



HAL
open science

CHARACTERISTIC FUNCTIONS ON THE BOUNDARY OF A PLANAR DOMAIN NEED NOT BE TRACES OF LEAST GRADIENT FUNCTIONS

Mickael dos Santos

► **To cite this version:**

Mickael dos Santos. CHARACTERISTIC FUNCTIONS ON THE BOUNDARY OF A PLANAR DOMAIN NEED NOT BE TRACES OF LEAST GRADIENT FUNCTIONS. 2015. hal-01216155v1

HAL Id: hal-01216155

<https://hal.science/hal-01216155v1>

Preprint submitted on 16 Oct 2015 (v1), last revised 28 Feb 2017 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

CHARACTERISTIC FUNCTIONS ON THE BOUNDARY OF A PLANAR DOMAIN NEED NOT BE TRACES OF LEAST GRADIENT FUNCTIONS

MICKAËL DOSSANTOS

ABSTRACT. Given a smooth bounded planar domain, we construct a compact set on the boundary s.t. its characteristic function is not the trace of a least gradient function. This generalizes the construction of Spradlin and Tamasan [ST14] on the disc.

CONTENTS

1. Introduction	1
2. Strategy of the proof	2
2.1. The model problem	2
2.2. Comments	4
3. Notations, definitions	4
4. Construction of the Cantor set \mathcal{K}	5
4.1. First step: main hypotheses	5
4.2. Step 2: Iterative construction	5
5. Construction of a sequence of functions used in conjunction with Coarea Formula	7
6. Basic properties for B_∞ and Ψ_N	9
7. Proof of Theorem 1	10
7.1. Upper bound	10
7.2. Saturation of the upper bound	10
7.3. Transverse argument	12
Appendices	13
Appendix A. Results related with the Cantor set \mathcal{K}	13
A.1. Two preliminary results	13
A.2. Proof of Proposition 12	14
Appendix B. Results related with the set B_∞	16
Appendix C. Adaptation of a result of Giusti in [Giu84]	17
References	19

1. INTRODUCTION

We let Ω be a bounded C^2 domain of \mathbb{R}^2 . For a function $h \in L^1(\partial\Omega, \mathbb{R})$, the least gradient problem with boundary datum h is the problem:

$$(1) \quad \min \left\{ \int_{\Omega} |Dw| \mid w \in BV(\Omega) \text{ and } \text{tr}_{\partial\Omega} w = h \right\}.$$

In the above minimization problem, $BV(\Omega)$ is the space of functions of bounded variation. It is the space of functions $w \in L^1(\Omega)$ having a distributional gradient Dw which is a bounded Radon measure.

If the minimization problem (1) admits solutions, such minimal functions are called *functions of least gradient*.

Sternberg, Williams and Ziemmer proved in [SWZ92] that if $h : \partial\Omega \rightarrow \mathbb{R}$ is a continuous map and if $\partial\Omega$ satisfies some geometric properties then Problem (1) admits a unique solution.

On the other hand, Spradlin and Tamasan [ST14] proved that, for the disc $\Omega = \{x \in \mathbb{R}^2 \mid |x| < 1\}$ [the disc satisfies the geometric hypotheses of [SWZ92]], we may find a function $h_0 \in L^1(\partial\Omega)$ which is not continuous

2010 *Mathematics Subject Classification.* 26B30, 35J56 .

s.t. Problem (1) has no solution. The function h_0 is the characteristic function of a Cantor set $\mathcal{K} \subset \mathbb{S}^1 = \{x \in \mathbb{R}^2 \mid |x| = 1\}$

The goal of this article is to extend the main result of [ST14] for a C^2 bounded open set $\Omega \subset \mathbb{R}^2$.

We prove the following theorem:

Theorem 1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded C^2 open set. Then there is a measurable set $\mathcal{K} \subset \partial\Omega$ s.t. the minimization problem*

$$(2) \quad \min \left\{ \int_{\Omega} |Dw| \mid w \in BV(\Omega) \text{ and } \text{tr}_{\partial\Omega} w = \mathbb{1}_{\mathcal{K}} \right\}$$

has no solution.

The above theorem has been proved when $\Omega = \mathbb{D}$ is the unit disk in [ST14].

The calculations in [ST14] are specific to the case $\Omega = \mathbb{D}$. The proof of Theorem 1 relies on new arguments for the construction of the Cantor set \mathcal{K} and in the strategy of the proof.

2. STRATEGY OF THE PROOF

2.1. The model problem. We present the strategy developed to prove Theorem 1 by proving its adaptation for $\Omega =]0, 1[^2$. Obviously this model case does not satisfy the hypothesis of regularity on Ω but it allows to make simple calculations.

We thus prove the following result:

Theorem 2. *Let $\tilde{\mathcal{K}}$ be a fat Cantor set [e.g. a Smith-Volterra-Cantor set] supported in $\left[\frac{1}{4}, \frac{3}{4}\right]$, then letting $\mathcal{K} = \tilde{\mathcal{K}} \times \{0\}$, Problem (2) does not admit a solution.*

This section is devoted to the proof of Theorem 2. The proof is done arguing by contradiction. We assume that there exists u_0 a solution of Problem (2). We obtain a contradiction in 3 Steps.

Step 1. Upper bound

Using a result in [Giu84] [Theorem 2.16 & Remark 2.17] we have

$$(3) \quad \min \left\{ \int_{\Omega} |Dw| \mid w \in BV(\Omega) \text{ and } \text{tr}_{\partial\Omega} w = h \right\} = \int_{\Omega} |Du_0| \leq \|\mathbb{1}_{\mathcal{K}}\|_{L^1(\partial\Omega)} = |\mathcal{K}|.$$

Here $|\mathcal{K}|$ is the length of \mathcal{K} .

Estimate (3) follows from Theorem 2.16 & Remark 2.17 in [Giu84]. By combining Theorem 2.16 & Remark 2.17 in [Giu84] we may prove that for all $\varepsilon > 0$ there exists a map $u_{\varepsilon} \in BV(\Omega)$ s.t.

$$\int_{\Omega} |Du_{\varepsilon}| \leq (1 + \varepsilon) \|\mathbb{1}_{\mathcal{K}}\|_{L^1(\partial\Omega)} \text{ and } \text{tr}_{\partial\Omega} u_{\varepsilon} = \mathbb{1}_{\mathcal{K}}.$$

The proof of this argument is presented in [Giu84] in a special context ⁽¹⁾. The adaptation for Ω a C^2 set is presented Appendix C. An easy adaptation of the argument may be done for $\Omega =]0, 1[^2$.

Step 2. Saturation of (3) [see (6)]

We fix $(u_n)_n \subset C^{\infty}(\Omega) \cap BV(\Omega)$ s.t. $\text{tr}_{\partial\Omega} u_n = \mathbb{1}_{\mathcal{K}}$ and $u_n \rightarrow u_0$ in BV-norm, i.e.,

$$u_n \xrightarrow{L^1} u_0 \text{ and } \int_{\Omega} |\nabla u_n| \rightarrow \int_{\Omega} |Du_0|.$$

The sequence is defined by convolution with u_0 and a Friedrichs mollifiers with compact support [see [Giu84]-Theorem 1.17].

We let also $B_{\infty} := \{(x, y) \in]0, 1[^2 \mid x \in \tilde{\mathcal{K}} \text{ and } y > 0\}$. The saturation of (3) consists in obtaining:

$$(4) \quad \int_{B_{\infty}} |\partial_2 u_n| \geq |\mathcal{K}|.$$

¹This result was initially proved by Gagliardo in [Gag57]. A short proof of " $\text{tr}W^{1,1} = L^1$ " may be found in [Mir15]. The key-point is here the quantitative form of the upper bound on $\int_{\Omega} |Du_{\varepsilon}|$ given by Remark 2.17 in [Giu84].

As observed by Mironescu, the key tool in the proof of (4) is Fubini Theorem combined with the Fundamental Theorem of calculus.

$$\begin{aligned}
 (5) \quad \int_{B_\infty} |\partial_2 u_n| &= \int_0^1 dx \int_0^1 |\partial_2 u_n| \mathbb{1}_{B_\infty} dy \\
 &\geq \int_0^1 dx \left| \int_0^1 \partial_2 u_n \mathbb{1}_{B_\infty} dy \right| \\
 &= \int_0^1 \mathbb{1}_{\tilde{\mathcal{K}}} \\
 (6) \quad &= |\tilde{\mathcal{K}}| = |\mathcal{K}|.
 \end{aligned}$$

Step 3. A transverse argument

In order to obtain a contradiction we are going to prove that [for sufficiently large n]

$$(7) \quad \int_{\Omega} |\nabla u_n| - \int_{B_\infty} |\partial_2 u_n| \geq \delta \text{ for some } \delta > 0 \text{ independent of } n.$$

From the BV-convergence we have

$$\begin{aligned}
 \int_{\Omega} |Du_0| &= \lim_n \int_{\Omega} |\nabla u_n| \\
 &= \lim_n \int_{B_\infty} |\partial_2 u_n| + \left(\int_{\Omega} |\nabla u_n| - \int_{B_\infty} |\partial_2 u_n| \right) \\
 [(6)\&\&(7)] \geq |\mathcal{K}| + \delta.
 \end{aligned}$$

The last estimate yields a contradiction with (3).

Thus it suffices to prove (7).

We make the following dichotomy: $u_0 \not\equiv 0$ in $\Omega \setminus B_\infty$ or $u_0 \equiv 0$ in $\Omega \setminus B_\infty$.

If $u_0 \not\equiv 0$ in $\Omega \setminus B_\infty$, since $\text{tr}_{\partial\Omega \setminus \mathcal{K}} u_0 = 0$, then u_0 can not be a constant in $\Omega \setminus B_\infty$. Thus

$$\delta := \frac{1}{2} \int_{\Omega \setminus B_\infty} |Du_0| > 0.$$

Therefore, for sufficiently large n ,

$$\int_{\Omega} |\nabla u_n| - \int_{B_\infty} |\partial_2 u_n| \geq \int_{\Omega \setminus B_\infty} |\nabla u_n| \geq \delta.$$

This prove Estimate (7).

We now assume that $u_0 \equiv 0$ in $\Omega \setminus B_\infty$. It is clear that we may fix $\frac{1}{4} < a < b < \frac{3}{4}$ and $0 < c < d < 1$ s.t.

$$\delta' := \frac{1}{2} \int_{]a,b[\times]c,d[} |u_0| > 0 \text{ and } (\{a,b\} \times]0,1[) \cap B_\infty = \emptyset.$$

Noting that $u_0 \equiv 0$ in $\Omega \setminus B_\infty$ and that the sequence $(u_n)_n$ is obtained by the convolution of u with a C^∞ -mollifier with compact support, it follows that for sufficiently large n we have $u_n \equiv 0$ in $\{a,b\} \times]c,d[$.

