^+(\mathcal{A})$ be a noncommutative map. For any $n \in \mathbb{N}$ and $a, c \in H^+(M_n(\mathcal{A}))$, the linear map*

$$M_n(\mathcal{A}) \ni b \mapsto (\Im f(a))^{-\frac{1}{2}} \Delta f(a, c) \left((\Im a)^{\frac{1}{2}} b (\Im c)^{\frac{1}{2}} \right) (\Im f(c))^{-\frac{1}{2}} \in M_n(\mathcal{A})$$

is a complete contraction. In particular,

$$\left\| (\Im f(a))^{-\frac{1}{2}} \Delta f(a, c)(b) (\Im f(c))^{-\frac{1}{2}} \right\| \leq \left\| (\Im a)^{-\frac{1}{2}} b (\Im c)^{-\frac{1}{2}} \right\|,$$

so that, by Equation (2), for $b = a - c$,

$$\left\| (\Im f(a))^{-\frac{1}{2}} (f(a) - f(c)) (\Im f(c))^{-\frac{1}{2}} \right\| \leq \left\| (\Im a)^{-\frac{1}{2}} (a - c) (\Im c)^{-\frac{1}{2}} \right\|.$$

The estimate will often be used under the equivalent form

$$\begin{aligned} & \left[(\Im f(a))^{-\frac{1}{2}} \Delta f(a, c)(b) (\Im f(c))^{-\frac{1}{2}} \right]^* \left[(\Im f(a))^{-\frac{1}{2}} \Delta f(a, c)(b) (\Im f(c))^{-\frac{1}{2}} \right] \\ (8) \quad & \leq \left\| (\Im a)^{-\frac{1}{2}} b (\Im c)^{-\frac{1}{2}} \right\|^2, \end{aligned}$$

which we give here for convenience.

Proof. For any C*-algebra \mathcal{A} , $\begin{pmatrix} u & v \\ v^* & w \end{pmatrix} \in M_2(\mathcal{A})$ is strictly positive iff $u > 0$, $w > 0$ and $v^*u^{-1}v < w$, or, equivalently, if $u > 0$, $w > 0$ and $u > vw^{-1}v^*$ (see [21]). Thus, $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in H^+(M_{2n}(\mathcal{A}))$ if and only if $a, c \in H^+(M_n(\mathcal{A}))$ and $b^*(\Im a)^{-1}b < 4\Im c$. This last relation is equivalent to $\left[(\Im a)^{-\frac{1}{2}}b(\Im c)^{-\frac{1}{2}} \right]^* \left[(\Im a)^{-\frac{1}{2}}b(\Im c)^{-\frac{1}{2}} \right] < 4$, or $\|(\Im a)^{-\frac{1}{2}}b(\Im c)^{-\frac{1}{2}}\| < 2$. Thus, as f maps the noncommutative upper half-plane into itself, and for any $b_0 \in M_n(\mathcal{A})$ there exists an $\varepsilon_{b_0} = \frac{2}{\|(\Im a)^{-\frac{1}{2}}b_0(\Im c)^{-\frac{1}{2}}\|} > 0$ such that

$$\begin{pmatrix} a & \varepsilon b_0 \\ 0 & c \end{pmatrix} \in H^+(M_{2n}(\mathcal{A})) \quad \text{for all } \varepsilon \in [0, \varepsilon_{b_0}),$$

and so

$$\begin{pmatrix} f(a) & \varepsilon \Delta f(a, c)(b_0) \\ 0 & f(c) \end{pmatrix} \in H^+(M_{2n}(\mathcal{A})) \quad \text{for all } \varepsilon \in [0, \varepsilon_{b_0}).$$

In particular, $\varepsilon \left\| (\Im f(a))^{-\frac{1}{2}} \Delta f(a, c)(b_0) (\Im f(c))^{-\frac{1}{2}} \right\| < 2$ for $\varepsilon < \frac{2}{\|(\Im a)^{-\frac{1}{2}}b_0(\Im c)^{-\frac{1}{2}}\|}$. Letting $\varepsilon \rightarrow \frac{2}{\|(\Im a)^{-\frac{1}{2}}b_0(\Im c)^{-\frac{1}{2}}\|}$ from below, we obtain

$$\left\| (\Im f(a))^{-\frac{1}{2}} \Delta f(a, c)(b_0) (\Im f(c))^{-\frac{1}{2}} \right\| \leq \|(\Im a)^{-\frac{1}{2}}b_0(\Im c)^{-\frac{1}{2}}\|.$$

As $b_0 \in M_n(\mathcal{A})$ has been chosen arbitrarily, we can replace it by $(\Im a)^{\frac{1}{2}}b(\Im c)^{\frac{1}{2}}$ to conclude that, as claimed

$$\left\| (\Im f(a))^{-\frac{1}{2}} \Delta f(a, c) \left((\Im a)^{\frac{1}{2}}b(\Im c)^{\frac{1}{2}} \right) (\Im f(c))^{-\frac{1}{2}} \right\| \leq \|b\|, \quad b \in M_n(\mathcal{A}).$$

□

Clearly, the same method can be used to obtain estimates involving $\Delta^j f$ for all $j \in \mathbb{N}$. However, for now their form seems to be too complicated when $j \geq 2$, and of no significant use for the purposes of this paper.

The following lemma will be useful when applying the above proposition to the proof of the main result (compare with the method used in [11, Remark 2.5]).

Lemma 3.2. *Assume that f is a noncommutative self-map of the noncommutative upper half-plane of \mathcal{A} . Let $v_1, v_2 > 0$ in \mathcal{A} . If*

$$\text{wo-}\lim_{y \downarrow 0} \frac{\Im f(\alpha + iyv_j)}{y} = c_j \in \mathcal{A}, \quad j \in \{1, 2\}$$

exist, then the set of limit points of $\Delta f(\alpha + zv_1, \alpha + \zeta v_2)(w)$ as $z, \zeta \rightarrow 0$ nontangentially is bounded uniformly in norm as w varies in the unit ball of \mathcal{A} .

