



# Characteristic functions on the boundary of a planar domain need not be traces of least gradient functions

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Mickael dos Santos. Characteristic functions on the boundary of a planar domain need not be traces of least gradient functions. 2017. hal-01216155v2

**HAL Id: hal-01216155**

**<https://hal.science/hal-01216155v2>**

Preprint submitted on 28 Feb 2017

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# CHARACTERISTIC FUNCTIONS ON THE BOUNDARY OF A PLANAR DOMAIN NEED NOT BE TRACES OF LEAST GRADIENT FUNCTIONS

MICKAËL DOSSANTOS

ABSTRACT. Given a smooth bounded planar domain  $\Omega$ , we construct a compact set on the boundary such that its characteristic function is not the trace of a least gradient function. This generalizes the construction of Spradlin and Tamasan [ST14] when  $\Omega$  is a disc.

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## 1. INTRODUCTION

We let  $\Omega$  be a bounded  $C^2$  domain of  $\mathbb{R}^2$ . For a function  $h \in L^1(\partial\Omega, \mathbb{R})$ , the least gradient problem with boundary datum  $h$  consists in deciding whether

$$(1) \quad \inf \left\{ \int_{\Omega} |Dw| ; w \in BV(\Omega) \text{ and } \text{tr}_{\partial\Omega} w = h \right\}$$

is achieved or not.

In the above minimization problem,  $BV(\Omega)$  is the space of functions of bounded variation. It is the space of functions  $w \in L^1(\Omega)$  having a distributional gradient  $Dw$  which is a bounded Radon measure.

If the *infimum* in (1) is achieved, minimal functions are called *functions of least gradient*.

Sternberg, Williams and Ziemmer proved in [SWZ92] that if  $h : \partial\Omega \rightarrow \mathbb{R}$  is a continuous map and if  $\partial\Omega$  satisfies a geometric properties then there exists a (unique) function of least gradient. For further use, we note that the geometric property is satisfied by Euclidean balls.

On the other hand, Spradlin and Tamasan [ST14] proved that, for the disc  $\Omega = \{x \in \mathbb{R}^2; |x| < 1\}$ , we may find a function  $h_0 \in L^1(\partial\Omega)$  which is not continuous s.t. the *infimum* in (1) is not achieved. The function  $h_0$  is the characteristic function of a Cantor type set  $\mathcal{K} \subset \mathbb{S}^1 = \{x \in \mathbb{R}^2; |x| = 1\}$

The goal of this article is to extend the main result of [ST14] to a general  $C^2$  bounded open set  $\Omega \subset \mathbb{R}^2$ .

We prove the following theorem.

**Theorem 1.** *Let  $\Omega \subset \mathbb{R}^2$  be a bounded  $C^2$  open set. Then there exists a measurable set  $\mathcal{K} \subset \partial\Omega$  such that the infimum*

$$(2) \quad \inf \left\{ \int_{\Omega} |Dw| ; w \in BV(\Omega) \text{ and } \text{tr}_{\partial\Omega} w = \mathbb{I}_{\mathcal{K}} \right\}$$

*is not achieved.*

The calculations in [ST14] are specific to the case  $\Omega = \mathbb{D}$ . The proof of Theorem 1 relies on new arguments for the construction of the Cantor set  $\mathcal{K}$  and the strategy of the proof.

## 2. STRATEGY OF THE PROOF

**2.1. The model problem.** We illustrate the strategy developed to prove Theorem 1 on the model case  $\mathcal{Q} = (0, 1)^2$ . Clearly, this model case does not satisfy the  $C^2$  assumption.

Nevertheless, the flatness of  $\partial\mathcal{Q}$  allows to get a more general counterpart of Theorem 1. Namely, the counterpart of Theorem 1 [see Proposition 1 below] is no more an existence result of a set  $\mathcal{K} \subset \partial\mathcal{Q}$  s.t. Problem (2) is not achieved. It is a non existence result of a least gradient function for  $h = \mathbb{I}_{\mathcal{M}}$  for *any* measurable domain  $\mathcal{M} \subset [0, 1] \times \{0\} \subset \partial\mathcal{Q}$  with positive Lebesgue measure.

We thus prove the following result whose strategy of the proof is due to Petru Mironescu.

**Proposition 1.** *[P. Mironescu] Let  $\tilde{\mathcal{M}} \subset [0, 1]$  be a measurable set with positive Lebesgue measure. Then the infimum in*

$$(3) \quad \inf \left\{ \int_{\mathcal{Q}} |Dw| ; w \in BV(\mathcal{Q}) \text{ and } \text{tr}_{\partial\mathcal{Q}} w = \mathbb{I}_{\tilde{\mathcal{M}} \times \{0\}} \right\}$$

*is not achieved.*

This section is devoted to the proof of Proposition 1. We fix a measurable set  $\tilde{\mathcal{M}} \subset [0, 1]$  with positive measure and we let  $h = \mathbb{I}_{\tilde{\mathcal{M}} \times \{0\}}$ . We argue by contradiction: we assume that there exists a minimizer  $u_0$  of (3). We obtain a contradiction in 3 steps.

### Step 1. Upper bound and lower bound

This first step consists in obtaining two estimates. The first estimate is the upper bound

$$(4) \quad \int_{\mathcal{Q}} |Du_0| \leq \|\mathbb{I}_{\tilde{\mathcal{M}} \times \{0\}}\|_{L^1(\partial\mathcal{Q})} = \mathcal{H}^1(\tilde{\mathcal{M}}).$$

Here,  $\mathcal{H}^1(\tilde{\mathcal{M}})$  is the length of  $\tilde{\mathcal{M}}$ .

Estimate (4) follows from Theorem 2.16 and Remark 2.17 in [Giu84]. Indeed, by combining Theorem 2.16 and Remark 2.17 in [Giu84] we may prove that for  $h \in L^1(\partial\Omega)$  and for all  $\varepsilon > 0$  there exists a map  $u_\varepsilon \in BV(\Omega)$  s.t.

$$\int_{\Omega} |Du_\varepsilon| \leq (1 + \varepsilon) \|h\|_{L^1(\partial\Omega)} \text{ and } \text{tr}_{\partial\Omega} u_\varepsilon = h.$$

The proof of this inequality when  $\Omega$  is a half space is presented in [Giu84]. It is easy to adapt the argument when  $\Omega = \mathcal{Q} = (0, 1)^2$ . The extension for a  $C^2$  set  $\Omega$  is presented in Appendix E.

**Step 2.** Optimality of (4) [see (5)]

The optimality of (4) is obtained *via* the following lemma.

**Lemma 2.** For  $u \in BV(\mathcal{Q})$  we have

$$\int_{\mathcal{Q}} |D_2 u| \geq \int_0^1 |\text{tr}_{\partial \mathcal{Q}} u(\cdot, 0) - \text{tr}_{\partial \mathcal{Q}} u(\cdot, 1)|.$$

Here, for  $k \in \{1, 2\}$  we denoted

$$\int_{\mathcal{Q}} |D_k u| = \sup \left\{ \int_{\mathcal{Q}} u \partial_k \xi; \xi \in C_c^1(\mathcal{Q}) \text{ and } |\xi| \leq 1 \right\}$$

where  $C_c^1(\mathcal{Q})$  are the set of real valued  $C^1$ -functions with compact support included in  $\mathcal{Q}$ .

Lemma 2 is proved in Appendix B.1.

From Lemma 2 we get

$$\int_{\mathcal{Q}} |D_2 u_0| \geq \int_0^1 |\text{tr}_{\partial \mathcal{Q}} u_0(\cdot, 0) - \text{tr}_{\partial \mathcal{Q}} u_0(\cdot, 1)| = \int_0^1 \mathbb{I}_{\tilde{\mathcal{M}} \times \{0\}} = \mathcal{H}^1(\tilde{\mathcal{M}}).$$

Since we have

$$(5) \quad \int_{\mathcal{Q}} |Du_0| := \sup \left\{ \int_{\mathcal{Q}} u \text{div}(\xi); \xi = (\xi_1, \xi_2) \in C_c^1(\mathcal{Q}, \mathbb{R}^2) \text{ and } \xi_1^2 + \xi_2^2 \leq 1 \right\} \geq \int_{\mathcal{Q}} |D_2 u_0| \geq \mathcal{H}^1(\tilde{\mathcal{M}}),$$

we get the optimality of (4).

**Step 3.** A transverse argument

From (4) and (5) we may prove

$$(6) \quad \int_{\mathcal{Q}} |D_1 u_0| = 0.$$

Equality (6) is a direct consequence of the following lemma.

**Lemma 3.** Let  $\Omega$  be a planar open set. If  $u \in BV(\Omega)$  is s.t.

$$\int_{\Omega} |Du| = \int_{\Omega} |D_2 u|,$$

then  $\int_{\Omega} |D_1 u| = 0$ .

Lemma 3 is proved in Appendix B.2.

In order to conclude we state an easy lemma.

**Lemma 4.** [Poincaré inequality] For  $u \in BV(\mathcal{Q})$  satisfying  $\text{tr}_{\partial \mathcal{Q}} u = 0$  in  $\{0\} \times [0, 1]$  we have

$$\int_{\mathcal{Q}} |u| \leq \int_{\mathcal{Q}} |D_1 u|.$$

Lemma 4 is proved in Appendix B.3.

Hence, from (6) and Lemma 4 we have  $u_0 = 0$  which is in contradiction with  $\text{tr}_{\partial \mathcal{Q}} u_0 = \mathbb{I}_{\tilde{\mathcal{M}} \times \{0\}}$  with  $\mathcal{H}^1(\tilde{\mathcal{M}}) > 0$ .

**2.2. Outline of the proof of Theorem 1.** The idea is to adapt the above construction and argument to the case of a general  $C^2$  domain  $\Omega$ . If  $\Omega$  has a flat or concave part  $\Gamma$  of the boundary  $\partial \Omega$ , then a rather straightforward variant of the above proof shows that  $\mathbb{I}_{\mathcal{M}}$ , where  $\mathcal{M}$  is a non trivial part of  $\Gamma$ , is not the trace of a least gradient function.

*Remark 5.* Things are more involved when  $\Omega$  is convex. For simplicity we illustrate this fact when  $\Omega = \mathbb{D} = \{x \in \mathbb{R}^2; |x| < 1\}$ . Let  $\mathcal{M} \subset \mathbb{S}^1 \cap \{(x, y) \in \mathbb{R}^2; x < 0\}$  be an arc whose endpoints are symmetric with respect to the  $x$ -axis. We let  $(x_0, -y_0)$  and  $(x_0, y_0)$  be the endpoints of  $\mathcal{M}$  [here  $x_0 \leq 0$  and  $y_0 > 0$ ].

We let  $\mathcal{C}$  be the chord of  $\mathcal{M}$ . On the one hand, if  $u \in C^1(\mathbb{D}) \cap W^{1,1}(\mathbb{D})$  is s.t.  $\text{tr}_{\mathbb{S}^1} u = \mathbb{I}_{\mathcal{M}}$  then, using the Fundamental Theorem of calculus, we have for  $-y_0 < y < y_0$

$$\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} |\partial_x u(x, y)| \geq 1.$$

Thus we easily get

$$\int_{\mathbb{D}} |\nabla u| \geq \int_{\mathbb{D}} |\partial_x u| \geq \int_{-y_0}^{y_0} dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} |\partial_x u(x, y)| \geq 2y_0 = \mathcal{H}^1(\mathcal{C}).$$

Consequently, with the help of a density argument [e.g. Lemma 17 in Appendix A] we obtain

$$\inf \left\{ \int_{\mathbb{D}} |Du| ; u \in BV(\mathbb{D}) \text{ and } \text{tr}_{\mathbb{S}^1} u = \mathbb{1}_{\mathcal{M}} \right\} \geq \mathcal{H}^1(\mathcal{C}).$$

On the other hand we let  $\omega := \{(x, y) \in \mathbb{R}^2 ; x < x_0\}$ . It is clear that  $u_0 = \mathbb{1}_{\omega} \in BV(\mathbb{D})$  and  $\text{tr}_{\mathbb{S}^1} u_0 = \mathbb{1}_{\mathcal{M}}$ . Moreover

$$\int_{\mathbb{D}} |Du_0| = \mathcal{H}^1(\mathcal{C}).$$

Consequently  $u_0$  is a function of least gradient. We may do the same argument for a domain  $\Omega$  as soon as we have a chord entirely contained in  $\Omega$ . This example suggest that for a convex set  $\Omega$ , the construction of a set  $\mathcal{K} \subset \partial\Omega$  s.t. (2) is not achieved has to be "sophisticated".

The strategy to prove Theorem 1 consists of constructing a special set  $\mathcal{K} \subset \partial\Omega$  [of Cantor type] and to associate to  $\mathcal{K}$  a set  $B_{\infty}$  [the analog of  $\tilde{\mathcal{M}} \times (0, 1)$  in the model problem] which "projects" onto  $\mathcal{K}$  and s.t., if  $u_0$  is a minimizer of (1), then

$$(7) \quad \int_{B_{\infty}} |\vec{X} \cdot Du_0| \geq \mathcal{H}^1(\mathcal{K}).$$

Here,  $\vec{X}$  is a vector field satisfying  $|\vec{X}| \leq 1$ . It is the curved analog of  $\vec{X} = \mathbf{e}_2$  used in the above proof.

By (7) [and Proposition 24 in Appendix E], if  $u_0$  is a minimizer, then

$$(8) \quad \int_{\Omega \setminus B_{\infty}} |Du_0| + \int_{B_{\infty}} (|Du_0| - |\vec{X} \cdot Du_0|) = 0.$$

We next establish a Poincaré type inequality implying that any  $u_0$  satisfying (8) and  $\text{tr}_{\partial\Omega \setminus \mathcal{K}} u = 0$  is 0, which is not possible.

The heart of the proof consists of constructing  $\mathcal{K}, B_{\infty}$  and  $\vec{X}$  [see Sections 4 and 5].

### 3. NOTATION, DEFINITIONS

The ambient space is the Euclidean plan  $\mathbb{R}^2$ . We let  $\mathcal{B}_{\text{can}}$  be the canonical basis of  $\mathbb{R}^2$ .

