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TITLE: Looking at old and new isometries between pc-sets in the Planet-4D Model

KEYWORDS: twelve tone, pc-sets, geometrical model, 4D, isometries, orientation

ABSTRACT: The Model introduced by (Baroin 2011b) is richer than the standard cyclic model (Kremer cycle). Beyond the usual T/I group of symmetries, some extra isometries appeared in actual visualizations of musical pieces. The present document elucidates how these isometries are in fact well-known symmetries which do not preserve shape in most previous models.

ACCOMPANYING FILES: Sequence 01.mp4 to Sequence 12.mp4, in low resolution. Full HD files will be uploaded for the online version of the paper.

The static images included in this pdf will be provided separately for the html version.

1 Quick sketch of the model

In this paper we focus on one motivation of the Planet-4D model (Baroin 2011b): to provide a visually intuitive geometric setting of pcs, in accord with our perception of the Euclidian (3D) world, and where major/minor third transpositions shall be elementary rotations. Before reading the mathematical description below, readers are invited to have a look at [Videos/Sequence 05.mp4] to get a feeling of the effectiveness of the model.

Every pitch class in equal temperament is labelled by an integer modulo 12. Their set, $\mathbb{Z}_{12}$, is isomorphic as a group or a graph with the product $\mathbb{Z}_3 \times \mathbb{Z}_4$ (so called 'torus of thirds'). Explicitly, the residue class of n modulo 12 is mapped to the pair $(n \mod 3, n \mod 4)$. Conversely, such a pair of residues $(a, b) \in \mathbb{Z}_3 \times \mathbb{Z}_4$ can be mapped to $4a - 3b$, which is well-defined modulo 12).[1]

Now both sets are turned into their complex representations (see Fig. 1):

$$\mathbb{Z}_3 \approx \{1, e^{2i\pi/3}, e^{-2i\pi/3}\} = \{1, -\frac{1}{2} + \frac{i\sqrt{3}}{2}, -\frac{1}{2} - \frac{i\sqrt{3}}{2}\} \quad \text{whereas} \quad \mathbb{Z}_4 \approx \{1, i, -1, -i\}$$

More precisely, each pair $(a, b) \in \mathbb{Z}_3 \times \mathbb{Z}_4$ is mapped to

$$(e^{2ai\pi/3}, e^{bi\pi/2}) = (\cos(2a\pi/3) + i\sin(2a\pi/3), \cos(b\pi/2) + i\sin(b\pi/2)).$$

The twelve pitch classes thus appear as elements of $\mathbb{C} \times \mathbb{C}$. For instance, C# is 1 in $\mathbb{Z}_{12}$, or equivalently the pair (1,1) in $\mathbb{Z}_3 \times \mathbb{Z}_4$, and turns into $(e^{2i\pi/3}, i)$ in $\mathbb{C} \times \mathbb{C}$.

The ambient space $\mathbb{C} \times \mathbb{C} = \mathbb{C}^2$ can be seen alternatively as $\mathbb{R}^4$ since any complex number is made of two real coordinates. Complex coordinates in $\mathbb{C}^2$ are more convenient for proofs and abstract descriptions, whereas real coordinates in $\mathbb{R}^4$ are preferred for the computation of movies. Whatever system is chosen, we will follow through the decomposition of $\mathbb{Z}_{12}$ into the product $\mathbb{Z}_3 \times \mathbb{Z}_4$ by projecting on the two orthogonal planes which sum up to the overall space, i.e. $\{(z, 0) \in \mathbb{C}^2\}$ or $\{(x, y, 0, 0) \in \mathbb{R}^4\}$ and $\{(0, z') \in \mathbb{C}^2\}$ or $\{(0, 0, x', y') \in \mathbb{R}^4\}$. Using the same example as above, we get for C# the following real coordinates in $\mathbb{R}^4$:

$$(\cos\frac{2\pi}{3}, \sin\frac{2\pi}{3}, \Re(i), \Im(i)) = (-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0, 1)$$

All these pairs $(z, z') \in \mathbb{C}^2$ satisfy $|z| = |z'| = 1$, z (resp. z') being a third (resp. fourth) root of unity.[2] Hence with the standard Hermitian norm in $\mathbb{C}^2$:

$$\|(z, z')\|_2 = \sqrt{|z|^2 + |z'|^2} = \sqrt{2} \qquad (S)$$

If one prefers real coordinates, i.e. $z = x + iy$ and $z' = x' + iy'$ then we get

$$x^2 + y^2 + x'^2 + y'^2 = 2$$

This is the equation of a hypersphere in 4D, a natural extension of the sphere $x^2 + y^2 + z^2 = R^2$ in 3D or the circle $x^2 + y^2 = r^2$ in 2D. Hence the 12 pcs lie on a 4D-polytope[3] on the hypersphere S with radius $\sqrt{2}$, the direct product of the equilateral triangle $\{1, e^{2i\pi/3}, e^{-2i\pi/3}\}$ by the square $\{1, i, -1, -i\}$.

