



HAL
open science

A mixed variational problem with applications in contact mechanics

Mircea Sofonea, Andaluzia Matei

► **To cite this version:**

Mircea Sofonea, Andaluzia Matei. A mixed variational problem with applications in contact mechanics. *Zeitschrift für Angewandte Mathematik und Physik*, 2015, 66 (6), pp.3573-3589. 10.1007/s00033-015-0573-3 . hal-01215236

HAL Id: hal-01215236

<https://hal.science/hal-01215236>

Submitted on 1 May 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A mixed variational problem with applications in contact mechanics

Mircea Sofonea and Andaluza Matei

Abstract. We provide an existence result in the study of an abstract mixed variational problem which involves a history-dependent operator and two nondifferentiable functionals depending on the solution. The proof relies on generalized saddle point theory and a fixed point argument. Then we consider a new mathematical model which describes the quasistatic frictionless contact process between a viscoplastic body and an obstacle. The contact is modeled with a multivalued normal compliance condition and unilateral constraint. We use our abstract result to prove the weak solvability of this contact problem.

Mathematics Subject Classification. Primary 49J40 · 74M15; Secondary 35M86 · 35M87.

Keywords. Mixed variational problem · Lagrange multiplier · History-dependent operator · Frictionless contact · Normal compliance · Unilateral constraint · Weak solution.

1. Introduction

Mixed variational problems provide a functional framework useful in the analysis and numerical simulation of a large number of nonlinear problems with unilateral constraints. For this reason, they are used in numerical analysis, optimization, theory of partial differential equations, solid mechanics and fluid mechanics, as well. The literature in the field has been growing rapidly in the last decades. Existence and uniqueness results in the study of stationary mixed variational problems with multipliers, together with various applications in Solid Mechanics, can be found in [4, 6, 8, 9, 14, 22] and the references therein. References concerning the analysis of mixed variational problems associated with contact problems include [2, 10–13, 16–20].

Recent results in the study of mixed variational problems were obtained in [24]. There, a system coupling a nonlinear variational equation involving a history-dependent operator with a time-dependent inequality for the multiplier was considered. The unique weak solvability of the problem was proved, and a continuous dependence result was obtained. The proofs were based on results on a class of generalized saddle point problems, various estimates and arguments of fixed point. Then, the abstract results were used in the study of a frictionless contact problem with normal compliance and unilateral constraint. In this way, the unique weak solvability of the problem was proved and the continuous dependence of the weak solution with respect to the viscoplastic constitutive function, the applied forces, the contact conditions and the initial data was obtained.

The current paper represents a continuation of [24], and its aim is twofold. The first one is to study the solvability of a new mixed variational problem involving a multiplier. To this end, as in [24], we use various estimates and fixed point arguments. Nevertheless, in contrast to [24], the present study is based on results on a different class of generalized saddle point problems. The problem we consider here has a different structure, since it is given by a system of two variational inequalities involving two nondifferentiable functionals which depend on the solution and on the history of the solution, respectively.

Therefore, in this paper, we work with different assumptions on the operators and functionals involved in the statement of the problem. The second aim is to show how our abstract result can be used in the analysis of mathematical models in contact mechanics. To this end, we consider the quasistatic process of contact between a viscoplastic body and a deformable foundation. As in [24], we describe the material's behavior with a rate-type viscoplastic constitutive law, and we model the contact with normal compliance and finite penetration. Nevertheless, we recall that the model used in [24] was focused on the memory effects of the foundation and, for this reason, it was assumed that the stiffness coefficient depends on the history of the penetration. In contrast, in the current paper we use a multivalued version of the normal compliance condition, which describes the rigid-elastic properties of the foundation, together with its possible hardening or softening. Considering such kind of model of contact leads to a new and interesting mathematical problem, governed by two nondifferentiable functionals.

Our manuscript is structured as follows. In Sect. 2 we introduce the abstract mixed variational problem we are interested in. We list the assumptions on the data, and then we state and prove an existence result, Theorem 2.1. In Sect. 3 we consider a mathematical model of contact together with its variational formulation. The weak solvability of the model is discussed in Sect. 4. There, we use the abstract result provided by Theorem 2.1 to prove that the model has at least a solution. The paper ends with Sect. 5 in which we present some concluding remarks.

2. An abstract existence result

Everywhere in this paper we use the notation \mathbb{N} for the set of positive integers and \mathbb{R}_+ for set of non-negative real numbers, i.e., $\mathbb{R}_+ = [0, \infty[$. Notation (x, y) will represent an element of the product of the sets X and Y , denoted $X \times Y$. Given a normed space $(X, \|\cdot\|_X)$, we use the notation $C(\mathbb{R}_+; X)$ for the space of continuous functions defined on \mathbb{R}_+ with values on X . Also, for a subset $K \subseteq X$, we use the symbol $C(\mathbb{R}_+; K)$ for the set of continuous functions defined on \mathbb{R}_+ with values in K . For an operator $\mathcal{L} : C(\mathbb{R}_+; X) \rightarrow C(\mathbb{R}_+; X)$, we write $\mathcal{L}\eta(t)$ for the value of the function $\mathcal{L}\eta$ at the point t , i.e., $\mathcal{L}\eta(t) = (\mathcal{L}\eta)(t)$. With these preliminaries, we recall the following fixed point result.

Theorem 2.1. *Let $(X, \|\cdot\|_X)$ be a real Banach space, and let $\mathcal{L} : C(\mathbb{R}_+; X) \rightarrow C(\mathbb{R}_+; X)$ be a nonlinear operator. Assume that there exists $m \in \mathbb{N}$ with the following property: for each $n \in \mathbb{N}$, there exist two constants $c_n \geq 0$ and $k_n \in [0, 1[$ such that for each $t \in [0, n]$,*

$$\|\mathcal{L}u(t) - \mathcal{L}v(t)\|_X^m \leq c_n \int_0^t \|u(s) - v(s)\|_X^m ds + k_n \|u(t) - v(t)\|_X^m \quad (2.1)$$

for all $u, v \in C(\mathbb{R}_+; X)$. Then the operator \mathcal{L} has a unique fixed point $\eta^* \in C(\mathbb{R}_+; X)$.

The proof of Theorem 2.1 can be found in [25] and is based on the properties of the Fréchet space $C(\mathbb{R}_+; X)$.

We now turn to the main result of this section. Let $(X, (\cdot, \cdot)_X, \|\cdot\|_X)$ and $(Y, (\cdot, \cdot)_Y, \|\cdot\|_Y)$ be two real Hilbert spaces, and let $(Z, \|\cdot\|_Z)$ be a normed space. We consider two operators $A : X \rightarrow X$, $\mathcal{S} : C(\mathbb{R}_+; X) \rightarrow C(\mathbb{R}_+; Z)$, two functionals $\varphi : Z \times X \rightarrow \mathbb{R}$, $j : X \times X \rightarrow \mathbb{R}$, a bilinear form $b : X \times Y \rightarrow \mathbb{R}$, a function $f : \mathbb{R}_+ \rightarrow X$, an element $h \in X$ and a set $\Lambda \subseteq Y$. With these data, we introduce the following problem.

Problem 1. Find two functions $u : \mathbb{R}_+ \rightarrow X$ and $\lambda : \mathbb{R}_+ \rightarrow \Lambda$ such that $u \in C(\mathbb{R}_+; X)$ and, for each $t \in \mathbb{R}_+$, the following inequalities hold:

$$\begin{aligned} & (Au(t), v - u(t))_X + \varphi(\mathcal{S}u(t), v) - \varphi(\mathcal{S}u(t), u(t)) + j(u(t), v) \\ & - j(u(t), u(t)) + b(v - u(t), \lambda(t)) \geq (f(t), v - u(t))_X \quad \forall v \in X, \end{aligned} \quad (2.2)$$

$$b(u(t), \mu - \lambda(t)) \leq b(h, \mu - \lambda(t)) \quad \forall \mu \in \Lambda. \quad (2.3)$$

Note that in the statement of Problem 1, the unknowns u and λ do not play a symmetric role, since the regularity $u \in C(\mathbb{R}_+; X)$ is required for the unknown u , but no regularity is required for the unknown λ . This situation arises from the structure of the problem. Indeed, from (2.2), it follows that the solution u cannot be outside the space $C(\mathbb{R}_+; X)$ where the operator \mathcal{S} is defined; in contrast, no restriction is needed for the unknown λ which is involved as second argument of the bilinear form b .