- a) The open ball centered at  $A \in \mathbb{R}^2$  with radius  $r > 0$  is denoted  $B(A, r)$ .
- b) A vector may be denoted by an arrow when it is defined by its endpoints (e.g.  $\overrightarrow{AB}$ ). It may be also denoted by a letter in bold font (e.g.  $\mathbf{u}$ ) or more simply by a Greek letter in normal font (e.g.  $\nu$ ). We let also  $|\mathbf{u}|$  be the Euclidean norm of the vector  $\mathbf{u}$ .
- c) For a vector  $\mathbf{u}$  we let  $\mathbf{u}^{\perp}$  be the direct orthogonal vector to  $\mathbf{u}$ , i.e., if  $\mathbf{u} = (x_1, x_2)$  then  $\mathbf{u}^{\perp} = (-x_2, x_1)$ .
- d) For  $A, B \in \mathbb{R}^2$ , the segment of endpoints  $A$  and  $B$  is denoted  $[AB] = \{A + t\overrightarrow{AB} ; t \in [0, 1]\}$  and  $\text{dist}(A, B) = |\overrightarrow{AB}|$  is the Euclidean distance.
- e) For a set  $U \subset \mathbb{R}^2$ , the topological interior of  $U$  is denoted by  $\overset{\circ}{U}$  and its topological closure is  $\overline{U}$ .
- f) For  $k \geq 1$ , a  $C^k$ -curve is the range of a  $C^k$  injective map from  $(0, 1)$  to  $\mathbb{R}^2$ . Note that, in this article,  $C^k$ -curves are not closed sets of  $\mathbb{R}^2$ .
- g) For  $\Gamma$  a  $C^1$ -curve,  $\mathcal{H}^1(\Gamma)$  is the 1-dimensional Hausdorff measure of  $\Gamma$ .
- h) For  $k \geq 1$ , a  $C^k$ -Jordan curve is the range of a  $C^k$  injective map from the unit circle  $\mathbb{S}^1$  to  $\mathbb{R}^2$ .
- i) For  $\Gamma$  a  $C^1$ -curve or a  $C^1$ -Jordan curve,  $\mathcal{C} = [AB]$  is a chord of  $\Gamma$  when  $A, B \in \overline{\Gamma}$  with  $A \neq B$ .
- j) If  $\Gamma$  is a  $C^1$ -Jordan curve then, for  $A, B \in \Gamma$  &  $A \neq B$ , the set  $\Gamma \setminus \{A, B\}$  admits exactly two connected components:  $\Gamma_1 \& \Gamma_2$ . These connected components are  $C^1$ -curves.

By smoothness of  $\Gamma$ , it is clear that there exists  $\eta_{\Gamma} > 0$  s.t. for  $0 < \text{dist}(A, B) < \eta_{\Gamma}$  there exists THE smallest connected components: we have  $\mathcal{H}^1(\Gamma_1) < \mathcal{H}^1(\Gamma_2)$  or  $\mathcal{H}^1(\Gamma_2) < \mathcal{H}^1(\Gamma_1)$ .

If  $0 < \text{dist}(A, B) < \eta_{\Gamma}$  we may define  $\widehat{AB}$  by:

$$(9) \quad \widehat{AB} \text{ is the closure of the smallest curve between } \Gamma_1 \text{ and } \Gamma_2.$$

- k) In this article  $\Omega \subset \mathbb{R}^2$  is a  $C^2$  bounded open set.

4. CONSTRUCTION OF THE CANTOR SET  $\mathcal{K}$ 

It is clear that, in order to prove Theorem 1, we may assume that  $\Omega$  is a connected set.

We fix  $\Omega \subset \mathbb{R}^2$  a bounded  $C^2$  open connected set. The set  $\mathcal{K} \subset \partial\Omega$  is a Cantor type set we will construct below.

**4.1. First step: localization of  $\partial\Omega$ .** From the regularity of  $\Omega$ , there exist  $\ell + 1$   $C^2$ -open sets,  $\omega_0, \dots, \omega_\ell$ , s.t.  $\Omega = \omega_0 \setminus \overline{\omega_1 \cup \dots \cup \omega_\ell}$  and

- $\omega_i$  is simply connected for  $i = 0, \dots, \ell$ ,
- $\overline{\omega_i} \subset \omega_0$  for  $i = 1, \dots, \ell$ ,
- $\overline{\omega_i} \cap \overline{\omega_j} = \emptyset$  for  $1 \leq i < j \leq \ell$ .

We let  $\Gamma = \partial\omega_0$ . The Cantor type set  $\mathcal{K}$  we construct "lives" on  $\Gamma$ . Note that  $\Gamma$  is a Jordan-curve.

Let  $M_0 \in \Gamma$  be s.t. the inner curvature of  $\Gamma$  at  $M_0$  is positive [the existence of  $M_0$  follows from the Gauss-Bonnet formula]. Then there exists  $r_0 \in (0, 1)$  s.t.  $[AB] \subset \overline{\Omega}$  and  $[AB] \cap \partial\Omega = \{A, B\}$ ,  $\forall A, B \in B(M_0, r_0) \cap \Gamma$ . Note that we may assume  $2r_0 < \eta_\Gamma$  [ $\eta_\Gamma$  is defined in Section 3-j].

We fix  $A, B \in B(M_0, r_0) \cap \Gamma$  s.t.  $A \neq B$ . We have:

- By the definition of  $M_0$  and  $r_0$ , the chord  $\mathcal{C}_0 := [AB]$  is included in  $\overline{\Omega}$ .
- We let  $\widehat{AB}$  be the closure of the smallest part of  $\Gamma$  which is delimited by  $A, B$  (see (9)). We may assume that  $\widehat{AB}$  is the graph of  $f \in C^2([0, \eta], \mathbb{R}^+)$  in the orthonormal frame  $\mathcal{R}_0 = (A, \mathbf{e}_1, \mathbf{e}_2)$  where  $\mathbf{e}_1 = \overrightarrow{AB}/|\overrightarrow{AB}|$ .
- The function  $f$  satisfies  $f(x) > 0$  for  $x \in (0, \eta)$  and  $f''(x) < 0$  for  $x \in [0, \eta]$ .

For further use we note that the length of the chord  $[AB]$  is  $\eta$  and that for intervals  $I, J \subset [0, \eta]$ , if  $I \subset J$  then

$$(10) \quad \begin{cases} \|f'_I\|_{L^\infty(I)} \leq \|f'_J\|_{L^\infty(J)} \\ \|f''_I\|_{L^\infty(I)} \leq \|f''_J\|_{L^\infty(J)} \end{cases}$$

where  $f|_I$  is the restriction of  $f$  to  $I$ .

Replacing the chord  $\mathcal{C}_0 = [AB]$  with a smaller chord of  $\widehat{AB}$  parallel to  $\mathcal{C}_0$ , we may assume that

$$(11) \quad 0 < \eta < \min \left\{ \frac{1}{2}; \frac{1}{16\|f''\|_{L^\infty([0, \eta])}^2}; \frac{1}{2\|f'\|_{L^\infty([0, \eta])}\|f''\|_{L^\infty([0, \eta])}} \right\}.$$

We may also assume that

- Letting  $D_0^+$  be the bounded open set s.t.  $\partial D_0^+ = [AB] \cup \widehat{AB}$  we have  $\Pi_{\partial\Omega}$ , the orthogonal projection on  $\partial\Omega$ , is well defined and of class  $C^1$  in  $D_0^+$ .
- We have

$$(12) \quad 1 + 4\|f''\|_{L^\infty}^2 \text{diam}(D_0^+) < \frac{16}{9}$$

where  $\text{diam}(D_0^+) = \sup\{\text{dist}(M, N); M, N \in D_0^+\}$ . [Here we used (10)]

**4.2. Step 2: Iterative construction.** We are now in position to construct the Cantor type set  $\mathcal{K}$  as a subset of  $\widehat{AB}$ . The construction is iterative.

The goal of the construction is to get at step  $N \geq 0$  a collection of  $2^N$  pairwise disjoint curves included in  $\widehat{AB}$  [denoted by  $\{K_1^N, \dots, K_{2^N}^N\}$ ] and their chords [denoted by  $\{\mathcal{C}_1^N, \dots, \mathcal{C}_{2^N}^N\}$ ].

The idea is standard: at the step  $N \geq 0$  we replace a curve  $\Gamma_0$  included in  $\widehat{AB}$  by two curves included in  $\Gamma_0$  (see Figure 1).

**Initialization.** We initialize the procedure by letting  $K_1^0 := \widehat{AB}$  and  $\mathcal{C}_1^0 = \mathcal{C}_0 = [AB]$ .

At step  $N \geq 0$  we have:

- A set of  $2^N$  curves included in  $\widehat{AB}$ ,  $\{K_1^N, \dots, K_{2^N}^N\}$ . The curves  $K_k^N$ 's are mutually disjoint. We let  $\mathcal{K}_N = \bigcup_{k=1}^{2^N} K_k^N$ .
- A set of  $2^N$  chords,  $\{\mathcal{C}_1^N, \dots, \mathcal{C}_{2^N}^N\}$  s.t. for  $k = 1, \dots, 2^N$ ,  $\mathcal{C}_k^N$  is the chord of  $K_k^N$ .

*Remark 6.* (1) Note that since the  $\mathcal{C}_k^N$ 's are chords of  $\widehat{AB}$  and since in the frame  $\mathcal{R}_0 = (A, \mathbf{e}_1, \mathbf{e}_2)$ ,  $\widehat{AB}$  is the graph of a function, none of the chords  $\mathcal{C}_k^N$  is vertical, *i.e.*, directed by  $\mathbf{e}_2$ .

Since the chords  $\mathcal{C}_k^N$  are not vertical, for  $k \in \{1, \dots, 2^N\}$ , we may define  $\nu_{\mathcal{C}_k^N}$  as the unit vector orthogonal to  $\mathcal{C}_k^N$  s.t.  $\nu_{\mathcal{C}_k^N} = \alpha \mathbf{e}_1 + \beta \mathbf{e}_2$  with  $\beta > 0$ .

- (2) For  $\eta$  satisfying (11), if we consider a chord  $\mathcal{C}_k^N$  and a straight line  $D$  orthogonal to  $\mathcal{C}_k^N$  and intersecting  $\mathcal{C}_k^N$ , then the straight line  $D$  intersect  $K_k^N$  at exactly one point. This fact is proved in Appendix C.1.

**Induction rules.** From step  $N \geq 0$  to step  $N + 1$  we follow the following rules:

- (1) For each  $k \in \{1, \dots, 2^N\}$ , we let  $\eta_k^N$  be the length of  $\mathcal{C}_k^N$ . Inside the chord  $\mathcal{C}_k^N$  we center a segment  $I_k^N$  of length  $(\eta_k^N)^2$ .
- (2) With the help of Remark 6.2, we may define two distinct points of  $K_k^N$  as the intersection of  $K_k^N$  with straight lines orthogonal to  $\mathcal{C}_k^N$  which pass to the endpoints of  $I_k^N$ .
- (3) These intersection points are the endpoints of a curve  $\tilde{K}_k^N$  included in  $K_k^N$ . We let  $K_{2k-1}^{N+1}$  and  $K_{2k}^{N+1}$  be the connected components of  $K_k^N \setminus \overline{\tilde{K}_k^N}$ . We let also
  - $\mathcal{C}_{2k-1}^{N+1}$  and  $\mathcal{C}_{2k}^{N+1}$  be the corresponding chords;
  - $\mathcal{K}_{N+1} = \bigcup_{k=1}^{2^{N+1}} K_k^{N+1}$ .

*Notation 7.* A natural terminology consists in defining the father and the sons of a chord or a curve:

- $\mathcal{F}(\mathcal{C}_{2k-1}^{N+1}) = \mathcal{F}(\mathcal{C}_{2k}^{N+1}) = \mathcal{C}_k^N$  is the *father* of the chords  $\mathcal{C}_{2k-1}^{N+1}$  and  $\mathcal{C}_{2k}^{N+1}$ .  
 $\mathcal{F}(K_{2k-1}^{N+1}) = \mathcal{F}(K_{2k}^{N+1}) = K_k^N$  is the *father* of the curves  $K_{2k-1}^{N+1}$  and  $K_{2k}^{N+1}$ .
- $\mathcal{S}(\mathcal{C}_k^N) = \{\mathcal{C}_{2k-1}^{N+1}, \mathcal{C}_{2k}^{N+1}\}$  is the set of sons of the chord  $\mathcal{C}_k^N$ , *i.e.*,  $\mathcal{F}(\mathcal{C}_{2k-1}^{N+1}) = \mathcal{F}(\mathcal{C}_{2k}^{N+1}) = \mathcal{C}_k^N$ .  
 $\mathcal{S}(K_k^N) = \{K_{2k-1}^{N+1}, K_{2k}^{N+1}\}$  is the set of sons of the curve  $K_k^N$ , *i.e.*,  $\mathcal{F}(K_{2k-1}^{N+1}) = \mathcal{F}(K_{2k}^{N+1}) = K_k^N$ .

The inductive procedure is represented in Figure 1.

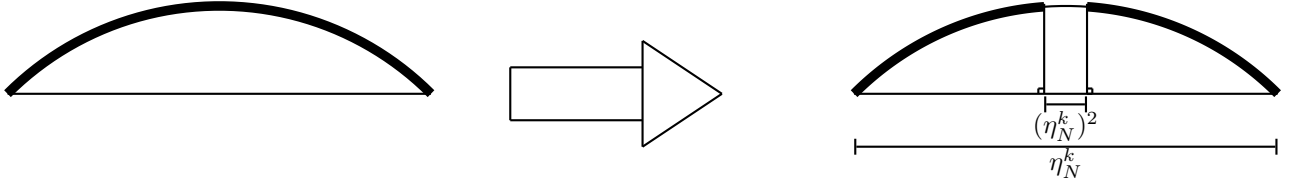


FIGURE 1. Induction step

In Figure 2&3 the two first iterations of the process are represented.