Consequently, from Lemma 3.2 in [ST14], we get [for sufficiently large n]

$$\int_{]a,b[\times]c,d[} |\partial_1 u_n| \geq \frac{2}{b-a} \int_{]a,b[\times]c,d[} |u_n| > \delta'.$$

Since $\int_{]a,b[\times]c,d[} |\partial_1 u_n| > \delta'$, $\int_{]a,b[\times]c,d[} |\partial_2 u_n| \leq 2|\mathcal{K}|$, by Lemma 3.3 in [ST14] we obtained:

$$\int_{]a,b[\times]c,d[} |\nabla u_n| \geq \int_{]a,b[\times]c,d[} |\partial_2 u_n| + \frac{\delta'^2}{4|\mathcal{K}| + \delta'}.$$

We thus get (7) with $\delta := \frac{\delta'^2}{4|\mathcal{K}| + \delta'}$.

2.2. Comments. To prove Theorem 1, we plan to follow the strategy presented above. The first main step in this strategy is the construction of a suitable Cantor set $\mathcal{K} \subset \partial\Omega$. Once \mathcal{K} is constructed we define B_∞ by bending lines orthogonal to $\partial\Omega$.

The construction of \mathcal{K} is done Section 4 and B_∞ is defined Section 5.

A second main step is to implementing the appropriate substitute of Fubini Theorem. This involves Coarea formula. The estimate we will device [the analog of (6) is done by Lemma 8] is obtained asymptotically by integration [via Coarea formula] along level sets of an appropriate function. The appropriate function is, roughly speaking, the projection on \mathcal{K} "along" connected components of B_∞ .

The plan of this article is the following:

- In Section 3 we specify some notations and definitions.
- In Section 4 we present the construction of the set $\mathcal{K} \subset \partial\Omega$.
- In Section 5 we construct the fundamental tool in the proof of Theorem 1. The Cantor set \mathcal{K} is obtained by a standard iterative process. It is defined as the intersection of set $\mathcal{K}_N \subset \partial\Omega$: $\mathcal{K} = \bigcap_{N \geq 1} \mathcal{K}_N$. In Section 5, for $N \geq 1$, we construct a Lipschitz map Ψ_N defined in a subset of Ω with value in \mathcal{K}_N . The main difficulty in this construction is to get a map with a sharp control of its Lipschitz semi-norm (see Lemma 6).
- In Section 6 we state some properties on B_∞ and Ψ_N : the topological interior of B_∞ is empty and we give an upper bound for the Lipschitz semi-norm of Ψ_N (see Lemma 6).
- Section 7 is dedicated to the proof of Theorem 1.
- For the convenience of the reader, the proof of some results are postponed to appendices.
 - In Appendix A we prove that the measure of the Cantor set \mathcal{K} is positive.
 - Appendix B is dedicated to some technical results about the set B_∞ .
 - In Appendix C we sketch the proof of Giusti to get the analog of (3).

3. NOTATIONS, DEFINITIONS

The ambient space is the euclidean plan \mathbb{R}^2 . We let \mathcal{B}_{can} be the canonical basis of its direction space.

- a) The open ball centered at $A \in \mathbb{R}^2$ with radius $r > 0$ is denoted by $B(A, r)$.
- b) A vector may be denoted by an arrow when it is defined with its extremities (e.g. \overrightarrow{AB}), it may be also denoted by a letter in bold font (e.g. \mathbf{u}) or more simply by a grec letter in normal font (e.g. ν).
We let also $|\mathbf{u}|$ be the usual euclidean norm of the vector \mathbf{u} .
- c) For a vector \mathbf{u} we let \mathbf{u}^\perp be the direct orthogonal vector to \mathbf{u} , i.e., if the coordinates of \mathbf{u} are (x_1, x_2) , then those of \mathbf{u}^\perp are $(-x_2, x_1)$.
- d) For $A, B \in \mathbb{R}^2$, the segment of extremities A and B is denoted by $[AB] = \{A + t\overrightarrow{AB} \mid t \in [0, 1]\}$ and $\text{dist}(A, B) = |\overrightarrow{AB}|$ is the usual euclidean distance.
- e) For a set $U \subset \mathbb{R}^2$, the topological interior of U is denoted by $\overset{\circ}{U}$ and its topological closure is \overline{U} .
- f) For $k \geq 1$, a C^k -curve is the range of a C^k injective map from $]0, 1[$ to \mathbb{R}^2 . Note that, in this article, C^k -curves are not closed sets of \mathbb{R}^2 .
- g) For Γ a C^1 -curve, $\mathcal{H}^1(\Gamma)$ is the 1-dimensional Hausdorff measure of Γ .
- h) For $k \geq 1$, a C^k -Jordan curve is the range of a C^k injective map from the unit circle \mathbb{S}^1 to \mathbb{R}^2 .
- i) For Γ a C^1 -curve or a C^1 -Jordan curve, $\mathcal{C} = [AB]$ is a chord of Γ when $A, B \in \overline{\Gamma}$ with $A \neq B$.
- j) If Γ is a C^1 -Jordan curve then, for $A, B \in \Gamma$ & $A \neq B$, the set $\Gamma \setminus \{A, B\}$ admits exactly two connected components: Γ_1 & Γ_2 . These connected components are C^1 -curves.

By smoothness of Γ , it is clear that there is $\eta_\Gamma > 0$ s.t. for $0 < \text{dist}(A, B) < \eta_\Gamma$ there exists THE smallest connected components: we have $\mathcal{H}^1(\Gamma_1) < \mathcal{H}^1(\Gamma_2)$ or $\mathcal{H}^1(\Gamma_2) < \mathcal{H}^1(\Gamma_1)$.

If $0 < \text{dist}(A, B) < \eta_\Gamma$ we may define \widehat{AB} by:

$$(8) \quad \widehat{AB} \text{ is the closure of the smallest curve between } \Gamma_1 \text{ and } \Gamma_2.$$

k) In this article $\Omega \subset \mathbb{R}^2$ is a C^2 bounded open set. By C^2 we mean that $\partial\Omega$ is of class C^2 :

- There exists a covering $\{U_1, \dots, U_n\}$ of $\partial\Omega$ by open sets, $\partial\Omega \subset \bigcup_{i=1}^n U_i$,
- For all $i \in \{1, \dots, n\}$ there is a C^2 diffeomorphism $\varphi_i : \overline{U_i} \rightarrow \overline{\mathbb{D}}$ s.t.

$$\left| \begin{array}{l} \varphi_i(U_i \cap \Omega) = \{(x_1, x_2) \in \mathbb{D} \mid x_2 > 0\} \\ \varphi_i(U_i \cap \partial\Omega) = \{(x_1, x_2) \in \mathbb{D} \mid x_2 = 0\} \end{array} \right. .$$

4. CONSTRUCTION OF THE CANTOR SET \mathcal{K}

It is clear that, up to consider a connected component of Ω , in order to prove Theorem 1 we may assume that Ω is a connected set.

We fix $\Omega \subset \mathbb{R}^2$ be a bounded C^2 open connected set. The set $\mathcal{K} \subset \partial\Omega$ of Theorem 1 is a Cantor set. It is constructed by a recursive way.

4.1. First step: main hypotheses. From the regularity of Ω , there are $\ell + 1$ C^2 -open sets, $\omega_0, \dots, \omega_\ell$, s.t. $\Omega = \omega_0 \setminus \overline{\omega_1 \cup \dots \cup \omega_\ell}$.

Moreover we may assume that:

- ω_i is simply connected for $i = 0, \dots, \ell$,
- $\overline{\omega_i} \subset \omega_0$ for $i = 1, \dots, \ell$,
- $\overline{\omega_i} \cap \overline{\omega_j} = \emptyset$ for $1 \leq i < j \leq \ell$.

We let $\Gamma = \partial\omega_0$. The Cantor set \mathcal{K} is constructed in Γ . Note that Γ is a Jordan-curve. From the Gauss-Bonnet Theorem we get the existence of $M_0 \in \Gamma$ and $r_0 \in]0, 1[$ s.t. for $A, B \in B(M_0, r_0) \cap \Gamma$, the chord $[AB] \subset \overline{\Omega}$ and $[AB] \cap \partial\Omega = \{A, B\}$. Note that we may assume $2r_0 < \eta_\Gamma$ [η_Γ is defined in Section 3-j]

We fix $A, B \in B(M_0, r_0) \cap \Gamma$ s.t. $A \neq B$. We have:

- By the definition of M_0 and r_0 , the chord $\mathcal{C}_0 := [AB]$ is included in $\overline{\Omega}$.
- We let \widehat{AB} be the closure of the smallest part of Γ which is delimited by A, B (see (8)). Up to replace r_0 by a smaller value we may assume that \widehat{AB} is the graph of $f \in C^2([0, \eta], \mathbb{R}^+)$ in the orthonormal frame $\mathcal{R}_0 = (A, \mathbf{e}_1, \mathbf{e}_2)$ where $\mathbf{e}_1 = \overrightarrow{AB}/|\overrightarrow{AB}|$.
- The function f is strictly concave and $f(x) > 0$ for $x \in]0, \eta[$.

Key Claim. Note that the length of the chord $[AB]$ is η and that for intervals $I, J \subset [0, \eta]$, if $I \subset J$ then

$$(9) \quad \|f''|_I\|_{L^\infty(I)} \leq \|f''|_J\|_{L^\infty(J)}$$

where $f|_I$ is the restriction of f to I .

Thus, up to replace the chord $\mathcal{C}_0 = [AB]$ by a smaller chord of \widehat{AB} parallel to \mathcal{C}_0 , and up to restrict the function f (and up to add a suitable constant to f) we may assume that

$$(10) \quad 0 < \eta < \min \left\{ \frac{1}{2}; \frac{1}{16\|f''\|_{L^\infty([0, \eta])}^2} \right\}.$$

We may also assume that

- Letting D_0^+ be the bounded open set s.t. $\partial D_0^+ = [AB] \cup \widehat{AB}$ we have $\Pi_{\partial\Omega}$, the orthogonal projection on $\partial\Omega$, which is well defined and of class C^2 in D_0^+ .
- Once again by (9) we have

$$(11) \quad 1 + 4\|f''\|_{L^\infty}^2 \text{diam}(D_0^+) < \frac{16}{9}$$

where $\text{diam}(D_0^+) = \sup\{\text{dist}(M, N) \mid M, N \in D_0^+\}$.