Proof. As shown in Proposition 3.1,

$$\begin{aligned} & \left\| (\Im f(\alpha + zv_1))^{-\frac{1}{2}} \Delta f(\alpha + zv_1, \alpha + \zeta v_2)(w) (\Im f(\alpha + \zeta v_2))^{-\frac{1}{2}} \right\| \\ & \leq \left\| (\Im zv_1)^{-\frac{1}{2}} w (\Im \zeta v_2)^{-\frac{1}{2}} \right\|. \end{aligned}$$

Dividing we obtain

$$\left\| \left[\frac{\Im f(\alpha + zv_1)}{\Im z} \right]^{-\frac{1}{2}} \Delta f(\alpha + zv_1, \alpha + \zeta v_2)(w) \left[\frac{\Im f(\alpha + \zeta v_2)}{\Im \zeta} \right]^{-\frac{1}{2}} \right\| \leq \|v_1^{-\frac{1}{2}} w v_2^{-\frac{1}{2}}\|.$$

Let $\varepsilon \geq 0$ be fixed, and denote $f_\varepsilon(a) = f(a) + \varepsilon a$, i.e. $f_\varepsilon = f + \varepsilon \text{Id}$. Since Id is completely positive, f_ε is still a noncommutative self-map of the noncommutative upper half-plane of \mathcal{A} , so that

$$\begin{aligned} & \left\| \left(\frac{\Im f(\alpha + zv_1)}{\Im z} + \varepsilon v_1 \right)^{-\frac{1}{2}} (\Delta f(\alpha + zv_1, \alpha + \zeta v_2)(w) + \varepsilon w) \times \right. \\ & \quad \left. \left(\frac{\Im f(\alpha + \zeta v_2)}{\Im \zeta} + \varepsilon v_2 \right)^{-\frac{1}{2}} \right\| \\ & \leq \left\| v_1^{-\frac{1}{2}} w v_2^{-\frac{1}{2}} \right\|. \end{aligned}$$

For simplicity, we denote $A_1(\Im z, \varepsilon) = \frac{\Im f(\alpha + zv_1)}{\Im z} + \varepsilon v_1$, $A_2(\Im \zeta, \varepsilon) = \frac{\Im f(\alpha + \zeta v_2)}{\Im \zeta} + \varepsilon v_2$, $W(z, \zeta, \varepsilon) = \Delta f(\alpha + zv_1, \alpha + \zeta v_2)(w) + \varepsilon w$, and $K = \left\| v_1^{-\frac{1}{2}} w v_2^{-\frac{1}{2}} \right\|^2$. The above is equivalent to

$$A_2(\Im \zeta, \varepsilon)^{-\frac{1}{2}} W(z, \zeta, \varepsilon)^* A_1(\Im z, \varepsilon)^{-1} W(z, \zeta, \varepsilon) A_2(\Im \zeta, \varepsilon)^{-\frac{1}{2}} \leq K 1.$$

As $A_j(\cdot, \varepsilon) \geq \varepsilon 1$, this implies

$$W(z, \zeta, \varepsilon)^* A_1(\Im z, \varepsilon)^{-1} W(z, \zeta, \varepsilon) \leq K A_2(\Im \zeta, \varepsilon) \leq K \|A_2(\Im \zeta, \varepsilon)\| 1.$$

We re-write this as

$$\left\| \left(A_1(\Im z, \varepsilon)^{-\frac{1}{2}} W(z, \zeta, \varepsilon) \right)^* \left(A_1(\Im z, \varepsilon)^{-\frac{1}{2}} W(z, \zeta, \varepsilon) \right) \right\| \leq K \|A_2(\Im \zeta, \varepsilon)\|.$$

Since in any C*-algebra, $\|x^*x\| = \|xx^*\| = \|x\|^2 = \|x^*\|^2$, we have

$$\left\| A_1(\Im z, \varepsilon)^{-\frac{1}{2}} W(z, \zeta, \varepsilon)^* W(z, \zeta, \varepsilon) A_1(\Im z, \varepsilon)^{-\frac{1}{2}} \right\| \leq K \|A_2(\Im \zeta, \varepsilon)\|,$$

so that

$$A_1(\Im z, \varepsilon)^{-\frac{1}{2}} W(z, \zeta, \varepsilon)^* W(z, \zeta, \varepsilon) A_1(\Im z, \varepsilon)^{-\frac{1}{2}} \leq K \|A_2(\Im \zeta, \varepsilon)\| 1.$$

We repeat the same argument to obtain

$$\|W(z, \zeta, \varepsilon)\|^2 \leq K \|A_1(\Im z, \varepsilon)\| \|A_2(\Im \zeta, \varepsilon)\|.$$

Let \mathcal{H} be the Hilbert space on which \mathcal{A} acts as a von Neumann algebra. By our hypothesis, $\lim_{y \downarrow 0} \frac{\langle \Im f(\alpha + iyv)\xi, \xi \rangle}{y} = \lim_{y \downarrow 0} \left\| \left(\frac{\Im f(\alpha + iyv)}{y} \right)^{\frac{1}{2}} \xi \right\|_2^2$ exists and equals $\langle c\xi, \xi \rangle$,

hence it is finite for any $\xi \in \mathcal{H}$. Thus, the family $\left\{ \left\| \frac{(\Im f(\alpha + iyv))^{1/2}}{\sqrt{y}} \xi \right\|_2 : y \in (0, 1) \right\}$ is bounded for any $\xi \in \mathcal{H}$. By the Banach-Steinhaus Theorem and the positivity of the operators $\frac{\Im f(\alpha + iyv)}{y}$, it follows that $\left\{ \left\| \frac{\Im f(\alpha + iyv)}{y} \right\| : y \in (0, 1) \right\}$ is a bounded set. Moreover, as it will be seen in the proof of Theorem 2.1, if z tends to zero nontangentially and $\lim_{y \downarrow 0} \frac{\langle \Im f(\alpha + iyv)\xi, \xi \rangle}{y}$ is finite, then $\left\{ \frac{\langle \Im f(\alpha + \Im z v)\xi, \xi \rangle}{\Im z} : |z| < 1, z \in \Gamma \right\}$ stays bounded for any closed cone $\Gamma \subset \mathbb{C}^+ \cup \{0\}$. The bound for c_j is $\|c_j\| \leq \limsup_{y \rightarrow 0} \left\| \frac{\Im f(\alpha + iyv_j)}{y} \right\|$. Thus, the set $\{\|W(z, \zeta, 0)\| : z, \zeta \in \Gamma, |z|, |\zeta| < 1\}$ is bounded for any closed cone $\Gamma \subset \mathbb{C}^+$ with vertex at zero. The lemma follows. \square

We note that the bounds depend exclusively on $c_j, v_j, j = 1, 2, w$.

