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Abstract

We study a class of one parameter (denoted by κ) family of quasilinear Schrödinger equations arising in the theory of superfluid film in plasma physics. Using variational techniques, we prove the orbital instability of solitary waves for small values of the parameter κ, which gives an answer to a question raised in [9].

1 Introduction

Many physical phenomena are described by quasilinear Schrödinger equations of the form

$$\begin{cases} iu_t + \Delta u + u\ell'(|u|^2)\Delta\ell(|u|^2) + f(|u|^2)u = 0 & \text{in } (0,\infty)\times\mathbb{R}^N, \\ u(0,x) = a_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

eq.schr0-intro

where ℓ and f are given functions, i is the imaginary unit, $N \geq 1$, $u : \mathbb{R}^N \to \mathbb{C}$ is a complex valued function. For example, the case $\ell(s) = \sqrt{1+s}$ is used to modelize the self-channeling of a high-power ultra short laser in matter (see [3, 10, 22]). If $\ell(s) = \sqrt{s}$, equation (1.1) appears in dissipative quantum mechanics ([12]). This model equation is also used in plasma physics and fluid mechanics ([11, 18]), in the theory of Heisenberg ferromagnets and magnons ([2]) and in condensed matter theory ([20]). However, little is known about

AMS classification scheme number: 35Q55, 35B35 the Cauchy problem (1.1) (see [6, 10, 13]) and the question of global well-posedness is still an open problem in many cases. In this direction, many efforts have been made to prove existence and stability of particular global solutions such as solitary waves (see [7, 8, 9, 19],). In particular, these problems were addressed in [9] where the following equation is studied (taking $\ell(s) = s,\ f(s) = s^{\frac{p-1}{2}}$)

$$iu_t + \Delta u + \kappa u \Delta |u|^2 + |u|^{p-1}u = 0 \quad \text{in } (0,\infty) \times \mathbb{R}^N \tag{1.2}$$ eq1

with $\kappa = 1$. In [9], a local well-posedness theory is proposed in Sobolev spaces H^s with s large and reads as follows.

cjs1 **Theorem 1** ([9]). *Let $N \geq 1$, $s = 2E(N/2) + 2$ ($E(z)$ stands for the integer part of z) and assume that $a_0 \in H^{s+2}(\mathbb{R}^N)$ and $f(s) = s^{\frac{p-1}{2}} \in C^{s+2}(\mathbb{R}_+)$. Then there exist $T > 0$ and a unique solution u to (1.2) satisfying*

$$u(0,x) = a_0(x),$$

$$u \in L^\infty(0,T; H^{s+2}(\mathbb{R}^N)) \cap C([0,T]; H^s(\mathbb{R}^N)),$$

and the conservation laws

$$\|u(t)\|_2 = \|a_0\|_2, \tag{1.3}$$ massc

$$E_\kappa(u(t)) = E_\kappa(a_0), \tag{1.4}$$ energyc

for all $t \in [0,T]$, where E_κ is defined by

$$E_\kappa(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{\kappa}{4}\int_{\mathbb{R}^N} |\nabla |u|^2|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1} dx. \tag{1.5}$$ enercla

Before going further, let us introduce some notations. For $\omega > 0$ and $\kappa > 0$, we say that $u_{\omega,\kappa}(t,x) = e^{i\omega t}\phi_{\omega,\kappa}(x)$ is a standing wave solution to (1.2) if $\phi_{\omega,\kappa}$ is a solution to

$$-\Delta\phi - \kappa\phi\Delta|\phi|^2 + \omega\phi = |\phi|^{p-1}\phi \quad \text{in } \mathbb{R}^N. \tag{1.6}$$ gs1

Let $m_{\omega,\kappa}$ be such that

$$m_{\omega,\kappa} = \inf\{S_{\omega,\kappa}(\phi) : \phi \text{ is a nontrivial weak solution of (1.6)}\}.$$

Here, $S_{\omega,\kappa}$ is the action associated with (1.6) and reads

$$S_{\omega,\kappa}(\phi) = E_\kappa(\phi) + \frac{\omega}{2}\int_{\mathbb{R}^N} |\phi|^2 dx$$

for $\phi \in X$, where E_κ is defined in (1.5), and

$$X = \{\phi \in H^1(\mathbb{R}^N, \mathbb{C}) : \int_{\mathbb{R}^N} |\nabla|\phi|^2|^2 dx < \infty\}.$$