4.2. Step 2: Iterative construction. We are now in position to construct the Cantor set \mathcal{K} as a subset of \widehat{AB} . The construction is iterative.

The goal of the construction is to get at Step $N \geq 0$ a collection of 2^N pairwise disjoint curves included in \widehat{AB} [denoted by $\{K_1^N, \dots, K_{2^N}^N\}$] and their chords [denoted by $\{\mathcal{C}_1^N, \dots, \mathcal{C}_{2^N}^N\}$].

The idea is standard: at the step $N \geq 0$ we replace a curve Γ_0 included in \widehat{AB} by two curves included in Γ_0 (see Figure 1).

Initialization. We initialize the procedure by letting $K_1^0 := \widehat{AB}$ and $\mathcal{C}_1^0 = \mathcal{C}_0 = [AB]$.

At step $N \geq 0$ we have:

- A set of 2^N curves included in \widehat{AB} , $\{K_1^N, \dots, K_{2^N}^N\}$. The curves K_k^N 's are mutually disjoint. We let $\mathcal{K}_N = \cup_{k=1}^{2^N} K_k^N$.
- A set of 2^N chords, $\{\mathcal{C}_1^N, \dots, \mathcal{C}_{2^N}^N\}$ s.t. for $k = 1, \dots, 2^N$, \mathcal{C}_k^N is the chord of K_k^N .

Remark 1. (1) Note that since the \mathcal{C}_k^N 's are chords of \widehat{AB} and since in the frame $\mathcal{R}_0 = (A, \mathbf{e}_1, \mathbf{e}_2)$, \widehat{AB} is the graph of a function, none of the chords \mathcal{C}_k^N are vertical, *i.e.*, directed by \mathbf{e}_2 .

Since the chords \mathcal{C}_k^N are not vertical, for $k \in \{1, \dots, 2^N\}$, we may define $\nu_{\mathcal{C}_k^N}$ as the unit vector orthogonal to \mathcal{C}_k^N s.t. $\nu_{\mathcal{C}_k^N} = \alpha \mathbf{e}_1 + \beta \mathbf{e}_2$ with $\beta > 0$.

- (2) Since the function $f \in C^2([0, \eta], \mathbb{R}^+)$ is strictly concave, considering a chord \mathcal{C}_k^N and a straight line D perpendicular to \mathcal{C}_k^N , the straight line D intersect K_k^N at exactly one points.

This fact may be easily checked by letting Γ^* be the range of \widehat{AB} with respect to the symmetry of axe $[AB]$ and by noting that the compact set delimited by $\widehat{AB} \cup \Gamma^*$ is a strictly convex domain.

Heredity rules. From Step $N \geq 0$ to Step $N + 1$ we follow the following rules:

- (1) For each $k \in \{1, \dots, 2^N\}$, we let η_k^N be the length of \mathcal{C}_k^N . Inside the chord \mathcal{C}_k^N we center a segment I_k^N of length $(\eta_k^N)^2$.
- (2) With the help of Remark 1.2, we may define two distinct points of K_k^N as the intersection of K_k^N with straight lines perpendicular to \mathcal{C}_k^N which pass by the extremities of I_k^N .
- (3) These intersection points are the extremities of a curve \widetilde{K}_k^N included in K_k^N . We let K_{2k-1}^{N+1} and K_{2k}^{N+1} the connected components of $K_k^N \setminus \widetilde{K}_k^N$. We let also
 - \mathcal{C}_{2k-1}^{N+1} and \mathcal{C}_{2k}^{N+1} their chords;
 - $\mathcal{K}_{N+1} = \cup_{k=1}^{2^{N+1}} K_k^{N+1}$.

Notation 2. A natural terminology consists in defining the father and the sons of a chord or a curve:

- $\mathcal{F}(\mathcal{C}_{2k-1}^{N+1}) = \mathcal{F}(\mathcal{C}_{2k}^{N+1}) = \mathcal{C}_k^N$ is the *father* of the chords \mathcal{C}_{2k-1}^{N+1} and \mathcal{C}_{2k}^{N+1} .
- $\mathcal{F}(K_{2k-1}^{N+1}) = \mathcal{F}(K_{2k}^{N+1}) = K_k^N$ is the *father* of the curves K_{2k-1}^{N+1} and K_{2k}^{N+1} .
- $\mathcal{S}(\mathcal{C}_k^N) = \{\mathcal{C}_{2k-1}^{N+1}, \mathcal{C}_{2k}^{N+1}\}$ is the set of sons of the chord \mathcal{C}_k^N , *i.e.*, $\mathcal{F}(\mathcal{C}_{2k-1}^{N+1}) = \mathcal{F}(\mathcal{C}_{2k}^{N+1}) = \mathcal{C}_k^N$.
- $\mathcal{S}(K_k^N) = \{K_{2k-1}^{N+1}, K_{2k}^{N+1}\}$ is the set of sons of the curve K_k^N , *i.e.*, $\mathcal{F}(K_{2k-1}^{N+1}) = \mathcal{F}(K_{2k}^{N+1}) = K_k^N$.

The heredity step is represented in Figure 1.

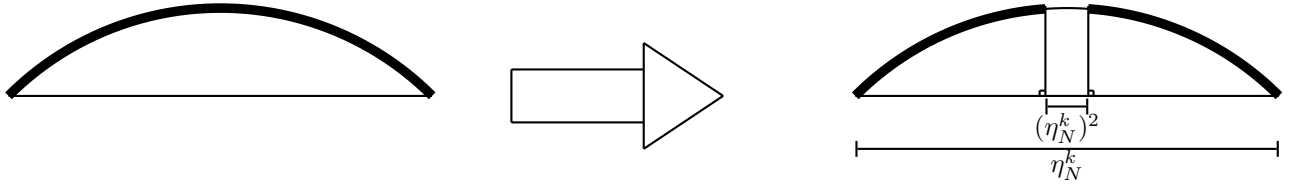


FIGURE 1. Heredity

In Figure 2&3 the two first iterations of the process are represented.

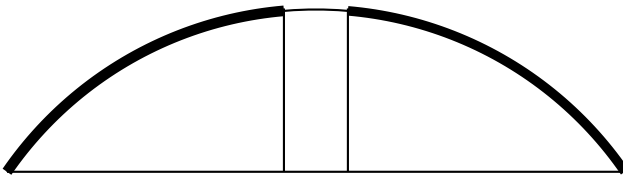


FIGURE 2. First iteration of the process

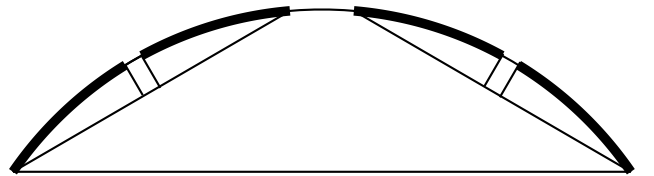


FIGURE 3. Second iteration of the process

We now define the Cantor set

$$(12) \quad \mathcal{K} = \bigcap_{N \geq 0} \overline{\mathcal{K}_N}.$$

The Cantor set \mathcal{K} is fat:

Proposition 3. *We have $\mathcal{H}^1(\mathcal{K}) > 0$.*

This proposition is proved in Appendix A.

5. CONSTRUCTION OF A SEQUENCE OF FUNCTIONS USED IN CONJUNCTION WITH COAREA FORMULA

A key argument in the proof of Theorem 1 is the use of the coarea formula to calculate a lower bound for (2). The coarea formula is applied to a function adapted to the set \mathcal{K} .

For $N = 0$ we let

- D_0^+ be the compact set delimited by $K_0 = \widehat{AB}$ and $\mathcal{C}_1^0 := [AB]$ the chord of K_0 .
- We recall that we fixed a frame $\mathcal{R}_0 = (A, \mathbf{e}_1, \mathbf{e}_2)$ where $\mathbf{e}_1 = \overrightarrow{AB}/|\overrightarrow{AB}|$. For $\sigma = (\sigma_1, 0) \in \mathcal{C}_1^0$, we define:

(13) I_σ is the connected component of $\{(\sigma_1, t) \in \Omega \mid t \leq 0\}$ which contains σ .

[I_σ is a vertical segment included in Ω].

- $D_0^- = \cup_{\sigma \in \mathcal{C}_1^0} I_\sigma$.
- We now define the maps

$$\begin{aligned} \tilde{\Psi}_0 : D_0^- &\rightarrow \mathcal{C}_1^0 \\ x &\mapsto \Pi_{\mathcal{C}_1^0}(x) \end{aligned}$$

and

$$\begin{aligned} \Psi_0 : D_0^- \cup D_0^+ &\rightarrow \mathcal{C}_1^0 \\ x &\mapsto \begin{cases} \Pi_{\partial\Omega}(x) & \text{if } x \in D_0^+ \\ \Pi_{\partial\Omega}[\tilde{\Psi}_0(x)] & \text{if } x \in D_0^- \end{cases} \end{aligned}$$

where $\Pi_{\partial\Omega}$ is the orthogonal projection on $\partial\Omega$ and $\Pi_{\mathcal{C}_1^0}$ is the orthogonal projection on \mathcal{C}_1^0 . Note that, in the frame \mathcal{R}_0 , for $x = (x_1, x_2) \in D_0^-$, we have $\Pi_{\mathcal{C}_1^0}(x) = (x_1, 0)$.

For $N = 1$ and $k \in \{1, 2\}$ we let:

- D_k^1 be the compact set delimited by K_k^1 and \mathcal{C}_k^1 ;
- T_k^1 be the compact right-angled triangle (with its interior) having \mathcal{C}_k^1 as side adjacent to the right angle and whose hypotenuse is included in \mathcal{C}_1^0 ;
- H_k^1 be the hypotenuse of T_k^1 .

We now define $D_1^- = \tilde{\Psi}_0^{-1}(H_1^1 \cup H_2^1)$, $T_1 = T_1^1 \cup T_2^1$ and $D_1^+ = D_1^1 \cup D_2^1$.

