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# Necessary and sufficient conditions for the existence of $\alpha$ -determinantal processes

Franck Maunoury \*

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## Abstract

We give necessary and sufficient conditions for existence and infinite divisibility of  $\alpha$ -determinantal processes. For that purpose we use results on negative binomial and ordinary binomial multivariate distributions.

**Keywords:** determinantal process, permanental process,  $\alpha$ -determinantal process,  $\alpha$ -permanental process, infinitely divisible, complete monotonicity, fermion process, boson process.

**AMS 2010 subject classifications:** 60G55, 60E07.

## 1 Introduction

Several authors have already established necessary and sufficient conditions for existence of  $\alpha$ -determinantal processes.

Macchi in [8] and Soshnikov in its survey paper [11] gave a necessary and sufficient condition for determinantal processes with self-adjoint kernels, which corresponds to the case  $\alpha = -1$ .

The same condition has also been established in a different way by Hough, Krishnapur, Peres and Virág in [7] in the case  $\alpha = -1$ . They have also given a sufficient condition of existence in the case  $\alpha = 1$  and self-adjoint kernel.

In the special case when the configurations are on a finite space, the paper of Vere-Jones [12] provides necessary and sufficient conditions for any value of  $\alpha$ .

Finally, Shirai and Takahashi have given sufficient conditions for the existence of an  $\alpha$ -determinantal process for any values of  $\alpha$ . However, in the case  $\alpha > 0$ , their sufficient condition (Condition B) in [9] does not work for the following example: the space is reduced to a single point space and the reference measure  $\lambda$  is a unit point mass. With their notations, the two kernels  $K$  and  $J_\alpha$  are respectively reduced to two real numbers  $k$  and  $j_\alpha$ , with

$$j_\alpha = \frac{k}{1 + \alpha k}$$

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We can choose  $\alpha > 0$  and  $k < 0$  such that  $j_\alpha > 0$ . Under these assumptions, Condition B is fulfilled but the obtained point process has a negative correlation function ( $\rho_1(x) = k$ ), which has to be excluded, since a correlation function is an almost everywhere non-negative function.

We are going to strengthen Condition B of Shirai and Takahashi and obtain a necessary and sufficient condition in the case  $\alpha > 0$ . This is presented in Theorem 1.

Besides, in the case  $\alpha < 0$ , we extend the result of Shirai and Takahashi to the case of non self-adjoint kernels and show that the obtained condition is also necessary (Theorems 4 and 5). Moreover, we show that  $-1/\alpha$  is necessarily an integer. This has been noticed by Vere-Jones in [13] in the case of configurations on a finite space.

We also give a necessary and sufficient condition for the infinite divisibility of an  $\alpha$ -determinantal process for all values of  $\alpha$ .

The main results are presented in Section 3. Section 2 introduces the needed notation. In Section 4, we write a multivariate version of a Shirai and Takahashi formulae on Fredholm determinant expansion. Sections 5 and 6 present the proofs of the results concerning respectively the cases  $\alpha > 0$  and  $\alpha < 0$ . The proofs concerning infinite divisibility are presented in Section 7.

## 2 Preliminaries

Let  $E$  be a locally compact Polish space. A locally finite configuration on  $E$  is an integer-valued positive Radon measure on  $E$ . It can also be identified with a set  $\{(M, \alpha_M) : M \in F\}$ , where  $F$  is a countable subset of  $E$  with no accumulation points (i.e. a discrete subset of  $E$ ) and, for each point in  $F$ ,  $\alpha_M$  is a non-null integer that corresponds to the multiplicity of the point  $M$  ( $M$  is a multiple point if  $\alpha_M \geq 2$ ).

Let  $\lambda$  be a Radon measure on  $E$ . Let  $\mathcal{X}$  be the space of the locally finite configurations of  $E$ . The space  $\mathcal{X}$  is endowed with the vague topology of measures, i.e. the smallest topology such that, for every real continuous function  $f$  with compact support, defined on  $E$ , the mapping

$$\mathcal{X} \ni \xi \mapsto \langle f, \xi \rangle = \sum_{x \in \xi} f(x) = \int f d\xi$$

is continuous. Details on the topology of the configuration space can be found in [1].

We denote by  $\mathcal{B}(\mathcal{X})$  the corresponding  $\sigma$ -algebra. A point process on  $E$  is a random variable with values in  $\mathcal{X}$ . We do not restrict ourselves to simple point processes, as the configurations in  $\mathcal{X}$  can have multiple points.

For a  $n \times n$  matrix  $A = (a_{[ij]})_{1 \leq i, j \leq n}$ , set:

$$\det_\alpha A = \sum_{\sigma \in \Sigma_n} \alpha^{n-\nu(\sigma)} \prod_{i=1}^n a_{i\sigma(i)}$$

where  $\Sigma_n$  is the set of all permutations on  $\{1, \dots, n\}$  and  $\nu(\sigma)$  is the number of cycles of the permutation  $\sigma$ .

For a relatively compact set  $\Lambda \subset E$ , the Janossy densities of a point process  $\xi$  w.r.t. a Radon measure  $\lambda$  are functions (when they exist)  $j_n^\Lambda : E^n \rightarrow [0, \infty)$  for  $n \in \mathbb{N}$ , such that

$$\begin{aligned} j_n^\Lambda(x_1, \dots, x_n) &= n! \mathbb{P}(\xi(\Lambda) = n) \pi_n^\Lambda(x_1, \dots, x_n) \\ j_0^\Lambda(\emptyset) &= \mathbb{P}(\xi(\Lambda) = 0), \end{aligned}$$

where  $\pi_n^\Lambda$  is the density with respect to  $\lambda^{\otimes n}$  of the ordered set  $(x_1, \dots, x_n)$ , obtained by first sampling  $\xi$ , given that there are  $n$  points in  $\Lambda$ , then choosing uniformly an order between the points.