We denote by $\mathcal{G}_{\omega,\kappa}$ the set of ground states, that is, the solutions ϕ to (1.6) satisfying

$$S_{\omega,\kappa}(\phi) = m_{\omega,\kappa}. \tag{1.7}$$

In [9], the authors prove the existence of regular radially symmetric ground state solution in any dimension considering Equation (1.6) with $\kappa = 1$. Uniqueness of ground states is also obtained in the one-dimensional case, while, in higher dimensions, this question seems to be more delicate. However, a partial result is given in [1] (see Theorem 1.2). Furthermore, for $3 + \frac{4}{N} < p < \frac{3N+2}{N-2}$, a blow-up result is presented in [9] which reads as follows.

cauchy **Theorem 2** ([9]). *Assume that $\kappa > 0$, $\omega > 0$,*

$$3 + \frac{4}{N} < p < \frac{3N+2}{N-2},$$

and that $f(\sigma) = \sigma^{\frac{p-1}{2}} \in C^{s+2}(\mathbb{R}^+)$. Let ϕ be a ground state solution of (1.6). Then, for all $\epsilon > 0$, there exists $a_0 \in H^{s+2}(\mathbb{R}^N)$ such that $\|a_0 - \phi\|_{H^{s+2}(\mathbb{R}^N)} < \epsilon$ and the solution $u(t)$ of (1.2) with $u(0) = a_0$ blows up in finite time in the $H^{s+2}(\mathbb{R}^N)$ norm.

It is then natural to investigate the situation when $1 < p < 3 + \frac{4}{N}$. For this case, in [9] the authors prove a stability result in a weak sense (namely stability in the set), leaving as an open problem the question of orbital stability. Introduce the stability issue for the minimizers of the problem :

$$m_\kappa(c) = \inf\{E_\kappa(u) : \ u \in X, \ \|u\|_2^2 = c\}, \tag{1.8}$$ defvalc

where the energy E_κ is defined in (1.5). The result is then following one.

stab **Theorem 3** ([9]). *Assume that $\kappa > 0$ and*

$$1 < p < 3 + \frac{4}{N},$$

and let $c > 0$ be such that $m_\kappa(c) < 0$. Then the set

$$\mathcal{G}_\kappa(c) = \{\phi \in X : E_\kappa(\phi) = m_\kappa(c), \ \|\phi\|_2^2 = c\}$$

is nonempty. Moreover, if $f(\sigma) = \sigma^{\frac{p-1}{2}} \in C^{s+2}(\mathbb{R}^+)$, then $\mathcal{G}_\kappa(c)$ is stable, that is : for every $\epsilon > 0$, there exists $\delta > 0$ such that, for any initial data $a_0 \in H^{s+2}(\mathbb{R}^N)$ such that $\inf_{\phi \in \mathcal{G}_\kappa(c)} \rho(a_0, \phi) < \delta$ the solution $u(t)$ of (1.2) *with initial condition $u(0) = a_0$ satisfies*

$$\sup_{0<t<T_0} \inf_{\phi \in \mathcal{G}_\kappa(c)} \rho(u(t), \phi) < \epsilon,$$

where $T_0 > 0$ is the existence time for $u(t)$, and we put

$$\rho(v, w) = \|v - w\|_{H^1} + \left| \int_{\mathbb{R}^N} |\nabla |v|^2|^2 dx - \int_{\mathbb{R}^N} |\nabla |w|^2|^2 dx \right| \tag{1.9}$$

rho

for v, $w \in X$.

Note that in [9], a discussion on the values of $m_\kappa(c)$ with respect to p and c is given. See also Definition 3.1 in [23] and Definition 4.1 in [5] for (1.9).

In this context, the aim of this paper is to give a partial answer to the conjecture raised in [9] concerning the orbital stability of the ground state solutions in the particular case $p = 3$ and $N = 3$, that is, we look for the following equation

$$iu_t + \Delta u + \kappa u \Delta |u|^2 + |u|^2 u = 0 \text{ in } (0, \infty) \times \mathbb{R}^3, \tag{1.10}$$

eq2

where κ denotes a positive parameter. Note that, in this article, the parameter κ will play a fundamental role in our analysis. The standing wave solution to (1.10) $u_{\omega,\kappa}(t, x) = e^{i\omega t}\phi_{\omega,\kappa}(x)$ is such that $\phi_{\omega,\kappa}$ solves

$$-\Delta \phi - \kappa \phi \Delta |\phi|^2 + \omega \phi = |\phi|^2 \phi \quad \text{in } \mathbb{R}^3. \tag{1.11}$$

gs2

Remark 1. Here, the assumption $p = 3$ is essential to ensure that $f(s) = s^{\frac{p-1}{2}}$ is regular enough to apply Theorem 1, and also remark that

$$1 + \frac{4}{N} < p = 3 < 3 + \frac{4}{N}$$

for the case $N = 3$. Since $s = 2E(3/2) + 2 = 4$ given in Theorem 1, the Cauchy problem for (1.10) is locally well-posed in $H^6(\mathbb{R}^3)$. It will be then interesting to develop a local existence theory in the energy space but it seems out of reach for the moment.