We first consider the map

$$\begin{aligned} \tilde{\Psi}_1 : T_1 \cup D_1^- &\rightarrow \mathcal{C}_1^1 \cup \mathcal{C}_2^1 \\ x &\mapsto \begin{cases} \Pi_{\mathcal{C}_k^1}(x) & \text{if } x \in T_k^1 \\ \Pi_{\mathcal{C}_k^1}[\tilde{\Psi}_0(x)] & \text{if } x \in D_1^- \end{cases} \end{aligned}$$

In Appendix B [Proposition 12], it is proved that the triangles T_1^1 and T_2^1 are disjoint. Thus the map $\tilde{\Psi}_1$ is well defined

By projecting $\mathcal{C}_1^1 \cup \mathcal{C}_2^1$ on $\partial\Omega$ we get

$$\begin{aligned} \Psi_1 : T_1 \cup D_1^- \cup D_1^+ &\rightarrow \mathcal{K}_1 \\ x &\mapsto \begin{cases} \Pi_{\partial\Omega}(x) & \text{if } x \in D_1^+ \\ \Pi_{\partial\Omega}[\tilde{\Psi}_1(x)] & \text{if } x \in T_1 \cup D_1^- \end{cases} \end{aligned}$$

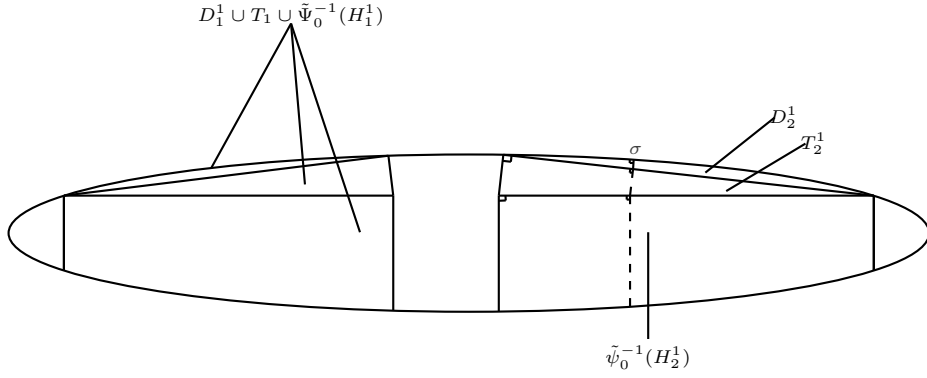


FIGURE 4. The sets defined at Step $N = 1$ and the level line [in dash] of Ψ_1 associated to σ

For $N \geq 1$, we first construct $\tilde{\Psi}_{N+1}$ and then Ψ_{N+1} is obtained from $\tilde{\Psi}_{N+1}$ and $\Pi_{\partial\Omega}$.
For $k \in \{1, \dots, 2^{N+1}\}$, we let

- D_k^{N+1} be the compact set delimited by K_k^{N+1} and \mathcal{C}_k^{N+1} [recall that \mathcal{C}_k^{N+1} is the chord associated to K_k^{N+1}];
- T_k^{N+1} be the right-angled triangle (with its interior) having \mathcal{C}_k^{N+1} as side adjacent to the right angle and whose hypotenuse is included in $\mathcal{F}(\mathcal{C}_k^{N+1})$. Here $\mathcal{F}(\mathcal{C}_k^{N+1})$ is the father of \mathcal{C}_k^{N+1} (see Notation 2);
- $H_k^{N+1} \subset \mathcal{F}(\mathcal{C}_k^{N+1})$ be the hypotenuse of T_k^{N+1} .

We denote $T_{N+1} = \bigcup_{k=1}^{2^{N+1}} T_k^{N+1}$, $D_{N+1}^- = \tilde{\Psi}_N^{-1} \left(\bigcup_{k=1}^{2^{N+1}} H_k^{N+1} \right)$ and $D_{N+1}^+ = \bigcup_{k=1}^{2^{N+1}} D_k^{N+1}$.

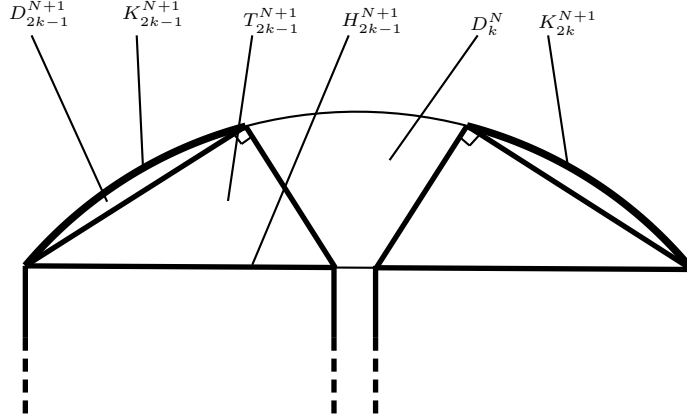


FIGURE 5. Heredity. The bold lines correspond to the new iteration

Remark 4. It is easy to check that for $N \geq 0$:

- (1) $T_{N+1} \subset D_N^+$,
- (2) if $x \in T_N$ then $x \notin T_{N'}$ for $N' \geq N + 1$ [here $T_0 = \emptyset$].

We now define

$$\begin{aligned} \tilde{\Psi}_{N+1} : T_{N+1} \cup D_{N+1}^- &\rightarrow \cup_{k=1}^{2^{N+1}} \mathcal{C}_k^{N+1} \\ x &\mapsto \begin{cases} \Pi_{\mathcal{C}_k^{N+1}}(x) & \text{if } x \in T_k^{N+1} \\ \Pi_{\mathcal{C}_k^{N+1}}[\tilde{\Psi}_N(x)] & \text{if } x \in \tilde{\Psi}_N^{-1}(\cup_{k=1}^{2^{N+1}} H_k^{N+1}) \end{cases}. \end{aligned}$$

In Appendix B [Proposition 12], it is proved that for $N \geq 1$, the triangles T_k^N for $k = 1, \dots, 2^N$ are mutually disjoint. Thus the map $\tilde{\Psi}_{N+1}$ is well defined is $\tilde{\Psi}_N$ makes sense.

And, as in the Initialization Step, we get Ψ_{N+1} from $\tilde{\Psi}_{N+1}$ by projecting $\cup_{k=1}^{2^{N+1}} \mathcal{C}_k^{N+1}$ on $\partial\Omega$:

$$\begin{aligned} \Psi_{N+1} : T_{N+1} \cup D_{N+1}^- \cup D_{N+1}^+ &\rightarrow \mathcal{K}_{N+1} \\ x &\mapsto \begin{cases} \Pi_{\partial\Omega}[\tilde{\Psi}_{N+1}(x)] & \text{if } x \in T_{N+1} \cup D_{N+1}^- \\ \Pi_{\partial\Omega}(x) & \text{if } x \in D_{N+1}^+ \end{cases}. \end{aligned}$$

It is easy to see that $\Psi_{N+1}(T_{N+1} \cup D_{N+1}^- \cup D_{N+1}^+) = \mathcal{K}_{N+1}$.

6. BASIC PROPERTIES FOR B_∞ AND Ψ_N

We first state some fundamental claims on the sets $B_N = T_N \cup D_N^+ \cup D_N^-$. It is easy to check that for $N \geq 0$ we have $B_{N+1} \subset B_N$ and $\mathcal{K} \subset \partial B_N$. Therefore we may define

$$B_\infty = \cap_{N \geq 0} \overline{B_N}$$

which is compact and it satisfies $\mathcal{K} \subset \partial B_\infty$.

We are going to prove:

Lemma 5. *The interior of B_∞ is empty.*

Proof of Lemma 5. From Proposition 12 in Appendix B combined with Hypothesis (10), we get two fundamental facts:

- (1) The triangles $T_1^N, \dots, T_{2^{N+1}}^N$ are mutually disjoint.
- (2) We have:

$$(14) \quad \mathcal{H}^1(H_k^{N+1}) < \frac{\mathcal{H}^1(\mathcal{F}(\mathcal{C}_k^N))}{2}.$$

For a non empty set $A \subset \mathbb{R}^2$ we let

$$\text{rad}(A) = \sup\{r \geq 0 \mid \exists x \in A \text{ s.t. } \overline{B(x, r)} \subset A\}.$$

Note that the topological interior of A is empty if and only if $\text{rad}(A) = 0$.

On the one hand, it is not difficult to check that

$$(15) \quad \text{rad}(B_N) = \text{rad}(B_N \cap D_N^-).$$

On the other hand, using (14) and the Incidence Theorem applied in a recursive way, we get that for $N \geq 1$:

$$(16) \quad \text{rad}(B_{N+1} \cap D_{N+1}^-) \leq \frac{\text{rad}(B_N \cap D_N^-)}{2}.$$

Consequently, by combining (15)&(16) we get the existence of $C_0 > 1$ s.t.

$$(17) \quad \text{rad}(B_N) \leq \frac{C_0}{2^N}.$$

Since $B_\infty = \cap_{N \geq 0} B_N$, from (17) we get that $\text{rad}(B_\infty) = 0$. The last estimate implies the expected result. \square

We now prove the key estimate for Ψ_N :

Lemma 6. *The function Ψ_N is $(1 + o_N(1))$ -Lipschitz.*

Proof. The functions $\tilde{\Psi}_N$ are obtained as compositions of orthogonal projections on straight lines and thus are 1-Lipschitz.

The projection $P_N := \Pi_{\partial\Omega}$ defined in $\overline{D_N^+}$ is $(1 + o_N(1))$ -Lipschitz. The functions Ψ_N are either the composition of $\tilde{\Psi}_N$ with P_N or $\Psi_N = P_N$. Consequently Ψ_N is $(1 + o_N(1))$ -Lipschitz. \square

7. PROOF OF THEOREM 1

We are now in position to prove Theorem 1. It is clear that we may assume that Ω is a connected set.

The proof of Theorem 1 is done arguing by contradiction. We assume that there exists a map $u_0 \in BV(\Omega)$ which is a solution of (2).

7.1. Upper bound. The first step in the demonstration is

$$(18) \quad \int_{\Omega} |Du_0| \leq \|\mathbb{I}_{\mathcal{K}}\|_{L^1(\partial\Omega)} = |\mathcal{K}|.$$

Estimate (18) is obtained by proving that for all $\varepsilon > 0$ there is $u_\varepsilon \in W^{1,1}(\Omega)$ s.t. $\text{tr}_{\partial\Omega} u_\varepsilon = \mathbb{I}_{\mathcal{K}}$ and

$$(19) \quad \|\nabla u_\varepsilon\|_{L^1(\Omega)} \leq (1 + \varepsilon) \|\text{tr}_{\partial\Omega} u_\varepsilon\|_{L^1(\Omega)} = (1 + \varepsilon) |\mathcal{K}|.$$

The last estimate is obtained by Theorem 2.16 & Remark 2.17 in [Giu84]. The proof of this result is done for special domains Ω . We sketch its adaptation for a C^2 domain in Appendix C.

With the help of (19), it suffices to use the minimality of u_0 to get (18).

7.2. Saturation of the upper bound. In order to have a contradiction we follow the strategy of Spradlin and Tamaskan in [ST14]. We fix a sequence $(u_n)_n \subset C^\infty(\Omega)$ defined by the convolution of u_0 and a Friedrichs mollifiers [with compact support]. This sequence of functions is defined in [Giu84]-Theorem 1.17. It satisfies $u_n \xrightarrow{BV} u_0$ and $\text{tr}_{\partial\Omega} u_n = \text{tr}_{\partial\Omega} u_0$.