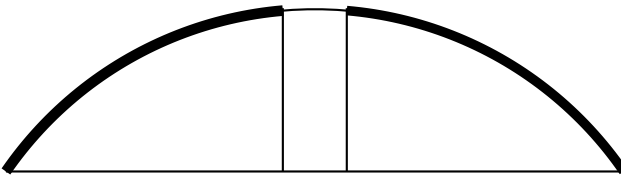


FIGURE 2. First iteration of the process

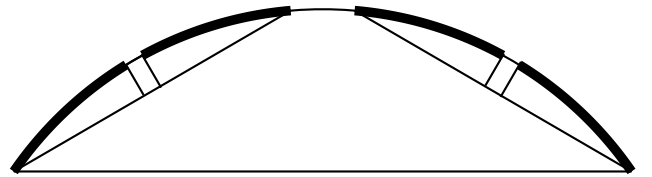


FIGURE 3. Second iteration of the process

We now define the Cantor type set

$$(13) \quad \mathcal{K} = \bigcap_{N \geq 0} \overline{\mathcal{K}_N}.$$

The Cantor type set  $\mathcal{K}$  is fat:

**Proposition 8.** *We have  $\mathcal{H}^1(\mathcal{K}) > 0$ .*

This proposition is proved in Appendix C.3.

## 5. CONSTRUCTION OF A SEQUENCE OF FUNCTIONS

A key argument in the proof of Theorem 1 is the use of the coarea formula to calculate a lower bound for (2). The coarea formula is applied to a function adapted to the set  $\mathcal{K}$ .

For  $N = 0$  we let

- $D_0^+$  be the compact set delimited by  $K_0 = \widehat{AB}$  and  $\mathcal{C}_1^0 := [AB]$  the chord of  $K_0$ .
- We recall that we fixed a frame  $\mathcal{R}_0 = (A, \mathbf{e}_1, \mathbf{e}_2)$  where  $\mathbf{e}_1 = \overrightarrow{AB}/|\overrightarrow{AB}|$ . For  $\sigma = (\sigma_1, 0) \in \mathcal{C}_1^0$ , we define:

$$(14) \quad I_\sigma \text{ is the connected component of } \{(\sigma_1, t) \in \Omega; t \leq 0\} \text{ which contains } \sigma.$$

$[I_\sigma]$  is a vertical segment included in  $\Omega$ .

- $D_0^- = \cup_{\sigma \in \mathcal{C}_1^0} I_\sigma$ .
- We now define the maps

$$\tilde{\Psi}_0 : \begin{array}{ccc} D_0^- & \rightarrow & \mathcal{C}_1^0 \\ x & \mapsto & \Pi_{\mathcal{C}_1^0}(x) \end{array}$$

and

$$\Psi_0 : \begin{array}{ccc} D_0^- \cup D_0^+ & \rightarrow & \mathcal{C}_1^0 \\ x & \mapsto & \begin{cases} \Pi_{\partial\Omega}(x) & \text{if } x \in D_0^+ \\ \Pi_{\partial\Omega}[\tilde{\Psi}_0(x)] & \text{if } x \in D_0^- \end{cases} \end{array}$$

where  $\Pi_{\partial\Omega}$  is the orthogonal projection on  $\partial\Omega$  and  $\Pi_{\mathcal{C}_1^0}$  is the orthogonal projection on  $\mathcal{C}_1^0$ . Note that, in the frame  $\mathcal{R}_0$ , for  $x = (x_1, x_2) \in D_0^-$ , we have  $\Pi_{\mathcal{C}_1^0}(x) = (x_1, 0)$ .

For  $N = 1$  and  $k \in \{1, 2\}$  we let:

- $D_k^1$  be the compact set delimited by  $K_k^1$  and  $\mathcal{C}_k^1$ ;
- $T_k^1$  be the compact right-angled triangle (with its interior) having  $\mathcal{C}_k^1$  as side adjacent to the right angle and whose hypotenuse is included in  $\mathcal{C}_1^0$ ;
- $H_k^1$  be the hypotenuse of  $T_k^1$ .

We now define  $D_1^- = \tilde{\Psi}_0^{-1}(H_1^1 \cup H_2^1)$ ,  $T_1 = T_1^1 \cup T_2^1$  and  $D_1^+ = D_1^1 \cup D_2^1$ .

We first consider the map

$$\tilde{\Psi}_1 : \begin{array}{ccc} T_1 \cup D_1^- & \rightarrow & \mathcal{C}_1^1 \cup \mathcal{C}_2^1 \\ x & \mapsto & \begin{cases} \Pi_{\mathcal{C}_k^1}(x) & \text{if } x \in T_k^1 \\ \Pi_{\mathcal{C}_k^1}[\tilde{\Psi}_0(x)] & \text{if } x \in D_1^- \end{cases} \end{array}$$

In Appendix D [Lemma 22 and Remark 23], it is proved that the triangles  $T_1^1$  and  $T_2^1$  are disjoint. Thus the map  $\tilde{\Psi}_1$  is well defined

By projecting  $\mathcal{C}_1^1 \cup \mathcal{C}_2^1$  on  $\partial\Omega$  we get

$$\Psi_1 : \begin{array}{ccc} T_1 \cup D_1^- \cup D_1^+ & \rightarrow & \mathcal{K}_1 \\ x & \mapsto & \begin{cases} \Pi_{\partial\Omega}(x) & \text{if } x \in D_1^+ \\ \Pi_{\partial\Omega}[\tilde{\Psi}_1(x)] & \text{if } x \in T_1 \cup D_1^- \end{cases} \end{array}$$

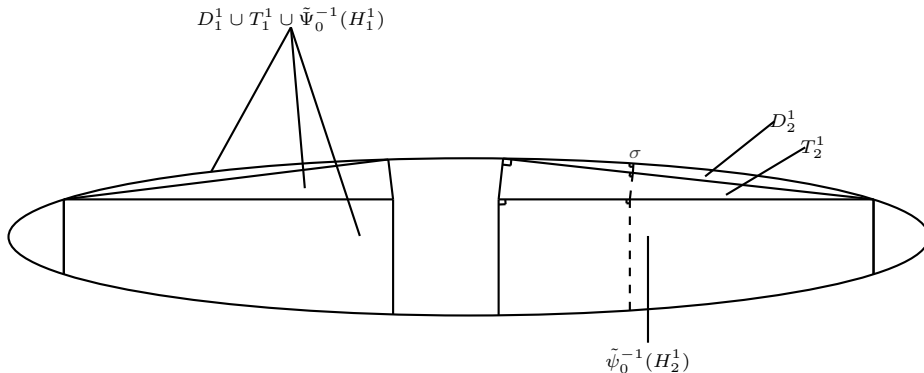


FIGURE 4. The sets defined at Step  $N = 1$  and the dashed level line of  $\Psi_1$  associated to  $\sigma \in \mathcal{K}_1$



For  $N \geq 1$ , we first construct  $\tilde{\Psi}_{N+1}$  and then  $\Psi_{N+1}$  is obtained from  $\tilde{\Psi}_{N+1}$  and  $\Pi_{\partial\Omega}$ .

For  $k \in \{1, \dots, 2^{N+1}\}$ , we let

- $D_k^{N+1}$  be the compact set delimited by  $K_k^{N+1}$  and  $\mathcal{C}_k^{N+1}$  [recall that  $\mathcal{C}_k^{N+1}$  is the chord associated to  $K_k^{N+1}$ ];
- $T_k^{N+1}$  be the right-angled triangle (with its interior) having  $\mathcal{C}_k^{N+1}$  as side adjacent to the right angle and whose hypotenuse is included in  $\mathcal{F}(\mathcal{C}_k^{N+1})$ . Here  $\mathcal{F}(\mathcal{C}_k^{N+1})$  is the father of  $\mathcal{C}_k^{N+1}$  (see Notation 7);
- $H_k^{N+1} \subset \mathcal{F}(\mathcal{C}_k^{N+1})$  be the hypotenuse of  $T_k^{N+1}$ .

We denote  $T_{N+1} = \bigcup_{k=1}^{2^{N+1}} T_k^{N+1}$ ,  $D_{N+1}^- = \tilde{\Psi}_N^{-1} \left( \bigcup_{k=1}^{2^{N+1}} H_k^{N+1} \right)$  and  $D_{N+1}^+ = \bigcup_{k=1}^{2^{N+1}} D_k^{N+1}$ .

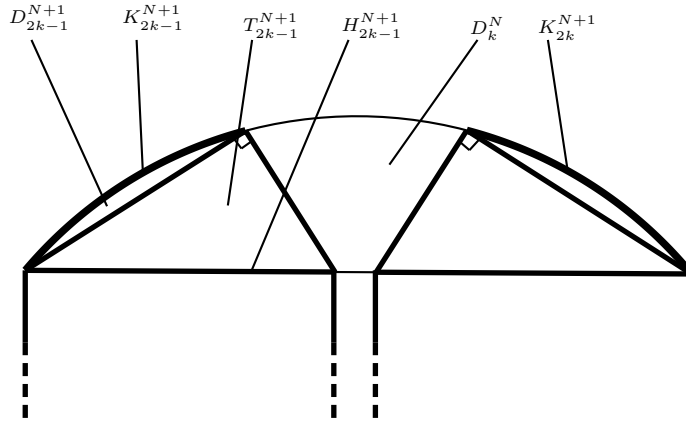


FIGURE 5. Induction. The bold lines correspond to the new iteration

*Remark 9.* It is easy to check that for  $N \geq 0$ :

- (1)  $T_{N+1} \subset D_N^+$ ,
- (2) if  $x \in T_N$  then  $x \notin T_{N'}$  for  $N' \geq N+1$  [here  $T_0 = \emptyset$ ].

We now define

$$\begin{aligned} \tilde{\Psi}_{N+1} : T_{N+1} \cup D_{N+1}^- &\rightarrow \bigcup_{k=1}^{2^{N+1}} \mathcal{C}_k^{N+1} \\ x &\mapsto \begin{cases} \Pi_{\mathcal{C}_k^{N+1}}(x) & \text{if } x \in T_k^{N+1} \\ \Pi_{\mathcal{C}_k^{N+1}}[\tilde{\Psi}_N(x)] & \text{if } x \in \tilde{\Psi}_N^{-1}(\bigcup_{k=1}^{2^{N+1}} H_k^{N+1}) \end{cases} . \end{aligned}$$

In Appendix D [Lemma 22 and Remark 23], it is proved that for  $N \geq 1$ , the triangles  $T_k^N$  for  $k = 1, \dots, 2^N$  are mutually disjoint. recursively, we find that all the  $\tilde{\Psi}_N$ 's are well-defined.

And, as in the Initialization Step, we get  $\Psi_{N+1}$  from  $\tilde{\Psi}_{N+1}$  by projecting  $\bigcup_{k=1}^{2^{N+1}} \mathcal{C}_k^{N+1}$  on  $\partial\Omega$ :

$$\begin{aligned} \Psi_{N+1} : T_{N+1} \cup D_{N+1}^- \cup D_{N+1}^+ &\rightarrow \mathcal{K}_{N+1} \\ x &\mapsto \begin{cases} \Pi_{\partial\Omega}[\tilde{\Psi}_{N+1}(x)] & \text{if } x \in T_{N+1} \cup D_{N+1}^- \\ \Pi_{\partial\Omega}(x) & \text{if } x \in D_{N+1}^+ \end{cases} . \end{aligned}$$

It is easy to see that  $\Psi_{N+1}(T_{N+1} \cup D_{N+1}^- \cup D_{N+1}^+) = \mathcal{K}_{N+1}$ .

## 6. BASIC PROPERTIES OF $B_\infty$ AND $\Psi_N$

**6.1. Basic properties of  $B_\infty$ .** We set  $B_N = T_N \cup D_N^+ \cup D_N^-$ . It is easy to check that for  $N \geq 0$  we have  $B_{N+1} \subset B_N$  and  $\mathcal{K} \subset \partial B_N$ . Therefore we may define

$$B_\infty = \bigcap_{N \geq 0} \overline{B_N}$$

which is compact and satisfies  $\mathcal{K} \subset \partial B_\infty$ .

We are going to prove:

**Lemma 10.** *The interior of  $B_\infty$  is empty.*

*Proof of Lemma 10.* From Lemma 22 [and Remark 23] in Appendix D combined with Hypothesis (11), we get two fundamental facts:

- (1) The triangles  $T_1^N, \dots, T_{2^{N+1}}^N$  are mutually disjoint.
- (2) We have:

$$(15) \quad \mathcal{H}^1(H_k^{N+1}) < \frac{\mathcal{H}^1(\mathcal{F}(\mathcal{C}_k^N))}{2}.$$

For a non empty set  $A \subset \mathbb{R}^2$  we let

$$\text{rad}(A) = \sup\{r \geq 0; \exists x \in A \text{ s.t. } \overline{B(x, r)} \subset A\}.$$

Note that the topological interior of  $A$  is empty if and only if  $\text{rad}(A) = 0$ .

On the one hand, it is not difficult to check that for sufficiently large  $N$

$$(16) \quad \text{rad}(B_N) = \text{rad}(B_N \cap D_N^-).$$

On the other hand, using (15) we obtain for  $N \geq 1$ :

$$(17) \quad \text{rad}(B_{N+1} \cap D_{N+1}^-) \leq \frac{\text{rad}(B_N \cap D_N^-)}{2}.$$

Consequently, by combining (16) and (17) we get the existence of  $C_0$  s.t.

$$(18) \quad \text{rad}(B_N) \leq \frac{C_0}{2^N}.$$

Since  $B_\infty = \cap_{N \geq 0} B_N$ , from (18) we get that  $\text{rad}(B_\infty) = 0$ . □

**6.2. Basic properties of  $\Psi_N$ .** We now prove the key estimate for  $\Psi_N$ :

**Lemma 11.** *There exists  $b_N = o_N(1)$  s.t. for  $N \geq 1$  and  $U$  a connected component of  $B_N$ , the restriction of  $\Psi_N$  to  $U$  is  $(1 + b_N)$ -Lipschitz.*

*Proof.* Let  $N \geq 1$  and  $U$  be a connected component of  $B_N$ . The restriction of  $\tilde{\Psi}_N$  to  $U \cap (T_N \cup D_N^-)$  is obtained as composition of orthogonal projections on straight lines and thus is 1-Lipschitz.