In the study of Problem 1, we consider the following assumptions.

$$\left\{ \begin{array}{l} \text{(a) There exists } m_A > 0 \text{ such that} \\ \quad (Au - Av, u - v)_X \geq m_A \|u - v\|_X^2 \quad \forall u, v \in X. \\ \text{(b) There exists } L_A > 0 \text{ such that} \\ \quad \|Au - Av\|_X \leq L_A \|u - v\|_X \quad \forall u, v \in X. \end{array} \right. \quad (2.4)$$

$$\left\{ \begin{array}{l} \text{For each } n \in \mathbb{N} \text{ there exists } s_n \geq 0 \text{ such that} \\ \quad \|\mathcal{S}u_1(t) - \mathcal{S}u_2(t)\|_Z \leq s_n \int_0^t \|u_1(s) - u_2(s)\|_X \, ds \\ \quad \forall u_1, u_2 \in C(\mathbb{R}_+; X), \quad \forall t \in [0, n]. \end{array} \right. \quad (2.5)$$

$$\left\{ \begin{array}{l} \text{(a) The function } \varphi(\eta, \cdot) : X \rightarrow \mathbb{R} \text{ is convex} \\ \quad \text{and Lipschitz continuous, for any } \eta \in Z. \\ \text{(b) There exists } \beta \geq 0 \text{ such that} \\ \quad \varphi(\eta_1, v_2) - \varphi(\eta_1, v_1) + \varphi(\eta_2, v_1) - \varphi(\eta_2, v_2) \\ \quad \leq \beta \|\eta_1 - \eta_2\|_X \|v_1 - v_2\|_X \quad \forall \eta_1, \eta_2 \in Z, \quad v_1, v_2 \in X. \end{array} \right. \quad (2.6)$$

$$\left\{ \begin{array}{l} \text{(a) The function } j(u, \cdot) : X \rightarrow \mathbb{R} \text{ is convex} \\ \quad \text{and Lipschitz continuous, for any } u \in X. \\ \text{(b) There exists } \alpha \geq 0 \text{ such that} \\ \quad j(u_1, v_2) - j(u_1, v_1) + j(u_2, v_1) - j(u_2, v_2) \\ \quad \leq \alpha \|u_1 - u_2\|_X \|v_1 - v_2\|_X \quad \forall u_1, u_2, v_1, v_2 \in X. \end{array} \right. \quad (2.7)$$

$$\alpha < m_A. \quad (2.8)$$

$$\left\{ \begin{array}{l} b : X \times Y \rightarrow \mathbb{R} \text{ is a bilinear form such that} \\ \text{(a) There exists } M_b > 0 : \\ \quad |b(v, \mu)| \leq M_b \|v\|_X \|\mu\|_Y \quad \forall v \in X, \mu \in Y. \\ \text{(b) There exists } b_0 > 0 : \\ \quad \inf_{\mu \in Y, \mu \neq 0_Y} \sup_{v \in X, v \neq 0_X} \frac{b(v, \mu)}{\|v\|_X \|\mu\|_Y} \geq b_0. \end{array} \right. \quad (2.9)$$

$$f \in C(\mathbb{R}_+; X), \quad h \in X. \quad (2.10)$$

$$\Lambda \text{ is a closed convex subset of } Y \text{ that contains } 0_Y. \quad (2.11)$$

On these assumptions, we have the following comments. First, (2.4) shows that A is a strongly monotone Lipschitz continuous operator. Next, following the terminology introduced in [26], (2.5) shows that \mathcal{S} is a history-dependent operator. Also (2.8) represents a smallness assumption which involves the operator A and the functional j . Finally, condition (2.9)(b) is the so-called inf-sup condition, used in the saddle point theory; see, for instance, [4, 6, 8, 9, 14] and the references therein.

The solvability of Problem 1 is given by the following result.

Theorem 2.1. *Assume (2.4)–(2.11). Then, Problem 1 has a solution (u, λ) , unique in u .*

We recall that as explained in page 3575, a couple of functions $u : \mathbb{R}_+ \rightarrow X$, $\lambda : \mathbb{R}_+ \rightarrow \Lambda$ is said to be a solution of Problem 1 iff (2.2)–(2.3) hold for each $t \in \mathbb{R}_+$ and, moreover, $u \in C(\mathbb{R}_+; X)$.

The proof of Theorem 2.1 will be carried out in several steps, based on the study of several intermediate problems and fixed point arguments. To provide it, we assume in what follows that (2.4)–(2.11) hold. In the first step, we consider the following elliptic problem.

Problem 2. Given $g, \theta \in X$ and $z \in Z$, find $(u, \lambda) \in X \times \Lambda$ such that

$$\begin{aligned} (Au, v - u)_X + \varphi(z, v) - \varphi(z, u) + j(\theta, v) - j(\theta, u) \\ + b(v - u, \lambda) \geq (g, v - u)_X \quad \forall v \in X, \end{aligned} \quad (2.12)$$

$$b(u, \mu - \lambda) \leq b(h, \mu - \lambda) \quad \forall \mu \in \Lambda. \quad (2.13)$$

We have the following existence result.

Lemma 2.2. *Problem 2 has a solution $(u, \lambda) \in X \times \Lambda$, unique in u . In addition, if (u_1, λ_1) and (u_2, λ_2) are two solutions of the problem corresponding to the data $g_1, \theta_1 \in X$, $z_1 \in Z$ and $g_2, \theta_2 \in X$, $z_2 \in Z$, respectively, then there exists $c > 0$ which depends only on A , φ and j such that*

$$\|u_1 - u_2\|_X \leq c(\|g_1 - g_2\|_X + \|z_1 - z_2\|_Z + \|\theta_1 - \theta_2\|_X). \quad (2.14)$$

Proof. Let $g, \theta \in X$, $z \in Z$, and let $\phi : X \rightarrow \mathbb{R}$ be the function defined as follows

$$\phi(v) = \varphi(z, v) + j(\theta, v) \quad \forall v \in X. \quad (2.15)$$

Note that the function ϕ depends on z and θ , but, for simplicity, we do not indicate explicitly this dependence. With this notation, it is easy to see that Problem 2 is equivalent to the problem of finding a couple $(u, \lambda) \in X \times \Lambda$ such that

$$(Au, v - u)_X + \phi(v) - \phi(u) + b(v - u, \lambda) \geq (g, v - u)_X \quad \forall v \in X, \quad (2.16)$$

$$b(u, \mu - \lambda) \leq b(h, \mu - \lambda) \quad \forall \mu \in \Lambda. \quad (2.17)$$

We claim that the problem (2.16)–(2.17) has a solution $(u, \lambda) \in X \times \Lambda$ which is unique in the first component. Indeed, let $\rho > 0$ and $\omega \in X$. Note that assumptions (2.6)(a) and (2.7)(a) imply that $v \mapsto \phi(v)$ is a convex lower semicontinuous function on X . Then, using assumption (2.9) and standard arguments for mixed variational problems (see, for instance, [5]), we deduce that there exists a couple $(u_\omega, \lambda_\omega) \in X \times \Lambda$, unique in the first component, such that

$$\begin{aligned} (u_\omega, v - u_\omega)_X + \rho\phi(v) - \rho\phi(u_\omega) + \rho b(v - u_\omega, \lambda_\omega) \\ \geq (\rho(g - A\omega) + \omega, v - u_\omega)_X \quad \forall v \in X, \end{aligned} \quad (2.18)$$

$$\rho b(u_\omega, \mu - \lambda_\omega) \leq \rho b(h, \mu - \lambda_\omega) \quad \forall \mu \in \Lambda. \quad (2.19)$$

The uniqueness of the solution of the problem (2.18)–(2.19) in the first component allows us to define an operator $T_\rho : X \rightarrow X$ by equality

$$T_\rho \omega := u_\omega. \quad (2.20)$$

We prove that this operator is a contraction for an appropriate value of ρ . Indeed, let $\omega_1, \omega_2 \in X$, and let $(u_{\omega_1}, \lambda_{\omega_1}), (u_{\omega_2}, \lambda_{\omega_2})$ be two solutions of problem (2.18)–(2.19) obtained for $\omega = \omega_1$ and $\omega = \omega_2$, respectively. Then, using a standard argument, we find that

$$\begin{aligned} (u_{\omega_1} - u_{\omega_2}, u_{\omega_1} - u_{\omega_2})_X &\leq \rho b(u_{\omega_2} - u_{\omega_1}, \lambda_{\omega_1} - \lambda_{\omega_2}) \\ &\quad + (\omega_1 - \omega_2 + \rho(A\omega_2 - A\omega_1), u_{\omega_1} - u_{\omega_2})_X, \end{aligned}$$

$$b(u_{\omega_1} - u_{\omega_2}, \lambda_{\omega_2} - \lambda_{\omega_1}) \leq 0,$$

which implies that

$$\|u_{\omega_1} - u_{\omega_2}\|_X \leq \|\omega_1 - \omega_2 - \rho(A\omega_1 - A\omega_2)\|_X.$$

Thus,

$$\|u_{\omega_1} - u_{\omega_2}\|_X^2 \leq \|\omega_1 - \omega_2\|_X^2 - 2\rho(A\omega_1 - A\omega_2, \omega_1 - \omega_2)_X + \rho^2\|A\omega_1 - A\omega_2\|_X^2$$

and, using (2.4)(a)–(b), we obtain

$$\|u_{\omega_1} - u_{\omega_2}\|_X^2 \leq (1 - 2\rho m_A + \rho^2 L_A^2)\|\omega_1 - \omega_2\|_X^2.$$

Therefore, we deduce that

$$\|T_\rho \omega_1 - T_\rho \omega_2\|_X \leq \sqrt{1 - 2\rho m_A + \rho^2 L_A^2} \|\omega_1 - \omega_2\|_X. \quad (2.21)$$

Assume now that $0 < \rho < \frac{2m_A}{L_A^2}$. Then, it is easy to see that

$$\sqrt{1 - 2\rho m_A + \rho^2 L_A^2} < 1,$$

and, using (2.21) and the Banach fixed point argument, we conclude that the operator T_ρ has a unique fixed point, denoted ω^* . Then, using (2.20) it is easy to see that the solution $(u_{\omega^*}, \lambda_{\omega^*})$ of problem (2.18)–(2.19) for $\omega = \omega^*$ is a solution of the problem (2.16)–(2.17).