It is more customary to consider the direct product $\mathbb{Z}_3 \times \mathbb{Z}_4$ as embedded in a product of circles, which is *topologically* a torus. But when the usual picture of a torus is drawn, distances are distorted (see

[1] NB: visually $\mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_{12}$ are seen as cyclic graphs, not groups. Their group structures only appear as rotations of the sets, and are embedded in the larger, non commutative groups of isometries in $\mathbb{R}^4$ in the last section.

[2] In the original model (Baroin 2011a, Baroin2011b, factors of respectively $1/\sqrt{3}, 1/\sqrt{2}$ are applied in order to scale all edges to unit size, but this does not change the isometry group studied below. We will stick to this simpler model which provides a better and easier understanding of the algebraic situation.

[3] It is a geometrical object with vertices, edges, faces, like a cube or a dodecahedron, but here in 4D-space.

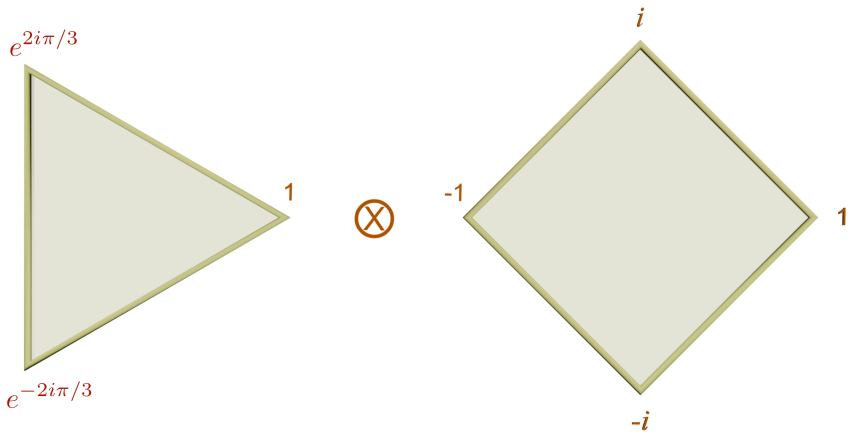

Figure 1: Product $\mathbb{Z}_3 \times \mathbb{Z}_4$ seen in $\mathbb{C} \times \mathbb{C}$

last picture in Fig. 2, where the diminished sevenths – the black squares inscribed in the red horizontal circle – have different sizes): it can be proved that there is no way to embed $\mathbb{Z}_3 \times \mathbb{Z}_4$ isometrically in 3D-space, which is one strong vindication of this 4D-model.[4] See [Videos/Sequence 01.mp4] for the construction[5] of a torus in $\mathbb{R}^3$, and [Videos/Sequence 02.mp4] for the embedding of the product in the hypersphere.

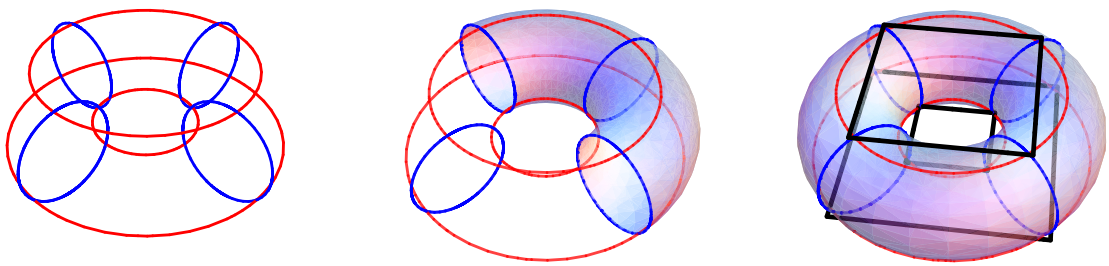

Figure 2: Embedding the torus of thirds in $\mathbb{R}^3$

2 Moving on the hypersphere

2.1 Moving in movies

The two basic moves are major third and minor third transpositions, which appear as rotations respectively in the two complex dimensions of $\mathbb{R}^4 = \mathbb{C}^2$. A video is, obviously, happening on a 2D-screen. The four coordinates of a pc (or pc-set, see below) in our 4D-model are first projected orthographically

[4] The model is drawn on the torus $T^2 = S^1 \times S^1$ defined by $\{(z, z') \in \mathbb{C}^2, |z| = |z'| = 1\}$, which is itself part of the sphere defined by equation (S). Thus is achieved an isometric embedding of the torus of thirds in 4D-space.

[5] We thank G. Mazzola for his kind permission.

to 3D,[6] then drawn as coloured 3D-objects (spheres, cubes...) and finally visualised on the 2D-screen of the computer. This projection induces a small amount of deformation (like the more familiar images of a tesseract, a 4D-hypercube) but this deformation is kept minimal because the model is user-centered, as opposed to most other models: in [Videos 03.mp4], the hypersphere rotates in front of the user, symbolized by a ball (the Planet-4D) centered on the screen.