There exists  $b_N = o_N(1)$  s.t. the projection  $P_N := \Pi_{\partial\Omega}$  defined in  $\overline{D_N^+}$  is  $(1 + b_N)$ -Lipschitz. The functions  $\Psi_N$  are either the composition of  $\tilde{\Psi}_N$  with  $P_N$  or  $\Psi_N = P_N$ . Consequently the restriction of  $\Psi_N$  to  $U$  is  $(1 + b_N)$ -Lipschitz. □

In the following we will not use  $\Psi_N$  but "its projection" on  $\mathbb{R}$ . For  $N \geq 1$  and  $k \in \{1, \dots, 2^N\}$ , we let  $B_k^N := \Psi_N^{-1}(K_k^N)$  and we define

$$\begin{aligned} \Pi_{k,N} : B_k^N &\rightarrow \mathbb{R} \\ x &\mapsto \mathcal{H}^1(\widehat{A\Psi_N(x)}) \end{aligned}$$

where  $\widehat{A\Psi_N(x)} \subset \widehat{AB}$  is defined by (9) as the smallest connected component of  $\partial\Omega \setminus \{A, \Psi_N(x)\}$  if  $\Psi_N(x) \neq A$  and  $\widehat{A\Psi_N(x)} = \{A\}$  otherwise.

**Lemma 12.** *For  $N \geq 1$  there exists  $c_N \in (0, 1)$  with  $c_N = o_N(1)$  s.t. for  $k \in \{1, \dots, 2^N\}$  the function  $\Pi_{k,N} : B_k^N \rightarrow \mathbb{R}$  is  $(1 + c_N)$ -Lipschitz.*

*Proof.* Let  $N \geq 1$ ,  $k \in \{1, \dots, 2^N\}$  and let  $x, y \in B_k^N$  be s.t.  $\Psi_N(x) \neq \Psi_N(y)$ . It is clear that we have

$$|\Pi_{k,N}(x) - \Pi_{k,N}(y)| = \mathcal{H}^1(\widehat{\Psi_N(y)\Psi_N(x)})$$

where  $\widehat{\Psi_N(y)\Psi_N(x)} \subset K_k^N$  is defined by (9) as the smallest connected component of  $\partial\Omega \setminus \{\Psi_N(y), \Psi_N(x)\}$ .

Moreover, from Lemma 20 in Appendix C.2, we have the existence of  $C \geq 1$  independent of  $N$  and  $k$  s.t. for  $x, y \in B_k^N$  s.t.  $\Psi_N(x) \neq \Psi_N(y)$  we have [denoting  $X := \Psi_N(x)$ ,  $Y := \Psi_N(y)$ ]

$$\text{dist}(X, Y) \leq \mathcal{H}^1(\widehat{XY}) \leq \text{dist}(X, Y) [1 + C \text{dist}(X, Y)]$$

and

$$\mathcal{H}^1(K_k^N) \leq \mathcal{H}^1(\mathcal{C}_k^N) [1 + C\mathcal{H}^1(\mathcal{C}_k^N)].$$

From Step 1 in the proof of Proposition 13 [Appendix C.3] we have

$$\max_{k=1,\dots,2^N} \mathcal{H}^1(\mathcal{C}_k^N) \leq \left(\frac{2}{3}\right)^N.$$

Thus letting  $a_N := \left(\frac{2}{3}\right)^N \left[1 + C\left(\frac{2}{3}\right)^N\right]$  we have  $a_N \rightarrow 0$  and since  $\widehat{XY} \subset K_k^N$  we get:

$$\text{dist}(X, Y) \leq \mathcal{H}^1(\widehat{XY}) \leq \mathcal{H}^1(K_k^N) \leq \mathcal{H}^1(\mathcal{C}_k^N) [1 + C\mathcal{H}^1(\mathcal{C}_k^N)] \leq a_N(1 + Ca_N).$$

Thus, letting  $\tilde{a}_N = \max\{a_N(1 + Ca_N), |b_N|\}$  where  $b_N$  is defined in Lemma 11, we get

$$\begin{aligned} \mathcal{H}^1(\widehat{XY}) &= |\Pi_{k,N}(x) - \Pi_{k,N}(y)| \leq \mathcal{H}^1([\Psi_N(y)\Psi_N(x)]) (1 + C\tilde{a}_N) \\ &\leq (1 + \tilde{a}_N) (1 + C\tilde{a}_N) |x - y|. \end{aligned}$$

Therefore, letting  $c_N$  be s.t.  $1 + c_N = (1 + \tilde{a}_N)(1 + C\tilde{a}_N)$  we have  $c_N = o_N(1)$ ,  $c_N$  is independent of  $k \in \{1, \dots, 2^N\}$  and  $\Pi_{k,N}$  is  $(1 + c_N)$ -Lipschitz.  $\square$

## 7. PROOF OF THEOREM 1

We are now in position to prove Theorem 1. This is done by contradiction. We assume that there exists a map  $u_0 \in BV(\Omega)$  which minimizes (2).

**7.1. Upper bound.** The first step in the proof is the estimate

$$(19) \quad \int_{\Omega} |Du_0| \leq \|\mathbb{I}_{\mathcal{K}}\|_{L^1(\partial\Omega)} = \mathcal{H}^1(\mathcal{K}).$$

This estimate is obtained by proving that for all  $\varepsilon > 0$  there exists  $u_{\varepsilon} \in W^{1,1}(\Omega)$  s.t.  $\text{tr}_{\partial\Omega} u_{\varepsilon} = \mathbb{I}_{\mathcal{K}}$  and

$$(20) \quad \|\nabla u_{\varepsilon}\|_{L^1(\Omega)} \leq (1 + \varepsilon) \|\text{tr}_{\partial\Omega} u_{\varepsilon}\|_{L^1(\Omega)} = (1 + \varepsilon) \mathcal{H}^1(\mathcal{K}).$$

Proposition 24 in Appendix E gives the existence of such  $u_{\varepsilon}$ 's.

Clearly (20) implies (19).

**7.2. Optimality of the upper bound.** In order to have a contradiction we follow the strategy of Spradlin and Tamasan in [ST14]. We fix a sequence  $(u_n)_n \subset C^1(\Omega)$  s.t.

$$(21) \quad u_n \in W^{1,1}(\Omega); u_n \rightarrow u \text{ in } L^1(\Omega); \int_{\Omega} |\nabla u_n| \rightarrow \int_{\Omega} |Du_0|; \text{tr}_{\partial\Omega} u_n = \text{tr}_{\partial\Omega} u_0.$$

Note that (21) implies

$$(22) \quad \int_F |\nabla u_n| \rightarrow \int_F |Du| \text{ for all } F \subset \Omega \text{ relatively closed set.}$$

Such a sequence can be obtained *via* partition of unity and smoothing; see the proof of Theorem 1.17 in [Giu84]. For the convenience of the reader a proof is presented in Appendix A [see Lemma 17].

For further use, let us note that the sequence  $(u_n)_n$  constructed in Appendix A satisfies the following additional property:

$$\left| \begin{array}{l} \text{If } u_0 = 0 \text{ outside a compact set } L \subset \overline{\Omega} \text{ and if } \omega \text{ is an open set} \\ \text{s.t. } \text{dist}(\omega, L) > 0 \text{ then, for large } n, u_n = 0 \text{ in } \omega \end{array} \right|.$$

For  $x \in B_0$  we let

$$(23) \quad V_0(x) = \begin{cases} \nu_{\Pi_{\partial\Omega}(x)} & \text{if } x \in D_0^+ \\ (0, 1) & \text{if } x \in D_0^- \end{cases},$$

and for  $N \geq 0$ ,  $x \in B_{N+1}$  we let

$$(24) \quad V_{N+1}(x) = \begin{cases} V_N(x) & \text{if } x \in B_N \setminus \overset{\circ}{T}^{N+1} \\ \nu_{\mathcal{C}_k^{N+1}} & \text{if } x \in \overset{\circ}{T}_k^{N+1} \end{cases},$$

where, for  $\sigma \in \partial\Omega$ ,  $\nu_{\sigma}$  is the normal outward of  $\Omega$  in  $\sigma$  and  $\nu_{\mathcal{C}_k^{N+1}}$  is defined in Remark 6.1.

We now prove the following lemma.

**Lemma 13.** When  $N \rightarrow \infty$  we may define  $V_\infty(x)$  a.e.  $x \in B_\infty$  by

$$(25) \quad \begin{aligned} V_\infty : B_\infty &\rightarrow \mathbb{R}^2 \\ x &\mapsto \lim_{N \rightarrow \infty} V_N(x) \end{aligned} .$$

Moreover, from dominated convergence, we have:

$$V_N \mathbb{I}_{B_N} \rightarrow V_\infty \mathbb{I}_{B_\infty} \text{ in } L^1(\Omega).$$

*Proof.* If  $x \in B_\infty \setminus \cup_{N \geq 1} T_N$ , then we have  $V_N(x) = V_0(x)$  for all  $N \geq 1$ . Thus  $\lim_{N \rightarrow \infty} V_N(x) = V_0(x)$ .

For a.e.  $x \in B_\infty \cap \cup_{N \geq 1} T_N$  there exists  $N_0 \geq 1$  s.t.  $x \in \overset{\circ}{T}_{N_0}$ . Therefore for all  $N > N_0$  we have  $V_N(x) = V_{N_0}(x)$ . Consequently  $\lim_{N \rightarrow \infty} V_N(x) = V_{N_0}(x)$ .  $\square$

This section is devoted to the proof of the following lemma:

**Lemma 14.** For all  $w \in C^\infty \cap W^{1,1}(\Omega)$  s.t.  $\text{tr}_{\partial\Omega} w = \mathbb{I}_K$  we have

$$\int_{B_\infty \cap \Omega} |\nabla w \cdot V_\infty| \geq \mathcal{H}^1(K)$$

where  $V_\infty$  is the vector field defined in (25).

*Remark 15.* Since  $|V_\infty(x)| = 1$  for a.e.  $x \in B_\infty$ , it is clear that Lemma 14 implies that for all  $n$  we have

$$\int_{B_\infty \cap \Omega} |\nabla u_n| \geq \mathcal{H}^1(K).$$

From (22) we have:

$$\int_{B_\infty \cap \Omega} |Du_0| \geq \mathcal{H}^1(K).$$

Section 7.3 is devoted to a sharper argument than above to get

$$\int_{B_\infty \cap \Omega} |\nabla u_n| \geq \int_{B_\infty \cap \Omega} |\nabla u_n \cdot V_\infty| + \delta$$

with  $\delta > 0$  is independent of  $n$ . The last estimate will imply  $\int_{B_\infty \cap \Omega} |Du_0| \geq \mathcal{H}^1(K) + \delta$  which will be the contradiction we are looking for.

*Proof of Lemma 14.* We will first prove that for  $w \in C^\infty \cap W^{1,1}(\Omega)$  s.t.  $\text{tr}_{\partial\Omega} w = \mathbb{I}_K$  we have

$$(26) \quad \int_{B_N \cap \Omega} |\nabla w \cdot V_N| \geq \frac{\mathcal{H}^1(K)}{1 + o_N(1)}.$$

where  $V_N$  is the vector field defined in (23) and (24).

Granted (26), we conclude as follows: if  $w \in C^\infty \cap W^{1,1}(\Omega)$  s.t.  $\text{tr}_{\partial\Omega} w = \mathbb{I}_K$ , then

$$\begin{aligned} \int_{B_\infty \cap \Omega} |\nabla w \cdot V_\infty| &= \lim_{N \rightarrow \infty} \int_{B_N \cap \Omega} |\nabla w \cdot V_N| \\ &\geq \lim_{N \rightarrow \infty} \frac{\mathcal{H}^1(K)}{1 + o_N(1)} = \mathcal{H}^1(K), \end{aligned}$$

by dominated convergence.

It remains to prove (26). We fix  $w \in C^\infty \cap W^{1,1}(\Omega)$  s.t.  $\text{tr}_{\partial\Omega} w = \mathbb{I}_K$ . Using the Coarea Formula we have for  $N \geq 1$  and  $k \in \{1, \dots, 2^N\}$ , with the help of Lemma 12, we have

$$\begin{aligned} (1 + c_N) \int_{B_N^{(k)} \cap \Omega} |\nabla w \cdot V_N| &\geq \int_{B_N^{(k)} \cap \Omega} |\nabla \Pi_{k,N}| |\nabla w \cdot V_N| \\ &= \int_{\mathbb{R}} dt \int_{\Pi_{k,N}^{-1}(\{t\}) \cap \Omega} |\nabla w \cdot V_N|. \end{aligned}$$

Here, if  $\Pi_{k,N}^{-1}(\{t\})$  is non trivial, then  $\Pi_{k,N}^{-1}(\{t\})$  is a polygonal line:

$$\Pi_{k,N}^{-1}(\{t\}) = I_{\sigma(t,k,N)} \cup I_{k,N,t}^1 \cup \dots \cup I_{k,N,t}^{N+1}$$

where

- $\sigma(t,k,N) \in [AB]$  is s.t.  $[AB] \cap \Pi_{k,N}^{-1}(\{t\}) = \{\sigma(t,k,N)\}$ ,
- $I_{\sigma(t,k,N)}$  is defined in (14),

- for  $l = 1, \dots, N$  we have  $I_{k,N,t}^l = \Pi_{k,N}^{-1}(\{t\}) \cap T_{N+1-l}$ ,
- $I_{k,N,t}^{N+1} = \Pi_{k,N}^{-1}(\{t\}) \cap D_N^+$ .