Here $u_n \xrightarrow{BV} u_0$ means $u_n \xrightarrow{L^1} u_0$ and $\int_{\Omega} |\nabla u_n| \rightarrow \int_{\Omega} |Du_0|$.

For $x \in B_0$ we let

$$(20) \quad V_0(x) = \begin{cases} \nu_{\Pi_{\partial\Omega}(x)} & \text{if } x \in D_0^+ \\ (0, 1) & \text{if } x \in D_0^- \end{cases},$$

and for $N \geq 0$, $x \in B_{N+1}$:

$$(21) \quad V_{N+1}(x) = \begin{cases} V_N(x) & \text{if } x \in B_N \setminus \overset{\circ}{T}^{N+1} \\ \nu_{\mathcal{C}_k^{N+1}} & \text{if } x \in \overset{\circ}{T}_k^{N+1} \end{cases},$$

where ν_σ is the normal outward of Ω in $\sigma \in \partial\Omega$ and $\nu_{\mathcal{C}_k^{N+1}}$ is defined in Remark 1.1.

We now prove the following lemma.

Lemma 7. *When $N \rightarrow \infty$ we may define*

$$(22) \quad \begin{array}{ccc} V_\infty : & B_\infty & \rightarrow \mathbb{R}^2 \\ & x & \mapsto \lim_{N \rightarrow \infty} V_N(x) \end{array}.$$

Moreover, from the Dominated Convergence Theorem, we have:

$$(23) \quad V_N \mathbb{I}_{B_N} \rightarrow V_\infty \mathbb{I}_{B_\infty} \text{ in } L^1(\Omega).$$

Proof. Let $x \in B_\infty$. Then $x \in B_N$ for all $N \geq 1$. In order to prove that $V_\infty(x)$ is well defined it suffices to note that, from Remark 4.2, if $x \in \overset{\circ}{T}_{N_0}$ for some $N_0 \geq 1$, then $x \notin T_N$ for $N > N_0$. Consequently, there is a dichotomy:

- $x \in B_\infty \setminus \bigcup_{N \geq 1} \overset{\circ}{T}_N$,
- $\exists! N_0 \geq 1$ s.t. $x \in \overset{\circ}{T}_{N_0}$.

If $x \in B_\infty \setminus \bigcup_{N \geq 1} \overset{\circ}{T}_N$, then we have $V_N(x) = V_0(x)$ for all $N \geq 1$. Thus $\lim_{N \rightarrow \infty} V_N(x) = V_0(x)$. Otherwise there exists a unique $N_0 \geq 1$ s.t. $x \in \overset{\circ}{T}_{N_0}$. Therefore for all $N > N_0$ we have $V_N(x) = V_{N_0}(x)$. Consequently $\lim_{N \rightarrow \infty} V_N(x) = V_{N_0}(x)$. \square

This section is dedicated for the proof of the lemma:

Lemma 8. *For all $w \in C^\infty \cap BV(\Omega)$ s.t. $\text{tr}_{\partial\Omega} w = \mathbb{I}_{\mathcal{K}}$ we have*

$$\int_{B_\infty \cap \Omega} |\nabla w \cdot V_\infty| \geq |\mathcal{K}|$$

where V_∞ is the vectorial field defined in (22).

Remark 9. Since $|V_\infty(x)| = 1$ for $x \in B_\infty$, it is clear that Lemma 8 implies that for all n we have

$$\int_{B_\infty \cap \Omega} |\nabla u_n| \geq |\mathcal{K}|.$$

Therefore, from the BV-convergence of u_n to u_0 we get

$$\int_{B_\infty \cap \Omega} |Du_0| \geq |\mathcal{K}|.$$

Section 7.3 is dedicated to a sharper argument than above to get

$$\int_{B_\infty \cap \Omega} |\nabla u_n| \geq \int_{B_\infty \cap \Omega} |\nabla u_n \cdot V_\infty| + \delta$$

with $\delta > 0$ is independent of n . The last estimate will imply $\int_{B_\infty \cap \Omega} |Du_0| \geq |\mathcal{K}| + \delta$ which will be the contradiction in the argumentation.

Proof. We first get that for $w \in C^\infty \cap BV(\Omega)$ s.t. $\text{tr}_{\partial\Omega} w = \mathbb{1}_{\mathcal{K}}$ we have

$$(24) \quad \int_{B_N \cap \Omega} |\nabla w \cdot V_N| \geq \frac{|\mathcal{K}|}{|\Psi_N|_{\text{Lip}(B_N)}^2}$$

where

$$|\Psi_N|_{\text{Lip}(B_N)} = \sup \left\{ \frac{|\Psi_N(x) - \Psi_N(y)|}{|x - y|} \mid x, y \in B_N, x \neq y \right\},$$

is the Lipschitz semi-norm of Ψ_N and V_N is the vectorial field defined in (20)&(21).

Once we get this estimate, we claim that for a fixed map $w \in C^\infty \cap BV(\Omega)$ s.t. $\text{tr}_{\partial\Omega} w = \mathbb{1}_{\mathcal{K}}$, from (23) and the Dominated Convergence Theorem, we have $\nabla w \cdot V_N \mathbb{1}_{B_N \cap \Omega} \rightarrow \nabla w \cdot V_\infty \mathbb{1}_{B_\infty \cap \Omega}$ in $L^1(\Omega)$.

Thus, with the help of Lemma 6 [$|\Psi_N|_{\text{Lip}(B_N)} = 1 + o_N(1)$] we have

$$\int_{B_\infty \cap \Omega} |\nabla w \cdot V_\infty| \geq |\mathcal{K}|.$$

We thus prove (24). We fix $w \in C^\infty \cap BV(\Omega)$ s.t. $\text{tr}_{\partial\Omega} w = \mathbb{1}_{\mathcal{K}}$. Using the Coarea Formula we have

$$\begin{aligned} |\Psi_N|_{\text{Lip}(B_N)}^2 \int_{B_N \cap \Omega} |\nabla w \cdot V_N| &\geq \int_{B_N \cap \Omega} |\text{jac} \Psi_N| |\nabla w \cdot V_N| \\ &= \int_{\widehat{AB}} d\sigma \int_{\Psi_N^{-1}(\{\sigma\}) \cap \Omega} |\nabla w \cdot V_N|. \end{aligned}$$

Here $\Psi_N^{-1}(\{\sigma\})$ is a polygonal chain:

$$\Psi_N^{-1}(\{\sigma\}) = I_{\tilde{\sigma}} \cup I_{N,\sigma}^1 \cup \dots \cup I_{N,\sigma}^{N+1}$$

where

- $\tilde{\sigma}$ is s.t. $[AB] \cap \tilde{\Psi}_N^{-1}(\{\sigma\}) = \{\tilde{\sigma}\}$,
- $I_{\tilde{\sigma}}$ is defined in (13),
- for $l = 1, \dots, N$ we have $I_{N,\sigma}^l = \Psi_N^{-1}(\{\sigma\}) \cap T_{N+1-l}$,
- $I_{N,\sigma}^{N+1} = \Psi_N^{-1}(\{\sigma\}) \cap D_N^+$.

From the Fundamental Theorem of calculus and from the definition of V_N , denoting

- $I_{\tilde{\sigma}} = [\alpha_0, \alpha_1]$ (where $\alpha_0 \in \partial\Omega$ and $\alpha_1 = \tilde{\sigma}$),
- $I_{N,\sigma}^l = [\alpha_l, \alpha_{l+1}]$, $l = 1, \dots, N+1$ and $\alpha_{N+2} = \sigma$,

we have for a.e. $\sigma \in \widehat{AB}$:

$$\int_{[\alpha_l, \alpha_{l+1}]} |\nabla w \cdot V_N| \geq |w(\alpha_{l+1}) - w(\alpha_l)|.$$

Here we used a little abuse of notation writing for $l = 0 \& N + 2$ $w(\alpha_l)$ instead of $\text{tr}_{\partial\Omega} w(\alpha_l)$.

Therefore for a.e. $\sigma \in \mathcal{K}_N$ we have

$$\int_{\Psi_N^{-1}(\{\sigma\}) \cap \Omega} |\nabla w \cdot V_N| \geq |\text{tr}_{\partial\Omega} w(\sigma) - \text{tr}_{\partial\Omega} w(\alpha_0)| = \mathbb{1}_{\mathcal{K}}(\sigma).$$

We may thus deduce that

$$|\Psi_N|_{\text{Lip}(B_N)}^2 \int_{B_N \cap \Omega} |\nabla w \cdot V_N| \geq \int_{AB} \mathbb{1}_{\mathcal{K}} = |\mathcal{K}|.$$

The last estimate clearly implies (24) and it ends the proof of Lemma 8. \square

7.3. Transverse argument. We assumed that there exists a map u_0 which solves Problem (2).

We make the following dichotomy:

- $u_0 \not\equiv 0$ in $\Omega \setminus B_\infty$;
- $u_0 \equiv 0$ in $\Omega \setminus B_\infty$.

We are going to prove that both cases give a contradiction.

7.3.1. *The case $u_0 \not\equiv 0$ in $\Omega \setminus B_\infty$.* We assume that $\int_{\Omega \setminus B_\infty} |u_0| > 0$. In this case, since $(\text{tr}_{\partial\Omega} u_0)|_{\partial\Omega \setminus \partial B_\infty} \equiv 0$, we have

$$\delta := \int_{\Omega \setminus B_\infty} |Du_0| > 0.$$

Recall that we fixed a sequence $(u_n)_n \subset C^\infty \cap BV(\Omega)$ s.t. $u_n \xrightarrow{BV} u_0$ & $\text{tr}_{\partial\Omega} u_n = \text{tr}_{\partial\Omega} u_0$.

Therefore, for sufficiently large n , we have

$$\int_{\Omega \setminus B_\infty} |\nabla u_n| > \frac{\delta}{2}.$$

Thus, from Lemma 8 we get [we use also $|V_\infty(x)| = 1$ for all $x \in B_\infty$]

$$\int_{\Omega} |\nabla u_n| \geq \int_{B_\infty} |\nabla u_n \cdot V_\infty| + \int_{\Omega \setminus B_\infty} |\nabla u_n| \geq |\mathcal{K}| + \frac{\delta}{2}.$$

This implies

$$\int_{\Omega} |Du_0| = \lim_n \int_{\Omega} |\nabla u_n| \geq |\mathcal{K}| + \frac{\delta}{2}$$

which is in contradiction with (18).