Let us use the results of Proposition 3.1 to give a short and elementary proof of Theorem 2.1.

Proof of Theorem 2.1. Assume equation (3) holds. By Proposition 3.1,

$$\left| \frac{f(z) - f(z')}{\sqrt{\Im f(z)\Im f(z')}} \right| \leq \left| \frac{z - z'}{\sqrt{\Im z\Im z'}} \right|, \quad z, z' \in \mathbb{C}^+.$$

This is equivalent to

$$(9) \quad \left| \frac{f(z) - f(z')}{z - z'} \right|^2 \leq \left| \frac{\Im f(z)\Im f(z')}{\Im z\Im z'} \right|, \quad z, z' \in \mathbb{C}^+, z \neq z'.$$

Consider a sequence $\{z'_n\}_{n \in \mathbb{N}} \subset \mathbb{C}^+$ converging to α such that $\lim_{n \rightarrow \infty} \frac{\Im f(z'_n)}{\Im z'_n} = c$. Clearly $\Im f(z'_n) \rightarrow 0$ as $n \rightarrow \infty$, and $\{\Re f(z'_n)\}_{n \in \mathbb{N}}$ is a bounded sequence in \mathbb{R} . Moreover, if $\{z_n\}_{n \in \mathbb{N}}$ and $\{z'_n\}_{n \in \mathbb{N}}$ are two arbitrary sequences converging to α along which $\Im f(z)/\Im z$ stays bounded, then $\{\Re(f(z_n) - f(z'_n))\}_{n \in \mathbb{N}}$ converges to zero. This implies that $\lim_{n \rightarrow \infty} f(z_n)$ exists for any sequence $\{z_n\}_{n \in \mathbb{N}}$ such that $\{\Im f(z_n)/\Im z_n\}_{n \in \mathbb{N}}$ is bounded and $\lim_{n \rightarrow \infty} z_n = \alpha$. We agree to call this limit $f(\alpha)$. Taking limit along z'_n in (9) we obtain

$$\left| \frac{f(z) - f(\alpha)}{z - \alpha} \right|^2 \leq c \frac{\Im f(z)}{\Im z}, \quad z \in \mathbb{C}^+.$$

Fix an $M \in [0, +\infty)$. Let $D_M = \{z \in \mathbb{C}^+ : |\Re z - \alpha| \leq M\Im z\}$. For any $z \in D_M$, this implies

$$\begin{aligned} (\Re f(z) - f(\alpha))^2 &= c \frac{\Im f(z)}{\Im z} |z - \alpha|^2 - (\Im f(z))^2 \\ &= \Im f(z) \left(\frac{c|z - \alpha|^2}{\Im z} - \Im f(z) \right) \\ &= \Im f(z) \left(c\Im z \frac{|\Re z - \alpha|^2}{(\Im z)^2} + \Im z - \Im f(z) \right) \\ &\leq \Im f(z) (c(M^2 + 1)\Im z - \Im f(z)). \end{aligned}$$

We conclude that $\Im f(z)/\Im z < c(M^2 + 1)$ for all $z \in D_M$ and thus $\lim_{z \xrightarrow{\triangleleft} \alpha} f(z) = f(\alpha)$.

Moreover, for $M = 0$ (i.e. z of the form $\alpha + iy$) we have $c \geq \Im f(\alpha + iy)/y$, which together with the definition of c implies $\lim_{y \downarrow 0} \frac{\Im f(\alpha + iy)}{y} = c$, so that

$$\frac{(\Re f(\alpha + iy) - f(\alpha))^2}{y^2} \leq c \left(c - \frac{\Im f(\alpha + iy)}{y} \right) \rightarrow 0 \quad \text{as } y \downarrow 0.$$

These two facts imply, via direct computation, that $\lim_{y \downarrow 0} \frac{f(\alpha + iy) - f(\alpha)}{iy} = c$. Since

$$\left| \frac{f(z) - f(\alpha)}{z - \alpha} \right|^2 \leq c \frac{\Im f(z)}{\Im z} \leq c^2(M^2 + 1), \quad z \in D_M, M \geq 0,$$

it follows straightforwardly that

$$\lim_{z \xrightarrow{\triangleleft} \alpha} \frac{f(z) - f(\alpha)}{z - \alpha} = c$$

(see for example [15, Exercise 5, Chapter I]).

Considering the classical definition of the derivative, the above directly implies that $\lim_{y \downarrow 0} f'(\alpha + iy) = c$. Relation (9) implies that $|f'(z)| \leq c(M^2 + 1)$ for $z \in D_M$, so, by the same [15, Exercise 5, Chapter I], $\lim_{z \xrightarrow{\triangleleft} \alpha} f'(z) = c$. This proves (1).

To prove (2), simply observe that

$$\left| \frac{f(\alpha + iy) - f(\alpha)}{iy} \right|^2 = \left| \frac{(\Re f(\alpha + iy) - f(\alpha))^2 + (\Im f(\alpha + iy))^2}{y^2} \right| \geq \frac{(\Im f(\alpha + iy))^2}{y^2},$$

so that $\liminf_{z \rightarrow \alpha} \frac{\Im f(z)}{\Im z} < \infty$. Part (2) follows now from part (1).

To prove part (3), we apply the classical mean value theorem to bound $\Im f(\alpha + iy)/y$. The result follows then from part (1). \square

We feel it necessary to reiterate that no claim to novelty is made for this proof, and we chose to write it down here for the sake of making the paper more self-contained.

4. PROOF OF THE MAIN RESULT

In this section we prove Theorem 2.2. The proof makes use quite often of the results, and sometimes of the proof, of Theorem 2.1. For the sake of simplicity, we will isolate some elements of the proof in separate lemmas.