Each pc is identified by a unique combination of three colours and four shapes. If for instance the colour does not change, it means that component z is constant i.e. that only z' changes, i.e. that we move by minor thirds.

This video shows both elementary rotations, respectively a minor and a major third transposition. A little time is required to get the hang of this visualisation, and especially to understand that these elementary rotations happen in two completely independent subspaces – one rotates the complex component z, the other rotates z', with the axis going through the centers of all triangles, or the centers of all squares, respectively.

As shown on the video, a semi-tone transposition (or a fifth) is thus a combination of two such rotations.

$$\begin{aligned}(z, z') &\mapsto (z.e^{2i\pi/3}, z') : \text{major third transposition}\\ (z, z') &\mapsto (z, i\, z') : \text{minor third transposition}\\ (z, z') &\mapsto (z.e^{2i\pi/3}, -i\, z') : \text{semitone transposition}\end{aligned}$$

The movies are build by interpolating between discrete states, thus providing continuous paths in the spirit of Mazzola's gestures which purport to answer D. Lewin question: "*If I am at s and wish to get to t, what characteristic gesture should I perform in order to arrive there?*" (Lewin 1987).

This model extends to chords (or pc-sets) by the simple expedient of projecting the barycenter G of all pcs involved on the surface of the sphere in P, see Fig. 3 or [Videos/Sequence 04.mp4]. The picture is in 3D for readability but the principle is the same in any dimension: in the example, a spherical triangle (i.e. a triad) is summarized by its center.

The famous progression of major-minor triads in Beethoven's Ninth Symphony in Cohn 1992 can be shown on the Hypersphere with this convention [Videos/Sequence 05.mp4]: the center of the screen shows triplets of numbers – the pcs – and in the center of the triangles, a label for the triad: C, a, F, d... and the smooth, short moves beetween them allow the observer to appreciate the parsimonious chord sequence chosen by the composer.

2.2 The usual isometries

Transpositions (musically speaking) are simply translations in $\mathbb{Z}_{12}$ or the isomorphic $\mathbb{Z}_3 \times \mathbb{Z}_4$. For instance, the semitone transposition T_1 is $x \mapsto x + 1$ in $\mathbb{Z}_{12}$ and $(a, b) \mapsto (a + 1, b - 1)$ in $\mathbb{Z}_3 \times \mathbb{Z}_4$. In the 4D model, this means rotating both complex components in $\mathbb{C}^2$ (up one major third, down one minor third), or equivalently rotating the two pairs of real coordinates in $\mathbb{R}^4$: the matrices of a semi-tone transposition are respectively

$$\begin{pmatrix} e^{2i\pi/3} & 0 \\ 0 & -i \end{pmatrix} \quad \text{in } \mathbb{C}^2 \text{ and} \quad \begin{pmatrix} -1/2 & -\sqrt{3}/2 & 0 & 0 \\ +\sqrt{3}/2 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \text{in } \mathbb{R}^4.$$

[6] The matrix is truncated from a 4×4 orthogonal matrix and has form $\begin{pmatrix} -\sin(t) & \cos(t) & 0 & 0 \\ -\cos(t)\sin(u) & -\sin(t)\sin(u) & \cos(u) & 0 \\ -\cos(t)\cos(u)\sin(v) & -\sin(t)\cos(u)\sin(v) & -\sin(u)\sin(v) & \cos(v) \end{pmatrix}$, as discussed in Zacharias 2000 where the effectiveness of orthographic projections is argued.

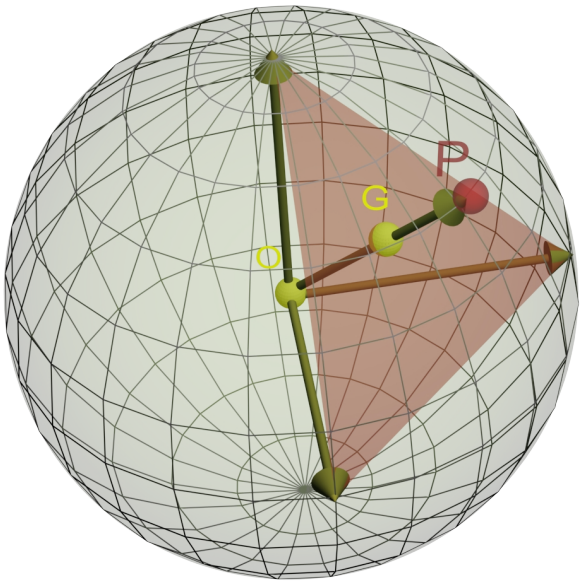

Figure 3: Projection of the barycenter of a triangle on the surface of the sphere

Geometrically (and visually), a transposition is thus composed of rotations of the two orthogonal planes whose direct sum makes up the space. A decomposition of T_2 as two minor thirds up and one major third down is shown on [Videos/Sequence 06.mp4]. See also fig. 4.