We first recall the notion of orbital stability we are interested in.

def-os **Definition 1.** We say that a standing wave solution $u_\omega(t,x) = e^{i\omega t}\phi_\omega$ of (1.10) is *orbitally stable* if for any $\epsilon > 0$ there exists $\delta > 0$ such that if $u_0 \in H^6(\mathbb{R}^3)$ and $\rho(u_0, \phi_\omega) < \delta$, then the solution $u(t)$ of (1.10) with $u(0) = u_0$ exists for all $t \geq 0$, and satisfies

$$\sup_{t\geq 0} \inf_{\theta\in\mathbb{R}, y\in\mathbb{R}^3} \rho(u(t), e^{i\theta}\phi_\omega(\cdot + y)) < \epsilon,$$

where ρ is defined in (1.9) with $N = 3$. Otherwise, $e^{i\omega t}\phi_\omega$ is called *orbitally unstable.*

Remark 2. In Definiton 1, we use the function ρ instead of the classical H^1-norm. This comes from the fact that the action $S_{\omega,\kappa}$ and the energy E_κ are not well-defined on $H^1(\mathbb{R}^3)$.

Our main result reads as follows.

main **Theorem 4.** *Assume that $p = 3$, $N = 3$, and let $\omega > 0$, $\kappa > 0$. Let $\phi_{\omega,\kappa}$ be a ground state of Equation* (1.11)*. Then, there exists $\kappa_0 > 0$ such that for all $\kappa \in (0, \kappa_0)$, the standing wave solution $u_{\omega,\kappa}(t,x) = e^{i\omega t}\phi_{\omega,\kappa}(x)$ to* (1.10) *is orbitally unstable in the sense of Definition 1.*

Remark 3. According to Theorem 4, it is then natural to think that the conjecture raised in [9], that is the orbital stability for ground states in the case $1 < p < 3 + \frac{4}{N}$ is false, which is not intuitive. We explain now why Theorem 3 and Theorem 4 are not contradictory. Take any solution v of the minimization problem (1.8) with $c = \|\phi_{\omega,\kappa}\|_2^2$. Then, by classical argument, there exists a Lagrange multiplier ω^* such that v solves

$$\Delta v - \kappa v \Delta |v|^2 + \omega^* v = |v|^2 v. \qquad (1.12)$$ lagrange

On one hand, we don't know if $\omega^* = \omega$ and on the other hand, it is not clear that v is a ground state of Equation (1.12). Moreover, we conjecture that $\phi_{\omega,\kappa} \notin \mathcal{G}_\kappa(c)$ with $c = \|\phi_{\omega,\kappa}\|_2^2$ and $\kappa \in (0, \kappa_0)$.

By the general theory of Grillakis, Shatah and Strauss (see [15, 16]), for fixed κ, the stability/instability issue of a standing wave $u_{\omega,\kappa}(t,x) = e^{i\omega t}\phi_{\omega,\kappa}$ is closely related to the monotonicity of the curve $\Lambda_\kappa \ : \ \omega \longrightarrow ||\phi_{\omega,\kappa}||_2^2$. Indeed, under some spectral properties of the elliptic operator appearing in Equation (1.11), one can say that, for fixed ω, $u_{\omega,\kappa}$ is stable (resp. unstable) if Λ_κ is strictly increasing (resp. decreasing) at ω. However, when dealing with quasilinear operator, it seems very delicate (may be impossible) to obtain the monoticity of Λ_κ. We thus need an alternative argument which is given by the following proposition which was developed in [21, 14].

altarg

Proposition 1. *Let $\phi_{\omega,\kappa}$ be a ground state of Equation* (1.11) *and denote $\phi^\lambda_{\omega,\kappa}(x) = \lambda^{3/2}\phi_{\omega,\kappa}(\lambda x)$. If $\partial^2_\lambda E_\kappa(\phi^\lambda_{\omega,\kappa})|_{\lambda=1} < 0$, then the standing wave solution $u_{\omega,\kappa}(t,x) = e^{i\omega t}\phi_{\omega,\kappa}$ of* (1.10) *is orbitally unstable in the sense of Definition 1.*

This paper is organized as follows. In Section 2, we prove Theorem 4 using Proposition 1. In Section 3 we give the proof of Proposition 1.

