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Molien formula for generic matrices

Let k be a (commutative) field whose characteristic is not 2. We set $S = k[a, b, c, d, e, f, g, h]$ a polynomial algebra in eight indeterminates and consider

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} e & f \\ g & h \end{bmatrix}.$$

We consider the following algebras by $R = \langle X, Y \rangle_{k\text{-alg.}} \subset M_2(S)$, $C = \langle \text{coeff of the char. pol. of } u \in R \rangle_{k\text{-alg.}} \subset S$ and $\bar{R} = \langle R, C \rangle_{k\text{-alg.}} \subset M_2(S)$. The algebra R is said to be the algebra of generic matrices on two generators, C is said to be the coefficient ring of R and \bar{R} is the characteristic ring of R .

In this context, we get the following results.

Proposition 1 – [FHL]—[FS].

- The bilinear Cayley-Hamilton identity:

$$XY + YX - \text{tr}(X)Y - \text{tr}(Y)X + \text{tr}(X)\text{tr}(Y) - \text{tr}(XY) = 0.$$

- We have $[R, R] = (XY - YX)\bar{R} = \bar{R}(XY - YX)$ and $R/[R, R]$ is a polynomial algebra in two indeterminates generated by the image of X and Y .
- $C = k[\text{tr}(X), \text{tr}(Y), \det(X), \det(Y), \text{tr}(XY)] = k[\text{tr}(X), \text{tr}(Y), \text{tr}(X^2), \text{tr}(Y^2), \text{tr}(XY)]$ is a polynomial algebra in five indeterminates
- \bar{R} is a free module of rang four over C with basis $1, X, Y, XY$.
- Let $G \subset \text{GL}_2(k)$ a finite subgroup. Then G acts on R . Moreover if $\text{car } k \nmid |G|$ and $G \subset \text{SL}_2(k)$ then R^G is a k algebra of finite type.

Molien formula for R .

In the first part of this note, we compute a Molien formula for the invariant of R and \bar{R} under a finite subgroup of $\text{GL}_2(k)$. In a second part we study the case of finite subgroup of $\text{SL}_2(k)$. They are parametrized by the Coxeter diagram of type A , D and E . We determine the Poincaré series of the invariants under groups of the A and E .

Let us first describe the action of $\text{GL}_2(k)$ on R , C and \bar{R} . The group $\text{GL}_2(k)$ acts on the right homogeneously on the k -algebra R in the following way : let $g = \begin{bmatrix} u & v \\ w & x \end{bmatrix} \in \text{GL}_2(k)$, we set

$$gX = uX + vY \quad \text{and} \quad gY = wX + xY$$

It is well defined since, g acts on $k[a, e]$, $k[b, f]$, $k[c, g]$ and $k[d, h]$ via $a \mapsto ua + ve$ et $e \mapsto wa + xe...$ Moreover the action of $\text{GL}_2(k)$ on R extends to C and \bar{R} .

Our first aim is to show the following relations on graded traces. The algebras R and \bar{R} are graded subalgebra of $M_2(S)$ (whose graduation is given via the identification with S^4). The algebra C is a graded subalgebra of S .

Proposition 2 – Graded traces. For $g \in \text{GL}_2(k)$, we have the following formulas.

$$\text{grtr}(g, C) = \frac{1}{\det(1 - gT) \det(1 - g_{S^2(V)}T^2)},$$

$$\text{grtr}(g, \bar{R}) = \frac{\det(1 + gT)}{\det(1 - gT) \det(1 - g_{S^2(V)}T^2)},$$

$$\text{grtr}(g, R) = \frac{1}{\det(1 - gT)} + \frac{\det(1 + gT) \det(g)T^2}{\det(1 - gT) \det(1 - g_{S^2(V)}T^2)} = \frac{\det(1 - g_{S^2(V)}T^2) + \det(1 + gT) \det(g)T^2}{\det(1 - gT) \det(1 - g_{S^2(V)}T^2)}.$$