In addition, we have uniqueness in the first component of the solution. Indeed, assume that (u_1, λ_1) and (u_2, λ_2) are solutions to problem (2.16)–(2.17). Then, standard arguments lead to

$$(Au_1 - Au_2, u_1 - u_2)_X \leq b(u_1 - u_2, \lambda_2 - \lambda_1) \leq 0$$

and, using (2.4)(a), we deduce that $u_1 = u_2$.

To conclude, we proved the existence of a solution to problem (2.16)–(2.17), unique in the first component. Therefore, using the equivalence between problems (2.16)–(2.17) and (2.12)–(2.13), we conclude the existence and uniqueness part of the Lemma 2.2.

We now turn to the proof of the estimate (2.14). To this end, consider the data $g_1, \omega_1 \in X, z_1 \in Z$ and $g_2, \theta_2 \in X, z_2 \in Z$. Then we have

$$\begin{aligned} (Au_i, v - u_i)_X + \varphi(z_i, v) - \varphi(z_i, u_i) \\ + j(\theta_i, v) - j(\theta_i, u_i) + b(v - u_i, \lambda_i) \geq (g_i, v - u_i)_X, \end{aligned}$$

$$b(u_i, \mu - \lambda_i) \leq b(h, \mu - \lambda_i)$$

for all $v \in X, \mu \in \Lambda$ and $i = 1, 2$. These inequalities imply that

$$\begin{aligned} (Au_1 - Au_2, u_1 - u_2)_X &\leq (g_1 - g_2, u_1 - u_2)_X + b(u_2 - u_1, \lambda_1 - \lambda_2) \\ &\quad + \varphi(z_1, u_2) - \varphi(z_1, u_1) + \varphi(z_2, u_1) - \varphi(z_2, u_2) \\ &\quad + j(\theta_1, u_2) - j(\theta_1, u_1) + j(\theta_2, u_1) - j(\theta_2, u_2), \end{aligned}$$

$$b(u_2 - u_1, \lambda_1 - \lambda_2) \leq 0.$$

We combine the previous inequalities, and then we use (2.4)(a), (2.6)(b), (2.7)(b) and (2.9)(a) to obtain

$$m_A \|u_1 - u_2\|_X \leq \|g_1 - g_2\|_X + \beta \|z_1 - z_2\|_Z + \alpha \|\theta_1 - \theta_2\|_X. \quad (2.22)$$

Inequality (2.22) shows that (2.14) holds with $c = \max\{1/m_A, \beta/m_A, \alpha/m_A\}$, which, clearly, depends only on A, φ, j . \square

In the second step, we consider an element $\eta \in C(\mathbb{R}_+; X)$ and introduce the notation $y_\eta := \mathcal{S}\eta \in C(\mathbb{R}_+; Z)$. Then we study the following time-dependent problem.

Problem 3. Find two functions $u_\eta : \mathbb{R}_+ \rightarrow X$ and $\lambda_\eta : \mathbb{R}_+ \rightarrow \Lambda$ such that $u_\eta \in C(\mathbb{R}_+; X)$ and, for each $t \in \mathbb{R}_+$, the following inequalities hold:

$$\begin{aligned} & (Au_\eta(t), v - u_\eta(t))_X + \varphi(y_\eta(t), v) - \varphi(y_\eta(t), u_\eta(t)) \\ & + j(\eta(t), v) - j(\eta(t), u_\eta(t)) + b(v - u_\eta(t), \lambda_\eta(t)) \\ & \geq (f(t), v - u_\eta(t))_X \quad \forall v \in X, \end{aligned} \quad (2.23)$$

$$b(u_\eta(t), \mu - \lambda_\eta(t)) \leq b(h, \mu - \lambda_\eta(t)) \quad \forall \mu \in \Lambda. \quad (2.24)$$

We proceed with the following result.

Lemma 2.3. *Problem 3 has a solution (u_η, λ_η) , unique in the first component. Moreover, if (u_1, λ_1) and (u_2, λ_2) are two solutions of Problem 3 corresponding to the data $\eta_1, \eta_2 \in C(\mathbb{R}_+; X)$, then for each positive integer n , we have*

$$\begin{aligned} \|u_1(t) - u_2(t)\|_X & \leq \frac{\beta s_n}{m_A} \int_0^t \|\eta_1(s) - \eta_2(s)\|_X ds \\ & + \frac{\alpha}{m_A} \|\eta_1(t) - \eta_2(t)\|_X \quad \forall t \in [0, n]. \end{aligned} \quad (2.25)$$

Proof. Let $t \in \mathbb{R}_+$ and denote $y_\eta(t) = z, \eta(t) = \theta, f(t) = g$. Then, Lemma 2.2 guarantees the existence of a couple $(u_\eta(t), \lambda_\eta(t))$, unique in the first component, which satisfies (2.23)–(2.24). Moreover, by (2.14) we deduce that for each $t_1, t_2 \in \mathbb{R}_+$, we have

$$\begin{aligned} & \|u_\eta(t_1) - u_\eta(t_2)\|_X \\ & \leq c (\|f(t_1) - f(t_2)\|_X + \|y_\eta(t_1) - y_\eta(t_2)\|_Z + \|\eta(t_1) - \eta(t_2)\|_X). \end{aligned}$$

Since f, η, y_η are continuous functions defined on \mathbb{R}_+ , by the previous inequality we deduce that $t \mapsto u_\eta(t) : \mathbb{R}_+ \rightarrow X$ is a continuous function. This proves the first part of the lemma.

We now turn to the proof of estimate (2.25). To this end, we fix $\eta_1, \eta_2 \in C(\mathbb{R}_+; X)$, $n \in \mathbb{N}$, and we consider an element $t \in [0, n]$. We write (2.23) and (2.24) with η_1 and η_2 and, as a result, we obtain two systems of the form (2.12)–(2.13) with $u = u_{\eta_i}(t)$, $z = y_{\eta_i}(t)$, $\theta = \eta_i(t)$ and $g = f(t)$, $i = 1, 2$. Therefore, using (2.22), we deduce that

$$m_A \|u_1(t) - u_2(t)\|_X \leq \beta \|y_{\eta_1}(t) - y_{\eta_2}(t)\|_Z + \alpha \|\eta_1(t) - \eta_2(t)\|_X.$$

Since $y_{\eta_i} = \mathcal{S}\eta_i$, the previous inequality yields

$$\|u_1(t) - u_2(t)\|_X \leq \frac{\beta}{m_A} \|\mathcal{S}\eta_1(t) - \mathcal{S}\eta_2(t)\|_Z + \frac{\alpha}{m_A} \|\eta_1(t) - \eta_2(t)\|_X. \quad (2.26)$$

Estimate (2.25) is now a consequence of inequalities (2.26) and (2.5). \square

In the next step, we use Lemma 2.3 and define the operator $\Theta : C(\mathbb{R}_+; X) \rightarrow C(\mathbb{R}_+; X)$ as follows:

$$\Theta\eta := u_\eta \quad \forall \eta \in C(\mathbb{R}_+; X). \quad (2.27)$$

We have the following result.

Lemma 2.4. *The operator Θ has a unique fixed point $\eta^* \in C(\mathbb{R}_+; X)$.*

Proof. Let $\eta_1, \eta_2 \in C(\mathbb{R}_+; X)$, and let u_1, u_2 be the first components of the corresponding solutions to Problem 3, obtained in Lemma 2.3. Let $n \in \mathbb{N}$, and let $t \in [0, n]$. Then, using the definition (2.27) of the operator Θ and estimate (2.25), we deduce that

$$\begin{aligned} \|\Theta\eta_1(t) - \Theta\eta_2(t)\|_X &\leq \frac{\beta s_n}{m_A} \int_0^t \|\eta_1(s) - \eta_2(s)\|_X \, ds \\ &\quad + \frac{\alpha}{m_A} \|\eta_1(t) - \eta_2(t)\|_X \quad \forall t \in [0, n]. \end{aligned} \quad (2.28)$$

Inequality (2.28) combined with the smallness assumption (2.8) allows us to apply Theorem 2.1 with $m = 1$ in order to conclude the proof of Lemma 2.4. \square

We are now in a position to provide the proof of Theorem 2.1.

Proof. Let η^* be the unique fixed point of the operator Θ . Then, writing (2.23)–(2.24) for $\eta = \eta^*$ and using the equalities $u_{\eta^*} = \eta^*$, $y_{\eta^*} = \mathcal{S}\eta^*$, it follows that the couple $(u_{\eta^*}, \lambda_{\eta^*})$ is a solution of Problem 1. The uniqueness of the solution in the first component follows from the uniqueness of the fixed point of the operator Θ , guaranteed by Lemma 2.4. \square

Note that Theorem 2.1 provides the existence of the solution to Problem 1 and its uniqueness in u . The uniqueness of the solution in λ and its regularity with respect to t are left open.