2 Proof of Theorem 4

proofmain

In this section, we prove Theorem 4. To this end, we first introduce the following scaling : for every $u \in X$ and $\lambda \in \mathbb{R}_+$, we denote

$$u^\lambda(x) = \lambda^{3/2}u(\lambda x), \quad x \in \mathbb{R}^3.$$

A direct computation gives

$$E_\kappa(u^\lambda) = \frac{\lambda^2}{2}\int_{\mathbb{R}^3} |\nabla u|^2dx + \kappa\frac{\lambda^5}{4}\int_{\mathbb{R}^3} |\nabla|u|^2|^2dx - \frac{\lambda^3}{4}\int_{\mathbb{R}^3} |u|^4dx,$$

$$\partial^2_\lambda E_\kappa(u^\lambda)|_{\lambda=1} = \int_{\mathbb{R}^3} |\nabla u|^2dx + 5\kappa\int_{\mathbb{R}^3} |\nabla|u|^2|^2dx - \frac{3}{2}\int_{\mathbb{R}^3} |u|^4dx.$$

Introducing

$$I_{\omega,\kappa}(u) = \int_{\mathbb{R}^3} |\nabla u|^2dx + \kappa\int_{\mathbb{R}^3} |\nabla|u|^2|^2dx + \omega\int_{\mathbb{R}^3} |u|^2dx - \int_{\mathbb{R}^3} |u|^4dx,$$

and

$$P_{\omega,\kappa}(u) = \frac{1}{3}\Big(\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2dx + \frac{\kappa}{4}\int_{\mathbb{R}^3} |\nabla|u|^2|^2dx\Big) + \frac{\omega}{2}\int_{\mathbb{R}^3} |u|^2dx - \frac{1}{4}\int_{\mathbb{R}^3} |u|^4dx,$$

we recall the classical identities associated with Equation (1.11).

poho

Proposition 2. *Any regular solution ϕ of Equation* (1.11) *satisfies*
i) $I_{\omega,\kappa}(\phi) = 0$,
ii) $P_{\omega,\kappa}(\phi) = 0$ (Pohozaev identity).

Proof. For i), multiply Equation (1.11) by $\overline{u}$ and integrate over $\mathbb{R}^3$. Since it is classical, we omit the details. For the Pohozaev identity ii), we refer to [9], Lemma 3.1. $\square$

Owning Proposition 2 i), for every regular solution ϕ of (1.11), since

$$\kappa \int_{\mathbb{R}^3} |\nabla|\phi|^2|^2 dx = -\int_{\mathbb{R}^3} |\nabla \phi|^2 dx - \omega \int_{\mathbb{R}^3} |\phi|^2 dx + \int_{\mathbb{R}^3} |\phi|^4 dx,$$

one has

$$\partial_\lambda^2 E_\kappa(\phi^\lambda)|_{\lambda=1} = -4 \int_{\mathbb{R}^3} |\nabla \phi|^2 dx - 5\omega \int_{\mathbb{R}^3} |\phi|^2 dx + \frac{7}{2} \int_{\mathbb{R}^3} |\phi|^4 dx.$$

Let $\phi_{\omega,\kappa}$ be a ground state of Equation (1.11). Note that the parameter κ plays an important role in the analysis. According to Proposition 1, the proof of Theorem 4 requires that

$$\partial_\lambda^2 E_\kappa(\phi_{\omega,\kappa}^\lambda)|_{\lambda=1} < 0. \tag{2.1}$$

id1

We then propose a perturbative argument based on the study of the case $\kappa = 0$. More precisely, let us introduce the classical cubic Schrödinger equation (obtained by taking $\kappa = 0$ in (1.10))

$$iu_t + \Delta u + |u|^2 u = 0 \quad \text{in } (0,\infty) \times \mathbb{R}^3, \tag{2.2}$$

eq3

and the associated stationary equation

$$-\Delta \phi + \omega \phi = |\phi|^2 \phi \quad \text{in } \mathbb{R}^3. \tag{2.3}$$

gs3

We set $E = E_0$ and $S_\omega = S_{\omega,0}$. It is then clear that E and S_ω are respectively the energy and the action associated with Equation (2.3). Note that for this equation, the situation is well-known (see [4]), that is, for $\omega > 0$, the standing wave $u_{\omega,0}(t,x) = e^{i\omega t}\phi_\omega(x)$, where ϕ_ω is a ground state of (2.3), is orbitally unstable.

stat1 **Lemma 1.** *Any regular solution ϕ to* (2.3) *satisfies*

$$\int_{\mathbb{R}^3} |\nabla \phi|^2 dx = \frac{3}{4} \int_{\mathbb{R}^3} |\phi|^4 dx, \;\; \omega \int_{\mathbb{R}^3} |\phi|^2 dx = \frac{1}{4} \int_{\mathbb{R}^3} |\phi|^4 dx.$$