Inversions are also rotations in $\mathbb{R}^4$. First notice that an inversion in $\mathbb{Z}_{12}$ is a map $z \mapsto c - z$, which is equivalent via isomorphism with $(x, y) \mapsto (a - x, b - y)$ in $\mathbb{Z}_3 \times \mathbb{Z}_4$. Now each of the components of this map

$$x \mapsto a - x \qquad y \mapsto b - y$$

is an axial symmetry in the equilateral triangle and the square components, respectively.

The direct product of those two symmetries is an orthogonal symmetry around the plane in $\mathbb{R}^4$ built up from the two axes (the eigenvalues are 1,1,-1,-1). This isometry has determinant +1, hence it is technically and visually a rotation in $\mathbb{R}^4$ (with angles 0 and π, so that mathematicians call it a half-turn in 4D) – remember the Beethoven movie. Geometrically, it is an orthogonal symmetry around the plane build from an axis of symmetry in the triangle and another one in the square.[7] It is well known that there are two kinds of inversions, those with two fixed points and those with none: in this model, this depends whether the second axis, in the square, passes through vertices or joins middlepoints of opposite sides. The two kinds are shown repectively on [Videos/Sequence 07.mp4] and [Videos/Sequence 08.mp4]. See also Fig. 5.

An interesting feature of the Planet-4D Model representation is the following:

Proposition 1 *All transformations of the T/I group are* orientation-preserving *isometries in* $\mathbb{R}^4$.

i.e. *inversions no longer reverse shapes*. For instance in Webern's fifth Bagatelle for string quartet, we see 014 *rotated* into 034 (inversion around 2), see [Videos/Sequence 09.mp4].

[7] It is difficult to draw in 4D but easy to imagine: the plane of fixed points of this symmetry intersects both component planes in two axes, which are axes of symmetry respectively of the triangle $\mathbb{Z}_3$ and of the square $\mathbb{Z}_4$.

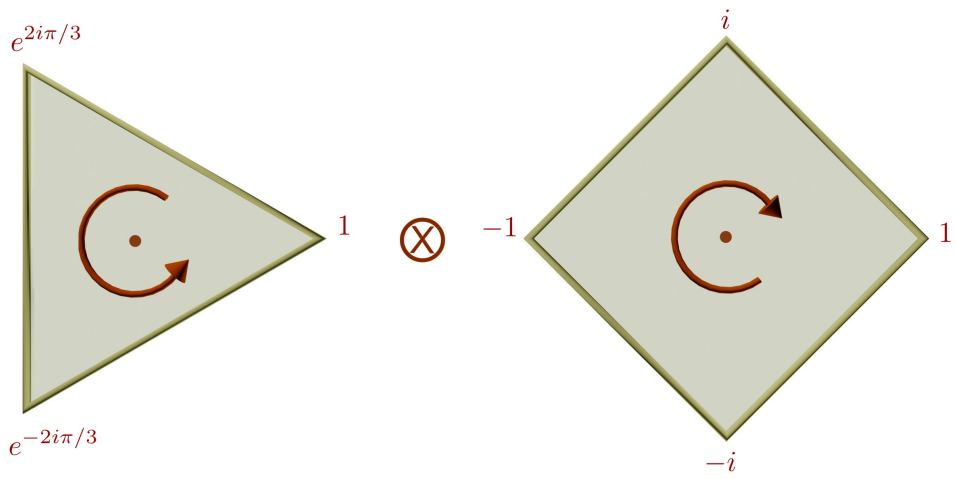

Figure 4: A product of rotations

2.3 The new isometries

During the defence of Baroin 2011a it was noticed that some isometries in 4D-space were not part of the usual T/I group.

A member of the jury then committed a master student to the search for all isometries of the Planet-4D Model, see Baixas 2012. We then discovered that these new isometries, preserving the model, are in fact well known, but even better disguised, transformations of pc-space. Since they had never before been seen as isometries (save once, cf. Thm. 2), and are not easily recognisable besides, we will call them "the new isometries" in the sequel. The result is as follows (with proof in the annex):

Theorem 1 *The Planet-4D Model has 48 isometries.*

To begin with, we can build 48 isometries as direct products of the 6 isometries of an equilateral triangle (the dihedral group D_3) and the 8 isometries of the square (the dihedral group D_4). The resulting

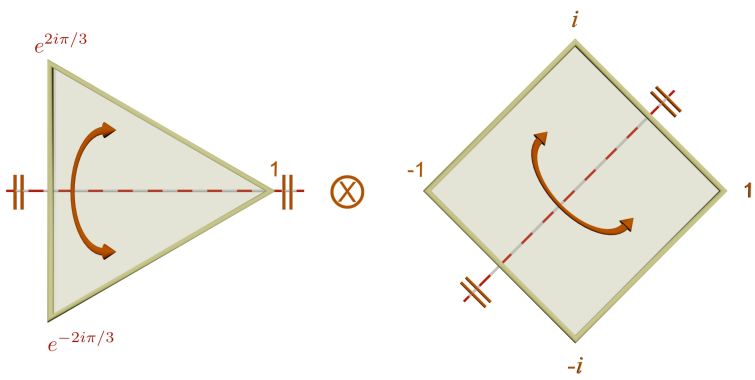

Figure 5: Product of axial symmetries

group is immediately twice larger than T/I, which is seen on $\mathbb{Z}_{12}$ as the dihedral group D_{12}.