From the Fundamental Theorem of calculus and from the definition of  $V_N$ , denoting

- $I_{\sigma(t,k,N)} = [M_0, M_1]$  [where  $M_0 \in \partial\Omega \setminus \widehat{AB}$  and  $M_1 = \sigma(t, k, N)$ ],
- $I_{k,N,t}^l = [M_l, M_{l+1}]$ ,  $l = 1, \dots, N+1$  and  $M_{N+2} \in K_k^N$ ,

we have for a.e.  $t \in \Pi_{k,N}(K_k^N)$  and using the previous notation,

$$\int_{[M_l, M_{l+1}]} |\nabla w \cdot V_N| \geq |w(M_{l+1}) - w(M_l)|.$$

Here we used the convention  $w(M_l) = \text{tr}_{\partial\Omega} w(M_l)$  for  $l = 0 \& N+2$ .

Therefore for a.e.  $t \in \Pi_{k,N}(K_k^N)$  we have

$$\int_{\Pi_{k,N}^{-1}(\{t\}) \cap \Omega} |\nabla w \cdot V_N| \geq |\text{tr}_{\partial\Omega} w(M_{N+2}) - \text{tr}_{\partial\Omega} w(M_0)| = \mathbb{1}_{\mathcal{K}}(M_{N+2}).$$

Since  $\mathcal{K} \subset \mathcal{K}_N = \cup_{k=1}^{2^N} K_k^N$ , we may thus deduce that

$$(1 + c_N) \int_{B_N \cap \Omega} |\nabla w \cdot V_N| = (1 + c_N) \sum_{k=1}^{2^N} \int_{B_N^{(k)} \cap \Omega} |\nabla w \cdot V_N| \geq \int_{\widehat{AB}} \mathbb{1}_{\mathcal{K}} = \mathcal{H}^1(\mathcal{K}).$$

The last estimate clearly implies (26) and completes the proof of Lemma 14.  $\square$

**7.3. Transverse argument.** We assumed that there exists a map  $u_0$  which solves Problem (2).

We investigate the following dichotomy:

- $u_0 \not\equiv 0$  in  $\Omega \setminus B_\infty$ ;
- $u_0 \equiv 0$  in  $\Omega \setminus B_\infty$ .

We are going to prove that both cases lead to a contradiction.

7.3.1. *The case  $u_0 \not\equiv 0$  in  $\Omega \setminus B_\infty$ .* We thus have  $\int_{\Omega \setminus B_\infty} |u_0| > 0$ . In this case, since  $(\text{tr}_{\partial\Omega} u_0)|_{\partial\Omega \setminus \partial B_\infty} \equiv 0$ , we have

$$(27) \quad \delta := \int_{\Omega \setminus B_\infty} |Du_0| > 0.$$

Estimate (27) is a direct consequence of the following lemma applied on each connected components of  $\Omega \setminus B_\infty$ .

**Lemma 16.** [Weak Poincaré lemma] *Let  $\omega \subset \mathbb{R}^2$  be an open connected set. Assume that there exist  $x_0 \in \partial\omega$  and  $r > 0$  s.t.  $\omega \cap B(x_0, r)$  is Lipschitz.*

*If  $u \in BV(\omega)$  satisfies  $\text{tr}_{\partial\omega \cap B(x_0, r)} u = 0$  and  $\int_\omega |Du| = 0$  then  $u = 0$ .*

Lemma 16 is proved in Appendix B.4.

Recall that we fixed a sequence  $(u_n)_n \subset C^1 \cap W^{1,1}(\Omega)$  satisfying (21).

In particular, for sufficiently large  $n$ , we have

$$\int_{\Omega \setminus B_\infty} |\nabla u_n| > \frac{\delta}{2}.$$

Thus, from Lemma 14 and the fact that  $|V_\infty(x)| = 1$  for a.e.  $x \in B_\infty$ ,

$$\int_\Omega |\nabla u_n| \geq \int_{B_\infty} |\nabla u_n \cdot V_\infty| + \int_{\Omega \setminus B_\infty} |\nabla u_n| \geq \mathcal{H}^1(\mathcal{K}) + \frac{\delta}{2}.$$

This implies

$$\int_\Omega |Du_0| = \lim_n \int_\Omega |\nabla u_n| \geq \mathcal{H}^1(\mathcal{K}) + \frac{\delta}{2}$$

which is in contradiction with (19).

7.3.2. *The case  $u_0 \equiv 0$  in  $\Omega \setminus B_\infty$ .* We first note that, since  $\text{tr}_{\partial D_0^+} u_0 \not\equiv 0$ , there exists a triangle  $T_k^{N_0}$  s.t.  $\int_{T_k^{N_0}} |u_0| > 0$ . We fix such a triangle  $T_k^{N_0}$  and we let  $\alpha$  be the vertex corresponding to the right angle.

We let  $\tilde{\mathcal{R}} = (\alpha, \tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2)$  be the direct orthonormal frame centered in  $\alpha$  where  $\tilde{\mathbf{e}}_2 = \nu_{\mathcal{C}_k^{N_0}} [\nu_{\mathcal{C}_k^{N_0}}]$  is defined Remark 6.1], i.e., the directions of the new frame are given by the side of the right-angle of  $T_k^{N_0}$ .

It is clear that for  $N \geq N_0$  we have  $V_N \equiv \tilde{\mathbf{e}}_2$  in  $T_k^{N_0}$ .

By construction of  $B_\infty$ ,  $T_k^{N_0} \cap B_\infty$  is a union of segments parallel to  $\tilde{\mathbf{e}}_2$ , i.e.  $\mathbb{I}_{B_\infty|T_k^{N_0}}(s, t)$  depends only on the first variable "s" in the frame  $\tilde{\mathcal{R}}$ .

Since  $\int_{T_k^{N_0}} |u_0| > 0$ , in the frame  $\tilde{\mathcal{R}}$ , we may find  $a, b, c, d \in \mathbb{R}$  s.t., considering the rectangle (whose sides are parallel to the direction of  $\tilde{\mathcal{R}}$ )

$$\mathcal{P} := \{\alpha + s\tilde{\mathbf{e}}_1 + t\tilde{\mathbf{e}}_2; (s, t) \in [a, b] \times [c, d]\} \subset T_k^{N_0}$$

we have

$$\int_{\mathcal{P}} |u_0| > 0.$$

Since from Lemma 10 the set  $B_\infty$  has an empty interior [and that  $\mathbb{I}_{B_\infty|T_k^{N_0}}(s, t)$  depends only on the first variable in the frame  $\tilde{\mathcal{R}}$ ], we may find  $a' < b'$  s.t.

- $[a', b'] \times [c, d] \subset [a, b] \times [c, d]$ ,
- $\mathcal{S} \cap B_\infty = \emptyset$  with  $\mathcal{S} := \{\alpha + s\tilde{\mathbf{e}}_1 + t\tilde{\mathbf{e}}_2; (s, t) \in \{a', b'\} \times [c, d]\}$
- $\delta := \int_{\mathcal{P}'} |u_0| > 0$  with  $\mathcal{P}' := \{\alpha + s\tilde{\mathbf{e}}_1 + t\tilde{\mathbf{e}}_2; (s, t) \in [a', b'] \times [c, d]\}$ .

Moreover, since  $\mathcal{S}$  and  $B_\infty$  are compact sets with empty intersection, we may find  $\mathcal{V}$ , an open neighborhood of  $\mathcal{S}$  s.t.  $\text{dist}(\mathcal{V}, B_\infty) > 0$ .

Noting that  $u_0 \equiv 0$  in  $\Omega \setminus B_\infty$ , from Lemma 17 [in Appendix A] it follows that for sufficiently large  $n$  we have

- $u_n \equiv 0$  in  $\mathcal{S}$ ,
- $\int_{\mathcal{P}'} |u_n| > \frac{\delta}{2}$ .

Consequently, from a standard Poincaré inequality

$$\int_{\mathcal{P}'} |\partial_{\tilde{\mathbf{e}}_1} u_n| \geq \frac{2}{b' - a'} \int_{\mathcal{P}'} |u_n| > \frac{\delta}{b' - a'} =: \delta'.$$

Therefore  $\int_{\mathcal{P}'} |\partial_{\tilde{\mathbf{e}}_1} u_n| > \delta'$ ,  $\int_{\mathcal{P}'} |\partial_{\tilde{\mathbf{e}}_2} u_n| \leq 2\mathcal{H}^1(\mathcal{K})$  and then by Lemma 3.3 in [ST14] we obtain:

$$\int_{\mathcal{P}'} |\nabla u_n| \geq \int_{\mathcal{P}'} |\partial_{\tilde{\mathbf{e}}_2} u_n| + \frac{\delta'^2}{4\mathcal{H}^1(\mathcal{K}) + \delta'}.$$

Thus, from Lemma 14, for sufficiently large  $n$ :

$$\int_{\Omega} |\nabla u_n| \geq \mathcal{H}^1(\mathcal{K}) + \frac{\delta'^2}{4\mathcal{H}^1(\mathcal{K}) + \delta'} - o_n(1).$$

From the convergence in  $BV$ -norm of  $u_n$  to  $u_0$  we have

$$\int_{\Omega} |Du_0| \geq \mathcal{H}^1(\mathcal{K}) + \frac{\delta'^2}{4\mathcal{H}^1(\mathcal{K}) + \delta'}.$$

Clearly this last assertion contradicts (19) and ends the proof of Theorem 1.

## APPENDICES

### APPENDIX A. A SMOOTHING RESULT

We first state a standard approximation lemma for  $BV$ -functions.

**Lemma 17.** *Let  $\Omega \subset \mathbb{R}^2$  be a bounded Lipschitz open set and let  $u \in BV(\Omega)$ . There exists a sequence  $(u_n)_n \subset C^1(\Omega)$  s.t.*

- (1)  $u_n \xrightarrow{\text{strictly}} u$  in the sense that  $u_n \rightarrow u$  in  $L^1(\Omega)$  and  $\int_{\Omega} |\nabla u_n| \rightarrow \int_{\Omega} |Du|$ ,  
(2)  $\text{tr}_{\partial\Omega} u_n = \text{tr}_{\partial\Omega} u$  for all  $n$ ,  
(3) for  $k \in \{1, 2\}$ ,

$$\int_{\Omega} |\partial_k u_n| \rightarrow \int_{\Omega} |D_k u| := \sup \left\{ \int_{\Omega} u \partial_k \xi; \xi \in C_c^1(\Omega, \mathbb{R}) \text{ and } |\xi| \leq 1 \right\}$$

- (4) If  $u = 0$  outside a compact set  $L \subset \overline{\Omega}$  and if  $\omega$  is an open set s.t.  $\text{dist}(\omega, L) > 0$  then, for large  $n$ ,  $u_n = 0$  in  $\omega$ .

*Proof.* The first assertion is quite standard. It is for example proved in [AG78] [Theorem 1]. We present below the classical example of sequence for such approximation result [we follow the presentation of [Giu84], Theorem 1.17].

Let  $\Omega \subset \mathbb{R}^2$  be a bounded Lipschitz open set and let  $u \in BV(\Omega)$ .

For  $n \geq 1$ , we let  $\varepsilon = 1/n$ . We may fix  $m \in \mathbb{N}^*$  sufficiently large s.t. letting for  $k \in \mathbb{N}$

$$\Omega_k = \left\{ x \in \Omega; \text{dist}(x, \partial\Omega) > \frac{1}{m+k} \right\}$$

we have

$$\int_{\Omega \setminus \Omega_0} |Du| < \varepsilon.$$

We fix now  $A_1 := \Omega_2$  and for  $i \in \mathbb{N} \setminus \{0, 1\}$  we let  $A_i = \Omega_{i+1} \setminus \overline{\Omega_{i-1}}$ . It is clear that  $(A_i)_{i \geq 1}$  is a covering of  $\Omega$  and that each point in  $\Omega$  belongs to at most three of the sets  $(A_i)_{i \geq 1}$ .

We let  $(\varphi_i)_{i \geq 1}$  be a partition of unity subordinate to the covering  $(A_i)_{i \geq 1}$ , i.e.,  $\varphi_i \in C_c^\infty(A_i)$ ,  $0 \leq \varphi_i \leq 1$  and  $\sum_{i \geq 1} \varphi_i = 1$  in  $\Omega$ .

We let  $\eta \in C_c^\infty(\mathbb{R}^2)$  be s.t.  $\text{supp}(\eta) \subset B(0, 1)$ ,  $\eta \geq 0$ ,  $\int \eta = 1$  and for  $x \in \mathbb{R}^2$   $\eta(x) = \eta(|x|)$ . For  $t > 0$  we let  $\eta_t = t^{-2} \eta(\cdot/t)$ .

As explained in [Giu84], for  $i \geq 1$ , we may choose  $\varepsilon_i \in (0, \varepsilon)$  sufficiently small s.t.

$$\begin{cases} \text{supp}(\eta_{\varepsilon_i} * (u\varphi_i)) \subset A_i \\ \int_{\Omega} |\eta_{\varepsilon_i} * (u\varphi_i) - u\varphi_i| < \frac{\varepsilon}{2^i} \\ \int_{\Omega} |\eta_{\varepsilon_i} * (u\nabla\varphi_i) - u\nabla\varphi_i| < \frac{\varepsilon}{2^i} \end{cases}.$$

Here  $*$  is the convolution operator.

Define

$$u_n := \sum_{i \geq 1} \eta_{\varepsilon_i} * (u\varphi_i).$$

In some neighborhood of each point  $x \in \Omega$  there are only finitely many nonzero terms in the sum defining  $u_n$ . Thus  $u_n$  is well defined and smooth in  $\Omega$ .

Moreover, we may easily check that

$$\|u_n - u\|_{L^1(\Omega)} + \left| \int_{\Omega} |Du| - \int_{\Omega} |\nabla u_n| \right| < \varepsilon \text{ [here } \varepsilon = 1/n].$$

Thus the previous estimate proves that  $(u_n)$  satisfies the first assertion, i.e.,  $u_n \xrightarrow{\text{strictly}} u$ .

As claimed in [Giu84] [Remark 2.12] we have  $\text{tr}_{\partial\Omega} u_n = \text{tr}_{\partial\Omega} u$  for all  $n$ . Thus the second assertion is satisfied.