Proof. From Proposition 2 taking $\kappa = 0$, ϕ satisfies

$$\begin{cases} \displaystyle\int_{\mathbb{R}^3} |\nabla\phi|^2 dx + \omega \int_{\mathbb{R}^3} |\phi|^2 dx = \int_{\mathbb{R}^3} |\phi|^4 dx, \\ \displaystyle\frac{1}{6}\int_{\mathbb{R}^3} |\nabla\phi|^2 dx + \frac{\omega}{2} \int_{\mathbb{R}^3} |\phi|^2 dx = \frac{1}{4}\int_{\mathbb{R}^3} |\phi|^4 dx \end{cases}$$

from which the result follows immediately by resolving this system. $\square$

Remark 4. Take any non-zero regular solution ϕ_ω of (2.3), then denoting $\phi_\omega^\lambda(x) = \lambda^{3/2}\phi_\omega(\lambda x)$ and using Lemma 1, one obtains directly

$$\begin{aligned} \partial_\lambda^2 E(\phi_\omega^\lambda) &= \int_{\mathbb{R}^3} |\nabla\phi_\omega|^2 dx - \frac{3}{2}\int_{\mathbb{R}^3} |\phi_\omega|^4 dx \\ &= -\frac{3}{4}\int_{\mathbb{R}^3} |\phi_\omega|^4 dx < 0. \end{aligned}$$

We then expect that (2.1) holds for small κ.

Before going further, let us introduce Theorem 1.1 of [1] which gives precise informations, for fixed ω, on the asymptotic behavior of the ground states of (1.11) as $\kappa \longrightarrow 0$.

wata **Theorem 5** ([1])**.** *Suppose $\omega > 0$, $\kappa > 0$, and let $\phi_{\omega,\kappa}$ be a ground state (positive and radial solution) of* (1.11) *. Let ϕ_ω be a ground state (positive and radial solution) of* (2.3)*. Then, $\phi_{\omega,\kappa} \to \phi_\omega$ in $H^1(\mathbb{R}^3)$ as $\kappa \to 0$.*

Equipped with Theorem 5 we are now able to prove that (2.1) holds for small κ.

Proof of Theorem 4. For fixed $\omega > 0$, let $\phi_{\omega,\kappa}$ be a ground state of (1.11) and ϕ_ω given by Theorem 5. By Sobolev embeddings, there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^3} |\phi_{\omega,\kappa} - \phi_\omega|^4 dx \le C||\phi_{\omega,\kappa} - \phi_\omega||_{H^1(\mathbb{R}^3)}^4 \underset{\kappa\to 0}{\longrightarrow} 0.$$

Then Theorem 5 and Lemma 1 ensure that

$$\begin{aligned} \partial_\lambda^2 E_\kappa(\phi_{\omega,\kappa}^\lambda)|_{\lambda=1} &= -4\int_{\mathbb{R}^3} |\nabla\phi_{\omega,\kappa}|^2 dx - 5\omega\int_{\mathbb{R}^3} |\phi_{\omega,\kappa}|^2 dx + \frac{7}{2}\int_{\mathbb{R}^3} |\phi_{\omega,\kappa}|^4 dx \\ &\underset{\kappa\to 0}{\longrightarrow} -4\int_{\mathbb{R}^3} |\nabla\phi_\omega|^2 dx - 5\omega\int_{\mathbb{R}^3} |\phi_\omega|^2 dx + \frac{7}{2}\int_{\mathbb{R}^3} |\phi_\omega|^4 dx \\ &= -\frac{3}{4}\int_{\mathbb{R}^3} |\phi_\omega|^4 dx < 0. \end{aligned}$$

Then by continuity of the curve $\kappa \to \partial_\lambda^2 E_\kappa(\phi_{\omega,\kappa}^\lambda)|_{\lambda=1}$, we deduce that there exists $\kappa_0 > 0$ such that for all $\kappa \in (0, \kappa_0)$, one has

$$\partial_\lambda^2 E_\kappa(\phi_{\omega,\kappa}^\lambda)|_{\lambda=1} < 0.$$

The proof of Theorem 4 follows from Proposition 1.