Proof of Theorem 2.2. For any $n \in \mathbb{N}$ and any state φ on $M_n(\mathcal{A})$, $z \mapsto \varphi(f(\alpha + zv))$ is a self-map of \mathbb{C}^+ whenever α is selfadjoint and $v > 0$ in $M_n(\mathcal{A})$. Thus, Theorem 2.1 applies to it. In particular, if \mathcal{H} is the Hilbert space on which the von Neumann algebra \mathcal{A} acts, the above holds for the vector state corresponding to any $\xi \in \oplus_{j=1}^n \mathcal{H}$ of L^2 -norm equal to one. For $n = 1$, our hypothesis guarantees that $\liminf_{z \rightarrow 0} \frac{\langle \Im f(\alpha + zv)\xi, \xi \rangle}{\Im z}$ is finite. Item (1) of Theorem 2.1 guarantees that $\lim_{y \downarrow 0} \frac{\langle \Im f(\alpha + iyv)\xi, \xi \rangle}{y} = \lim_{y \downarrow 0} \left\| \left(\frac{\Im f(\alpha + iyv)}{y} \right)^{\frac{1}{2}} \xi \right\|_2^2$ exists and equals the above \liminf , hence it is finite for any $\xi \in \mathcal{H}$. As in the proof of Lemma 3.2 the Banach-Steinhaus Theorem and the positivity of the operators $\frac{\Im f(\alpha + iyv)}{y}$ guarantee that $\left\{ \left\| \frac{\Im f(\alpha + iyv)}{y} \right\| : y \in (0, 1) \right\}$ is a bounded set. Moreover, the existence of the limits $\lim_{y \downarrow 0} \frac{\langle \Im f(\alpha + iyv)\xi, \xi \rangle}{y}$ for all $\xi \in \mathcal{H}$ implies, via polarization, the existence of

$$\lim_{y \downarrow 0} \frac{\langle \Im f(\alpha + iyv)\xi, \eta \rangle}{y}, \quad \xi, \eta \in \mathcal{H}.$$

We conclude the existence of a bounded operator $0 \leq c = c(v) \in \mathcal{A}$ such that

$$\lim_{y \downarrow 0} \frac{\langle \Im f(\alpha + iyv)\xi, \eta \rangle}{y} = \langle c\xi, \eta \rangle, \quad \xi, \eta \in \mathcal{H}.$$

The bound for c is $\|c\| \leq \limsup_{y \rightarrow 0} \left\| \frac{\Im f(\alpha + iyv)}{y} \right\|$. On the other hand, as seen in the proof of Theorem 2.1, $\Im \langle f(\alpha + iyv)\xi, \xi \rangle \leq y \langle c\xi, \xi \rangle$ for all $y > 0$. Since f takes values in $H^+(\mathcal{A})$, applying this relation to $y = 1$ guarantees that $c > 0$.

We show next that the limit $\text{wo-}\lim_{y \downarrow 0} f(\alpha + iyv) = f(\alpha)$ exists in \mathcal{A} (i.e. does not depend on v) and is selfadjoint. Indeed, consider again any state φ on \mathcal{A} and define $z \mapsto \varphi(f(\alpha + zv))$. We have seen that this is a self-map of \mathbb{C}^+ to which Theorem 2.1 applies. Thus, there exists a number $k = k(\varphi, \alpha, v) \in \mathbb{R}$ such that $\lim_{z \rightarrow 0} \varphi(f(\alpha + zv)) = k$. We recall the estimate from Proposition 3.1

$$\left| \frac{\varphi(f(\alpha + zv)) - \varphi(f(\alpha + z'v))}{z - z'} \right|^2 \leq \frac{\varphi(\Im f(\alpha + zv))\varphi(\Im f(\alpha + z'v))}{\Im z \Im z'}.$$

In this estimate we take $z' = i$ and let $z = iy$ tend to zero. We obtain

$$|k(\varphi, \alpha, v) - \varphi(f(\alpha + iv))|^2 \leq \varphi(c)\varphi(\Im f(\alpha + iv)).$$

Obviously, $|\varphi(f(\alpha + iv))| \leq \|f(\alpha + iv)\|$, a constant value. Thus,

$$|k(\varphi, \alpha, v)| \leq \|f(\alpha + iv)\| + \sqrt{\|c\| \|\Im f(\alpha + iv)\|},$$

for any state φ on \mathcal{A} . By applying as before this result to vector states and using polarization, we find an operator $f_v(\alpha) \in \mathcal{A}$ such that

$$\langle f_v(\alpha)\xi, \eta \rangle = \lim_{y \downarrow 0} \langle f(\alpha + iy)\xi, \eta \rangle, \quad \xi, \eta \in \mathcal{H}.$$

Since $\|x\| = \sup\{|\varphi(x)| : \varphi \text{ state on } \mathcal{A}\}$, the estimate

$$\|f_v(\alpha)\| \leq 4 \left(\|f(\alpha + iv)\| + \sqrt{\|c\| \|\Im f(\alpha + iv)\|} \right)$$

holds. Since for any state φ , $k(\varphi, \alpha, v) = \lim_{y \downarrow 0} \varphi(f(\alpha + iyv)) \in \mathbb{R}$, it follows that $f_v(\alpha) = f_v(\alpha)^*$. The fact that $f_v(\alpha)$ does not depend on v follows from Proposition 3.1 and Lemma 3.2: indeed,

$$\begin{aligned} & \left\| (\Im f(\alpha + iy_1v))^{-\frac{1}{2}} (f(\alpha + iy_1v) - f(\alpha + iy_2v)) (\Im f(\alpha + iy_2v))^{-\frac{1}{2}} \right\| \\ & \leq \left\| (y_1v)^{-\frac{1}{2}} (iy_1v - iy_2v) (y_2v)^{-\frac{1}{2}} \right\| \end{aligned}$$

is equivalent to

$$\begin{aligned} & \left\| \left(\frac{\Im f(\alpha + iy_1v)}{y_1} \right)^{-\frac{1}{2}} (f(\alpha + iy_1v) - f(\alpha + iy_2v)) \left(\frac{\Im f(\alpha + iy_2v)}{y_2} \right)^{-\frac{1}{2}} \right\| \\ & \leq \left\| v^{-\frac{1}{2}} \right\| \|y_1v - y_2v\|. \end{aligned}$$

We obtain as in the proof of Lemma 3.2

$$(10) \quad \begin{aligned} & \|f(\alpha + iy_1v) - f(\alpha + iy_2v)\| \\ & \leq \left\| v^{-\frac{1}{2}} \right\| \|y_1v - y_2v\| \sqrt{\left\| \frac{\Im f(\alpha + iy_1v)}{y_1} \right\| \left\| \frac{\Im f(\alpha + iy_2v)}{y_2} \right\|}. \end{aligned}$$

As seen above, the two factors under the square root are bounded. Thus, we conclude.