We now prove the third assertion. Since  $u_n \rightarrow u$  in  $L^1(\Omega)$ , by inferior semi continuity we easily get for  $k \in \{1, 2\}$

$$\int_{\Omega} |D_k u| \leq \liminf_{n \rightarrow \infty} \int_{\Omega} |\partial_k u_n|.$$

We now prove  $\int_{\Omega} |D_k u| \geq \limsup_{n \rightarrow \infty} \int_{\Omega} |\partial_k u_n|$ .

Let  $\xi \in C_c^1(\Omega, \mathbb{R})$  with  $|\xi| \leq 1$ . Since  $\eta$  is a symmetric mollifier and  $\sum \varphi_i = 1$  we have

$$\begin{aligned}
\int_{\Omega} u_n \partial_k \xi &= \sum_{i \geq 1} \int_{\Omega} \eta_{\varepsilon_i} * (u \varphi_i) \partial_k \xi \\
&= \sum_{i \geq 1} \int_{\Omega} u \varphi_i \partial_k (\eta_{\varepsilon_i} * \xi) \\
&= \sum_{i \geq 1} \int_{\Omega} u \partial_k [\varphi_i (\eta_{\varepsilon_i} * \xi)] - \sum_{i \geq 1} \int_{\Omega} u \partial_k \varphi_i (\eta_{\varepsilon_i} * \xi) \\
&= \sum_{i \geq 1} \int_{\Omega} u \partial_k [\varphi_i (\eta_{\varepsilon_i} * \xi)] - \sum_{i \geq 1} \int_{\Omega} \xi [\eta_{\varepsilon_i} * (u \partial_k \varphi_i) - u \partial_k \varphi_i].
\end{aligned}$$

On the one hand we have [note that  $\varphi_i(\eta_{\varepsilon_i} * \xi) \in C_c^1(A_i)$  and  $|\varphi_i(\eta_{\varepsilon_i} * \xi)| \leq 1$ ]

$$\begin{aligned}
\left| \sum_{i \geq 1} \int_{\Omega} u \partial_k [\varphi_i (\eta_{\varepsilon_i} * \xi)] \right| &= \left| \int_{A_1} u \partial_k [\varphi_i (\eta_{\varepsilon_i} * \xi)] + \sum_{i \geq 2} \int_{A_i} u \partial_k [\varphi_i (\eta_{\varepsilon_i} * \xi)] \right| \\
&\leq \int_{\Omega} |D_k u| + \sum_{i \geq 2} \int_{A_i} |D_k u| \\
&\leq \int_{\Omega} |D_k u| + 3 \int_{\Omega \setminus \Omega_0} |D_k u| \\
&\leq \int_{\Omega} |D_k u| + 3\varepsilon.
\end{aligned}$$

Here we used that each point in  $\Omega$  belongs to at most three of the sets  $(A_i)_{i \geq 1}$ , for  $i \geq 2$  we have  $A_i \subset \Omega \setminus \Omega_0$  and

$$\int_{\Omega \setminus \Omega_0} |D_k u| \leq \int_{\Omega \setminus \Omega_0} |Du| < \varepsilon.$$

On the other hand, since for  $i \geq 1$   $\int_{\Omega} |\eta_{\varepsilon_i} * (u \nabla \varphi_i) - u \nabla \varphi_i| < \frac{\varepsilon}{2^i}$ , we get

$$\left| \sum_{i \geq 1} \int_{\Omega} \xi [\eta_{\varepsilon_i} * (u \partial_k \varphi_i) - u \partial_k \varphi_i] \right| \leq \sum_{i \geq 1} \int_{\Omega} |\eta_{\varepsilon_i} * (u \partial_k \varphi_i) - u \partial_k \varphi_i| < \varepsilon.$$

Consequently

$$\sup \left\{ \int_{\Omega} u_n \partial_k \xi ; \xi \in C_c^1(\Omega, \mathbb{R}) \text{ and } |\xi| \leq 1 \right\} = \int_{\Omega} |\partial_k u_n| \leq \int_{\Omega} |D_k u| + 4\varepsilon$$

and thus  $\limsup_n \int_{\Omega} |\partial_k u_n| \leq \int_{\Omega} |D_k u|$ . This inequality in conjunction with  $\liminf_n \int_{\Omega} |\partial_k u_n| \geq \int_{\Omega} |D_k u|$  proves the third assertion of Lemma 17.

The last assertion of Lemma 17 is a direct consequence of the definition of the  $u_n$ 's.  $\square$

## APPENDIX B. PROOFS OF LEMMA 2, LEMMA 3, LEMMA 4 AND LEMMA 16

**B.1. Proof of Lemma 2.** Let  $u \in BV(\mathcal{Q})$ . We prove that

$$\int_{\mathcal{Q}} |D_2 u| \geq \int_0^1 |\text{tr}_{\partial \mathcal{Q}} u(\cdot, 0) - \text{tr}_{\partial \mathcal{Q}} u(\cdot, 1)|.$$

From Lemma 17, there exists  $(u_n)_n \subset C^1(\mathcal{Q})$  s.t.  $\text{tr}_{\partial \mathcal{Q}} u_n = \text{tr}_{\partial \mathcal{Q}} u$ ,  $u_n \xrightarrow{\text{strictly}} u$  and

$$\int_{\mathcal{Q}} |\partial_2 u_n| \rightarrow \int_{\mathcal{Q}} |D_2 u|.$$



From Fubini's theorem and the Fundamental theorem of calculus we have

$$\begin{aligned}
\int_{\mathcal{Q}} |\partial_2 u_n| &= \int_0^1 dx_1 \int_0^1 |\partial_2 u_n(x_1, x_2)| dx_2 \\
&\geq \int_0^1 dx_1 \left| \int_0^1 \partial_2 u_n(x_1, x_2) dx_2 \right| \\
&= \int_0^1 dx_1 |\operatorname{tr}_{\partial \mathcal{Q}} u_n(x_1, 1) - \operatorname{tr}_{\partial \mathcal{Q}} u_n(x_1, 0)| \\
&= \int_0^1 |\operatorname{tr}_{\partial \mathcal{Q}} u(\cdot, 1) - \operatorname{tr}_{\partial \mathcal{Q}} u(\cdot, 0)|.
\end{aligned}$$

Since  $\int_{\mathcal{Q}} |\partial_2 u_n| \rightarrow \int_{\mathcal{Q}} |D_2 u|$ , Lemma 2 is proved.

**B.2. Proof of Lemma 3.** Let  $\Omega$  be a planar open set. Let  $u \in BV(\Omega)$  be s.t.

$$\int_{\Omega} |Du| = \int_{\Omega} |D_2 u|.$$

We prove that  $\int_{\Omega} |D_1 u| = 0$ . We argue by contradiction and we assume that  $\int_{\Omega} |D_1 u| > 0$ , i.e., there exists  $\xi \in C_c^1(\Omega)$  s.t.  $|\xi| \leq 1$  and

$$\eta := \int_{\Omega} u \partial_1 \xi > 0.$$

Let  $(\xi_n)_n \subset C_c^1(\Omega)$  be s.t.  $|\xi_n| \leq 1$  and

$$\eta_n := \int_{\Omega} u \partial_2 \xi_n \rightarrow \int_{\Omega} |D_2 u|.$$

For  $(\alpha, \beta) \in \{x \in \mathbb{R}^2; |x| \leq 1\}$  we let  $\xi_{\alpha, \beta}^{(n)} = (\alpha \xi, \beta \xi_n) \in C_c^1(\Omega, \mathbb{R}^2)$ . Clearly,  $|\xi_{\alpha, \beta}^{(n)}| \leq 1$  and

$$(28) \quad \int_{\Omega} |Du| \geq \int_{\Omega} u \operatorname{div}(\xi_{\alpha, \beta}^{(n)}) = \alpha \eta + \beta \eta_n.$$

If we maximize the right hand side of (28) w.r.t.  $(\alpha, \beta) \in \{x \in \mathbb{R}^2; |x| \leq 1\}$ , then we find with  $(\alpha, \beta) =$

$$\left( \frac{\eta}{\sqrt{\eta^2 + \eta_n^2}}, \frac{\eta_n}{\sqrt{\eta^2 + \eta_n^2}} \right)$$

$$\int_{\Omega} |Du| \geq \sqrt{\eta^2 + \eta_n^2} \xrightarrow{n \rightarrow \infty} \sqrt{\eta^2 + \left( \int_{\Omega} |Du| \right)^2} > \int_{\Omega} |Du|.$$

This is a contradiction.

**B.3. Proof of Lemma 4.** Let  $u \in BV(\mathcal{Q})$  satisfying  $\operatorname{tr}_{\partial \mathcal{Q}} u = 0$  in  $\{0\} \times [0, 1]$ . We are going to prove that

$$\int_{\mathcal{Q}} |u| \leq \int_{\mathcal{Q}} |D_1 u|.$$

Let  $(u_n)_n \subset C^1(\Omega)$  be given by Lemma 17. Using the Fundamental theorem of calculus we have for  $(x_1, x_2) \in \mathcal{Q}$

$$|u_n(x_1, x_2)| \leq \int_0^{x_1} |\partial_1 u_n(t, x_2)| dt \leq \int_0^1 |\partial_1 u_n(t, x_2)| dt.$$

Therefore, from Fubini's theorem, we get

$$\int_{\mathcal{Q}} |u_n| \leq \int_{\mathcal{Q}} dx_1 dx_2 \int_0^1 |\partial_1 u_n(t, x_2)| dt = \int_0^1 dx_2 \int_0^1 |\partial_1 u_n(t, x_2)| dt = \int_{\mathcal{Q}} |\partial_1 u_n|.$$

It suffices to see that  $\int_{\mathcal{Q}} |u_n| \rightarrow \int_{\mathcal{Q}} |u|$  and  $\int_{\mathcal{Q}} |\partial_1 u_n| \rightarrow \int_{\mathcal{Q}} |D_1 u|$  to get the result.

**B.4. Proof of Lemma 16.** Let  $\omega \subset \mathbb{R}^2$  be an open connected set. Assume there exist  $x_0 \in \partial \omega$  and  $r > 0$  s.t.  $\omega \cap B(x_0, r)$  is Lipschitz.

Let  $u \in BV(\omega)$  satisfying  $\operatorname{tr}_{\partial \omega \cap B(x_0, r)} u = 0$  and  $\int_{\omega} |Du| = 0$ . We are going to prove that  $u = 0$ . On the one hand, since  $\int_{\omega} |Du| = 0$ , we get  $u = C$  with  $C \in \mathbb{R}$  a constant. We thus have  $\operatorname{tr}_{\partial \omega \cap B(x_0, r)} u = C$ . Consequently  $C = 0$  and  $u \equiv 0$ .

APPENDIX C. RESULTS RELATED TO THE CANTOR SET  $\mathcal{K}$ 

**C.1. Justification of Remark 6.(1).** We prove the following lemma:

**Lemma 18.** *Let  $\eta > 0$  and let  $f \in C^2([0, \eta], \mathbb{R})$  be s.t.  $\eta < \frac{1}{2\|f'\|_{L^\infty([0, \eta])}\|f''\|_{L^\infty([0, \eta])}}$ . We denote  $C_f$  the graph of  $f$  in an orthonormal frame  $\mathcal{R}_0$ .*

*For  $0 \leq a < b \leq \eta$ , denoting  $\mathcal{C}$  the chord  $[(a, f(a)), (b, f(b))]$ , for any straight line  $D$  orthogonal to  $\mathcal{C}$  s.t.  $D \cap \mathcal{C} \neq \emptyset$ , the straight line  $D$  intersect  $C_{f,a,b}$  at exactly one points where  $C_{f,a,b}$  is the part of  $C_f$  delimited by  $(a, f(a))$  and  $(b, f(b))$ .*

*Remark 19.* We may state an analog result with  $f \in C^1$  where we use the modulus of continuity of  $f'$  instead of  $\|f''\|_\infty$  in the hypothesis.

*Proof.* The key point here is uniqueness. Indeed, for  $0 \leq a < b \leq \eta$  and  $\mathcal{C}, D$  as in the lemma, we may easily prove that  $C_{f,a,b} \cap D \neq \emptyset$  by solving an equation. [We do not use  $\eta < (2\|f'\|_{L^\infty([0, \eta])}\|f''\|_{L^\infty([0, \eta])})^{-1}$  for the existence]

In contrast with the existence of an intersection point, its uniqueness is valid only for  $\eta$  not too large. To prove uniqueness we argue by contradiction and we consider  $f$  and  $\eta$  as in lemma and we assume that there exist two points  $0 \leq a < b \leq \eta$  s.t. there exist  $a \leq x < y \leq b$  s.t. the segments  $[(x, f(x)), (y, f(y))]$  and  $[(a, f(a)), (b, f(b))]$  are orthogonal. Note that with this hypothesis the straight line  $D := ((x, f(x)), (y, f(y)))$  is orthogonal to the chord  $\mathcal{C} := [(a, f(a)), (b, f(b))]$ .

So we get

$$\frac{f(y) - f(x)}{y - x} = -\frac{b - a}{f(b) - f(a)}.$$

From the Mean Value Theorem, there exist  $c \in (x, y)$  and  $\tilde{c} \in (a, b)$  s.t.  $f'(c) = -\frac{1}{f'(\tilde{c})}$ . Consequently

$$(29) \quad f'(c) \times [f'(\tilde{c}) - f'(c)] = -1 - [f'(c)]^2.$$

From the hypothesis  $\eta < (2\|f'\|_{L^\infty([0, \eta])}\|f''\|_{L^\infty([0, \eta])})^{-1}$ , we have

$$|f'(\tilde{c}) - f'(c)| \leq \eta\|f''\|_{L^\infty([0, \eta])} < \frac{1}{2\|f'\|_{L^\infty([0, \eta])}}.$$

Therefore, we get

$$|f'(c) \times [f'(\tilde{c}) - f'(c)]| < \frac{1}{2}$$

which is in contradiction with (29).  $\square$

**C.2. Two preliminary results.** We first prove a standard result which states that the length of a *small* chord is a *good* approximation for the length of a curve.