3 Proof of Proposition 1.

sect3

In this section, we give the proof of Proposition 1 which follows from that of Theorem 3 of [21] (see also [14]). We begin with the variational characterization of the ground states of Equation (1.11). For convenience, we introduce for $v \in X$,

$$R(v) = \frac{1}{4}\int_{\mathbb{R}^3} |\nabla v|^2 dx + \frac{\omega}{4}\int_{\mathbb{R}^3} |v|^2 dx.$$

Note that $R(v) = S_{\omega,\kappa}(v) - \frac{1}{4} I_{\omega,\kappa}(v)$.

varia **Lemma 2.** *Let $\phi_{\omega,\kappa} \in \mathcal{G}_{\omega,\kappa}$. Then*

$$\begin{aligned} i) \quad m_{\omega,\kappa} &= \inf\{R(v) \ : \ v \in X,\ I_{\omega,\kappa}(v) = 0\} \\ &= \inf\{R(v) \ : \ v \in X,\ I_{\omega,\kappa}(v) \leq 0\}, \\ ii) \quad S_{\omega,\kappa}(\phi_{\omega,\kappa}) &= \inf\{S_{\omega,\kappa}(v) \ : \ v \in X,\ R(v) = R(\phi_{\omega,\kappa})\}. \end{aligned}$$

Proof. Denote

$$m_{\omega,\kappa}^1 = \inf\{R(v) \ : \ v \in X,\ I_{\omega,\kappa}(v) = 0\},$$

$$m_{\omega,\kappa}^2 = \inf\{R(v) \ : \ v \in X,\ I_{\omega,\kappa}(v) \leq 0\}.$$

By Lemma 3.4 of [9], one has

$$S_{\omega,\kappa}(\phi_{\omega,\kappa}) = m_{\omega,\kappa} = \inf\{S_{\omega,\kappa}(v) \ : \ v \in X,\ I_{\omega,\kappa}(v) = 0\}.$$

Take any $v \in H^1(\mathbb{R}^3)$ such that $I_{\omega,\kappa}(v) = 0$. Then, since $S_{\omega,\kappa}(v) = R(v)$, we have $m_{\omega,\kappa}^1 = m_{\omega,\kappa}$. Moreover it is clear that $m_{\omega,\kappa}^2 \leq m_{\omega,\kappa}^1$. Let us prove the converse inequality. Let $v \in X$ such that $I_{\omega,\kappa}(v) < 0$. We claim that there exists $\lambda_v \in (0,1)$ such that $I_{\omega,\kappa}(\lambda_v v) = 0$. Indeed, for $\lambda \in \mathbb{R}$,

$$I_{\omega,\kappa}(\lambda v) = \lambda^2 \int_{\mathbb{R}^3} |\nabla v|^2 dx + \lambda^4 \kappa \int_{\mathbb{R}^3} |\nabla |v|^2|^2 dx + \lambda^2 \omega \int_{\mathbb{R}^3} |v|^2 dx - \lambda^4 \int_{\mathbb{R}^3} |v|^4 dx,$$

which provides

$$
\begin{aligned}
\partial_\lambda I_{\omega,\kappa}(\lambda v)|_{\lambda=1} &= 2\int_{\mathbb{R}^3}|\nabla v|^2dx + 4\kappa\int_{\mathbb{R}^3}|\nabla|v|^2|^2dx + 2\omega\int_{\mathbb{R}^3}|v|^2dx - 4\int_{\mathbb{R}^3}|v|^4dx \\
&\leq 4(\int_{\mathbb{R}^3}|\nabla v|^2dx + \kappa\int_{\mathbb{R}^3}|\nabla|v|^2|^2dx + \omega\int_{\mathbb{R}^3}|v|^2dx - \int_{\mathbb{R}^3}|v|^4dx \\
&= 4I_{\omega,\kappa}(v) < 0.
\end{aligned}
$$

Since $I_{\omega,\kappa}(v) < 0$, $I_{\omega,\kappa}(0) = 0$ and $\partial_\lambda I_{\omega,\kappa}(\lambda v) > 0$ for small λ, the claim is proved. Then

$$S_{\omega,\kappa}(\lambda_v v) = R(\lambda_v v) \geq R(\phi_{\omega,\kappa}) = S_{\omega,\kappa}(\phi_{\omega,\kappa}) = m_{\omega\kappa},$$

and thus, since $R(v) \geq R(\lambda_v v)$, one has $m_{\omega\kappa} = m^2_{\omega\kappa}$. Now take $v \in X$ such that $R(v) = R(\phi_{\omega,\kappa})$. If $I_{\omega,\kappa}(v) < 0$, then

$$S_{\omega,\kappa}(v) < R(v) = R(\phi_{\omega,\kappa}) = m_{\omega,\kappa},$$

a contradiction with i). Then $I_{\omega,\kappa}(v) \geq 0$, from which it follows that

$$S_{\omega,\kappa}(v) \geq R(v) = R(\phi_{\omega,\kappa}) = m_{\omega,\kappa}.$$