7.3.2. *The case $u_0 \equiv 0$ in $\Omega \setminus B_\infty$.* We first note that, since $\text{tr}_{\partial D_0^+} u_0 \not\equiv 0$, there exists a triangle $T_k^{N_0}$ s.t. $\int_{T_k^{N_0}} |u_0| > 0$. We fix such a triangle $T_k^{N_0}$ and we let α be its right-angle.

We let $\tilde{\mathcal{R}} = (\alpha, \tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2)$ be the direct orthonormal frame centered in α where $\tilde{\mathbf{e}}_2 = \nu_{\mathcal{C}_k^{N_0}}$ [$\nu_{\mathcal{C}_k^{N_0}}$ is defined Remark 1.1], i.e., the directions of the new frame are given by the side of the right-angle of $T_k^{N_0}$.

It is clear that for $N \geq N_0$ we have $V_N \equiv \tilde{\mathbf{e}}_2$ in $T_k^{N_0}$.

By construction of B_∞ , $T_k^{N_0} \cap B_\infty$ is a union of segments directed by $\tilde{\mathbf{e}}_2$, i.e. $\mathbb{1}_{B_\infty|_{T_k^{N_0}}}(s, t)$ depends only on the first variable "s" in the frame $\tilde{\mathcal{R}}$.

Since $\int_{T_k^{N_0}} |u_0| > 0$, in the frame $\tilde{\mathcal{R}}$, we may find $a, b, c, d \in \mathbb{R}$ s.t., considering the rectangle (whose sides are parallel to the direction of $\tilde{\mathcal{R}}$)

$$\mathcal{P} := \{\alpha + s\tilde{\mathbf{e}}_1 + t\tilde{\mathbf{e}}_2 \mid (s, t) \in [a, b] \times [c, d]\} \subset T_k^{N_0}$$

we have

$$\int_{\mathcal{P}} |u_0| > 0.$$

Since from Lemma 5 the set B_∞ has an empty interior [and that $\mathbb{1}_{B_\infty|_{T_k^{N_0}}}(s, t)$ depends only on the first variable in the frame $\tilde{\mathcal{R}}$], we may find a', b' s.t.

- $[a', b'] \times [c, d] \subset [a, b] \times [c, d]$,
- $\mathcal{S} \cap B_\infty = \emptyset$ with $\mathcal{S} := \{\alpha + s\tilde{\mathbf{e}}_1 + t\tilde{\mathbf{e}}_2 \mid (s, t) \in [a', b'] \times [c, d]\}$
- $\delta := \int_{\mathcal{P}'} |u_0| > 0$ with $\mathcal{P}' := \{\alpha + s\tilde{\mathbf{e}}_1 + t\tilde{\mathbf{e}}_2 \mid (s, t) \in [a', b'] \times [c, d]\}$.

Moreover, since B_∞ is a compact set, we may find an open neighborhood of \mathcal{S} which do not intersect B_∞ .

Noting that $u \equiv 0$ in $\Omega \setminus B_\infty$ and that the sequence $(u_n)_n$ is obtained by the convolution of u_0 with a C^∞ -mollifier with compact support, it follows that for sufficiently large n we have

- $u_n \equiv 0$ in \mathcal{S} ,
- $\int_{\mathcal{P}'} |u_n| > \frac{\delta}{2}$.

Consequently, from Lemma 3.2 in [ST14], we get [for sufficiently large n]

$$\int_{\mathcal{P}'} |\partial_{\mathbf{e}_1} u_n| \geq \frac{2}{b' - a'} \int_{\mathcal{P}'} |u_n| > \frac{\delta}{b' - a'} =: \delta'.$$

Therefore $\int_{\mathcal{P}'} |\partial_{\mathbf{e}_1} u_n| > \delta'$, $\int_{\mathcal{P}'} |\partial_{\mathbf{e}_2} u_n| \leq 2|\mathcal{K}|$ and then by Lemma 3.3 in [ST14] we obtain:

$$\int_{\mathcal{P}'} |\nabla u_n| \geq \int_{\mathcal{P}'} |\partial_{\mathbf{e}_2} u_n| + \frac{\delta'^2}{4|\mathcal{K}| + \delta'}.$$

Thus, from Lemma 8, for sufficiently large n :

$$\int_{\Omega} |\nabla u_n| \geq |\mathcal{K}| + \frac{\delta'^2}{4|\mathcal{K}| + \delta'} - o_n(1).$$

From the convergence in BV -norm of u_n to u_0 we have

$$\int_{\Omega} |Du_0| \geq |\mathcal{K}| + \frac{\delta'^2}{4|\mathcal{K}| + \delta'}.$$

Clearly this last assertion contradicts (18) and ends the proof of Theorem 1.

APPENDICES

APPENDIX A. RESULTS RELATED WITH THE CANTOR SET \mathcal{K}

A.1. Two preliminary results. We first prove a standard result which states that the length of a *small* chords is a *good* approximation for the length of a curve.

Lemma 10. *Let $0 < \eta < 1$ and let $f \in C^2([0, \eta], \mathbb{R}^+)$. We fix an orthonormal frame and we denote C_f the graph of f in the orthonormal frame. Let $A = (a, f(a))$, $B = (b, f(b)) \in C_f$ (with $0 \leq a < b \leq \eta$) and let $\mathcal{C} = [AB]$ be the chord of C_f joining A and B . We denote \widehat{AB} the arc of C_f with extremities A and B .*

We have

$$\mathcal{H}^1(\mathcal{C}) \leq \mathcal{H}^1(\widehat{AB}) \leq \mathcal{H}^1(\mathcal{C}) \{1 + (b - a)\|f''\|_{L^\infty} [2\|f'\|_{L^\infty} + \|f''\|_{L^\infty}(b - a)]\}.$$

Proof. The estimate $\mathcal{H}^1(\mathcal{C}) \leq \mathcal{H}^1(\widehat{AB})$ is standard, we thus prove the second inequality.

On the one hand

$$\mathcal{H}^1(\mathcal{C}) = \sqrt{(a - b)^2 + [f(a) - f(b)]^2} = (b - a) \sqrt{1 + \left(\frac{f(a) - f(b)}{a - b}\right)^2}.$$

On the other hand

$$\mathcal{H}^1(\widehat{AB}) = \int_a^b \sqrt{1 + f'^2}.$$

With the help of the Mean Value Theorem, there is $c \in]a, b[$ s.t.

$$\frac{f(a) - f(b)}{a - b} = f'(c).$$

Applying once again the Mean Value Theorem [to f'], for $x \in [a, b]$ there is c_x between c and x s.t.

$$f'(x) = f'(c) + f''(c_x)(x - c).$$

Consequently for $x \in [a, b]$ we have:

$$\begin{aligned} \sqrt{1 + f'(x)^2} &= \sqrt{1 + [f'(c) + f''(c_x)(x - c)]^2} \\ &= \sqrt{1 + f'(c)^2} \sqrt{1 + \frac{2f'(c)f''(c_x)(x - c) + f''(c_x)^2(x - c)^2}{1 + f'(c)^2}} \\ &\leq \sqrt{1 + \left(\frac{f(a) - f(b)}{a - b}\right)^2} [1 + 2\|f'\|_{L^\infty}\|f''\|_{L^\infty}(b - a) + \|f''\|_{L^\infty}^2(b - a)^2]. \end{aligned}$$

Thus we have

$$\begin{aligned} \mathcal{H}^1(\widehat{AB}) &= \int_a^b \sqrt{1 + f'(x)^2} dx \\ &\leq (b - a) \sqrt{1 + \left(\frac{f(a) - f(b)}{a - b}\right)^2} [1 + 2\|f'\|_{L^\infty}\|f''\|_{L^\infty}(b - a) + \|f''\|_{L^\infty}^2(b - a)^2] \\ &= \mathcal{H}^1(\mathcal{C}) \{1 + (b - a)\|f''\|_{L^\infty}[2\|f'\|_{L^\infty} + \|f''\|_{L^\infty}(b - a)]\}. \end{aligned}$$

□

We now state another technical lemma which give an upper bound for the height of the curve w.r.t. its chord.

Lemma 11. *Let $0 \leq a < b \leq \eta$, $f \in C^2([0, \eta], \mathbb{R}^+)$ be a strictly concave function and let C_f be the graph of f in an orthonormal frame. Let $A = (a, f(a))$ and $B = (b, f(b))$ be two points of C_f .*

For $C \in [AB]$ we denote by \tilde{C} the intersection point of C_f with the line orthogonal to $[AB]$ passing by C [from Remark 1.2, \tilde{C} is well defined].

We have

$$\mathcal{H}^1([C\tilde{C}]) \leq (b - a)^2 \|f''\|_{L^\infty}.$$

Proof. We first note that if we let \tilde{x} be the abscissa of \tilde{C} and if we let P be the point of $[AB]$ having \tilde{x} for abscissa, then $\mathcal{H}^1([P\tilde{C}]) \geq \mathcal{H}^1([C\tilde{C}])$ since the triangle $CP\tilde{C}$ is right-angled in C .

From the Mean Value Theorem, there exist

- $c_x \in]a, \tilde{x}[$ s.t. $f'(c_x) = \frac{f(\tilde{x}) - f(a)}{\tilde{x} - a}$
- $c \in]a, b[$ s.t. $f'(c) = \frac{f(b) - f(a)}{b - a}$
- \tilde{c} between c and c_x s.t. $f''(\tilde{c}) = \frac{f'(c_x) - f'(c)}{c_x - c}$

Moreover

$$\begin{aligned} \mathcal{H}^1([P\tilde{C}]) &= \left| f(\tilde{x}) - \left[(\tilde{x} - a) \frac{f(b) - f(a)}{b - a} + f(a) \right] \right| \\ &= |\tilde{x} - a| \left| \frac{f(\tilde{x}) - f(a)}{\tilde{x} - a} - \frac{f(b) - f(a)}{b - a} \right| \\ &= |\tilde{x} - a| |f'(c_x) - f'(c)| \\ &= |\tilde{x} - a| |c_x - c| |f''(\tilde{c})| \\ &\leq (b - a)^2 \|f''\|_{L^\infty} \end{aligned}$$

which gives the result. □

A.2. Proof of Proposition 12. We prove that

$$(25) \quad \liminf_{N \rightarrow \infty} \mathcal{H}^1(\mathcal{K}_N) > 0.$$

Step 1. We prove that $\max_{k=1, \dots, 2^N} \mathcal{H}^1(\mathcal{C}_k^N) \leq \left(\frac{2}{3}\right)^N$

For $N \geq 1$ we let $\{K_k^N \mid k = 1, \dots, 2^N\}$ be the set of the connected components of \mathcal{K}_N . We let \mathcal{C}_k^N be the chord of K_k^N and we define $\mu_N = \max_{k=1, \dots, 2^N} \mathcal{H}^1(\mathcal{C}_k^N)$. Note that by (10) we have $\mu_0 < 1$.