**Lemma 20.** *Let  $0 < \eta < 1$  and let  $f \in C^2([0, \eta], \mathbb{R}^+)$ . We fix an orthonormal frame and we denote  $C_f$  the graph of  $f$  in the orthonormal frame. Let  $A = (a, f(a)), B = (b, f(b)) \in C_f$  (with  $0 \leq a < b \leq \eta$ ) and let  $\mathcal{C} = [AB]$  be the chord of  $C_f$  joining  $A$  and  $B$ . We denote  $\widehat{AB}$  the arc of  $C_f$  with endpoints  $A$  and  $B$ .*

*We have*

$$\mathcal{H}^1(\mathcal{C}) \leq \mathcal{H}^1(\widehat{AB}) \leq \mathcal{H}^1(\mathcal{C}) \{1 + (b - a)\|f''\|_{L^\infty} [2\|f'\|_{L^\infty} + \|f''\|_{L^\infty}(b - a)]\}.$$

*Proof.* The estimate  $\mathcal{H}^1(\mathcal{C}) \leq \mathcal{H}^1(\widehat{AB})$  is standard, we thus prove the second inequality.

On the one hand

$$\mathcal{H}^1(\mathcal{C}) = \sqrt{(a - b)^2 + [f(a) - f(b)]^2} = (b - a) \sqrt{1 + \left(\frac{f(a) - f(b)}{a - b}\right)^2}.$$

On the other hand

$$\mathcal{H}^1(\widehat{AB}) = \int_a^b \sqrt{1 + f'^2}.$$

With the help of the Mean Value Theorem, there exists  $c \in (a, b)$  s.t.

$$\frac{f(a) - f(b)}{a - b} = f'(c).$$

Applying once again the Mean Value Theorem [to  $f'$ ], for  $x \in [a, b]$  there exists  $c_x$  between  $c$  and  $x$  s.t.

$$f'(x) = f'(c) + f''(c_x)(x - c).$$

Consequently for  $x \in [a, b]$  we have:

$$\begin{aligned} \sqrt{1 + f'(x)^2} &= \sqrt{1 + [f'(c) + f''(c_x)(x - c)]^2} \\ &= \sqrt{1 + f'(c)^2} \sqrt{1 + \frac{2f'(c)f''(c_x)(x - c) + f''(c_x)^2(x - c)^2}{1 + f'(c)^2}} \\ &\leq \sqrt{1 + \left(\frac{f(a) - f(b)}{a - b}\right)^2} [1 + 2\|f'\|_{L^\infty}\|f''\|_{L^\infty}(b - a) + \|f''\|_{L^\infty}^2(b - a)^2]. \end{aligned}$$

Thus we have

$$\begin{aligned} \mathcal{H}^1(\widehat{AB}) &= \int_a^b \sqrt{1 + f'(x)^2} dx \\ &\leq (b - a) \sqrt{1 + \left(\frac{f(a) - f(b)}{a - b}\right)^2} [1 + 2\|f'\|_{L^\infty}\|f''\|_{L^\infty}(b - a) + \|f''\|_{L^\infty}^2(b - a)^2] \\ &= \mathcal{H}^1(\mathcal{C}) \{1 + (b - a)\|f''\|_{L^\infty}[2\|f'\|_{L^\infty} + \|f''\|_{L^\infty}(b - a)]\}. \end{aligned}$$

□

We now state another technical lemma which gives an upper bound for the height of the curve w.r.t. its chord.

**Lemma 21.** *Let  $0 \leq a < b \leq \eta$ ,  $f \in C^2([0, \eta], \mathbb{R}^+)$  be a strictly concave function and let  $C_f$  be the graph of  $f$  in an orthonormal frame. Let  $A = (a, f(a))$  and  $B = (b, f(b))$  be two points of  $C_f$ .*

*Assume that we have  $\eta < \frac{1}{2\|f'\|_{L^\infty([0, \eta])}\|f''\|_{L^\infty([0, \eta])}}$  in order to define for  $C \in [AB]$  [with the help of Lemma 18]  $\tilde{C}$  as the unique intersection point of  $C_f$  with the line orthogonal to  $[AB]$  passing by  $C$ .*

*We have*

$$\mathcal{H}^1([C\tilde{C}]) \leq \frac{(b - a)^2\|f''\|_{L^\infty}}{8}.$$

*Proof.* Let  $0 \leq a < b \leq \eta$ ,  $f \in C^2([0, \eta], \mathbb{R}^+)$  be as in Lemma 21.

We consider the function

$$\begin{aligned} g : [0, \eta] &\rightarrow \mathbb{R} \\ x &\mapsto f(x) - \left[ \frac{f(b) - f(a)}{b - a}(x - a) + f(a) \right]. \end{aligned}$$

It is clear that  $g$  is non negative since  $f$  is strictly concave.

For  $C \in [AB]$ , we let  $\tilde{C}$  be as in Lemma 21. Then we have

$$\sup_{C \in [AB]} \mathcal{H}^1([C\tilde{C}]) = \max_{[0, \eta]} g.$$

Thus, it suffices to prove  $\max_{[0, \eta]} g \leq \frac{(b - a)^2\|f''\|_{L^\infty}}{8}$ .

Since  $g$  is  $C^1$  and  $g(a) = g(b) = 0$ , there exists  $c \in (a, b)$  s.t.

$$g(c) = \max_{[0, \eta]} g \text{ and } g'(c) = 0.$$

Let  $t \in \{a, b\}$  be s.t.  $|t - c| \leq \frac{b - a}{2}$ . Using a Taylor expansion, there exists  $\tilde{c}$  between  $c$  and  $t$  s.t.

$$0 = g(t) = g(c) + (t - c)g'(c) + \frac{(t - c)^2}{2}g''(\tilde{c}).$$

Thus

$$0 \leq \max_{[0, \eta]} g = g(c) = -\frac{(t - c)^2}{2}g''(\tilde{c}) \leq \frac{(b - a)^2\|f''\|_{L^\infty}}{8}.$$

The last inequality completes the proof.

□

C.3. **Proof of Proposition 13.** We prove that

$$(30) \quad \liminf_{N \rightarrow \infty} \mathcal{H}^1(\mathcal{K}_N) > 0.$$

**Step 1.** We prove that  $\max_{k=1, \dots, 2^N} \mathcal{H}^1(\mathcal{C}_k^N) \leq \left(\frac{2}{3}\right)^N$

For  $N \geq 1$  we let  $\{K_k^N; k = 1, \dots, 2^N\}$  be the set of the connected components of  $\mathcal{K}_N$ . We let  $\mathcal{C}_k^N$  be the chord of  $K_k^N$  and we define  $\mu_N = \max_{k=1, \dots, 2^N} \mathcal{H}^1(\mathcal{C}_k^N)$ . Note that by (11) we have  $\mu_0 < 1$ .

We first prove that for  $N \geq 0$  we have

$$(31) \quad \mu_{N+1} \leq \frac{2}{3} \mu_N.$$

By induction (31) implies [since to  $\mu_0 < 1$ ]

$$(32) \quad \mu_N \leq \left(\frac{2}{3}\right)^N.$$

In order to get (31), we prove that for  $N \geq 1$  and  $K_k^N$  a connected component of  $\mathcal{K}_N$  and  $\mathcal{C}_k^N$  its chord, we have

$$(33) \quad \mathcal{H}^1(\mathcal{C}) \leq \frac{2\mathcal{H}^1(\mathcal{C}_k^N)}{3} \text{ for } \mathcal{C} \in \mathcal{S}(\mathcal{C}_k^N)$$

[see Notation 7 for  $\mathcal{S}(\cdot)$ , the set of sons of a chord].

Let  $N \geq 1$ . For  $k \in \{1, \dots, 2^N\}$ , we let  $K_k^N$  be a connected component of  $\mathcal{K}_N$ . We let  $K_{2k-1}^{N+1}, K_{2k}^{N+1} \in \mathcal{S}(K_k^N)$  be the curve obtained from  $K_k^N$  in the induction step.

For  $\tilde{k} \in \{2k-1, 2k\}$ , we let  $\mathcal{C}_{\tilde{k}}^{N+1}$  be the chords of  $K_k^{N+1}$ .

In the frame  $\mathcal{R}_0$ , we may define four points of  $\Gamma$ ,  $(a_1, f(a_1)), (b_1, f(b_1)), (a_2, f(a_2)), (b_2, f(b_2))$ , with  $0 < a_1 < b_1 < a_2 < b_2 < \eta$  s.t.:

- the endpoints of  $K_{2k-1}^{N+1}$  are  $(a_1, f(a_1)) \& (b_1, f(b_1))$ ;
- the endpoints of  $K_{2k}^{N+1}$  are  $(a_2, f(a_2)) \& (b_2, f(b_2))$ ;
- the endpoints of  $K_k^N$  are  $(a_1, f(a_1)) \& (b_2, f(b_2))$ .

In the frame  $\mathcal{R}_0$  we let also  $(\alpha_1, \beta_1), (\alpha_2, \beta_2)$  be the coordinates of the points of  $\mathcal{C}_k^N$  s.t. for  $l \in \{1, 2\}$ , the triangles whose vertices are  $\{(a_l, f(a_l)); (b_l, f(b_l)); (\alpha_l, \beta_l)\}$  are right angled in  $(\alpha_l, \beta_l)$ .

We denote

- $\mathcal{I}_1$  the segment  $[(b_1, f(b_1)); (\alpha_1, \beta_1)]$ ;
- $\mathcal{I}_2$  the segment  $[(a_2, f(a_2)); (\alpha_2, \beta_2)]$ .

From the construction of  $K_{2k-1}^{N+1} \& K_{2k}^{N+1}$  and from Pythagorean theorem we have for  $l = 1, 2$

$$\mathcal{H}^1(\mathcal{C}_{2k-2+l}^{N+1})^2 = \mathcal{H}^1(\mathcal{I}_l)^2 + \left( \frac{\mathcal{H}^1(\mathcal{C}_k^N) - \mathcal{H}^1(\mathcal{C}_k^N)^2}{2} \right)^2.$$

Using Lemma 21 we get that

$$\mathcal{H}^1(\mathcal{I}_l) \leq (b_2 - a_1)^2 \|f''\|_{L^\infty}.$$

On the other hand we have obviously  $b_2 - a_1 \leq \mathcal{H}^1(\mathcal{C}_k^N)$ . Consequently we get

$$\begin{aligned} \mathcal{H}^1(\mathcal{C}_{2k-2+l}^{N+1})^2 &\leq \mathcal{H}^1(\mathcal{C}_k^N)^4 \|f''\|_{L^\infty}^2 + \left( \frac{\mathcal{H}^1(\mathcal{C}_k^N) - \mathcal{H}^1(\mathcal{C}_k^N)^2}{2} \right)^2 \\ &\leq \mathcal{H}^1(\mathcal{C}_k^N)^4 \|f''\|_{L^\infty}^2 + \frac{\mathcal{H}^1(\mathcal{C}_k^N)^2}{4}. \end{aligned}$$

Therefore

$$\mathcal{H}^1(\mathcal{C}_{2k-2+l}^{N+1}) \leq \frac{\mathcal{H}^1(\mathcal{C}_k^N)}{2} \sqrt{1 + 4\|f''\|_{L^\infty}^2 \mathcal{H}^1(\mathcal{C}_k^N)^2},$$

thus using (12) we get

$$\mathcal{H}^1(\mathcal{C}_{2k-2+l}^{N+1}) \leq \frac{2\mathcal{H}^1(\mathcal{C}_k^N)}{3}.$$

The last estimate gives (33) and thus (32) holds.

**Step 2.** We prove that  $\liminf_{N \rightarrow \infty} \sum_{k=1}^{2^N} \mathcal{H}^1(\mathcal{C}_k^N) > 0$

For  $N \geq 1$ , we let

$$c_N = \sum_{k=1}^{2^N} \mathcal{H}^1(\mathcal{C}_k^N).$$

The main ingredient in this step consists in noting that, a son of  $\mathcal{C}_k^N$  is an hypotenuse of a right angled triangle which admits a cathetus of length  $\frac{\mathcal{H}^1(\mathcal{C}_k^N) - \mathcal{H}^1(\mathcal{C}_k^N)^2}{2}$ .

Consequently we have

$$\mathcal{H}^1(\mathcal{C}_{2k-1}^{N+1}) + \mathcal{H}^1(\mathcal{C}_{2k}^{N+1}) \geq \mathcal{H}^1(\mathcal{C}_k^N) - \mathcal{H}^1(\mathcal{C}_k^N)^2.$$

Thus, summing the previous inequality for  $k = 1, \dots, 2^N$  we get

$$c_{N+1} = \sum_{k=1}^{2^N} \mathcal{H}^1(\mathcal{C}_{2k-1}^{N+1}) + \mathcal{H}^1(\mathcal{C}_{2k}^{N+1}) \geq \sum_{k=1}^{2^N} \mathcal{H}^1(\mathcal{C}_k^N) [1 - \mathcal{H}^1(\mathcal{C}_k^N)] \geq c_N (1 - \mu_N) \geq c_N \left[ 1 - \left( \frac{2}{3} \right)^N \right].$$

By induction for  $N \geq 2$

$$c_N \geq c_1 \prod_{k=1}^{N-1} \left[ 1 - \left( \frac{2}{3} \right)^k \right] = c_1 \times \exp \left[ \sum_{k=1}^{N-1} \ln \left[ 1 - \left( \frac{2}{3} \right)^k \right] \right].$$

It is clear that  $\liminf_N \sum_{l=1}^{N-1} \ln \left[ 1 - \left( \frac{2}{3} \right)^k \right] > -\infty$ , thus  $\liminf_N c_N > 0$ .

**Step 3.** We prove (30).

Since for  $K_k^N$ , a connected component of  $\mathcal{K}_N$ , and  $\mathcal{C}_k^N$  its chord, we have  $\mathcal{H}^1(K_k^N) \geq \mathcal{H}^1(\mathcal{C}_k^N)$ , from Step 2 we get (30).

#### APPENDIX D. A FUNDAMENTAL INGREDIENT IN THE CONSTRUCTION OF THE $\tilde{\Psi}_N$ 'S

In this section we use the notation of Sections 4 and 5.