Notice that

1. A product of two rotations is a transposition in $\mathbb{Z}_{12}$. There are 3×4 such transpositions.
2. A product of two axial symmetries is an inversion in $\mathbb{Z}_{12}$. There are 3×4 such inversions.

   This takes care of the traditional group T/I and its 24 elements, all rotations (i.e. orientation-preserving isometries) in $\mathbb{R}^4$. The new isometries have the following form:

3. A product of a symmetry by a rotation (or the reverse), which reverses orientation. There are $3 \times 4 + 3 \times 4 = 24$ such isometries.

Here is the 4D-matrix of one of the 24 new isometries, which rotates the triangle and flips the square around one diagonal: $\begin{pmatrix} -1/2 & \sqrt{3}/2 & 0 & 0 \\ -\sqrt{3}/2 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

In the discrete torus of thirds $\mathbb{Z}_3 \times \mathbb{Z}_4$, this is equivalent to $(x, y) \mapsto (x - 1 \mod 3, 1 - y \mod 4)$. An illustration can be found on Fig. 6.

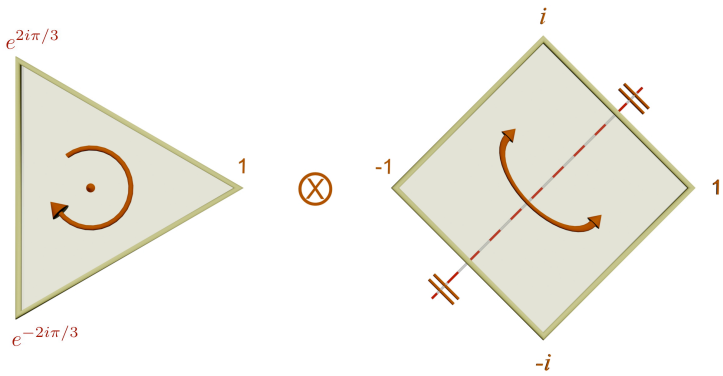

Figure 6: Rotation times axial symmetry

The proof of the theorem checks that the list is exhaustive, all possible isometries of the model pertain to one or the other kind enumerated above.

Major or minor triads are helpful in understanding this larger group: the original T/I group acts simply transitively on these 24 triads, i.e. there is exactly one way to transform one given triad into another one by way of a T/I transformation (this is one of the most famous G.I.S. as defined in Lewin 1987).

But applying any of the new isometries turns these triads into so-called *major-minor trichords*, i.e. translates of 014 or 034, i.e. **3-3** in Forte's classification.[8]

For instance (see [Videos/Sequence 10.mp4]), the C major triad is made of the three pairs

$$C = (0, 0),\ E = (1, 0),\ G = (1, -1) \text{ in } \mathbb{Z}_3 \times \mathbb{Z}_4, \text{ which are turned into } B = (2, 3),\ C = (0, 0),\ E\flat = (0, 3)$$

by the transformation $(x, y) \mapsto (2+x, 3-y)$. Of course the tonal quality is destroyed, as is the intervallic content in $\mathbb{Z}_{12}$ – see for instance what becomes of the aforementioned sequence of triads in Beethoven's Ninth in [Videos/Sequence 11.mp4].

[8] In Noll 2013 these chords are called "jets" and "sharks" in memory of West Side Story.

Some useful features of the new group are given below (Theorem 2 below will explain why they may look familiar to most readers):

Proposition 2 *The product of two new isometries is an old one (an element of the T/I group). More specifically, applying twice one of the new transforms yields either a major third or a tritone transposition.*[9] *The product of a new isometry by any old one is a new one.*

This comes from the characterization of T/I (resp. the new isometries) by the value 1 (resp -1) of their determinant. T/I is a normal subgroup (with index 2) in the 48-element group of all isometries of the 4D-model.

We tried to transform a diatonic or pentatonic scale, the better to grasp the meaning of these new transforms, and noticed a striking fact:

The transform of a diatonic or pentatonic scale by any of the new (indirect) isometries is a chromatic sequence and vice versa.

For instance, $(x, y) \mapsto (x-1, 3-y)$ turns the pentatonic CGDAE or $(0, 0), (1, 3), (2, 2), (0, 1), (1, 0)$ into $(2, 3), (0, 0), (1, 1), (2, 2), (0, 3)$, that is to say is chromatic BCC#DD# (see [Videos/Sequence 12.mp4]).

This means that conversely, the image of a chromatic sequence by one of the new isometries is a fifth-generated scale.