Hence ii) is proved. □

Following [21], we introduce, for $\delta > 0$ and $\phi_{\omega,\kappa} \in \mathcal{G}_{\omega,\kappa}$, the following set

$$N_\delta(\phi_{\omega,\kappa}) = \left\{ v \in X \ : \ \inf_{\theta\in\mathbb{R},\ y\in\mathbb{R}^3} \rho(v, e^{i\theta}\phi_{\omega,\kappa}(\cdot + y)) < \delta \right\}.$$

maj **Lemma 3.** *Let $\phi_{\omega,\kappa} \in \mathcal{G}_{\omega,\kappa}$. If $\partial^2_\lambda E_\kappa(\phi_{\omega,\kappa})|_{\lambda=1} < 0$, there exist positive constants ε and δ satisfying the following property : for any $v \in N_\delta(\phi_{\omega,\kappa})$ with $||v||_2 = ||\phi_{\omega,\kappa}||_2$, there exists $\lambda(v) \in (1-\varepsilon, 1+\varepsilon)$ such that*

$$E_\kappa(\phi_{\omega,\kappa}) \leq E_\kappa(v) + (\lambda(v) - 1)Q(v),$$

where

$$Q(v) = \int_{\mathbb{R}^3}|\nabla u|^2dx + \frac{5}{4}\kappa\int_{\mathbb{R}^3}|\nabla|u|^2|^2dx - \frac{3}{4}\int_{\mathbb{R}^3}|u|^4dx.$$

Proof. First, remark that $Q(v) = \partial_\lambda E_\kappa(v^\lambda)|_{\lambda=1}$, and recall that $v^\lambda(x) = \lambda^{3/2}v(\lambda x)$. By the continuity of $\partial^2_\lambda E_\kappa(v^\lambda)$ in λ and the fact that $\rho(v, \phi_{\omega,\kappa}) \leq \delta$, one can find $\varepsilon > 0$ and $\delta > 0$ such that $\partial_\lambda^2 E_\kappa(v^\lambda) < 0$ for all $\lambda \in (1-\varepsilon, 1+\varepsilon)$ and $v \in N_\delta(\phi_{\omega,\kappa})$. A second order Taylor expansion at $\lambda = 1$ provides

$$E_\kappa(v^\lambda) \leq E_\kappa(v) + (\lambda - 1)Q(v), \quad \lambda \in (1-\varepsilon, 1+\varepsilon), \quad v \in N_\delta(\phi_{\omega,\kappa}). \tag{3.1}$$ maj1

For any $v \in N_\delta(\phi_{\omega,\kappa})$ satisfying $||v||_2 = ||\phi_{\omega,\kappa}||_2$, take

$$\lambda(v) = \left(\frac{\int_{\mathbb{R}^3} |\nabla \phi_{\omega,\kappa}|^2 dx}{\int_{\mathbb{R}^3} |\nabla v|^2 dx} \right)^{\frac{1}{2}},$$

so that $R(v^{\lambda(v)}) = R(\phi_{\omega,\kappa})$. Furthermore, since $||v^{\lambda(v)}||_2 = ||v||_2 = ||\phi_{\omega,\kappa}||_2$, using Lemma 2, we have

$$\begin{aligned} E_\kappa(v^{\lambda(v)}) &= S_{\omega,\kappa}(v^{\lambda(v)}) - \frac{\omega}{2}||v^{\lambda(v)}||_2^2 \\ &\geq S_{\omega,\kappa}(\phi_{\omega,\kappa}) - \frac{\omega}{2}||\phi_{\omega,\kappa}||_2^2 = E_\kappa(\phi_{\omega,\kappa}). \end{aligned} \tag{3.2}$$ maj2

Combining (3.1) and (3.2), we obtain the desire result. □

Definition 2. Let δ be the constant of Lemma 3. We define

$$\mathcal{A} = \left\{ v \in N_\delta(\phi_{\omega,\kappa});\ E_\kappa(v) < E_\kappa(\phi_{\omega,\kappa}),\ ||v||_2 = ||\phi_{\omega,\kappa}||_2,\ Q(v) < 0 \right\}.$$

Moreover, for any $a_0 \in H^6(\mathbb{R}^3) \cap N_\delta(\phi_{\omega,\kappa})$, we introduce the exit time $T(a_0)$ from $N_\delta(\phi_{\omega,\kappa})$ by

$$T(a_0) = \sup \left\{ T \in (0, T_0)\ :\ u(t) \in N_\delta(\phi_{\omega,\kappa}),\ 0 \leq t \leq T \right\},$$

where $u(t)$ is the solution to (1.10) with initial data a_0, and T_0 is the existence time of $u(t)$.

virolo1 **Lemma 4.** *Let $\phi_{\omega,\kappa} \in \mathcal{G}_{\omega,\kappa}$. If $\partial_\lambda^2 E_\kappa(\phi_{\omega,\kappa})|_{\lambda=1} < 0$, then for any $a_0 \in H^6(\mathbb{R}^3) \cap \mathcal{A}$, one can find $\varepsilon_0 > 0$ depending only on a_0 such that for all $t \in [0, T(a_0))$, $Q(u(t)) \leq -\varepsilon_0$.*