Remark 4.1. *This result is similar to results in [5, 14, 25]. We observe that this essentially improves the convergence to norm convergence, without requiring norm convergence in formula (5).*

We discuss next the question of the derivatives.

Lemma 4.2. *Let f be as in Theorem 2.2. Fix $\alpha = \alpha^* \in \mathcal{A}$, $v_1, v_2 > 0$ in \mathcal{A} , and $b \in \mathcal{A}$ of norm $\|b\|^2 \mathbf{1} < v_1^{1/2} v_2 v_1^{1/2}$. Then*

$$\lim_{y \downarrow 0} \frac{1}{y} \left\| f \begin{pmatrix} \alpha + iyv_1 & iyb \\ 0 & \alpha + iyv_2 \end{pmatrix} - f \begin{pmatrix} \alpha + iyv_1 & \frac{iyb}{2} \\ \frac{iyb^*}{2} & \alpha + iyv_2 \end{pmatrix} \right\| = 0.$$

Proof. Observe first that $\|b\|^2 1 < 4v_1^{1/2}v_2v_1^{1/2}$ implies $\begin{pmatrix} \alpha + iyv_1 & iyb \\ 0 & \alpha + iyv_2 \end{pmatrix} > 0$ for all $y > 0$. We use the same trick as in Lemma 3.2. For simplicity, denote

$$\mathfrak{D} = f \begin{pmatrix} \alpha + iyv_1 & iyb \\ 0 & \alpha + iyv_2 \end{pmatrix} - f \begin{pmatrix} \alpha + iyv_1 & \frac{iyb}{2} \\ \frac{iyb^*}{2} & \alpha + iyv_2 \end{pmatrix}.$$

Proposition 3.1 applied to a and c equal to the two arguments of the function f in the formula of \mathfrak{D} above give

$$\begin{aligned} & \Im f \left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + iy \begin{pmatrix} v_1 & b \\ 0 & v_2 \end{pmatrix} \right)^{-\frac{1}{2}} \mathfrak{D}^* \Im f \left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + iy \begin{pmatrix} v_1 & \frac{b}{2} \\ \frac{b^*}{2} & v_2 \end{pmatrix} \right)^{-1} \\ & \quad \times \mathfrak{D} \Im f \left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + iy \begin{pmatrix} v_1 & b \\ 0 & v_2 \end{pmatrix} \right)^{-\frac{1}{2}} \\ & \leq \left\| \begin{pmatrix} yv_1 & \frac{yb}{2} \\ \frac{yb^*}{2} & yv_2 \end{pmatrix}^{-\frac{1}{2}} \begin{pmatrix} 0 & -iyb \\ \frac{iyb^*}{2} & 0 \end{pmatrix} \begin{pmatrix} yv_1 & \frac{yb}{2} \\ \frac{yb^*}{2} & yv_2 \end{pmatrix}^{-\frac{1}{2}} \right\|^2, \end{aligned}$$

for all $y > 0$ (we have kept the y 's on the right hand side for transparency of the method). As in the proof of Lemma 3.2, we “multiply out” the imaginary parts of f on the left to obtain

$$\begin{aligned} \mathfrak{D} \mathfrak{D}^* & \leq \|\mathfrak{D}\|^2 1 \\ & \leq \left\| y \Im f \left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + iy \begin{pmatrix} v_1 & \frac{b}{2} \\ \frac{b^*}{2} & v_2 \end{pmatrix} \right) \right\| \left\| \begin{pmatrix} v_1 & \frac{b}{2} \\ \frac{b^*}{2} & v_2 \end{pmatrix}^{-\frac{1}{2}} \begin{pmatrix} 0 & -ib \\ \frac{ib^*}{2} & 0 \end{pmatrix} \right. \\ & \quad \left. \times \begin{pmatrix} v_1 & \frac{b}{2} \\ \frac{b^*}{2} & v_2 \end{pmatrix}^{-\frac{1}{2}} \right\|^2 \left\| \frac{1}{y} \Im f \left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + iy \begin{pmatrix} v_1 & b \\ 0 & v_2 \end{pmatrix} \right) \right\|. \end{aligned}$$

The last factor on the right hand side is bounded by the hypothesis, formula (1), Lemma 3.2 and the above arguments. The first factor needs not apriori be zero, but it is clearly bounded. However, if this factor is nonzero, consider \mathcal{H} to be the Hilbert space on which \mathcal{A} acts as a von Neumann algebra. Then there exists a vector $\xi \in \mathcal{H}^2$ of norm one such that $\lim_{y \downarrow 0} y \varphi_\xi \left(\Im f \left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + iy \begin{pmatrix} v_1 & \frac{b}{2} \\ \frac{b^*}{2} & v_2 \end{pmatrix} \right) \right)$ exists and belongs to $(0, +\infty)$, so that necessarily

$$\left\| \Im f \left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + iy \begin{pmatrix} v_1 & \frac{b}{2} \\ \frac{b^*}{2} & v_2 \end{pmatrix} \right) \right\| \rightarrow +\infty, \quad y \rightarrow 0.$$

(Recall that we have denoted by φ_ξ the vector state corresponding to ξ : $\varphi_\xi(a) = \langle a\xi, \xi \rangle$.) But then $2\|\Im \mathfrak{D}\| = \|\mathfrak{D} - \mathfrak{D}^*\| \leq 2\|\mathfrak{D}\|$ is unbounded as y tends to zero, so that

$$\begin{aligned} & \|\Im \mathfrak{D}\|^2 \leq \|\mathfrak{D}\|^2 1 \\ & \leq \left\| y \Im f \left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + iy \begin{pmatrix} v_1 & \frac{b}{2} \\ \frac{b^*}{2} & v_2 \end{pmatrix} \right) \right\| \left\| \begin{pmatrix} v_1 & \frac{b}{2} \\ \frac{b^*}{2} & v_2 \end{pmatrix}^{-\frac{1}{2}} \begin{pmatrix} 0 & -ib \\ \frac{ib^*}{2} & 0 \end{pmatrix} \right. \\ (11) \quad & \left. \times \begin{pmatrix} v_1 & \frac{b}{2} \\ \frac{b^*}{2} & v_2 \end{pmatrix}^{-\frac{1}{2}} \right\|^2 \left\| \frac{1}{y} \Im f \left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + iy \begin{pmatrix} v_1 & b \\ 0 & v_2 \end{pmatrix} \right) \right\|, \end{aligned}$$

making the right hand side unbounded, a contradiction. Thus,

$$\lim_{y \rightarrow 0} \left\| y \mathfrak{S}f \left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + iy \begin{pmatrix} v_1 & \frac{b}{2} \\ \frac{b^*}{2} & v_2 \end{pmatrix} \right) \right\| = 0,$$

and thus, by above,

$$\lim_{y \rightarrow 0} \left\| \mathfrak{S}f \left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + iy \begin{pmatrix} v_1 & \frac{b}{2} \\ \frac{b^*}{2} & v_2 \end{pmatrix} \right) \right\| = 0.$$