Proof. Let us start by computing the graded trace of an element $g \in \text{GL}_2(k)$ acting on C . To do this, we remark that since $\text{char } k \neq 2$, we have $\det(X) = \frac{1}{2}(\text{tr}(X)^2 - \text{tr}(X^2))$. So

$$C = k[\text{tr}(X), \text{tr}(Y), \text{tr}(X^2), \text{tr}(XY), \text{tr}(Y^2)] = k[\text{tr}(X), \text{tr}(Y)] \otimes k[\text{tr}(X^2), \text{tr}(XY), \text{tr}(Y^2)].$$

where $\text{GL}_2(k)$ acts diagonally on the tensor product. So if we set $V = \langle X, Y \rangle_{k\text{-v.s.}}$, then the $\text{GL}_2(k)$ -module C is isomorphic to $S(V) \otimes S(S^2(V))$. So the usually Molien formula gives the graded trace of $g \in \text{GL}_2(k)$ acting on C

$$\text{grtr}(g, C) = \frac{1}{\det(1 - gT) \det(1 - g_{S^2(V)}T^2)}.$$

Using the relation $\bar{R} = C \oplus CX \oplus CY \oplus CXY$, we get

$$\text{grtr}(g, \bar{R}) = \text{grtr}(g, C)(1 + \text{tr}(g)T + \det(g)T^2).$$

Indeed, if we choose $(c_i)_i$ a k -basis of C , then $(c_i X)$ is a basis of CX and the component of $g(c_i X)$ is $u\alpha$ where α is the component of gc_i in the basis c_i . With the same argument for CY , we get the term $\text{tr}(g)T$. Let now study CXY , we have $gXY = (uX + vY)(wX + xY)$. Using the Cayley-Hamilton bilinear formula, we get that the component of gXY on XY is $(ux - vw) = \det g$. Moreover $(1 + \text{tr}(g)T + \det(g)T^2) = \det(1 + gT)$. Finally, we have

$$\text{grtr}(g, \bar{R}) = \text{grtr}(g, C) \det(1 + gT) = \frac{\det(1 + gT)}{\det(1 - gT) \det(1 - g_{S^2(V)}T^2)}.$$

Let $J = [R, R]$. Now we use the fact that we have the following exact sequence of graded $\text{GL}_2(k)$ -modules

$$0 \longrightarrow J \longrightarrow R \longrightarrow R/J = S(V) \longrightarrow 0$$

So that $\text{grtr}(g, R) = \text{grtr}(g, R/J) + \text{grtr}(g, J)$. But $\text{grtr}(g, R/J) = 1/\det(1 - gT)$ and $J = (XY - YX)\bar{R}$ (see proposition 1). Thus we get $\text{grtr}(g, J) = \det(g)T^2 \text{grtr}(g, \bar{R})$. Finally we obtain the following expression :

$$\text{grtr}(g, R) = \frac{1}{\det(1 - gT)} + \frac{\det(1 + gT) \det(g)T^2}{\det(1 - gT) \det(1 - g_{S^2(V)}T^2)} = \frac{\det(1 - g_{S^2(V)}T^2) + \det(1 + gT) \det(g)T^2}{\det(1 - gT) \det(1 - g_{S^2(V)}T^2)}.$$

For $g = \text{id}$, we get back the Poincaré series for R, C and \bar{R} . \square

Let us now suppose that $\text{car } k = 0$. The action of any finite subgroup G of $\text{GL}_2(k)$ is semisimple and the Reynold operator $|G|^{-1} \sum_{g \in G} g$ is the projection on the invariants. So we get the following formula for the Poincaré series of invariants

Proposition 3 – Poincaré series for invariants.

$$P_{C^G} = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1 - gT) \det(1 - g_{S^2(C^2)}T^2)}$$

$$P_{\bar{R}^G} = \frac{1}{|G|} \sum_{g \in G} \frac{\det(1 + gT)}{\det(1 - gT) \det(1 - g_{S^2(C^2)}T^2)}$$

$$P_{R^G} = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1 - gT)} + \sum_{g \in G} \frac{\det(1 + gT) \det(g)T^2}{\det(1 - gT) \det(1 - g_{S^2(C^2)}T^2)}$$

Type A_n case: the cyclic group \mathbb{U}_n .