Actually this should not come as a surprise, since the result (in the discrete case) can be read in Mazzola 2002, as usual:

Theorem 2 *The isometries of the torus of thirds $\mathbb{Z}_3 \times \mathbb{Z}_4$ with the 'natural' metric are the affine transforms in the isomorphic ring $\mathbb{Z}_{12}$, e.g. the*

$$x \mapsto a\,x + b \in \mathbb{Z}_{12}, \quad a \in \{1, 5, 7, 11\}, \quad b \in \mathbb{Z}_{12}$$

So the group of isometries of the Planet 4D-model is actually isomorphic with the rather better known affine group on $\mathbb{Z}_{12}$, which extends T/I with the multiplicative operators $M5$ and $M7$ (see for instance Rahn 1980). An independent proof of this is to be found in the annex.

There is another little known result (actually we have not found it stated anywhere) that clarifies the relationship between preserving geometric shapes and the affine group:

Theorem 3 *Any permutation of $\mathbb{Z}_n$ that induces a permutation of orbits under T (or even T/I) must be an affine transformation $x \mapsto a\,x + b$ (with a invertible modulo n).*

"Inducing a permutation of orbits" means that if (say) the image of one particular minor triad is a shark, then the image of *any* major or minor triad must be either a shark or a jet, i.e. *a given geometrical shape is always mapped to the same (other) shape*. This result is actually elementary (proof in annex).

This gives another presentation of the 48 element group, as a semi-direct product $K \rtimes \mathbb{Z}_{12}$ of the Klein group with four elements (the a part in $a\,x + b$) by $\mathbb{Z}_{12}$ acting by translation (the b part in $a\,x + b$), instead of a direct product of two dihedral groups $D_3 \times D_4$ which is the way we constructed it in the first place.

Another way to look at its group structure is as an extension of the traditional dihedral group, $T/I \rtimes \mathbb{Z}_2$ in the sense of Popoff 2013, a group whose action is simply transitive on the union of triads (major/minor) and jets and sharks, wherein the "second dimension" specifies the "shape", i.e. whether one goes to a triad, or to a jet/shark. Considering that there are other, non isomorphic overgroups of T/I, this gives a perspective on the meaningfulness of Thm. 2.

[9] This is reminiscent of Robert Peck's search for roots of inversion Peck 2010: the example provides a square root of a major third transposition, which is not a whole tone transposition.

2.4 Conclusion

So was this worthwhile ? We think it is:

- Mazzola's theorem deals with an abstract feature. Here the result is embodied in physical space, it is made visible, within the grasp of our senses.
- All of those 48-element groups ($D_3 \times D_4$, Aff_{12}, $K \rtimes \mathbb{Z}_{12}$, $D_{12} \rtimes \mathbb{Z}_2$) are isomorphic. But this algebraic truth does not deter from trying to visualise "this" group in the most intuitive way possible. The first one, represented as a group of isometries in Euclidean space, is certainly closest to human intuition (for instance, the so-called 'natural' metric on a Torus is much harder to visualize.[10]).
- Among all possible overgroups of T/I, this group of isometries happens to naturally isomorphic with another known musical group.
- Last but not least, the Planet-4D Model features *continuous* moves. This is obvious on the movies, wherein a move from one pitch class (or chord) to another is not a discontinuous jump but a smooth *gesture*, an instance of the notion defined by the same Mazzola in Mazzola 2007.

These isometries existed already, but only as an abstract/conceptual result, which, like sleeping beauties, needed to be awakened in the physical (4D-)world for us to appreciate their exquisite charm.

Acknowledgements

The authors thank Moreno Andreatta for pointing out the question of these new isometries, Guillaume Baixas for his master thesis describing the whole group, and the anonymous reviewers for their helpful comments and critiques.

3 Annex

3.1 Proof of Thm. 1

We provide a simpler proof than Baixas 2012.

The new isometries are described in the main text. We prove here that the list is now complete. The 48 isometries described henceforth make up a subgroup of the whole group. We endeavour to prove that this subgroup is equal to the whole.

Firstly notice that the Planet-4D Model is a polytope with 12 vertices contains three squares and four equilateral triangles, obtained by moving only one component (either k or ℓ) of the generic point $(e^{2ik\pi/3}, e^{2i\ell\pi/4})$ in complex notation. Next we observe that any isometry of this polytope must send a square on a square and a triangle on a triangle. To be completely rigorous, it must be checked that there are no other squares (resp. equilateral triangles) in the model than these ones, which readily appears from the mutual distances between vertices (see Fig. 7).