However, more can be concluded from (11): dividing by y^2 , one obtains

$$\begin{aligned} \frac{\|\mathfrak{S}\mathfrak{D}\|^2}{y^2} &= \left\| \frac{1}{y} \mathfrak{S}f \left(\begin{pmatrix} \alpha + iyv_1 & iyb \\ 0 & \alpha + iyv_2 \end{pmatrix} \right) - \frac{1}{y} \mathfrak{S}f \left(\begin{pmatrix} \alpha + iyv_1 & \frac{iyb}{2} \\ \frac{iyb^*}{2} & \alpha + iyv_2 \end{pmatrix} \right) \right\|^2 \\ &\leq \left\| \frac{1}{y} \mathfrak{S}f \left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + iy \begin{pmatrix} v_1 & \frac{b}{2} \\ \frac{b^*}{2} & v_2 \end{pmatrix} \right) \right\| \left\| \begin{pmatrix} v_1 & \frac{b}{2} \\ \frac{b^*}{2} & v_2 \end{pmatrix}^{-\frac{1}{2}} \begin{pmatrix} 0 & -ib \\ \frac{ib^*}{2} & 0 \end{pmatrix} \right\| \\ &\quad \times \left\| \begin{pmatrix} v_1 & \frac{b}{2} \\ \frac{b^*}{2} & v_2 \end{pmatrix}^{-\frac{1}{2}} \right\|^2 \left\| \frac{1}{y} \mathfrak{S}f \left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + iy \begin{pmatrix} v_1 & b \\ 0 & v_2 \end{pmatrix} \right) \right\|. \end{aligned}$$

We know from our hypothesis and Lemma 3.2 that the set of real positive numbers $\left\{ \left\| \frac{1}{y} \mathfrak{S}f \left(\begin{pmatrix} \alpha + iyv_1 & iyb \\ 0 & \alpha + iyv_2 \end{pmatrix} \right) \right\| : y \in (0, 1) \right\}$ is bounded. If we assume that the set $\left\{ \left\| \frac{1}{y} \mathfrak{S}f \left(\begin{pmatrix} \alpha + iyv_1 & \frac{iyb}{2} \\ \frac{iyb^*}{2} & \alpha + iyv_2 \end{pmatrix} \right) \right\| : y \in (0, 1) \right\}$ is unbounded and choose a sequence $\{y_n\}_{n \in \mathbb{N}}$ converging to zero so that the strictly positive real number

$$\ell := \lim_{n \rightarrow \infty} \left\| \frac{1}{y_n} \mathfrak{S}f \left(\begin{pmatrix} \alpha + iy_n v_1 & iy_n b \\ 0 & \alpha + iy_n v_2 \end{pmatrix} \right) \right\| \text{ exists, and}$$

$$\left\| \frac{1}{y_n} \mathfrak{S}f \left(\begin{pmatrix} \alpha + iy_n v_1 & \frac{iy_n b}{2} \\ \frac{iy_n b^*}{2} & \alpha + iy_n v_2 \end{pmatrix} \right) \right\| = +\infty,$$

then

$$\begin{aligned} &\left\| \frac{1}{y_n} \mathfrak{S}f \left(\begin{pmatrix} \alpha + iy_n v_1 & \frac{iy_n b}{2} \\ \frac{iy_n b^*}{2} & \alpha + iy_n v_2 \end{pmatrix} \right) \right\| - \left\| \frac{1}{y_n} \mathfrak{S}f \left(\begin{pmatrix} \alpha + iy_n v_1 & iy_n b \\ 0 & \alpha + iy_n v_2 \end{pmatrix} \right) \right\| \\ &\leq \left\| \frac{1}{y_n} \mathfrak{S}f \left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + iy_n \begin{pmatrix} v_1 & \frac{b}{2} \\ \frac{b^*}{2} & v_2 \end{pmatrix} \right) \right\|^{\frac{1}{2}} \left\| \begin{pmatrix} v_1 & \frac{b}{2} \\ \frac{b^*}{2} & v_2 \end{pmatrix}^{-\frac{1}{2}} \begin{pmatrix} 0 & -ib \\ \frac{ib^*}{2} & 0 \end{pmatrix} \right\| \\ &\quad \times \left\| \begin{pmatrix} v_1 & \frac{b}{2} \\ \frac{b^*}{2} & v_2 \end{pmatrix}^{-\frac{1}{2}} \right\|^{\frac{1}{2}} \left\| \frac{1}{y_n} \mathfrak{S}f \left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + iy_n \begin{pmatrix} v_1 & b \\ 0 & v_2 \end{pmatrix} \right) \right\|^{\frac{1}{2}} \end{aligned}$$

becomes

$$\begin{aligned} & \left\| \frac{\mathfrak{S}f \left(\begin{pmatrix} \alpha + iy_n v_1 & \frac{iy_n b}{2} \\ \frac{iy_n b^*}{2} & \alpha + iy_n v_2 \end{pmatrix} \right)}{y_n} \right\|^{\frac{1}{2}} - \frac{\left\| \frac{1}{y_n} \mathfrak{S}f \left(\begin{pmatrix} \alpha + iy_n v_1 & iy_n b \\ 0 & \alpha + iy_n v_2 \end{pmatrix} \right) \right\|^{\frac{1}{2}}}{\left\| \frac{1}{y_n} \mathfrak{S}f \left(\begin{pmatrix} \alpha + iy_n v_1 & \frac{iy_n b}{2} \\ \frac{iy_n b^*}{2} & \alpha + iy_n v_2 \end{pmatrix} \right) \right\|^{\frac{1}{2}}} \\ & \leq \left\| \begin{pmatrix} v_1 & \frac{b}{2} \\ \frac{b^*}{2} & v_2 \end{pmatrix}^{-\frac{1}{2}} \begin{pmatrix} 0 & \frac{-ib}{2} \\ \frac{ib^*}{2} & 0 \end{pmatrix} \begin{pmatrix} v_1 & \frac{b}{2} \\ \frac{b^*}{2} & v_2 \end{pmatrix}^{-\frac{1}{2}} \right\| \\ & \quad \times \left\| \frac{1}{y_n} \mathfrak{S}f \left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + iy_n \begin{pmatrix} v_1 & b \\ 0 & v_2 \end{pmatrix} \right) \right\|^{\frac{1}{2}}; \end{aligned}$$