Let $G = \langle \text{diag}(\zeta, \zeta^{-1}), \zeta \in \mathbb{U}_n \rangle$ be the cyclic group of order n where \mathbb{U}_n is the group of n^{th} roots of unity in

C. We compute the Poincaré series of \bar{R}^G and R^G .

We have to compute

$$P_{\bar{R}^G} = \frac{1}{n} \left(\sum_{\zeta \in \mathbb{U}_n} \frac{(1 + \zeta T)(1 + \zeta^{-1}T)}{(1 - \zeta T)(1 - \zeta^{-1}T)(1 - \zeta^2 T^2)(1 - T^2)(1 - \zeta^{-2} T^2)} \right) = \frac{1}{n} \frac{1}{1 - T^2} \sum_{\zeta \in \mathbb{U}_n} \frac{1}{(1 - \zeta T)^2 (1 - \zeta^{-1} T)^2}$$

and

$$P_{R^G} = \frac{1}{n} \sum_{\zeta \in \mathbb{U}_n} \frac{1}{(1 - \zeta T)(1 - \zeta^{-1}T)} + T^2 P_{\bar{R}^G} = \frac{1}{n} \sum_{\zeta \in \mathbb{U}_n} \frac{1}{(1 - \zeta T)(1 - \zeta^{-1}T)} + \frac{T^2}{1 - T^2} \frac{1}{n} \sum_{\zeta \in \mathbb{U}_n} \frac{1}{(1 - \zeta T)^2 (1 - \zeta^{-1} T)^2}$$

For this, we will need the few computations shown in the following lemmas

Lemma 4 For $\ell \in \mathbb{Z}$, we have

$$\sum_{k \geq -\ell} T^k = \frac{T^{-\ell}}{1 - T} \quad \text{and} \quad \sum_{k \geq -\ell} k T^k = \frac{-\ell T^{-\ell}}{1 - T} + \frac{T^{-\ell+1}}{(1 - T)^2} = \frac{-\ell T^{-\ell} + (\ell + 1)T^{-\ell+1}}{(1 - T)^2}.$$

Proof. The first equality is just obtain by factorizing $T^{-\ell}$. Moreover, let $f = \sum_{k \geq -\ell} T^k$, then we have

$$\sum_{k \geq -\ell} k T^k = T f'.$$

It now suffices to derive the first equality to get the second one.

Lemma 5 We have

$$\sum_{k \geq 0} T^k = \frac{1}{1 - T}, \quad \sum_{k \geq 0} k T^k = \frac{T}{(1 - T)^2} \quad \text{and} \quad \sum_{k \geq 0} k^2 T^k = \frac{T + T^2}{(1 - T)^3}.$$

Proof. The first two equalities are special cases of the preceding lemma. For the third one, let $f = \sum_{k \geq 0} k T^k$.

Then

$$\sum_{k \geq 0} k^2 T^k = T f'.$$

It now suffices to derive the second equality to get the third one.

From now on, our main aim is to compute

$$\Phi(T) = \frac{1}{n} \sum_{\zeta \in \mathbb{U}_n} \frac{1}{(1 - \zeta T)^2 (1 - \zeta^{-1} T)^2}.$$

But we have

$$\Phi(T) = \frac{1}{(1-x)^2} \Big|_{x=\zeta T} \quad \frac{1}{(1-y)^2} \Big|_{y=\zeta^{-1} T}$$

and $1/(1-u)^2 = (1/(1-u))' = (\sum u^i)'$. So

$$\frac{1}{(1-x)^2 (1-y)^2} = \sum_{(i,j) \in \mathbb{N}^2} i j x^{i-1} y^{j-1}$$

and

$$\Phi(T) = \sum_{(i,j) \in \mathbb{N}^2} (i+1)(j+1) T^{i+j} \left(\frac{1}{n} \sum_{\zeta \in \mathbb{U}_n} \zeta^{i-j} \right).$$

But

$$\frac{1}{n} \sum_{\zeta \in \mathbb{U}_n} \zeta^{i-j} = 1$$

whenever $i-j = kn$ for some $k \in \mathbb{Z}$ and 0 otherwise. So

$$\Phi(T) = \sum_{(k,j) \in \mathbb{N}, k \geq -j/n} (j+kn+1)(j+1) T^{2j+kn}.$$

which can also be written as

$$\Phi(T) = \sum_{\ell \in \mathbb{N}} \left(\sum_{j=\ell n}^{\ell n+n-1} (j+1)^2 T^{2j} \right) \left(\sum_{k \geq -\ell} T^{kn} \right) + \sum_{\ell \in \mathbb{N}} \left(\sum_{j=\ell n}^{\ell n+n-1} (j+1) T^{2j} \right) \left(\sum_{k \geq -\ell} kn T^{kn} \right)$$

Before going on the computation, we need the following lemma.