3. A viscoplastic contact model

In this section, we provide an application of the abstract result presented in Sect. 2, in the study of a mathematical model which describes the frictionless contact between a viscoplastic body and a foundation. Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be a domain with a regular boundary $\partial\Omega = \Gamma$ partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 , such that $meas\Gamma_1 > 0$, and denote by \mathbb{S}^d the space of symmetric tensors of second order on \mathbb{R}^d . Then, the classical formulation of the contact model is the following.

Problem 4. Find a displacement field $\mathbf{u} : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma} : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{S}^d$ such that

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{E}\varepsilon(\dot{\mathbf{u}}(t)) + \mathcal{G}(\boldsymbol{\sigma}(t), \varepsilon(\mathbf{u}(t))) \quad \text{in } \Omega, \quad (3.1)$$

$$\text{Div } \boldsymbol{\sigma}(t) + \mathbf{f}_0(t) = \mathbf{0} \quad \text{in } \Omega, \quad (3.2)$$

$$\mathbf{u}(t) = \mathbf{0} \quad \text{on } \Gamma_1, \quad (3.3)$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \mathbf{f}_2(t) \quad \text{on } \Gamma_2, \quad (3.4)$$

$$\boldsymbol{\sigma}_\tau(t) = \mathbf{0} \quad \text{on } \Gamma_3, \quad (3.5)$$

for all $t \in \mathbb{R}_+$, there exists $\xi : \Gamma_3 \times \mathbb{R}_+ \rightarrow \mathbb{R}$, which satisfies

$$\left. \begin{aligned} u_\nu(t) &\leq g, \quad \sigma_\nu(t) + p(u_\nu(t)) + \xi(t) \leq 0, \\ (u_\nu(t) - g) (\sigma_\nu(t) + p(u_\nu(t)) + \xi(t)) &= 0, \\ 0 &\leq \xi(t) \leq F(u_\nu(t)), \\ \xi(t) &= 0 \quad \text{if } u_\nu(t) < 0, \\ \xi(t) &= F(u_\nu(t)) \quad \text{if } u_\nu(t) > 0 \end{aligned} \right\} \quad \text{on } \Gamma_3, \quad (3.6)$$

for all $t \in \mathbb{R}_+$ and, moreover,

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0 \quad \text{in } \Omega. \quad (3.7)$$

Details on the constructions of various models of contact similar with Problem 4 could be found in the books [7, 23, 27] and the references therein. Here we restrict ourselves to present a brief descriptions of the equations and conditions (3.1)–(3.7) and to underline the traits of novelties of this model.

First, Eq. (3.1) represents the viscoplastic constitutive law of the material; here \mathcal{E} represents the elasticity tensor, \mathcal{G} is a viscoplastic constitutive function, and the dot above represents the derivative with respect to the time. Equation (3.2) is the equilibrium equation in which \mathbf{f}_0 denotes the density of body forces; we use it here since the process is assumed to be quasistatic. Conditions (3.3) and (3.4) are the displacement and traction boundary conditions, respectively, in which $\boldsymbol{\nu}$ denotes the unit outward normal to Γ . Condition (3.5) shows that the tangential stress on the contact surface, denoted $\boldsymbol{\sigma}_\tau$, vanishes; we use it here since we assume that the contact process is frictionless. Finally, (3.7) represents the initial conditions in which \mathbf{u}_0 and $\boldsymbol{\sigma}_0$ denote the initial displacement and the initial stress field, respectively.

We now provide some comments on condition (3.6) in which the constant $g > 0$ represents a given bound for the penetration, p is a prescribed positive function which vanishes for a negative argument, F is a given positive function, and u_ν , σ_ν represent the normal displacement and the normal stress, respectively. This condition was introduced for the first time in [28], in the case when F does not depend on u_ν . There, the arguments used to obtain this condition were presented, together with some mechanical interpretation. Here we restrict ourselves to recall that condition (3.6) with F given describes the following features of the contact: when there is separation between the body's surface and the foundation, then the normal stress vanishes; the penetration arises only if the absolute value of the normal stress reaches the critical value F ; when there is penetration, the contact follows a normal compliance condition of the form

$$-\sigma_\nu(t) = p(u_\nu(t)) + F, \quad (3.8)$$

but up to the bound g and then, when this limit is reached, the contact follows a Signorini-type unilateral condition with the gap g . For this reason, following [28], we refer to condition (3.6) as a *multivalued normal compliance contact condition with unilateral constraint*. It can be interpreted physically as follows. The foundation is assumed to be made of a hard material covered by a thin layer of a soft material with thickness g . The soft material has a rigid-elastic behavior, i.e., is deformable, allows penetration, but only if the absolute value of the normal stress arrives to the yield value F ; then, when this limit is reached, the contact with this layer is modeled with normal compliance, as shown in equality (3.8). Here p is a positive function which describes the dependence of the normal stress with respect to the penetration. The hard material is perfectly rigid, and, therefore, it does not allow penetration; the contact with this material is modeled with the Signorini contact condition.

Note that, in contrast to [28], in this paper we consider the contact condition in the case when the yield value F depends on the normal displacement, i.e., $F = F(u_\nu(t))$. This dependence could describe the hardening and the softening properties of the foundation. Considering the dependence $F = F(u_\nu(t))$ makes the contact problem more general and leads to a new and interesting mathematical model, as shown in the rest of this paper.

We now proceed with further notation and preliminaries which are needed in the study of Problem 4. Everywhere below the indices i, j, k, l run between 1 and d and, unless stated otherwise, the summation convention over repeated indices is used. We denote by “ \cdot ” and $\|\cdot\|$ the inner product and norm on \mathbb{R}^d and \mathbb{S}^d and use standard notation for the Lebesgue and Sobolev spaces associated with Ω and Γ . Moreover, $\mathbf{x} = (x_i)$ will represent a typical point in $\Omega \cup \Gamma$, and index that follows a comma will denote the partial derivative with respect to the corresponding component of the spatial variable, e.g., $u_{i,j} = \partial u_i / \partial x_j$. In addition, we consider the spaces

$$\begin{aligned} V &= \{ \mathbf{v} = (v_i) \in H^1(\Omega)^d : \mathbf{v} = \mathbf{0} \text{ on } \Gamma_1 \}, \\ Q &= \{ \boldsymbol{\tau} = (\tau_{ij}) \in L^2(\Omega)^{d \times d} : \tau_{ij} = \tau_{ji} \}, \end{aligned}$$

which are real Hilbert spaces with the inner products

$$(\mathbf{u}, \mathbf{v})_V = \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}) \cdot \boldsymbol{\varepsilon}(\mathbf{v}) \, dx, \quad (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_{\Omega} \boldsymbol{\sigma} \cdot \boldsymbol{\tau} \, dx.$$

Here $\boldsymbol{\varepsilon}$ represents the deformation operator, i.e.,

$$\boldsymbol{\varepsilon}(\mathbf{v}) = (\varepsilon_{ij}(\mathbf{v})), \quad \varepsilon_{ij}(\mathbf{v}) = \frac{1}{2} (v_{i,j} + v_{j,i}) \quad \forall \mathbf{v} \in H^1(\Omega)^d.$$

The associated norms on these space will be denoted $\|\cdot\|_V$ and $\|\cdot\|_Q$, respectively.

For an element $\mathbf{v} \in V$, we still denote \mathbf{v} for the trace of \mathbf{v} on the boundary. Moreover, we use the notation v_ν and \mathbf{v}_τ for the normal and tangential components of \mathbf{v} on Γ , given by $v_\nu = \mathbf{v} \cdot \boldsymbol{\nu}$ and $\mathbf{v}_\tau = \mathbf{v} - v_\nu \boldsymbol{\nu}$, respectively. In addition, the Sobolev trace theorem yields that exists a positive constant c_{tr} which depends only on Ω , Γ_1 and Γ_3 such that

$$\|\mathbf{v}\|_{L^2(\Gamma_3)^d} \leq c_{tr} \|\mathbf{v}\|_V \quad \forall \mathbf{v} \in V. \quad (3.9)$$

We also consider the space

$$S = \{ \mathbf{w} = \mathbf{v}|_{\Gamma_3} : \mathbf{v} \in V \}$$

where, here and below, $\mathbf{v}|_{\Gamma_3}$ denotes the restriction of the trace of the element $\mathbf{v} \in V$ to Γ_3 . Recall that S can be organized as a Hilbert space in a canonical way, as explained in [1, 15]. We denote by D the dual of the space S , and we use $\langle \cdot, \cdot \rangle_{\Gamma_3}$ for the duality pairing between D and S . In addition, when $\boldsymbol{\mu} \in D$ and $\mathbf{v} \in V$, we shall write $\langle \boldsymbol{\mu}, \mathbf{v} \rangle_{\Gamma_3}$ instead of $\langle \boldsymbol{\mu}, \mathbf{v}|_{\Gamma_3} \rangle_{\Gamma_3}$.