by letting $n \rightarrow \infty$, we obtain

$$\infty - 0 \leq \ell \left\| \begin{pmatrix} v_1 & \frac{b}{2} \\ \frac{b^*}{2} & v_2 \end{pmatrix}^{-\frac{1}{2}} \begin{pmatrix} 0 & \frac{-ib}{2} \\ \frac{ib^*}{2} & 0 \end{pmatrix} \begin{pmatrix} v_1 & \frac{b}{2} \\ \frac{b^*}{2} & v_2 \end{pmatrix}^{-\frac{1}{2}} \right\|,$$

an obvious contradiction. \square

The previous lemma implies more: since $\left\| \frac{1}{y} \mathfrak{S}f \left(\begin{pmatrix} \alpha + iy v_1 & iy b \\ 0 & \alpha + iy v_2 \end{pmatrix} \right) \right\|$ is bounded as $y \in (0, 1)$, it follows immediately from the lemma that

$$\liminf_{y \downarrow 0} \frac{1}{y} \varphi \left(\mathfrak{S}f \left(\begin{pmatrix} \alpha + iy_n v_1 & \frac{iy_n b}{2} \\ \frac{iy_n b^*}{2} & \alpha + iy_n v_2 \end{pmatrix} \right) \right) < \infty,$$

for all states φ on $M_2(\mathcal{A})$, and so, as proved above,

$$\lim_{y \downarrow 0} \frac{1}{y} \mathfrak{S}f \left(\begin{pmatrix} \alpha + iy_n v_1 & \frac{iy_n b}{2} \\ \frac{iy_n b^*}{2} & \alpha + iy_n v_2 \end{pmatrix} \right) := C > 0 \text{ in } M_2(\mathcal{A}).$$

In particular,

$$\begin{aligned} C &= \lim_{y \downarrow 0} \frac{1}{y} \mathfrak{S}f \left(\begin{pmatrix} \alpha + iy v_1 & iy b \\ 0 & \alpha + iy v_2 \end{pmatrix} \right) \\ &= \lim_{y \downarrow 0} \begin{pmatrix} \frac{\mathfrak{S}f(\alpha + iy v_1)}{y} & \frac{1}{2} \Delta f(\alpha + iy v_1, \alpha + iy v_2)(b) \\ \frac{1}{2} \Delta f(\alpha + iy v_1, \alpha + iy v_2)(b)^* & \frac{\mathfrak{S}f(\alpha + iy v_2)}{y} \end{pmatrix}. \end{aligned}$$

We conclude that $\lim_{y \downarrow 0} \Delta f(\alpha + iy v_1, \alpha + iy v_2)(b)$ exists and is uniformly bounded as $b \in \mathcal{A}$ stays in a bounded subset of \mathcal{A} . Clearly the limit depends linearly on b , since each of $\Delta f(\alpha + iy v_1, \alpha + iy v_2)(b)$ does. In particular, if $v_1 = v_2 = v$, $\Delta f(\alpha + iy v, \alpha + iy v)(b) = f'(\alpha + iy v)(b)$ has a limit as $y \rightarrow 0$, as claimed in part (1) of Theorem 2.2. Let now in addition $b = v/4$. For any state φ on \mathcal{A} and $v > 0$, $z \mapsto \varphi(f(\alpha + zv))$ is a self-map of \mathbb{C}^+ which satisfies the conditions of Theorem 2.1 at $z = 0$. Thus, $\lim_{y \downarrow 0} \varphi(f'(\alpha + iy v)(v)) = \lim_{y \downarrow 0} \frac{\varphi(\mathfrak{S}f(\alpha + iy v))}{y}$, so that indeed

$$\lim_{y \downarrow 0} f'(\alpha + iy v)(v) = \lim_{y \downarrow 0} \frac{\mathfrak{S}f(\alpha + iy v)}{y} = c(v) > 0.$$

It is essential to observe that Lemma 4.2 (and hence its conclusion above) applies equally well to elements $v_j, b \in M_n(\mathcal{A})$ for all $n \in \mathbb{N}$ (the proof is independent of dimension). This concludes the proof of part (1).

We prove next part (1') of Theorem 2.2. We show first that Let $v, w > 0$ be fixed. Recall that we have shown in the proof of part (1) that $\lim_{t \downarrow 0} f'(\alpha + ity_1v + ity_2w)$ exists pointwise. Our hypothesis that

$$\lim_{\substack{y_1, y_2 \rightarrow 0 \\ y_1, y_2 \in F_M}} (\varphi(f'(\alpha + iy_1v + iy_2w)(v)), \varphi(f'(\alpha + iy_1v + iy_2w)(w)))$$

exists and is finite for any state φ on \mathcal{A} implies that $f'(\alpha + iy_1v + iy_2w)(v), f'(\alpha + iy_1v + iy_2w)(w)$ have a weak limit as $(y_1, y_2) \downarrow (0, 0)$ in F_M . Note that the function $(z, \zeta) \mapsto \varphi(f(\alpha + zv + \zeta w))$ extends, with values in \mathbb{C}^+ , to the strictly larger cone $\{(z, \zeta) \in \mathbb{C}^2: \Im(zv + \zeta w) > 0\}$. In particular, $\{(z, 0): z \in \mathbb{C}^+\} \cup \{(0, \zeta): \zeta \in \mathbb{C}^+\} \subset \{(z, \zeta) \in \mathbb{C}^2: \Im(zv + \zeta w) > 0\}$. The following lemma is well-known. We provide here a proof (following [15, Exercise 5, Chapter I]) for the convenience of the reader.