Lemma 6 – Some computation. For $\ell \in \mathbb{N}$, we have

$$\sum_{i=\ell n+1}^{\ell n+n} T^i = \frac{T^{\ell n+1} - T^{\ell n+n+1}}{1-T}$$

$$\sum_{i=\ell n}^{\ell n+n-1} (i+1) T^i = \sum_{i=\ell n+1}^{\ell n+n} i T^{i-1} = \frac{(\ell n+1) T^{\ell n} - (\ell n+n+1) T^{\ell n+n}}{1-T} + \frac{T^{\ell n+1} - T^{\ell n+n+1}}{(1-T)^2}$$

$$\sum_{i=\ell n}^{\ell n+n-1} (i+1)^2 T^i = \sum_{i=\ell n+1}^{\ell n+n} i^2 T^{i-1} = \frac{(\ell n+1)^2 T^{\ell n} - (\ell n+n+1)^2 T^{\ell n+n}}{1-T} + \frac{(2\ell n+3) T^{\ell n+1} - (2\ell n+2n+3) T^{\ell n+n+1}}{(1-T)^2} + 2 \frac{T^{\ell n+2} - T^{(\ell+1)n+2}}{(1-T)^3}$$

Proof. Let $u = \sum_{i=\ell n+1}^{\ell n+n} T^i$. The second expression is just u' and the third is $(Tu)'' = Tu'' + u'$.

Using the preceding lemma and lemma 4, we obtain

$$\Phi(T) = \frac{1 + 3T^2 + \dots + (2n-1)T^{2n-2} + T^n((2n-1) + (2n-3)T^2 + \dots + T^{2n-2})}{(1-T^n)^3(1-T^2)}$$

which can also be written as

$$\Phi(T) = \frac{(1-T^n)(1 + 3T^2 + \dots + (2n-1)T^{2n-2}) + 2nT^n(1 + T^2 + \dots + T^{2n-2})}{(1-T^n)^3(1-T^2)}$$

or

$$\Phi(T) = \frac{(1 + 3T^2 + \dots + (2n-1)T^{2n-2})(1-T^2) + 2nT^n(1+T^n)}{(1-T^n)^2(1-T^2)^2}$$

and also

$$\Phi(T) = \frac{1 + 2nT^n + T^{2n} + 2T^2 + 2T^4 + \dots + 2T^{2n-2}}{(1-T^n)^2(1-T^2)^2}$$

Finally we get the following result

Proposition 7 We have

$$P_{RG} = \Phi(T) = \frac{1 + 2nT^n + T^{2n} + 2T^2 + 2T^4 + \dots + 2T^{2n-2}}{(1-T^n)^2(1-T^2)^3}$$

and

$$P_{RG} = \frac{1 - T^2 + T^4 + 2nT^{n+2} - T^{2n} + 3T^{2n+2} - T^{2n+4} + 2T^4 + 2T^6 + \dots + 2T^{2n}}{(1-T^n)^2(1-T^2)^3}$$

Proof. The expression

$$\frac{1}{n} \sum_{\zeta \in \bar{\mathbb{U}}_n} \frac{1}{(1 - \zeta T)(1 - \zeta^{-1} T)}$$

is the Poincaré series of the fixed points of the action of G on $S(\mathbb{C}^2)$ which is described as

$$S(\mathbb{C}^2)^G = k[u^n, v^n] \oplus uvk[u^n, v^n] \oplus \cdots \oplus (uv)^{n-1}k[u^n, v^n]$$

which gives the Poincaré series

$$\frac{1 + T^n}{(1 - T^2)(1 - T^n)}.$$

The Exceptional Group Cases.

Sage computations give the following properties

Proposition 8 – The E6 case. We have the following formulas

$$P_{CG} = \frac{T^{16} - T^{12} + 2T^{10} + 8T^8 + 2T^6 - T^4 + 1}{(1 - T^4)^3(1 - T^6)^2}$$

$$P_{\bar{R}G} = \frac{T^{16} + T^{14} - T^{12} + 4T^{10} + 14T^8 + 4T^6 - T^4 + T^2 + 1}{(1 - T^4)^2(1 - T^6)^2(1 - T^2)}$$

$$P_{RG} = \frac{-T^{20} + 2T^{18} + 3T^{16} - 2T^{14} + T^{12} + 14T^{10} + 7T^8 - T^4 + 1}{(1 - T^4)^2(1 - T^6)^2(1 - T^2)}$$