Assume now that $\boldsymbol{\sigma} \in Q$ is a regular function. Then we use the notation σ_ν and $\boldsymbol{\sigma}_\tau$ for the normal and the tangential traces, i.e., $\sigma_\nu = (\boldsymbol{\sigma} \boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma} \boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$. Moreover, we denote by Div the divergence of the field $\boldsymbol{\sigma}$, i.e., $\text{Div} \boldsymbol{\sigma} = (\sigma_{ij,j})$.

We now turn to the variational formulation of the mechanical problem (3.1)–(3.7), and, to this end, we assume that the elasticity tensor \mathcal{E} and the nonlinear constitutive function \mathcal{G} satisfy the following conditions.

$$\begin{cases} \text{(a)} & \mathcal{E} = (\mathcal{E}_{ijkl}) : \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d. \\ \text{(b)} & \mathcal{E}_{ijkl} = \mathcal{E}_{klji} = \mathcal{E}_{jikl} \in L^\infty(\Omega), \quad 1 \leq i, j, k, l \leq d. \\ \text{(c)} & \text{There exists } m_{\mathcal{E}} > 0 \quad \text{such that} \\ & \mathcal{E} \boldsymbol{\tau} \cdot \boldsymbol{\tau} \geq m_{\mathcal{E}} \|\boldsymbol{\tau}\|^2 \quad \forall \boldsymbol{\tau} \in \mathbb{S}^d, \text{ a.e. in } \Omega. \end{cases} \quad (3.10)$$

$$\left\{ \begin{array}{l} \text{(a) } \mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \rightarrow \mathbb{S}^d. \\ \text{(b) There exists } L_G > 0 \text{ such that} \\ \quad \|\mathcal{G}(\mathbf{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1) - \mathcal{G}(\mathbf{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2)\| \\ \quad \leq L_G (\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|) \\ \quad \quad \forall \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\ \text{(c) The mapping } \mathbf{x} \mapsto \mathcal{G}(\mathbf{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}) \text{ is measurable in } \Omega, \\ \quad \text{for any } \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\ \text{(d) The mapping } \mathbf{x} \mapsto \mathcal{G}(\mathbf{x}, \mathbf{0}, \mathbf{0}) \text{ belongs to } Q. \end{array} \right. \quad (3.11)$$

The body forces and the tractions have the regularity

$$\mathbf{f}_0 \in C(\mathbb{R}_+; L^2(\Omega)^d), \quad \mathbf{f}_2 \in C(\mathbb{R}_+; L^2(\Gamma_2)^d) \quad (3.12)$$

and, recall, the bound g is positive, i.e.,

$$g > 0. \quad (3.13)$$

Moreover, the normal compliance function and the surface yield function are such that

$$\left\{ \begin{array}{l} \text{(a) } p : \Gamma_3 \times \mathbb{R} \rightarrow \mathbb{R}_+. \\ \text{(b) There exists } L_p > 0 \text{ such that} \\ \quad |p(\mathbf{x}, r_1) - p(\mathbf{x}, r_2)| \leq L_p |r_1 - r_2| \\ \quad \quad \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_3. \\ \text{(c) } (p(\mathbf{x}, r_1) - p(\mathbf{x}, r_2))(r_1 - r_2) \geq 0 \\ \quad \quad \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_3. \\ \text{(d) The mapping } \mathbf{x} \mapsto p(\mathbf{x}, r) \text{ is measurable on } \Gamma_3, \\ \quad \text{for any } r \in \mathbb{R}. \\ \text{(e) } p(\mathbf{x}, r) = 0 \quad \forall r < 0, \text{ a.e. } \mathbf{x} \in \Gamma_3; \end{array} \right. \quad (3.14)$$

$$\left\{ \begin{array}{l} \text{(a) } F : \Gamma_3 \times \mathbb{R} \rightarrow \mathbb{R}_+. \\ \text{(b) There exists } L_F > 0 \text{ such that} \\ \quad |F(\mathbf{x}, r_1) - F(\mathbf{x}, r_2)| \leq L_F |r_1 - r_2| \\ \quad \quad \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_3. \\ \text{(c) The mapping } \mathbf{x} \mapsto F(\mathbf{x}, r) \text{ is measurable on } \Gamma_3, \\ \quad \text{for any } r \in \mathbb{R}. \\ \text{(d) The mapping } \mathbf{x} \mapsto F(\mathbf{x}, 0) \text{ belongs to } L^2(\Gamma_3). \end{array} \right. \quad (3.15)$$

Finally, we assume that

$$\mathbf{u}_0 \in V, \quad \boldsymbol{\sigma}_0 \in Q \quad (3.16)$$

and, in addition,

$$\text{there exists } \tilde{\boldsymbol{\theta}} \in V \text{ such that } \tilde{\theta}_\nu = 1 \text{ a.e. on } \Gamma_3. \quad (3.17)$$

Next, we define the sets $K \subset V$ and $\Lambda \subset D$, the operator $P : V \rightarrow V$, the functional $j : V \times V \rightarrow \mathbb{R}$, the bilinear form $b : V \times D \rightarrow \mathbb{R}$ and the function $\mathbf{f} : \mathbb{R}_+ \rightarrow V$ by equalities

$$K = \{ \mathbf{v} \in V : v_\nu \leq 0 \text{ a.e. on } \Gamma_3 \}, \quad (3.18)$$

$$\Lambda = \{ \boldsymbol{\mu} \in D : \langle \boldsymbol{\mu}, \mathbf{v} \rangle_{\Gamma_3} \leq 0 \quad \forall \mathbf{v} \in K \}, \quad (3.19)$$

$$(P\mathbf{u}, \mathbf{v})_V = \int_{\Gamma_3} p(u_\nu) v_\nu \, da \quad \forall \mathbf{u}, \mathbf{v} \in V, \quad (3.20)$$

$$j(\mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} F(u_\nu) v_\nu^+ \, da \quad \forall \mathbf{u}, \mathbf{v} \in V, \quad (3.21)$$

$$b(\mathbf{v}, \boldsymbol{\mu}) = \langle \boldsymbol{\mu}, \mathbf{v} \rangle_{\Gamma_3} \quad \forall \mathbf{v} \in V, \boldsymbol{\mu} \in D, \quad (3.22)$$

$$(\mathbf{f}(t), \mathbf{v})_V = \int_{\Omega} \mathbf{f}_0(t) \cdot \mathbf{v} \, dx + \int_{\Gamma_2} \mathbf{f}_2(t) \cdot \mathbf{v} \, da \quad (3.23)$$

$$\forall \mathbf{v} \in V, t \in \mathbb{R}_+.$$

Let \mathbf{u} and $\boldsymbol{\sigma}$ are regular functions which verify (3.1)–(3.7), and let $t \in \mathbb{R}_+$, $\mathbf{v} \in V$, $\boldsymbol{\mu} \in \Lambda$. Then, using (3.1) and (3.7), we find that

$$\boldsymbol{\sigma}(t) = \mathcal{E} \boldsymbol{\varepsilon}(\mathbf{u}(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\mathbf{u}(s))) \, ds + \boldsymbol{\sigma}_0 - \mathcal{E} \boldsymbol{\varepsilon}(\mathbf{u}_0). \quad (3.24)$$

Next, an integration by part combined with the equation of equilibrium (3.2) yields

$$\begin{aligned} & (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}(t)))_Q \\ &= (\mathbf{f}_0(t), \mathbf{v} - \mathbf{u}(t))_{L^2(\Omega)^d} + \int_{\Gamma} \boldsymbol{\sigma}(t) \boldsymbol{\nu} \cdot (\mathbf{v} - \mathbf{u}(t)) \, da. \end{aligned} \quad (3.25)$$

Then, since $\mathbf{v} - \mathbf{u}(t) = \mathbf{0}$ on Γ_1 , using (3.4), (3.5) and (3.23) we obtain that

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}(t)))_Q = (\mathbf{f}(t), \mathbf{v} - \mathbf{u}(t))_V + \int_{\Gamma_3} \sigma_\nu(t) (v_\nu - u_\nu(t)) \, da. \quad (3.26)$$

We now introduce the multiplier $\boldsymbol{\lambda}(t) \in D$ defined by

$$\langle \boldsymbol{\lambda}(t), \mathbf{w} \rangle_{\Gamma_3} = - \int_{\Gamma_3} (\sigma_\nu(t) + p(u_\nu(t)) + \xi(t)) w_\nu \, da \quad \forall \mathbf{w} \in S. \quad (3.27)$$

Then, (3.22) and (3.27) imply that

$$- \int_{\Gamma_3} \sigma_\nu(t) (v_\nu - u_\nu(t)) \, da = b(\mathbf{v} - \mathbf{u}(t), \boldsymbol{\lambda}(t)) + \int_{\Gamma_3} (p(u_\nu(t)) + \xi(t)) (v_\nu - u_\nu(t)) \, da,$$

and combining this equality with (3.26) yields

$$\begin{aligned} & (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}(t)))_Q + \int_{\Gamma_3} p(u_\nu(t)) (v_\nu - u_\nu(t)) \, da \\ &+ \int_{\Gamma_3} \xi(t) (v_\nu - u_\nu(t)) \, da + b(\mathbf{v} - \mathbf{u}(t), \boldsymbol{\lambda}(t)) = (\mathbf{f}(t), \mathbf{v} - \mathbf{u}(t))_V. \end{aligned} \quad (3.28)$$