We first prove that for $N \geq 0$ we have

$$(26) \quad \mu_{N+1} \leq \frac{2}{3}\mu_N.$$

With the help of a standard recursive argument, we may prove that (26) implies [because $\mu_0 < 1$]

$$(27) \quad \mu_N \leq \left(\frac{2}{3}\right)^N.$$

In order to get (26), we prove that for $N \geq 1$ and K_k^N a connected component of \mathcal{K}_N and \mathcal{C}_k^N its chord, we have

$$(28) \quad \mathcal{H}^1[\mathcal{C}] \leq \frac{2\mathcal{H}^1(\mathcal{C}_k^N)}{3} \text{ for } \mathcal{C} \in \mathcal{S}(\mathcal{C}_k^N)$$

[see Notations 2 for $\mathcal{S}(\cdot)$, the set of sons of a chord].

Let $N \geq 1$, for $k \in \{1, \dots, 2^N\}$ we let K_k^N be a connected component of \mathcal{K}_N . We let $K_{2k-1}^{N+1}, K_{2k}^{N+1} \in \mathcal{S}(K_k^N)$ be the curve obtained from K_k^N in the heredity step.

For $\tilde{k} \in \{2k-1, 2k\}$, we let $\mathcal{C}_{\tilde{k}}^{N+1}$ be the chord of $K_{\tilde{k}}^{N+1}$.

In the frame \mathcal{R}_0 , we may define four points of Γ , $(a_1, f(a_1)), (b_1, f(b_1)), (a_2, f(a_2)), (b_2, f(b_2))$, with $0 < a_1 < b_1 < a_2 < b_2 < \eta$ s.t.:

- the extremities of K_{2k-1}^{N+1} are $(a_1, f(a_1)) \& (b_1, f(b_1))$;
- the extremities of K_{2k}^{N+1} are $(a_2, f(a_2)) \& (b_2, f(b_2))$;
- the extremities of K_k^N are $(a_1, f(a_1)) \& (b_2, f(b_2))$.

In the frame \mathcal{R}_0 we let also $(\alpha_1, \beta_1), (\alpha_2, \beta_2)$ be the coordinates of the points of \mathcal{C}_k^N s.t. for $l \in \{1, 2\}$, the triangles whose vertices are $\{(a_l, f(a_l)); (b_l, f(b_l)); (\alpha_l, \beta_l)\}$ are right angled in (α_l, β_l) .

We denote

- \mathcal{I}_1 the segment $[(b_1, f(b_1)); (\alpha_1, \beta_1)]$;
- \mathcal{I}_2 the segment $[(a_2, f(a_2)); (\alpha_2, \beta_2)]$.

From the construction of $K_{2k-1}^{N+1} \& K_{2k}^{N+1}$ and from Pythagore theorem we have for $l = 1, 2$

$$\mathcal{H}^1(\mathcal{C}_{2k-2+l}^{N+1})^2 = \mathcal{H}^1(\mathcal{I}_l)^2 + \left(\frac{\mathcal{H}^1(\mathcal{C}_k^N) - \mathcal{H}^1(\mathcal{C}_k^N)^2}{2} \right)^2.$$

Using Lemma 11 we get that

$$\mathcal{H}^1(\mathcal{I}_l) \leq (b_2 - a_1)^2 \|f''\|_{L^\infty}.$$

On the other hand we have obviously $b_2 - a_1 \leq \mathcal{H}^1(\mathcal{C}_k^N)$. Consequently we get

$$\begin{aligned} \mathcal{H}^1(\mathcal{C}_{2k-2+l}^{N+1})^2 &\leq \mathcal{H}^1(\mathcal{C}_k^N)^4 \|f''\|_{L^\infty}^2 + \left(\frac{\mathcal{H}^1(\mathcal{C}_k^N) - \mathcal{H}^1(\mathcal{C}_k^N)^2}{2} \right)^2 \\ &\leq \mathcal{H}^1(\mathcal{C}_k^N)^4 \|f''\|_{L^\infty}^2 + \frac{\mathcal{H}^1(\mathcal{C}_k^N)^2}{4}. \end{aligned}$$

Therefore

$$\mathcal{H}^1(\mathcal{C}_{2k-2+l}^{N+1}) \leq \frac{\mathcal{H}^1(\mathcal{C}_k^N)}{2} \sqrt{1 + 4\|f''\|_{L^\infty}^2 \mathcal{H}^1(\mathcal{C}_k^N)^2},$$

thus using (11) we get

$$\mathcal{H}^1(\mathcal{C}_{2k-2+l}^{N+1}) \leq \frac{2\mathcal{H}^1(\mathcal{C}_k^N)}{3}.$$

The last estimate gives (28) and thus (27) holds.

Step 2. We prove that $\liminf_{N \rightarrow \infty} \sum_{k=1}^{2^N} \mathcal{H}^1(\mathcal{C}_k^N) > 0$

For $N \geq 1$, we let

$$c_N = \sum_{k=1}^{2^N} \mathcal{H}^1(\mathcal{C}_k^N).$$

The main ingredient in this step consists in noting that, a son of \mathcal{C}_k^N is an hypotenuse of a right angled triangle which admits a side with length $\frac{\mathcal{H}^1(\mathcal{C}_k^N) - \mathcal{H}^1(\mathcal{C}_k^N)^2}{2}$.

Consequently we have

$$\mathcal{H}^1(\mathcal{C}_{2k-1}^{N+1}) + \mathcal{H}^1(\mathcal{C}_{2k}^{N+1}) \geq \mathcal{H}^1(\mathcal{C}_k^N) - \mathcal{H}^1(\mathcal{C}_k^N)^2.$$

Thus, summing the previous inequality for $k = 1, \dots, 2^N$ we get

$$\begin{aligned} c_{N+1} &= \sum_{k=1}^{2^N} \mathcal{H}^1(\mathcal{C}_{2k-1}^{N+1}) + \mathcal{H}^1(\mathcal{C}_{2k}^{N+1}) \\ &\geq \sum_{k=1}^{2^N} \mathcal{H}^1(\mathcal{C}_k^N) [1 - \mathcal{H}^1(\mathcal{C}_k^N)] \\ &\geq c_N (1 - \mu_N) \\ &\geq c_N \left[1 - \left(\frac{2}{3}\right)^N \right]. \end{aligned}$$

By a standard recursive argument we obtain for $N \geq 2$

$$\begin{aligned} c_N &\geq c_1 \prod_{k=1}^{N-1} \left[1 - \left(\frac{2}{3}\right)^k \right] \\ &= c_1 \times \exp \left[\sum_{k=1}^{N-1} \ln \left[1 - \left(\frac{2}{3}\right)^k \right] \right]. \end{aligned}$$

It is clear that $\liminf_N \sum_{l=1}^{N-1} \ln \left[1 - \left(\frac{2}{3}\right)^k \right] > -\infty$, thus $\liminf_N c_N > 0$.

Step 3. We prove (25).

Since for K_k^N , a connected component of \mathcal{K}_N , and \mathcal{C}_k^N its chord, we have $\mathcal{H}^1(K_k^N) \geq \mathcal{H}^1(\mathcal{C}_k^N)$, from Step 2 we get (25).

APPENDIX B. RESULTS RELATED WITH THE SET B_∞

In this section we use the notations of Section 4.

Proposition 12. *Let $\gamma \subset \widehat{AB}$ be a curve and let \mathcal{C} be its chord. We let γ_1, γ_2 be the curves included in γ obtained by the heredity construction represented Figure 1 [section 4.2]. For $l = 1, 2$, we denote also by \mathcal{C}_l the chord of γ_l and by T_l the right-angled triangle having \mathcal{C}_l as side of the right-angle and having its hypotenuse included in \mathcal{C} .*

If $\mathcal{H}^1(\mathcal{C}) < (4\|f''\|_{L^\infty}^2)^{-2}$, then the hypotenuses of the triangles T_1 and T_2 have their length strictly lower than $\frac{\mathcal{H}^1(\mathcal{C})}{2}$. And in particular the triangles T_1 and T_2 are disjoint.

Proof. We model the statement by denoting M and Q the extremities of γ and N and P are points s.t.:

- M, N are the extremities of γ_1
- P, Q are the extremities of γ_2 .

We denote $\delta = \mathcal{H}^1([MQ])$.

We fix an orthonormal frame $\tilde{\mathcal{R}}$ with the origin in M , with the x -axe (MQ) and s.t. N, P, Q have respectively for coordinates (x_1, y_1) , (x_2, y_2) and $(x_3, 0)$ where $0 < x_1 < x_2 < x_3$ and $y_1, y_2 > 0$.

By construction we have

$$x_1 = \frac{\delta - \delta^2}{2}, \quad x_2 = \frac{\delta + \delta^2}{2} \quad \text{and} \quad x_3 = \delta.$$

Moreover, from Lemma 11 we have [recall that \widehat{AB} is the graph of a function f in an other orthonormal frame]:

$$0 < y_1, y_2 \leq \delta^2 \|f''\|_{L^\infty}.$$

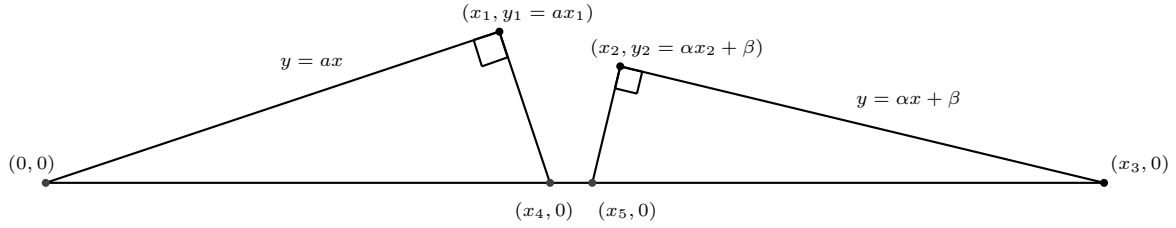


FIGURE 6. Modelisation of the situation

From these points, in Section 4.2, we defined two right-angled triangles having their hypotenuses included in the x -axes.