Proposition 9 – The E7 case. We have the following formulas

$$P_{CG} = \frac{T^{24} + T^{20} - 2T^{18} + 4T^{16} + 2T^{14} + 12T^{12} + 2T^{10} + 4T^8 - 2T^6 + T^4 + 1}{(1 - T^6)^2(1 - T^8)^2(1 - T^4)}$$

$$P_{\bar{R}G} = \frac{T^{24} + T^{22} + T^{20} - T^{18} + 6T^{16} + 6T^{14} + 20T^{12} + 6T^{10} + 6T^8 - T^6 + T^4 + T^2 + 1}{(1 - T^6)^2(1 - T^8)^2(1 - T^2)}$$

$$P_{RG} = \frac{-T^{28} + 2T^{26} + T^{24} + 3T^{22} - 2T^{20} + 5T^{18} + 4T^{16} + 20T^{14} + 8T^{12} + 7T^{10} - T^6 + T^4 + 1}{(1 - T^6)^2(1 - T^8)^2(1 - T^2)}$$

Proposition 10 – The E8 case. We have the following formulas

$$P_{CG} = \frac{T^{32} + 2T^{30} + 2T^{28} - 2T^{24} - 4T^{22} + 2T^{20} + 16T^{18} + 26T^{16} + 16T^{14} + 2T^{12} - 4T^{10} - 2T^8 + 2T^4 + 2T^2 + 1}{(1 - T^6)^2(1 - T^{10})^2(1 - T^4)(1 + T^2)^2}$$

$$P_{\bar{R}G} = \frac{T^{32} + 3T^{30} + 4T^{28} + 2T^{26} - 2T^{24} - 6T^{22} + 4T^{20} + 30T^{18} + 48T^{16} + 30T^{14} + 4T^{12} - 6T^{10} - 2T^8 + 2T^6 + 4T^4 + 3T^2 + 1}{(1 - T^6)^2(1 - T^{10})^2(1 - T^4)(1 + T^2)}$$

$$P_{RG} = \frac{-T^{36} + 4T^{32} + 7T^{30} + 4T^{28} - 2T^{26} - 8T^{24} + 26T^{20} + 48T^{18} + 34T^{16} + 8T^{14} - 4T^{12} - 2T^{10} + T^6 + 2T^4 + 2T^2 + 1}{(1 - T^6)^2(1 - T^{10})^2(1 - T^4)(1 + T^2)}$$

This function computes the matrix of m acting on $S^2(V)$

where m acts on the two dimensional space V

def S2(m):

A=m.base_ring()

MS=MatrixSpace(A,3)

T=MS([i for i in range(9)])

T[0]=[m[0][0]^2, 2*m[0][0]*m[0][1], m[0][1]^2]

T[1]=[m[0][0]*m[1][0], m[1][0]*m[0][1] + m[0][0]*m[1][1], m[0][1]*m[1][1]]

T[2]=[m[1][0]^2, 2*m[1][0]*m[1][1], m[1][1]^2]

return T

This function computes the graded trace of m (a 2 by 2 invertible matrix)

acting on \mathbb{C} as a power series

def trgrC(m):

A=m.base_ring()

SF.<T>=PowerSeriesRing(A)

MS=MatrixSpace(SF,2)

sor=1/(det(1-m*T)*det(1-S2(m)*T^2))