Finally we consider an arbitrary isometry preserving the polytope. Say it turns a triangle T_1 into triangle T_2, meaning in complex components that for any k, $\{(e^{2ik\pi/3}, u)\}$ is turned into some $\{(e^{2ik'\pi/3}, v)\}$, $u, v \in \{1, -1, i, -i\}$ indexing the two triangles. By rotating the second component without changing the first (i.e. just rotating v to u), we can turn T_2 back to T_1, so T_1 is fixed as a set. Next we fix all *points* of T_1 by applying the appropriate isometry of the first dihedral group (i.e. just moving the $e^{2ik\pi/3}$ around). We now have an isometry which fixes the first component. Since it is isometric in $\mathbb{C}^2$, it induces an isometry of the second component which describes a square. Applying the inverse isometry which

[10] The torus has first to be flattened on a plane like pizza dough and mentally prolonged periodically, both horizontally and vertically.

	$\{1, 1\}$	$\{1, i\}$	$\{1, -1\}$	$\{1, -i\}$	$\left\{e^{\frac{2i\pi}{3}}, 1\right\}$	$\left\{e^{\frac{2i\pi}{3}}, i\right\}$	$\left\{e^{\frac{2i\pi}{3}}, -1\right\}$	$\left\{e^{\frac{2i\pi}{3}}, -i\right\}$	$\left\{e^{-\frac{2i\pi}{3}}, 1\right\}$	$\left\{e^{-\frac{2i\pi}{3}}, i\right\}$	$\left\{e^{-\frac{2i\pi}{3}}, -1\right\}$	$\left\{e^{-\frac{2i\pi}{3}}, -i\right\}$
$\{1, 1\}$	0	$\sqrt{2}$	2	$\sqrt{2}$	$\sqrt{3}$	$\sqrt{5}$	$\sqrt{7}$	$\sqrt{5}$	$\sqrt{3}$	$\sqrt{5}$	$\sqrt{7}$	$\sqrt{5}$
$\{1, i\}$	$\sqrt{2}$	0	$\sqrt{2}$	2	$\sqrt{5}$	$\sqrt{3}$	$\sqrt{5}$	$\sqrt{7}$	$\sqrt{5}$	$\sqrt{3}$	$\sqrt{5}$	$\sqrt{7}$
$\{1, -1\}$	2	$\sqrt{2}$	0	$\sqrt{2}$	$\sqrt{7}$	$\sqrt{5}$	$\sqrt{3}$	$\sqrt{5}$	$\sqrt{7}$	$\sqrt{5}$	$\sqrt{3}$	$\sqrt{5}$
$\{1, -i\}$	$\sqrt{2}$	2	$\sqrt{2}$	0	$\sqrt{5}$	$\sqrt{7}$	$\sqrt{5}$	$\sqrt{3}$	$\sqrt{5}$	$\sqrt{7}$	$\sqrt{5}$	$\sqrt{3}$
$\left\{e^{\frac{2i\pi}{3}}, 1\right\}$	$\sqrt{3}$	$\sqrt{5}$	$\sqrt{7}$	$\sqrt{5}$	0	$\sqrt{2}$	2	$\sqrt{2}$	$\sqrt{3}$	$\sqrt{5}$	$\sqrt{7}$	$\sqrt{5}$
$\left\{e^{\frac{2i\pi}{3}}, i\right\}$	$\sqrt{5}$	$\sqrt{3}$	$\sqrt{5}$	$\sqrt{7}$	$\sqrt{2}$	0	$\sqrt{2}$	2	$\sqrt{5}$	$\sqrt{3}$	$\sqrt{5}$	$\sqrt{7}$
$\left\{e^{\frac{2i\pi}{3}}, -1\right\}$	$\sqrt{7}$	$\sqrt{5}$	$\sqrt{3}$	$\sqrt{5}$	2	$\sqrt{2}$	0	$\sqrt{2}$	$\sqrt{7}$	$\sqrt{5}$	$\sqrt{3}$	$\sqrt{5}$
$\left\{e^{\frac{2i\pi}{3}}, -i\right\}$	$\sqrt{5}$	$\sqrt{7}$	$\sqrt{5}$	$\sqrt{3}$	$\sqrt{2}$	2	$\sqrt{2}$	0	$\sqrt{5}$	$\sqrt{7}$	$\sqrt{5}$	$\sqrt{3}$
$\left\{e^{-\frac{2i\pi}{3}}, 1\right\}$	$\sqrt{3}$	$\sqrt{5}$	$\sqrt{7}$	$\sqrt{5}$	$\sqrt{3}$	$\sqrt{5}$	$\sqrt{7}$	$\sqrt{5}$	0	$\sqrt{2}$	2	$\sqrt{2}$
$\left\{e^{-\frac{2i\pi}{3}}, i\right\}$	$\sqrt{5}$	$\sqrt{3}$	$\sqrt{5}$	$\sqrt{7}$	$\sqrt{5}$	$\sqrt{3}$	$\sqrt{5}$	$\sqrt{7}$	$\sqrt{2}$	0	$\sqrt{2}$	2
$\left\{e^{-\frac{2i\pi}{3}}, -1\right\}$	$\sqrt{7}$	$\sqrt{5}$	$\sqrt{3}$	$\sqrt{5}$	$\sqrt{7}$	$\sqrt{5}$	$\sqrt{3}$	$\sqrt{5}$	2	$\sqrt{2}$	0	$\sqrt{2}$
$\left\{e^{-\frac{2i\pi}{3}}, -i\right\}$	$\sqrt{5}$	$\sqrt{7}$	$\sqrt{5}$	$\sqrt{3}$	$\sqrt{5}$	$\sqrt{7}$	$\sqrt{5}$	$\sqrt{3}$	$\sqrt{2}$	2	$\sqrt{2}$	0

Figure 7: Distances between pcs on the Planet-4D Model

lies in the second dihedral group D_4 (i.e. moving around the four $e^{2i\ell\pi/4}$) we have all 12 pcs fixed. Any isometry of $\mathbb{C}^2$ which fixes these 12 points must be identity.