The first triangle admits for vertices the origin $(0, 0)$, (x_1, y_1) and a point of the x -axis $(x_4, 0)$. This triangle is right angled in (x_1, y_1) . In the frame $\tilde{\mathcal{R}}$, one of the side of the right-angle is included in the line parametrized by the cartesian equation $y = ax$. Since $\delta \leq 1/2$

$$|a| = \left| \frac{y_1}{x_1} \right| \leq \frac{2\delta^2 \|f''\|_{L^\infty}}{\delta - \delta^2} \leq 4\|f''\|_{L^\infty} \delta.$$

The second triangle admits for vertices (x_2, y_2) , $(x_3, 0)$ and a point of the x -axes $(x_5, 0)$. This triangle is right-angled in (x_2, y_2) . In the frame $\tilde{\mathcal{R}}$, one of the side of the right-angle is included in the line parametrized by the cartesian equation $y = \alpha x + \beta$ where

$$|\alpha| = \left| \frac{y_2}{x_2 - x_3} \right| \leq \frac{2\delta^2 \|f''\|_{L^\infty}}{\delta - \delta^2} \leq 4\|f''\|_{L^\infty} \delta.$$

The proof of the proposition consists in obtaining

$$x_4 < \frac{x_3}{2} \text{ and } x_3 - x_5 < \frac{x_3}{2}.$$

We get the first estimate. With the help of Pythagore Theorem we have

$$x_1^2 + y_1^2 + (x_1 - x_4)^2 + y_1^2 = x_4^2.$$

By noting that $y_1 = ax_1$ we have

$$x_4 = (1 + a^2)x_1.$$

Thus:

$$\begin{aligned} x_4 < \frac{x_3}{2} &\iff (1 + a^2) \frac{\delta - \delta^2}{2} < \frac{\delta}{2} \\ &\iff (1 + 16\|f''\|_{L^\infty}^2 \delta^2)(1 - \delta) < 1 \\ &\iff \delta - \delta^2 < \frac{1}{16\|f''\|_{L^\infty}^2} \\ &\iff \delta < \frac{1}{16\|f''\|_{L^\infty}^2}. \end{aligned}$$

Following the same strategy we get that if $\delta < \frac{1}{16\|f''\|_{L^\infty}^2}$ then $x_3 - x_5 < \frac{x_3}{2}$. \square

APPENDIX C. ADAPTATION OF A RESULT OF GIUSTI IN [Giu84]

In this appendix we present briefly the proof of Theorem 2.16&Remark 2.17 in [Giu84]. The argument we present below follows the proof of Theorem 2.15 in [Giu84].

Proposition 13. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set of class C^2 and let $h \in L^1(\partial\Omega)$. For all $\varepsilon > 0$ there exists $u_\varepsilon \in W^{1,1}(\Omega)$ s.t. $\text{tr}_{\partial\Omega} u_\varepsilon = h$ and*

$$\|u_\varepsilon\|_{W^{1,1}(\Omega)} := \|u_\varepsilon\|_{L^1(\Omega)} + \|\nabla u_\varepsilon\|_{L^1(\Omega)} \leq (1 + \varepsilon)\|h\|_{L^1(\Omega)}.$$

Proof. We sketch the proof of Proposition 13. Let $h \in L^1(\partial\Omega)$ and let $\varepsilon > 0$ be sufficiently small s.t.

$$(1 + \varepsilon^2)^2 + \varepsilon^2 + \varepsilon^4 < 1 + \frac{\varepsilon}{2} \text{ and } (1 + \varepsilon^2)\varepsilon^2 < \frac{\varepsilon}{2}.$$

Step 1. We may consider $\eta > 0$ sufficiently small s.t. in $\Omega_\eta := \{x \in \Omega \mid \text{dist}(x, \partial\Omega) < \eta\}$ we have:

(1) The function

$$\begin{aligned} d : \Omega_\eta &\rightarrow]0, \eta[\\ x &\mapsto \text{dist}(x, \partial\Omega) \end{aligned}$$

is of class C^2 and satisfy $|\nabla d| \geq 1/2$.

(2) The orthogonal projection on $\partial\Omega$, $\Pi_{\partial\Omega}$, is Lipschitz.

We now fix a sequence $(h_k)_k \subset C^\infty(\partial\Omega)$ s.t. $h_k \xrightarrow{L^1} h$. We may assume that (up to replace the first term and to consider an extraction):

- (1) $h_0 \equiv 0$
- (2) $\sum_{k \geq 0} \|h_{k+1} - h_k\|_{L^1} \leq (1 + \varepsilon^2)\|h\|_{L^1}$.

And finally we fix a decreasing sequence $(t_k)_k \subset \mathbb{R}_+^*$ s.t.

- (1) $t_0 < \min(\eta, \varepsilon^2)$ is sufficiently small s.t.
 - $4t_0 \max(1; \|\nabla \Pi_{\partial\Omega}\|_{L^\infty}) \times \max(1, \sup_k \|h_k\|_{L^1}) < \min(\varepsilon^2, \varepsilon^2\|h\|_{L^1})$,
 - for $\varphi \in L^1(\partial\Omega)$ we have for $s \in]0, t_0[$

$$\int_{d^{-1}(\{s\})} |\varphi \circ \Pi_{\partial\Omega}(x)| \leq (1 + \varepsilon^2) \int_{\partial\Omega} |\varphi(x)|.$$

- (2) For $k \geq 1$ we have $t_k \leq \frac{t_0 \|h\|_{L^1}}{2^k (1 + \|\nabla h_k\|_{L^\infty} + \|\nabla h_{k+1}\|_{L^\infty})}$.

Step 2. We define

$$\begin{aligned} u_\varepsilon : \Omega &\rightarrow \mathbb{R} \\ x &\mapsto \begin{cases} \frac{d(x) - t_{k+1}}{t_k - t_{k+1}} h_k \circ \Pi_{\partial\Omega}(x) + \frac{t_k - d(x)}{t_k - t_{k+1}} h_{k+1} \circ \Pi_{\partial\Omega}(x) & \text{if } d(x) \in [t_{k+1}, t_k[\\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

We may easily check that u_ε is locally Lipschitz and thus weakly differentiable.

From the coarea formula and a standard change of variable we have

$$\begin{aligned} \|u_\varepsilon\|_{L^1} &\leq 2 \int_{\{d \leq t_0\}} |u_\varepsilon| |\nabla d| dx \\ &= 2 \int_0^{t_0} \int_{d^{-1}(\{s\})} |u_\varepsilon| dx \\ &\leq 2 \sum_{k \geq 0} \int_{t_{k+1}}^{t_k} ds \int_{d^{-1}(\{s\})} |u_\varepsilon| dx \\ &\leq 2 \sum_{k \geq 0} \int_{t_{k+1}}^{t_k} ds \int_{d^{-1}(\{s\})} [|h_k \circ \Pi_{\partial\Omega}(x)| + |h_{k+1} \circ \Pi_{\partial\Omega}(x)|] dx \\ &\leq 2(1 + \varepsilon^2) \sum_{k \geq 0} \int_{t_{k+1}}^{t_k} ds \int_{\partial\Omega} [|h_k(x)| + |h_{k+1}(x)|] dx \\ &\leq 2(1 + \varepsilon^2) \sum_{k \geq 0} (t_k - t_{k+1}) (\|h_k\|_{L^1} + \|h_{k+1}\|_{L^1}) \\ &\leq 4(1 + \varepsilon^2) t_0 \sup_k \|h_k\|_{L^1} \\ &\leq (1 + \varepsilon^2) \varepsilon^2 \|h\|_{L^1} \\ &\leq \frac{\varepsilon}{2} \|h\|_{L^1}. \end{aligned}$$

We now estimate $\|\nabla u_\varepsilon\|_{L^1}$. It is easy to check that if $d(x) \in]t_{k+1}, t_k[$ then we have

$$|\nabla u_\varepsilon(x)| \leq |\nabla d(x)| \left[\frac{|h_k \circ \Pi_{\partial\Omega}(x) - h_{k+1} \circ \Pi_{\partial\Omega}(x)|}{t_k - t_{k+1}} + 2\|\nabla \Pi_{\partial\Omega}\|_{L^\infty} [|\nabla h_k| \circ \Pi_{\partial\Omega}(x) + |\nabla h_{k+1}| \circ \Pi_{\partial\Omega}(x)] \right].$$

Consequently we get

$$\begin{aligned} \|\nabla u_\varepsilon\|_{L^1} &\leq (1 + \varepsilon^2) \sum_{k \geq 0} \left\{ \int_{t_{k+1}}^{t_k} \frac{\|h_{k+1} - h_k\|_{L^1}}{t_k - t_{k+1}} + 2\|\nabla \Pi_{\partial\Omega}\|_{L^\infty} (t_k - t_{k+1}) (\|\nabla h_{k+1}\|_{L^1} + \|\nabla h_k\|_{L^1}) \right\} \\ &\leq (1 + \varepsilon^2) [(1 + \varepsilon^2)\|h\|_{L^1} + 2\|\nabla \Pi_{\partial\Omega}\|_{L^\infty} t_0 \|h\|_{L^1}] \\ &\leq (1 + \varepsilon^2) [(1 + \varepsilon^2) + \varepsilon^2] \|h\|_{L^1} \\ &\leq (1 + \varepsilon/2) \|h\|_{L^1}. \end{aligned}$$

Consequently $u_\varepsilon \in W^{1,1}(\Omega)$ and $\|u_\varepsilon\|_{W^{1,1}} \leq (1 + \varepsilon) \|h\|_{L^1}$.

In order to end the proof it suffices to check that $\text{tr}_{\partial\Omega}(u_\varepsilon) = h$. The justification of this property follows the argument of Lemma 2.4 in [Giu84]. \square

Acknowledgements. The author would like to thank Petru Mironescu for fruitful discussions.

REFERENCES

- [Gag57] E. Gagliardo, *Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili*, Rendiconti del seminario matematico della università di Padova **27** (1957), 284–305.
- [Giu84] E. Giusti, *Minimal surfaces and functions of bounded variation*, no. 80, Springer Science & Business Media, 1984.
- [Mir15] P. Mironescu, *Note on Gagliardo's theorem "tr $W^{1,1} = L^1$ "*, Annals of the University of Bucharest (mathematical series) **6 (LXIV)** (2015), no. 1, 99–103.
- [ST14] G. Spradlin and A. Tamasan, *Not all traces on the circle come from functions of least gradient in the disk*, Indiana University Mathematics Journal **63** (2014), no. 3, 1819–1837.
- [SWZ92] P. Sternberg, G. Williams, and W. Ziemer, *Existence, uniqueness, and regularity for functions of least gradient.*, Journal für die reine und angewandte Mathematik **430** (1992), 35–60.

M. DOS SANTOS, UNIVERSITÉ PARIS EST-CRÉTEIL, 61 AVENUE DU GÉNÉRAL DE GAULLE, 94010 CRÉTEIL CEDEX,
E-mail address: mickael.dos-santos@u-pec.fr