Next, using (3.6) it is easy to see that

$$\xi(t) (v_\nu - u_\nu(t)) \leq F(u_\nu(t)) (v_\nu^+ - u_\nu^+(t)) \quad \text{a.e. on } \Gamma_3$$

which implies that

$$\int_{\Gamma_3} \xi(t) (v_\nu - u_\nu(t)) \, da \leq \int_{\Gamma_3} F(u_\nu(t)) (v_\nu^+ - u_\nu^+(t)) \, da. \quad (3.29)$$

We now combine (3.28), (3.29), and then we use notation (3.20), (3.21). As a result, we find that

$$\begin{aligned} & (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}(t)))_Q + (P\mathbf{u}(t), \mathbf{v} - \mathbf{u}(t))_V \\ &+ j(\mathbf{u}(t), \mathbf{v}) - j(\mathbf{u}(t), \mathbf{u}(t)) + b(\mathbf{v} - \mathbf{u}(t), \boldsymbol{\lambda}(t)) \geq (\mathbf{f}(t), \mathbf{v} - \mathbf{u}(t))_V. \end{aligned} \quad (3.30)$$

On the other hand, using (3.6) and definitions (3.18), (3.19), (3.27), we deduce that $\boldsymbol{\lambda}(t) \in \Lambda$. Moreover, using assumption (3.17) and the definition (3.22) of the bilinear form b , it is easy to see that

$$\begin{aligned} b(\mathbf{u}(t), \boldsymbol{\mu} - \boldsymbol{\lambda}(t)) &= b(\mathbf{u}(t) - g\tilde{\boldsymbol{\theta}}, \boldsymbol{\mu} - \boldsymbol{\lambda}(t)) + b(g\tilde{\boldsymbol{\theta}}, \boldsymbol{\mu} - \boldsymbol{\lambda}(t)) \\ &= \langle \boldsymbol{\mu} - \boldsymbol{\lambda}(t), \mathbf{u}(t) - g\tilde{\boldsymbol{\theta}} \rangle_{\Gamma_3} + b(g\tilde{\boldsymbol{\theta}}, \boldsymbol{\mu} - \boldsymbol{\lambda}(t)) \end{aligned}$$

and, therefore,

$$\begin{aligned} b(\mathbf{u}(t), \boldsymbol{\mu} - \boldsymbol{\lambda}(t)) &= \langle \boldsymbol{\mu}, \mathbf{u}(t) - g\tilde{\boldsymbol{\theta}} \rangle_{\Gamma_3} \\ &\quad - \langle \boldsymbol{\lambda}(t), \mathbf{u}(t) - g\tilde{\boldsymbol{\theta}} \rangle_{\Gamma_3} + b(g\tilde{\boldsymbol{\theta}}, \boldsymbol{\mu} - \boldsymbol{\lambda}(t)). \end{aligned} \tag{3.31}$$

In addition, (3.6) and (3.17) imply that

$$\mathbf{u}(t) - g\tilde{\boldsymbol{\theta}} \in K, \quad \langle \boldsymbol{\lambda}(t), \mathbf{u}(t) \rangle_{\Gamma_3} = \langle \boldsymbol{\lambda}(t), g\tilde{\boldsymbol{\theta}} \rangle_{\Gamma_3}.$$

Therefore,

$$\langle \boldsymbol{\mu}, \mathbf{u}(t) - g\tilde{\boldsymbol{\theta}} \rangle_{\Gamma_3} \leq 0 \quad \text{and} \quad \langle \boldsymbol{\lambda}(t), \mathbf{u}(t) - g\tilde{\boldsymbol{\theta}} \rangle_{\Gamma_3} = 0. \tag{3.32}$$

We combine now (3.31) and (3.32) to deduce that

$$b(\mathbf{u}(t), \boldsymbol{\mu} - \boldsymbol{\lambda}(t)) \leq b(g\tilde{\boldsymbol{\theta}}, \boldsymbol{\mu} - \boldsymbol{\lambda}(t)). \tag{3.33}$$

We now gather equality (3.24), and inequalities (3.30), (3.33) to obtain the following variational formulation of Problem 4.

Problem 5. Find a displacement field $\mathbf{u} : \mathbb{R}_+ \rightarrow V$, a stress field $\boldsymbol{\sigma} : \mathbb{R}_+ \rightarrow Q$ and a multiplier $\boldsymbol{\lambda} : \mathbb{R}_+ \rightarrow \Lambda$ such that $\mathbf{u} \in C(\mathbb{R}_+; V)$, $\boldsymbol{\sigma} \in C(\mathbb{R}_+; Q)$ and

$$\boldsymbol{\sigma}(t) = \mathcal{E} \boldsymbol{\varepsilon}(\mathbf{u}(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\mathbf{u}(s))) ds + \boldsymbol{\sigma}_0 - \mathcal{E} \boldsymbol{\varepsilon}(\mathbf{u}_0), \tag{3.34}$$

$$\begin{aligned} &(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}(t)))_Q + (P\mathbf{u}(t), \mathbf{v} - \mathbf{u}(t))_V \\ &\quad + j(\mathbf{u}(t), \mathbf{v}) - j(\mathbf{u}(t), \mathbf{u}(t)) + b(\mathbf{v} - \mathbf{u}(t), \boldsymbol{\lambda}(t)) \\ &\geq (\mathbf{f}(t), \mathbf{v} - \mathbf{u}(t))_V \quad \forall \mathbf{v} \in V, \end{aligned} \tag{3.35}$$

$$b(\mathbf{u}(t), \boldsymbol{\mu} - \boldsymbol{\lambda}(t)) \leq b(g\tilde{\boldsymbol{\theta}}, \boldsymbol{\mu} - \boldsymbol{\lambda}(t)) \quad \forall \boldsymbol{\mu} \in \Lambda, \tag{3.36}$$

for all $t \in \mathbb{R}_+$.

It follows from here that a triple of functions $(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\lambda}) : \mathbb{R}_+ \rightarrow V \times Q \times \Lambda$ represents a solution of Problem 5 iff (3.34)–(3.36) hold for each $t \in \mathbb{R}_+$ and, moreover, $\mathbf{u} \in C(\mathbb{R}_+; V)$, $\boldsymbol{\sigma} \in C(\mathbb{R}_+; Q)$. Such a solution will be also called a weak solution of Problem 4. Note that, in the study of Problem 5, the unknowns \mathbf{u} and $\boldsymbol{\sigma}$ (on the one hand) and $\boldsymbol{\lambda}$ (on the other hand) do not play a symmetric role. The reason arises from arguments similar to those explained in page 3.

Note that Problem 5 represents a mixed variational formulation which couples a nonlinear implicit integral equation for the stress field, (3.34), a time-dependent quasivariational equation for the displacement field, (3.35), and a first-order time-dependent variational inequality for the multiplier, (3.36). This formulation is quite different to that in Problem 1. Nevertheless, we shall see in the next section that we can associate with Problem 5 a mixed variational formulation of the form (2.2)–(2.3) and, therefore, the analysis of Problem 5 can be carried out by using the abstract result obtained in Section 2 of this paper.

4. Existence of the solution

In the study of Problem 5, we have the following existence result.

Theorem 4.1. *Under the assumptions (3.10)–(3.17), there exists $d_0 > 0$ which depends only on \mathcal{E} , Ω , Γ_1 and Γ_3 such that if $L_F < d_0$, then Problem 5 has at least one solution $(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\lambda})$, unique in \mathbf{u} and $\boldsymbol{\sigma}$.*

We conclude from above that under assumptions (3.10)–(3.17), Problem 4 has at least one weak solution $(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\lambda})$, unique in \mathbf{u} and $\boldsymbol{\sigma}$, if the Lipschitz constant of the surface yield function F is small enough.

The proof of Theorem 4.1 will be carried out in several steps. To present them, we assume in what follows that (3.10)–(3.17) holds. The first step is given by the following existence and uniqueness result, already obtained in [3], that we present here with the details, for the convenience of the reader.

Lemma 4.2. *For each function $\mathbf{u} \in C(\mathbb{R}_+; V)$, there exists a unique function $\mathcal{S}\mathbf{u} \in C(\mathbb{R}_+; Q)$ such that*

$$\mathcal{S}\mathbf{u}(t) = \int_0^t \mathcal{G}(\mathcal{S}\mathbf{u}(s) + \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}(s)), \boldsymbol{\varepsilon}(\mathbf{u}(s))) ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_0) \quad \forall t \in \mathbb{R}_+. \quad (4.1)$$

Moreover, the operator $\mathcal{S} : C(\mathbb{R}_+; V) \rightarrow C(\mathbb{R}_+; Q)$ satisfies the following property: for every $n \in \mathbb{N}$, there exists $s_n > 0$ such that

$$\|\mathcal{S}\mathbf{u}_1(t) - \mathcal{S}\mathbf{u}_2(t)\|_Q \leq s_n \int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V ds \quad (4.2)$$

$$\forall \mathbf{u}_1, \mathbf{u}_2 \in C(\mathbb{R}_+; V), \quad \forall t \in [0, n].$$

Proof. Let $\mathbf{u} \in C(\mathbb{R}_+; V)$ and consider the operator $\mathcal{L} : C(\mathbb{R}_+; Q) \rightarrow C(\mathbb{R}_+; Q)$ defined as follows

$$\mathcal{L}\boldsymbol{\tau}(t) := \int_0^t \mathcal{G}(\boldsymbol{\tau}(s) + \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}(s)), \boldsymbol{\varepsilon}(\mathbf{u}(s))) ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_0) \quad (4.3)$$

$$\forall \boldsymbol{\tau} \in C(\mathbb{R}_+; Q), \quad t \in \mathbb{R}_+.$$

The operator \mathcal{L} depends on \mathbf{u} , but for simplicity, we do not indicate explicitly this dependence.