We have proved that our initial isometry, when composed with some appropriate combination of the 48 isometries listed before, is reduced to identity: this proves that this isometry is the product of their inverses, e.g. a member of the 48-element subgroup, and hence this subgroup is the whole group.

3.2 Proof of Thm. 2 on affine transforms

We make explicit the isomorphism between $\mathbb{Z}_3 \times \mathbb{Z}_4$ and $\mathbb{Z}_{12}$, e.g.

$$\Phi : (a, b) \mapsto 4a - 3b \mod 12 \quad \text{and conversely} \quad \Phi^{-1}(x) = (x \mod 3, x \mod 4)$$

One easily checks that this makes sense (a (resp. b) being defined modulo 3 (resp. 4), $4a$ (resp. $3b$) is defined modulo 12) and is an injective group morphism, bijective since both groups on either end have the same cardinality. Finally,

$$(4a - 3b \mod 3, 4a - 3b \mod 4) = (a, b) \in \mathbb{Z}_3 \times \mathbb{Z}_4$$

Now the T/I group is generated by transposition $\mathcal{T} : x \mapsto x + 1$ and inversion $\mathcal{I} : x \mapsto -x$, which are turned by Φ into

$$\widetilde{\mathcal{T}} : (a, b) \mapsto (a + 1, b - 1) \qquad \widetilde{\mathcal{I}} : (a, b) \mapsto (-a, -b)$$

This generates the group of transforms $(a, b) \mapsto \pm(a - k \mod 3, b + k \mod 4), k \in \mathbb{Z}_{12}$, a group with 24 elements.

The new elements which are neither transpositions nor inversions change orientation, e.g. (a, b) turns into either $(-a, b)$ or $(a, -b)$. Thus the whole group is made of transforms

$$(a, b) \mapsto (\pm a - k, \pm b + k), k \in \mathbb{Z}_{12}$$

with independent signums for a, b. Back in $\mathbb{Z}_{12}$, if we drop the arbitrary transposition factor k for clarity, we have the correspondence shown on Table 1, since, for instance, $5 \times (4a - 3b \mod 3, 4a - 3b \mod 4) = (20\,a \mod 3, -15\,b \mod 4) = (-a, b)$.

We have recovered exactly the $4 \times 12 = 48$ affine transformations in $\mathbb{Z}_{12}$.

Table 1: Isomorphism between isometries of $\mathbb{Z}_3 \times \mathbb{Z}_4$ and $\mathbb{Z}_{12}$

$(a,b) \mapsto (a,b)$	$x \mapsto x$
$(a,b) \mapsto (-a,-b)$	$x \mapsto -x$
$(a,b) \mapsto (-a,b)$	$x \mapsto 5x$
$(a,b) \mapsto (a,-b)$	$x \mapsto 7x$

3.3 Proof of Thm. 3 on affine transforms and shape

Actually it is enough to reduce the hypothesis to the preservation of dyads, i.e. we assume that σ is a permutation of $\mathbb{Z}_n$ such that *for any dyad* (a,b) *with interval class* $i = |b-a|$*, the image dyad always spans the same interval* $j = |\sigma(b) - \sigma(a)|$ (for instance, any semi-tone is always turned into a fifth – or fourth).

Consider dyad $(0,1)$ and let its image be $(\sigma(0), \sigma(1)) = (\alpha, \beta)$ with $\beta - \alpha = \delta$. Consider also dyad $(1,2)$ with image $(\sigma(1) = \beta, \sigma(2) = \gamma)$: if we work with orbits under T only, then necessarily $\gamma - \beta = \delta$ too. If we work with T/I, then it could happen that the interval is reversed, i.e. $\gamma - \beta = -\delta$. But since $\beta - \alpha = \delta$ this would lead to $\alpha = \gamma$ and σ would no longer be bijective.

Hence $\sigma(2) = \gamma = \gamma - \beta + \beta - \alpha + \alpha = \alpha + 2\delta$. Similarly or by easy induction,

$$\sigma(k) = \alpha + k\delta$$

(confusing $k \in \mathbb{Z}$ and its class modulo n without danger).

This is an affine map. Since σ is bijective, it is necessary and sufficient that δ be invertible in $\mathbb{Z}_n$ and hence we have an affine transformation.

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