return sor

```

# This function computes the graded trace of m (a 2 by 2 invertible matrix)
# acting on Rbarre as a power series
def trgrRbarre(m):
A=m.base_ring()
SF.<T>=PowerSeriesRing(A)
MS=MatrixSpace(SF,2)
sor=det(1+m*T)*trgrC(m)
return sor

# This function computes the graded trace of m (a 2 by 2 invertible matrix)
# acting on R as a power series
def trgrGEN(m):
A=m.base_ring()
SF.<T>=PowerSeriesRing(A)
MS=MatrixSpace(SF,2)
sor=1/det(1-m*T) + det(m)*T^2*trgrRbarre(m)
return sor

# This function computes the graded trace of m (a 2 by 2 invertible matrix)
# acting on C as a rational function of T
def trgrfrC(m):
A=m.base_ring()
Pol.<T>=PolynomialRing(A)
Fr=FractionField(Pol)
MS=MatrixSpace(Fr,2)
sor=1/(det(1-m*T)*det(1-S2(m)*T^2))
return sor

# This function computes the graded trace of m (a 2 by 2 invertible matrix)
# acting on Rbarre as a rational function of T
def trgrfrRbarre(m):
A=m.base_ring()
Pol.<T>=PolynomialRing(A)
Fr=FractionField(Pol)
MS=MatrixSpace(Fr,2)
sor=det(1+m*T)*trgrfrC(m)
return sor

# This function computes the graded trace of m (a 2 by 2 invertible matrix)
# acting on R as a rational function of T
def trgrfrGEN(m):
A=m.base_ring()
Pol.<T>=PolynomialRing(A)
Fr=FractionField(Pol)
MS=MatrixSpace(Fr,2)
sor=1/det(1-m*T) + det(m)*T^2*trgrfrRbarre(m)
return sor

#E6 binary tetrahedral group of order 24
kE6.<zE6>=CyclotomicField(4)
MSE6=MatrixSpace(kE6,2)
g1E6=MSE6([zE6,0,0,zE6^(-1)])
g2E6=MSE6([0,zE6,zE6,0])
g3E6=MSE6([(1-zE6)^(-1), zE6*(1-zE6)^(-1), (1-zE6)^(-1), -zE6*(1-zE6)^(-1)])
gens=[g1E6,g2E6,g3E6]
E6=MatrixGroup(gens);
SE6=list(Set(E6))
PoincareE6C=1/E6.order() *sum([trgrfrC(MSE6(m)) for m in SE6])
print("Poincaré series of the invariant of C is")
print(PoincareE6C)
PoincareE6Rbarre=1/E6.order() *sum([trgrfrRbarre(MSE6(m)) for m in SE6])
print("Poincaré series of the invariant of Rbarre is")
print(PoincareE6Rbarre)
PoincareE6GEN=1/E6.order() *sum([trgrfrGEN(MSE6(m)) for m in SE6])
print("Poincaré series of the invariant of Rbarre is")

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print(PoincareE6GEN)

#E7 binary tetrahedral group of order 48
kE7.<zE7>=CyclotomicField(8)
MSE7=MatrixSpace(kE7,2)
g1E7=MSE7([zE7,0,0,zE7^(-1)])
g2E7=MSE7([0,zE7^2,zE7^2,0])
g3E7=MSE7([(1-zE7^2)^(-1), zE7^2*(1-zE7^2)^(-1), (1-zE7^2)^(-1), -zE7^2*(1-zE7^2)^(-1)])
gens=[g1E7,g2E7,g3E7]
E7=MatrixGroup(gens)
SE7=list(Set(E7))
PoincareE7C=1/E7.order()*sum([trgrfrC(MSE7(m)) for m in SE7])
print("Poincaré series of the invariant of C is")
print(PoincareE7C)
PoincareE7Rbarre=1/E7.order()*sum([trgrfrRbarre(MSE7(m)) for m in SE7])
print("Poincaré series of the invariant of Rbarre is")
print(PoincareE7Rbarre)
PoincareE7GEN=1/E7.order()*sum([trgrfrGEN(MSE7(m)) for m in SE7])
print("Poincaré series of the invariant of R is")
print(PoincareE7GEN)

#E8 binary tetrahedral group of order 120
kE8.<zE8>=CyclotomicField(20)
MSE8=MatrixSpace(kE8,2)
g1E8=MSE8([zE8^2,0,0,zE8^(-2)])
a=2*(zE8^4+zE8^(-4)) + 1
g2E8=MSE8([(zE8^4-zE8^(16))/a,(zE8^8-zE8^(12))/a,(zE8^8-zE8^(12))/a,(-zE8^4+zE8^(16))/a])
gens=[g1E8,g2E8]
E8=MatrixGroup(gens)
SE8=list(Set(E8))
PoincareE8C=1/E8.order()*sum([trgrfrC(MSE8(m)) for m in SE8])
print("Poincaré series of the invariant of C is")
print(PoincareE8C)
PoincareE8Rbarre=1/E8.order()*sum([trgrfrRbarre(MSE8(m)) for m in SE8])
print("Poincaré series of the invariant of Rbarre is")
print(PoincareE8Rbarre)
PoincareE8GEN=1/E8.order()*sum([trgrfrGEN(MSE8(m)) for m in SE8])
print("Poincaré series of the invariant of R is")
print(PoincareE8GEN)

```

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