Lemma 4.3. *Assume that $\Gamma \subseteq \mathbb{C}^2$ is open, connected, contains $(i\mathbb{R}_+) \times (i\mathbb{R}_+)$ and for any $r, s > 0, (z_1, z_2) \in \Gamma$, we have $(sz_1, rz_2) \in \Gamma$. Let $g: \Gamma \rightarrow \mathbb{C}$ be analytic. Consider a second cone $(i\mathbb{R}_+) \times (i\mathbb{R}_+) \subset \Gamma' \subseteq \Gamma$ satisfying the same conditions as Γ and such that $g(\Gamma' \cap \mathbb{D}^2) \subset \mathbb{C}$ is a bounded set. Assume that there is a $c \in \mathbb{C}$ such that*

$$\lim_{y_1, y_2 \rightarrow 0} g(iy_1, iy_2) = c.$$

Then for any convex cone Γ'' such that $\overline{\Gamma''} \subset \Gamma'$, we have

$$\lim_{\substack{z_1, z_2 \rightarrow 0 \\ z_1, z_2 \in \Gamma''}} g(z_1, z_2) = c.$$

Proof. The function $g|_{\Gamma' \cap \mathbb{D}^2}$ is bounded. For any $r, s > 0$, define $g_{(r,s)}: \Gamma \rightarrow \mathbb{C}$ by $g_{(r,s)}(z, \zeta) = g(rz, s\zeta)$. Clearly $g_{(r,s)}$ is well-defined and analytic, by Γ 's definition. Moreover, $\{g_{(r,s)}|_{\Gamma' \cap \mathbb{D}^2}\}_{0 < r, s < 1}$ is clearly bounded by hypothesis, so it forms a normal family. For any sequence $(r_n, s_n) \rightarrow (0, 0)$ along which $g_{(r_n, s_n)}$ converges in the topology of uniform convergence on compacts, the limit, call it g_0 , is bounded on $\Gamma' \cap \mathbb{D}^2$ and satisfies $g_0(i\mathbb{R}_+ \times i\mathbb{R}_+ \cap \mathbb{D}^2) = \{c\}$. Since $i\mathbb{R}_+ \times i\mathbb{R}_+ \cap \mathbb{D}^2$ is a set of uniqueness for holomorphic functions¹ on $\Gamma \cap \mathbb{D}^2$, it follows that $g_0 \equiv c$. Thus, any limit point of $\{g_{(r,s)}|_{\Gamma' \cap \mathbb{D}^2}\}_{0 < r, s < 1}$ at $(0, 0)$ is the constant function c . Now, assume that there is a cone Γ'' as in the hypothesis along which the limit of g at zero is not c . That means that there is a sequence $(z_n^{(1)}, z_n^{(2)}) \in \Gamma''$ converging to zero and $\varepsilon_0 > 0$ so that $|g(z_n^{(1)}, z_n^{(2)}) - c| > \varepsilon_0$. Take $r_n, s_n > 0$ such that $|z_n^{(1)}/r_n| = |z_n^{(2)}/s_n| = 1/2$ (note that $r_n, s_n \rightarrow 0$). Then $\{(z_n^{(1)}/r_n, z_n^{(2)}/s_n): n \in \mathbb{N}\}$ has a compact closure in $\Gamma'' \subset \Gamma'$ and yet $g_{(r_n, s_n)}$ does not converge uniformly to c on this set, a contradiction. \square

The set $\{(z, \zeta) \in \mathbb{C}^2: \Im(zv + \zeta w) > 0\}$ satisfies all the hypotheses imposed by the above lemma on Γ , and $\varphi(f'(\alpha + zv + \zeta w)(v))$ is bounded along any path $t(zv + \zeta w)$

¹For a domain $D \subset \mathbb{C}^2$, we call $B \subset D$ a set of uniqueness if for any analytic functions h, k on D such that $h|_B \equiv k|_B$, it follows that $h \equiv k$ on D . It is easy to see that any open subset of $\mathbb{R} \times \mathbb{R}$ in an open simply connected domain $D \subset \mathbb{C}^2$ is a set of uniqueness. Indeed, by translation and rescaling, we may assume that $D \supseteq \mathbb{D}^2$ and $B = (-1, 1) \times (-1, 1)$. For any fixed $x_1 \in (-1, 1)$, consider the one-variable analytic functions $z_2 \mapsto h(x_1, z_2), k(x_1, z_2)$. Since they agree on $(-1, 1)$, they must agree on $\{z_2: (x_1, z_2) \in D\}$. Thus h and k agree on $\{(z_1, z_2) \in D: z_1 \in (-1, 1) \text{ or } z_2 \in (-1, 1)\}$. But now if $h(z_1^0, z_2^0) \neq k(z_1^0, z_2^0)$ for some $(z_1^0, z_2^0) \in \mathbb{D}^2$, then we fix z_1^0 and let z_2 vary in \mathbb{D} to conclude that the one-variable functions $z_2 \mapsto h(z_1^0, z_2), k(z_1^0, z_2)$ coincide on $(-1, 1)$ but not on \mathbb{D} , an obvious contradiction.

with z and ζ such that $\Im(zv + \zeta w)^{-1}$ stays uniformly bounded. Thus, the above lemma applies to show that $\lim_{y \downarrow 0} \varphi(f'(\alpha + iyw)(v)) = \lim_{y \downarrow 0} \varphi(f'(\alpha + iyv)(v))$ to show that the limit of $f'(\alpha + iyv)$ as $y \downarrow 0$ does not depend on v and is positive. Applying this same result to $M_n(\mathcal{A})$ and recalling the properties of noncommutative functions guarantee complete positivity for $f'(\alpha)$. To conclude the proof of (1'), simply observe that $\|\Delta f(\alpha + iyv_1, \alpha + iyv_2)(b) - f'(\alpha + iyv_1)(b)\|$ converges to zero as $y \downarrow 0$.

The proof of (2) is much simpler. Indeed, the existence of the limit $\lim_{y \downarrow 0} f'(\alpha + iyv)$ implies the existence of the limit $\lim_{y \downarrow 0} \varphi(f'(\alpha + iyv)(v))$ for all states φ on \mathcal{A} . An application of Theorem 2.1 and of parts (1) and (1') of Theorem 2.2 allows us to conclude. \square

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