**Lemma 22.** *Let  $\gamma \subset \widehat{AB}$  be a curve and let  $\mathcal{C}$  be its chord. We let  $\gamma_1, \gamma_2$  be the curves included in  $\gamma$  obtained by the induction construction represented Figure 1 [section 4.2]. For  $l = 1, 2$ , we denote also by  $\mathcal{C}_l$  the chord of  $\gamma_l$  and by  $T_l$  the right-angled triangle having  $\mathcal{C}_l$  as side of the right-angle and having its hypotenuse included in  $\mathcal{C}$ .*

*If  $\mathcal{H}^1(\mathcal{C}) < \min\{2^{-1}, (4\|f''\|_{L^\infty}^2)^{-2}\}$ , then the hypotenuses of the triangles  $T_1$  and  $T_2$  have their length strictly lower than  $\frac{\mathcal{H}^1(\mathcal{C})}{2}$ . And in particular the triangles  $T_1$  and  $T_2$  are disjoint.*

*Remark 23.* From (11), we know that  $\mathcal{C}_0 = \mathcal{C}_1^0$  is s.t.  $\mathcal{H}^1(\mathcal{C}_1^0) < \min\{2^{-1}, (4\|f''\|_{L^\infty}^2)^{-2}\}$ . From (31) we have that for  $N \geq 1$  and  $k \in \{1, \dots, 2^N\}$  we have  $\mathcal{H}^1(\mathcal{C}_k^N) < \mathcal{H}^1(\mathcal{C}_1^0) < \min\{2^{-1}, (4\|f''\|_{L^\infty}^2)^{-2}\}$ .

Therefore with the help of Lemma 22, for  $N \geq 1$ , the triangles  $T_k^N$ 's are pairwise disjoint.

*Proof.* We model the statement by denoting  $\{M, Q\}$  the set of endpoints of  $\gamma$  and  $N$  and  $P$  are points s.t.:

- $M, N$  are the endpoints of  $\gamma_1$
- $P, Q$  are the endpoints of  $\gamma_2$ .

We denote  $\delta := \mathcal{H}^1([MQ]) = \mathcal{H}^1(\mathcal{C}) < \min\{2^{-1}, (4\|f''\|_{L^\infty}^2)^{-2}\}$ .

We fix an orthonormal frame  $\tilde{\mathcal{R}}$  with the origin in  $M$ , with the  $x$ -axis  $(MQ)$  and s.t.  $N, P, Q$  have respectively for coordinates  $(x_1, y_1)$ ,  $(x_2, y_2)$  and  $(x_3, 0)$  where  $0 < x_1 < x_2 < x_3$  and  $y_1, y_2 > 0$ .

By construction we have

$$x_1 = \frac{\delta - \delta^2}{2}, x_2 = \frac{\delta + \delta^2}{2} \text{ and } x_3 = \delta.$$

Moreover, arguing as in the proof of Lemma 21 we have [recall that  $\widehat{AB}$  is the graph of a function  $f$  in an other orthonormal frame]:

$$0 < y_1, y_2 \leq \delta^2 \|f''\|_{L^\infty}.$$

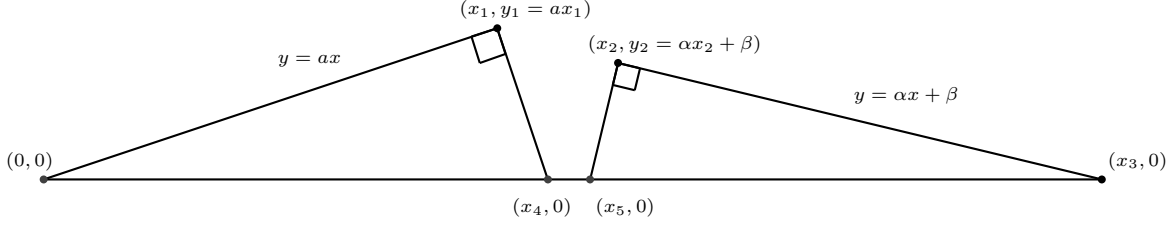


FIGURE 6. Model problem

From these points, in Section 4.2, we defined two right-angled triangles having their hypotenuses contained in the  $x$ -axis.

The first triangle admits for vertices the origin  $(0, 0)$ ,  $(x_1, y_1)$  and a point of the  $x$ -axis  $(x_4, 0)$ . This triangle is right angled in  $(x_1, y_1)$ . In the frame  $\tilde{\mathcal{R}}$ , one of the side of the right-angle is included in the line parametrized by the cartesian equation  $y = ax$ . Since  $\delta \leq 1/2$

$$|a| = \left| \frac{y_1}{x_1} \right| \leq \frac{2\delta^2 \|f''\|_{L^\infty}}{\delta - \delta^2} \leq 4\|f''\|_{L^\infty}\delta.$$

The second triangle admits for vertices  $(x_2, y_2)$ ,  $(x_3, 0)$  and a point of the  $x$ -axis  $(x_5, 0)$ . This triangle is right-angled in  $(x_2, y_2)$ . In the frame  $\tilde{\mathcal{R}}$ , one of the side of the right-angle is included in the line parametrized by the cartesian equation  $y = \alpha x + \beta$  where

$$|\alpha| = \left| \frac{y_2}{x_2 - x_3} \right| \leq \frac{2\delta^2 \|f''\|_{L^\infty}}{\delta - \delta^2} \leq 4\|f''\|_{L^\infty}\delta.$$

The proof of the proposition consists in obtaining

$$x_4 < \frac{x_3}{2} \text{ and } x_3 - x_5 < \frac{x_3}{2}.$$

We get the first estimate. With the help of Pythagorean theorem we have

$$x_1^2 + y_1^2 + (x_1 - x_4)^2 + y_1^2 = x_4^2.$$

By noting that  $y_1 = ax_1$  we have

$$x_4 = (1 + a^2)x_1.$$

Thus:

$$\begin{aligned} x_4 < \frac{x_3}{2} &\iff (1 + a^2) \frac{\delta - \delta^2}{2} < \frac{\delta}{2} \\ &\iff (1 + 16\|f''\|_{L^\infty}^2 \delta^2)(1 - \delta) < 1 \\ &\iff \delta - \delta^2 < \frac{1}{16\|f''\|_{L^\infty}^2} \\ &\iff \delta < \frac{1}{16\|f''\|_{L^\infty}^2}. \end{aligned}$$

Following the same strategy we get that if  $\delta < \frac{1}{16\|f''\|_{L^\infty}^2}$  then  $x_3 - x_5 < \frac{x_3}{2}$ . □

#### APPENDIX E. ADAPTATION OF A RESULT OF GIUSTI IN [Giu84]

In this appendix we present briefly the proof of Theorem 2.16 and Remark 2.17 in [Giu84]. The argument we present below follows the proof of Theorem 2.15 in [Giu84].

**Proposition 24.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded open set of class  $C^2$  and let  $h \in L^1(\partial\Omega)$ . For all  $\varepsilon > 0$  there exists  $u_\varepsilon \in W^{1,1}(\Omega)$  s.t.  $\text{tr}_{\partial\Omega} u_\varepsilon = h$  and*

$$\|u_\varepsilon\|_{W^{1,1}(\Omega)} := \|u_\varepsilon\|_{L^1(\Omega)} + \|\nabla u_\varepsilon\|_{L^1(\Omega)} \leq (1 + \varepsilon)\|h\|_{L^1(\Omega)}.$$

*Proof.* We sketch the proof of Proposition 24. Let  $h \in L^1(\partial\Omega)$  and let  $\varepsilon > 0$  be sufficiently small s.t.

$$(1 + \varepsilon^2)^2 + \varepsilon^2 + \varepsilon^4 < 1 + \frac{\varepsilon}{2} \text{ and } (1 + \varepsilon^2)\varepsilon^2 < \frac{\varepsilon}{2}.$$

**Step 1.** We may consider  $\eta > 0$  sufficiently small s.t. in  $\Omega_\eta := \{x \in \Omega; \text{dist}(x, \partial\Omega) < \eta\}$  we have:

(1) The function

$$\begin{aligned} d : \Omega_\eta &\rightarrow (0, \eta) \\ x &\mapsto \text{dist}(x, \partial\Omega) \end{aligned}$$

is of class  $C^1$  and satisfies  $|\nabla d| \geq 1/2$ ,

(2) The orthogonal projection on  $\partial\Omega$ ,  $\Pi_{\partial\Omega}$ , is Lipschitz.

We now fix a sequence  $(h_k)_k \subset C^\infty(\partial\Omega)$  s.t.  $h_k \xrightarrow{L^1} h$ . We may assume that (up to replace the first term and to consider an extraction):

- (1)  $h_0 \equiv 0$ ,
- (2)  $\sum_{k \geq 0} \|h_{k+1} - h_k\|_{L^1} \leq (1 + \varepsilon^2)\|h\|_{L^1}$ .

And finally we fix a decreasing sequence  $(t_k)_k \subset \mathbb{R}_+^*$  s.t.

- (1)  $t_0 < \min(\eta, \varepsilon^2)$  is sufficiently small s.t.
  - $4t_0 \max(1; \|\nabla \Pi_{\partial\Omega}\|_{L^\infty}) \times \max(1, \sup_k \|h_k\|_{L^1}) < \min(\varepsilon^2, \varepsilon^2\|h\|_{L^1})$ ,
  - for  $\varphi \in L^1(\partial\Omega)$  we have for  $s \in (0, t_0)$

$$\int_{d^{-1}(\{s\})} |\varphi \circ \Pi_{\partial\Omega}(x)| \leq (1 + \varepsilon^2) \int_{\partial\Omega} |\varphi(x)|.$$

- (2) For  $k \geq 1$  we have  $t_k \leq \frac{t_0 \|h\|_{L^1}}{2^k (1 + \|\nabla h_k\|_{L^\infty} + \|\nabla h_{k+1}\|_{L^\infty})}$ .

**Step 2.** We define

$$\begin{aligned} u_\varepsilon : \Omega &\rightarrow \mathbb{R} \\ x &\mapsto \begin{cases} \frac{d(x) - t_{k+1}}{t_k - t_{k+1}} h_k \circ \Pi_{\partial\Omega}(x) + \frac{t_k - d(x)}{t_k - t_{k+1}} h_{k+1} \circ \Pi_{\partial\Omega}(x) & \text{if } d(x) \in [t_{k+1}, t_k] \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

We may easily check that  $u_\varepsilon$  is locally Lipschitz and thus weakly differentiable.

From the coarea formula and a standard change of variable we have

$$\begin{aligned} \|u_\varepsilon\|_{L^1} &\leq 2 \int_{\{d \leq t_0\}} |u_\varepsilon| |\nabla d| \\ &\leq 2 \int_0^{t_0} ds \int_{d^{-1}(\{s\})} |u_\varepsilon| dx \\ &\leq 2 \sum_{k \geq 0} \int_{t_{k+1}}^{t_k} ds \int_{d^{-1}(\{s\})} |u_\varepsilon| dx \\ &\leq 2 \sum_{k \geq 0} \int_{t_{k+1}}^{t_k} ds \int_{d^{-1}(\{s\})} [|h_k \circ \Pi_{\partial\Omega}(x)| + |h_{k+1} \circ \Pi_{\partial\Omega}(x)|] dx \\ &\leq 2(1 + \varepsilon^2) \sum_{k \geq 0} \int_{t_{k+1}}^{t_k} ds \int_{\partial\Omega} [|h_k(x)| + |h_{k+1}(x)|] dx \\ &\leq 2(1 + \varepsilon^2) \sum_{k \geq 0} (t_k - t_{k+1}) (\|h_k\|_{L^1} + \|h_{k+1}\|_{L^1}) \\ &\leq 4(1 + \varepsilon^2) t_0 \sup_k \|h_k\|_{L^1} \\ &\leq (1 + \varepsilon^2) \varepsilon^2 \|h\|_{L^1} \\ &\leq \frac{\varepsilon}{2} \|h\|_{L^1}. \end{aligned}$$

We now estimate  $\|\nabla u_\varepsilon\|_{L^1}$ . It is easy to check that if  $d(x) \in (t_{k+1}, t_k)$  then we have

$$|\nabla u_\varepsilon(x)| \leq |\nabla d(x)| \left[ \frac{|h_k \circ \Pi_{\partial\Omega}(x) - h_{k+1} \circ \Pi_{\partial\Omega}(x)|}{t_k - t_{k+1}} + 2\|\nabla \Pi_{\partial\Omega}\|_{L^\infty} [|\nabla h_k| \circ \Pi_{\partial\Omega}(x) + |\nabla h_{k+1}| \circ \Pi_{\partial\Omega}(x)] \right].$$

Consequently we get

$$\begin{aligned} \|\nabla u_\varepsilon\|_{L^1} &\leq (1 + \varepsilon^2) \sum_{k \geq 0} \left\{ \int_{t_{k+1}}^{t_k} \frac{\|h_{k+1} - h_k\|_{L^1}}{t_k - t_{k+1}} + 2\|\nabla \Pi_{\partial\Omega}\|_{L^\infty} (t_k - t_{k+1}) (\|\nabla h_{k+1}\|_{L^1} + \|\nabla h_k\|_{L^1}) \right\} \\ &\leq (1 + \varepsilon^2) [(1 + \varepsilon^2)\|h\|_{L^1} + 2\|\nabla \Pi_{\partial\Omega}\|_{L^\infty} t_0 \|h\|_{L^1}] \\ &\leq (1 + \varepsilon^2) [(1 + \varepsilon^2) + \varepsilon^2] \|h\|_{L^1} \\ &\leq (1 + \varepsilon/2) \|h\|_{L^1}. \end{aligned}$$

Consequently  $u_\varepsilon \in W^{1,1}(\Omega)$  and  $\|u_\varepsilon\|_{W^{1,1}} \leq (1 + \varepsilon) \|h\|_{L^1}$ .

In order to end the proof it suffices to check that  $\text{tr}_{\partial\Omega}(u_\varepsilon) = h$ . The justification of this property follows the argument of Lemma 2.4 in [Giu84].  $\square$

**Acknowledgements.** The author would like to thank Petru Mironescu for fruitful discussions.

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