Let $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \in C(\mathbb{R}_+; Q)$ and let $t \in \mathbb{R}_+$. Then, using (4.3) and (3.11), we have

$$\begin{aligned} & \|\mathcal{L}\boldsymbol{\tau}_1(t) - \mathcal{L}\boldsymbol{\tau}_2(t)\|_Q \\ &= \left\| \int_0^t \mathcal{G}(\boldsymbol{\tau}_1(s) + \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}(s)), \boldsymbol{\varepsilon}(\mathbf{u}(s))) ds - \int_0^t \mathcal{G}(\boldsymbol{\tau}_2(s) + \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}(s)), \boldsymbol{\varepsilon}(\mathbf{u}(s))) ds \right\|_Q \\ &\leq L_{\mathcal{G}} \int_0^t \|\boldsymbol{\tau}_1(s) - \boldsymbol{\tau}_2(s)\|_Q ds. \end{aligned}$$

Next, we use Theorem 2.1 to see that \mathcal{L} has a unique fixed point in $C(\mathbb{R}_+; Q)$, denoted $\mathcal{S}\mathbf{u}$. And, finally, we combine (4.3) with equality $\mathcal{L}(\mathcal{S}\mathbf{u}) = \mathcal{S}\mathbf{u}$ to see that (4.1) holds.

To proceed, let $\mathbf{u}_1, \mathbf{u}_2 \in C(\mathbb{R}_+; V)$, $n \in \mathbb{N}$, and let $t \in [0, n]$. Then, using (4.1) and taking into account (3.10)–(3.11), we write

$$\begin{aligned}
& \|\mathcal{S}\mathbf{u}_1(t) - \mathcal{S}\mathbf{u}_2(t)\|_Q \\
& \leq k \left(\int_0^t \|\mathcal{S}\mathbf{u}_1(s) - \mathcal{S}\mathbf{u}_2(s)\|_Q ds + \int_0^t \|\varepsilon(\mathbf{u}_1(s)) - \varepsilon(\mathbf{u}_2(s))\|_Q ds \right) \\
& = k \left(\int_0^t \|\mathcal{S}\mathbf{u}_1(s) - \mathcal{S}\mathbf{u}_2(s)\|_Q ds + \int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V ds \right),
\end{aligned}$$

where k is a positive constant which depends on \mathcal{G} and \mathcal{E} . Using now a Gronwall argument, we deduce that

$$\|\mathcal{S}\mathbf{u}_1(t) - \mathcal{S}\mathbf{u}_2(t)\|_Q \leq k e^{nk} \int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V ds.$$

This inequality shows that (4.2) holds with $s_n = k e^{nk}$. \square

Next, we define the operator $A : V \rightarrow V$ and the functional $\varphi : Q \times V \rightarrow \mathbb{R}$ by equalities

$$(A\mathbf{v}, \mathbf{w})_V := (\mathcal{E}\varepsilon(\mathbf{v}), \varepsilon(\mathbf{w}))_Q + (P\mathbf{v}, \mathbf{w})_V \quad \forall \mathbf{v}, \mathbf{w} \in V, \quad (4.4)$$

$$\varphi : Q \times V \rightarrow \mathbb{R} \quad \varphi(\boldsymbol{\sigma}, \mathbf{v}) := (\boldsymbol{\sigma}, \varepsilon(\mathbf{v}))_Q \quad \forall \boldsymbol{\sigma} \in Q, \mathbf{v} \in V. \quad (4.5)$$

The next step is given by the following equivalence result.

Lemma 4.3. *The triple $(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\lambda})$ is a solution of Problem 5 if and only if*

$$\boldsymbol{\sigma}(t) = \mathcal{E}\varepsilon(\mathbf{u}(t)) + \mathcal{S}\mathbf{u}(t), \quad (4.6)$$

$$\begin{aligned}
& (A\mathbf{u}(t), \mathbf{v} - \mathbf{u}(t))_V + \varphi(\mathcal{S}\mathbf{u}(t), \mathbf{v}) - \varphi(\mathcal{S}\mathbf{u}(t), \mathbf{u}(t)) \\
& + j(\mathbf{u}(t), \mathbf{v}) - j(\mathbf{u}(t), \mathbf{u}(t)) + b(\mathbf{v} - \mathbf{u}(t), \boldsymbol{\lambda}(t)) \\
& \geq (\mathbf{f}(t), \mathbf{v} - \mathbf{u}(t))_V \quad \forall \mathbf{v} \in V,
\end{aligned} \quad (4.7)$$

$$b(\mathbf{u}(t), \boldsymbol{\mu} - \boldsymbol{\lambda}(t)) \leq b(g\tilde{\boldsymbol{\theta}}, \boldsymbol{\mu} - \boldsymbol{\lambda}(t)) \quad \forall \boldsymbol{\mu} \in \Lambda, \quad (4.8)$$

for all $t \in \mathbb{R}_+$.

Proof. Assume that $(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\lambda})$ is a solution of Problem 5 and let $t \in \mathbb{R}_+$. Then, using (3.34), we have

$$\boldsymbol{\sigma}(t) - \mathcal{E}\varepsilon(\mathbf{u}(t)) = \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s) - \mathcal{E}\varepsilon(\mathbf{u}(s)) + \mathcal{E}\varepsilon(\mathbf{u}(s)), \varepsilon(\mathbf{u}(s))) ds + \boldsymbol{\sigma}_0 - \mathcal{E}\varepsilon(\mathbf{u}_0)$$

and using the definition (4.1) of the operator \mathcal{S} , we obtain (4.6). Inequality (4.7) follows now from (3.35) combined with the definition of A and φ . Finally (4.8) coincides with (3.36).

Conversely, assume that $(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\lambda})$ satisfies (4.6)–(4.8). Then by (4.6) and the definition (4.1) of $\mathcal{S}\mathbf{u}(t)$, we obtain (3.34). Moreover, using (4.6), (4.7) and the definition of A and φ , we obtain (3.35), which concludes the proof. \square

We now proceed with the following existence and uniqueness result.

Lemma 4.4. *There exists $d_0 > 0$ which depends only on \mathcal{E} , Ω , Γ_1 and Γ_3 such that if $L_F < d_0$, then the problem (4.7)–(4.8) has a unique solution $(\mathbf{u}, \boldsymbol{\lambda})$, unique in \mathbf{u} .*

Proof. We apply Theorem 2.1, with $X = V$, $Y = D$, $Z = Q$. To this end, we use assumptions (3.10) and (3.14) to see that the operator A defined by (4.4) verifies condition (2.4) with

$$m_A = m_{\mathcal{E}}. \quad (4.9)$$

Moreover, inequality (4.2) shows that the operator \mathcal{S} satisfies condition (2.5) and, obviously, the map φ defined by (4.5) verifies (2.6).

We turn now on the functional j defined by (3.21). It is easy to see that j satisfies condition (2.7)(a). Moreover, using (3.9) and (3.15), we find that

$$\begin{aligned} & j(\mathbf{u}_1, \mathbf{v}_2) - j(\mathbf{u}_1, \mathbf{v}_1) + j(\mathbf{u}_2, \mathbf{v}_1) - j(\mathbf{u}_2, \mathbf{v}_2) \\ &= \int_{\Gamma_3} (F(u_{1\nu}) - F(u_{2\nu}))(v_{2\nu}^+ - v_{1\nu}^+) \, da \\ &\leq \int_{\Gamma_3} |F(u_{1\nu}) - F(u_{2\nu})| \|\mathbf{v}_1 - \mathbf{v}_2\| \, da \\ &\leq L_F \int_{\Gamma_3} \|\mathbf{u}_1 - \mathbf{u}_2\| \|\mathbf{v}_1 - \mathbf{v}_2\| \, da \\ &\leq c_{\text{tr}}^2 L_F \|\mathbf{u}_1 - \mathbf{u}_2\|_V \|\mathbf{v}_1 - \mathbf{v}_2\|_V \end{aligned}$$

for all $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2 \in V$. We conclude from here that j satisfies condition (2.7)(b) with

$$\alpha = c_{\text{tr}}^2 L_F. \quad (4.10)$$

To continue, as shown, e.g., in [17], the bilinear form $b(\cdot, \cdot)$ is continuous and satisfies the ‘‘inf-sup’’ condition, i.e., there exists $b_0 > 0$ which depends only on Ω, Γ_1 and Γ_3 such that

$$\inf_{\boldsymbol{\mu} \in D, \boldsymbol{\mu} \neq \mathbf{0}_D} \sup_{\mathbf{v} \in V, \mathbf{v} \neq \mathbf{0}_V} \frac{b(\mathbf{v}, \boldsymbol{\mu})}{\|\mathbf{v}\|_V \|\boldsymbol{\mu}\|_D} \geq b_0.$$

We conclude from here that condition (2.9) holds. Also, taking into account (3.12) and (3.23), it follows that $\mathbf{f} \in C(\mathbb{R}_+, V)$. Finally, assumption (3.17) implies that $g\tilde{\boldsymbol{\theta}} \in V$. Therefore, the condition (2.10) holds, too.