Proof. Starting from $a_0 \in H^6(\mathbb{R}^3) \cap \mathcal{A}$, we first introduce

$$\varepsilon_1 = E_\kappa(\phi_{\omega,\kappa}) - E_\kappa(a_0).$$

By conservation of energy (see Theorem 1), one has $E_\kappa(u(t)) = E_\kappa(a_0)$ and then Lemma 3 provides

$$\varepsilon_1 = E_\kappa(\phi_{\omega,\kappa}) - E_\kappa(a_0) = E_\kappa(\phi_{\omega,\kappa}) - E_\kappa(u(t)) \leq (\lambda(u(t)) - 1)Q(u(t)),$$

as long as $u(t) \in N_\delta(\phi_{\omega,\kappa})$. Then for $0 \leq t < T(a_0)$, one can see that $Q(u(t)) \neq 0$, which provides, by the continuity of $t \longrightarrow Q(u(t))$ and $Q(a_0) < 0$, that $Q(u(t)) < 0$. By Lemma 3, one has $|1 - \lambda(v(t))| \leq \varepsilon$ for all $0 \leq t < T(a_0)$ from which we derive

$$-Q(u(t)) \geq \frac{\varepsilon_1}{1-\lambda(u(t))} \geq \frac{\varepsilon_1}{\varepsilon}, \quad t \in [0, T(a_0)).$$

We then set $\varepsilon_0 = \varepsilon_1/\varepsilon$, which ends the proof of Lemma 4. □

Before giving the proof of Proposition 1, we present a virial-type identity for Equation (1.10) (see Lemma 3.2 of [9]).

virial **Lemma 5.** *Let $a_0 \in H^6(\mathbb{R}^3)$ satisfy $|x|a_0 \in L^2(\mathbb{R}^3)$. Then the solution $u(t)$ of Equation* (1.10) *with $u(0) = a_0$ satisfies*

$$\frac{d^2}{dt^2}\int_{\mathbb{R}^3} |x|^2 |u(t,x)|^2 dx = 8Q(u(t)), \ \ \forall\, t \in [0, T_0),$$

where T_0 is the existence time of $u(t)$.

Proof of Proposition 1. First remark that, since $||\phi^\lambda_{\omega,\kappa}||_2 = ||\phi_{\omega,\kappa}||_2$,

$$Q(\phi_{\omega,\kappa}) = \partial_\lambda E_\kappa(\phi^\lambda_{\omega,\kappa})|_{\lambda=1} = \partial_\lambda S_{\omega,\kappa}(\phi^\lambda_{\omega,\kappa})|_{\lambda=1} = 0.$$

From $\partial^2_\lambda E_\kappa(\phi^\lambda_{\omega,\kappa})|_{\lambda=1} < 0$, we deduce, by a Taylor expansion of order 2, that for $\lambda > 1$ sufficiently close to 1,

$$E_\kappa(\phi^\lambda_{\omega,\kappa}) < E_\kappa(\phi_{\omega,\kappa}), \ \ Q(\phi^\lambda_{\omega,\kappa}) = \lambda\partial_\lambda E_\kappa(\phi^\lambda_{\omega,\kappa}) < 0.$$

Moreover it is obvious that $\lim_{\lambda\to 1}\rho(\phi^\lambda_{\omega,\kappa}, \phi_{\omega,\kappa}) = 0$, which provides that for $\lambda > 1$ sufficiently close to 1, $\phi^\lambda_{\omega,\kappa} \in \mathcal{A}$. Since $\phi_{\omega,\kappa} \in H^6(\mathbb{R}^3)$ and $|x|\phi_{\omega,\kappa} \in L^2(\mathbb{R}^3)$, by Lemma 5, one can write

$$\frac{d^2}{dt^2}||xu_\lambda(t)||_2^2 = 8Q(u_\lambda(t)), \ \ 0 \leq t \leq T(\phi^\lambda_{\omega,\kappa}), \tag{3.3}$$ fin1

where $u_\lambda(t)$ is the solution of Equation (1.10) with $u_\lambda(0) = \phi^\lambda_{\omega,\kappa}$. Applying Lemma 4, one can find $\varepsilon_\lambda > 0$ such that

$$Q(u_\lambda(t)) \leq -\varepsilon_\lambda, \ \ 0 \leq t \leq T(\phi^\lambda_{\omega,\kappa}). \tag{3.4}$$ fin2

As a consequence of (3.3) and (3.4), we conclude that $T(\phi^\lambda_{\omega,\kappa}) < +\infty$. □

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