Let

$$d_0 := \frac{m_{\mathcal{E}}}{c_{\text{tr}}^2} \quad (4.11)$$

which, clearly, depends only on $\mathcal{E}, \Omega, \Gamma_1$ and Γ_3 . Assume that $L_F < d_0$. Then, using (4.9)–(4.11), it follows that condition (2.8) is satisfied. We now apply Theorem 2.1 to conclude the proof of the lemma. \square

We now have all the ingredients to provide the proof of Theorem 4.1.

Proof. Let d_0 be defined by (4.11) and assume that $L_F < d_0$. Under this condition, Lemma 4.4 implies that there exists a couple of functions $(\mathbf{u}, \boldsymbol{\lambda})$, unique in \mathbf{u} , such that (4.7)–(4.8) hold, for all $t \in \mathbb{R}_+$. Define $\boldsymbol{\sigma} : \mathbb{R}_+ \rightarrow Q$ by equality

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \mathcal{S}\mathbf{u}(t) \quad \forall t \in \mathbb{R}_+$$

and note that, obviously, $\boldsymbol{\sigma} \in C(\mathbb{R}_+; Q)$. Then, we deduce that the triple $(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\lambda})$ represents a solution to problem (4.6)–(4.8) which shows the existence part of the theorem. The uniqueness part follows from the uniqueness in \mathbf{u} of the solution of the system (4.7)–(4.8), guaranteed by Lemma 4.4. \square

5. Conclusions

This paper provides an existence and uniqueness result in the study of a new mixed variational problem with history-dependent operators. This result was presented in an abstract framework and was motivated by the development of the mathematical theory of contact mechanics, which requires new mathematical tools needed for the study of the contact problems. In particular, the theory requires results for new classes of variational inequalities, hemivariational inequalities and mixed variational problems, under specific assumptions on functionals and operators. It is clear from the contents of the books [7, 21, 23, 27] that considerable progress has been made in the development of the mathematical theory of contact mechanics. However, a short analysis shows that plenty remains to be done in the future. We now briefly present three open problems, related to the contents of this paper, which could represent a subject for future research. Any progress in their study will open avenues for new advances and applications.

The first one would be to consider abstract evolutionary versions of Problem 1 in which the derivatives of the unknown u are involved. This would open the way to the study of dynamic contact problems with viscoplastic materials of the form (3.1) for which, to the best of our knowledge, there are no results in the literature. A second open problem would be to relax the assumptions (2.4)–(2.11) in the study of the system (2.2)–(2.3). For instance, it would be interesting to avoid the convexity of the function j and to obtain a version of Theorem 2.1 in the case when j is locally Lipschitz with respect to its second argument. This would provide a mathematical tool in the study of quasistatic viscoplastic problems with nonmonotone normal compliance and unilateral constraints. Finally, since the solutions of Problem 1 are defined on the positive real line, it would be of interest to study their asymptotic behavior as $t \rightarrow \infty$.

Acknowledgments

This work was supported by the LEA Math-Mode project CNRS-IMAR. The work of Andaluza Matei was also supported by a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0257.

References

1. Adams, R.A.: Sobolev Spaces. Academic Press, New York (1975)
2. Amdouni, S., Hild, P., Lleras, V., Moakher, M., Renard, Y.: A stabilized Lagrange multiplier method for the enriched finite-element approximation of contact problems of cracked elastic bodies. *ESAIM Math. Model. Numer. Anal.* **2**(46), 813–839 (2012)
3. Barboteu, M., Matei, A., Sofonea, M.: Analysis of quasistatic viscoplastic contact problems with normal compliance. *Q. J. Mech. Appl. Math.* **65**, 555–579 (2012)
4. Céa, J.: Optimization. Théorie et Algorithmes. Dunod, Paris (1971)
5. Ciurcea, R., Matei, A.: Solvability of a mixed variational problem. *Ann. Univ. Craiova* **36**, 105–111 (2009)
6. Ekeland, I., Temam, R.: Convex Analysis and Variational Problems. Classics in Applied Mathematics 28 SIAM, Philadelphia (1999)
7. Han, W., Sofonea, M.: Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity. Studies in Advanced Mathematics 30, American Mathematical Society, Providence, RI–International Press, Sommerville (2002)
8. Haslinger, J., Hlaváček, I., Nečas, J.: Numerical methods for unilateral problems in solid mechanics. In: Lions J.-L., Ciarlet, P.G. (eds.) Handbook of Numerical Analysis, vol. IV, pp. 313–485. North-Holland, Amsterdam (1996)
9. Hlaváček, I., Haslinger, J., Nečas, J., Lovíšek, J.: Solution of Variational Inequalities in Mechanic. Springer, New York (1988)
10. Hüeber, S., Wohlmuth, B.: An optimal a priori error estimate for nonlinear multibody contact problems. *SIAM J. Numer. Anal.* **43**, 156–173 (2005)
11. Hüeber, S., Matei, A., Wohlmuth, B.: A mixed variational formulation and an optimal a priori error estimate for a frictional contact problem in elasto-piezoelectricity. *Bull. Math. Soc. Math. Roumanie* **48**, 209–232 (2005)

12. Hübner, S., Matei, A., Wohlmuth, B.: Efficient algorithms for problems with friction. *SIAM J. Sci. Comput.* **29**, 70–92 (2007)
13. Hild, P., Renard, Y.: A stabilized Lagrange multiplier method for the finite element approximation of contact problems in elastostatics. *Numer. Math.* **115**, 101–129 (2010)
14. Lions, J.-L., Glowinski, R., Trémolières, R.: *Numerical Analysis of Variational Inequalities*. North-Holland, Amsterdam (1981)
15. Lions, J.-L., Magenes, E.: *Problèmes aux limites non homogènes*. Dunod, Paris (1968)
16. Matei, A., Ciurcea, R.: Weak solvability for a class of contact problems. *Ann. Acad. Romanian Sci. Ser. Math. Appl.* **2**, 25–44 (2010)
17. Matei, A., Ciurcea, R.: Contact problems for nonlinearly elastic materials: weak solvability involving dual Lagrange multipliers. *Aust. N. Z. Ind. Appl. Math. J.* **52**, 160–178 (2010)
18. Matei, A.: On the solvability of mixed variational problems with solution-dependent sets of Lagrange multipliers. *Proc. R. Soc. Edinb. Sect. A Math.* **143**, 1047–1059 (2013)
19. Matei, A., Ciurcea, R.: Weak solutions for contact problems involving viscoelastic materials with long memory. *Math. Mech. Solids* **16**, 393–405 (2011)
20. Matei, A.: An evolutionary mixed variational problem arising from frictional contact mechanics. *Math. Mech. Solids* **19**, 225–241 (2014)
21. Migórski S., Ochal A., Sofonea M.: *Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems*, vol. 26. *Advances in Mechanics and Mathematics*. Springer, New York (2013)
22. Reddy, B.D.: Mixed variational inequalities arising in elastoplasticity. *Nonlinear Anal. Theory Methods Appl.* **19**, 1071–1089 (1992)
23. Shillor, M., Sofonea, M., Telega, J.J.: *Models and Analysis of Quasistatic Contact*. *Lecture Notes in Physics* 655, Springer, Berlin (2004)
24. Sofonea, M., Matei, A.: History-dependent mixed variational problems in contact mechanics. *J. Global Optim.* **61**, 591–614 (2015)
25. Sofonea, M., Avramescu, C., Matei, A.: A Fixed point result with applications in the study of viscoplastic frictionless contact problems. *Commun. Pure Appl. Anal.* **7**, 645–658 (2008)
26. Sofonea, M., Matei, A.: History-dependent quasivariational inequalities arising in contact mechanics. *Eur. J. Appl. Math.* **22**, 471–491 (2011)
27. Sofonea, M., Matei, A.: *Mathematical Models in Contact Mechanics*. *London Mathematical Society Lecture Note Series* 398, Cambridge University Press, Cambridge (2012)
28. Sofonea, M., Pătrulescu, F.: Penalization of history-dependent variational inequalities. *Eur. J. Appl. Math.* **25**, 155–176 (2014)

Mircea Sofonea
 Laboratoire de Mathématiques et Physique
 Université de Perpignan Via Domitia
 52 Avenue Paul Alduy
 66860 Perpignan
 France
 e-mail: sofonea@univ-perp.fr

Andaluzia Matei
 Department of Mathematics
 University of Craiova
 A.I. Cuza 13
 200585 Craiova
 Romania
 e-mail: andaluziamatei@inf.ucv.ro