For  $\Lambda_1, \dots, \Lambda_n$  disjoint subsets included in  $\Lambda$ ,  $\int_{\Lambda_1 \times \dots \times \Lambda_n} j_n^\Lambda(x_1, \dots, x_n) \lambda(dx_1) \dots \lambda(dx_n)$  is the probability that there is exactly one point in each subset  $\Lambda_i$  ( $1 \leq i \leq n$ ), and no other point elsewhere.

We recall that we have the following formula, for a non-negative measurable function  $f$  with support in a relatively compact set  $\Lambda \subset E$ :

$$\mathbb{E}(f(\xi)) = f(\emptyset) j_0^\Lambda(\emptyset) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} f(x_1, \dots, x_n) j_n^\Lambda(x_1, \dots, x_n) \lambda(dx_1) \dots \lambda(dx_n).$$

For  $n \in \mathbb{N}$  and  $a \in \mathbb{R}$ , we denote  $a^{(n)} = \prod_{i=0}^{n-1} (a - i)$ .

The correlation functions (also called joint intensities) of a point process  $\xi$  w.r.t. a Radon measure  $\lambda$  are functions (when they exist)  $\rho_n : E^n \rightarrow [0, \infty)$  for  $n \geq 1$ , such that for any family of mutually disjoint relatively compact subsets  $\Lambda_1, \dots, \Lambda_d$  of  $E$  and for any non-null integers  $n_1, \dots, n_d$  such that  $n_1 + \dots + n_d = n$ , we have

$$\mathbb{E} \left( \prod_{i=1}^d \xi(\Lambda_i)^{(n_i)} \right) = \int_{\Lambda_1^{n_1} \times \dots \times \Lambda_d^{n_d}} \rho_n(x_1, \dots, x_n) \lambda(dx_1), \dots, \lambda(dx_n).$$

Intuitively, for a simple point process,  $\rho_n(x_1, \dots, x_n) \lambda(dx_1) \dots \lambda(dx_n)$  is the infinitesimal probability that there is at least one point in the vicinity of each  $x_i$  (each vicinity having an infinitesimal volume  $\lambda(dx_i)$  around  $x_i$ ),  $1 \leq i \leq n$ .

Let  $\alpha$  be a real number and  $K$  a kernel from  $E^2$  to  $\mathbb{R}$  or  $\mathbb{C}$ . An  $\alpha$ -determinantal point process, with kernel  $K$  with respect to  $\lambda$  (also called  $\alpha$ -permanental point process) is defined, when it exists, as a point process with the following correlation functions  $\rho_n$ ,  $n \in \mathbb{N}$  with respect to  $\lambda$ :

$$\rho_n(x_1, \dots, x_n) = \det_\alpha(K(x_i, x_j))_{1 \leq i, j \leq n}.$$

We denote by  $\mu_{\alpha, K, \lambda}$  the probability distribution of such a point process.

We exclude the case of a point process almost surely reduced to the empty configuration.

The case  $\alpha = -1$  corresponds to a determinantal process and the case  $\alpha = 1$  to a permanental process. The case  $\alpha = 0$  corresponds to the Poisson point process. We suppose in the following that  $\alpha \neq 0$ .

We will always assume that the kernel  $K$  defines a locally trace class integral operator  $\mathcal{K}$  on  $L^2(E, \lambda)$ . Under this assumption, one obtains an equivalent definition for the  $\alpha$ -determinantal process, using the following Laplace functional formula:

$$\mathbb{E}_{\mu_{\alpha, K, \lambda}} \left[ \exp \left( - \int_E f d\xi \right) \right] = \text{Det} \left( \mathcal{I} + \alpha \mathcal{K} [1 - e^{-f}] \right)^{-1/\alpha} \quad (1)$$

where  $f$  is a compactly-supported non-negative function on  $E$ ,  $\mathcal{K}[1 - e^{-f}]$  stands for  $\sqrt{1 - e^{-f}}\mathcal{K}\sqrt{1 - e^{-f}}$ ,  $\mathcal{I}$  is the identity operator on  $L^2(E, \lambda)$  and  $\text{Det}$  is the Fredholm determinant. Details on the link between the correlation function and the Laplace functional of an  $\alpha$ -determinantal process can be found in the chapter 4 of [9]. Some explanations and useful formula on the Fredholm determinant are given in chapter 2.1 of [9].

For a subset  $\Lambda \subset E$ , set:  $\mathcal{K}_\Lambda = p_\Lambda \mathcal{K} p_\Lambda$ , where  $p_\Lambda$  is the orthogonal projection operator from  $L^2(E, \lambda)$  to the subspace  $L^2(\Lambda, \lambda)$ .

For two subsets  $\Lambda, \Lambda' \subset E$ , set:  $\mathcal{K}_{\Lambda\Lambda'} = p_\Lambda \mathcal{K} p_{\Lambda'}$ , and denote by  $K_{\Lambda\Lambda'}$  its kernel. We have for any  $x, y \in E$ ,  $K_{\Lambda\Lambda'}(x, y) = \mathbb{1}_\Lambda(x) \mathbb{1}_{\Lambda'}(y) K(x, y)$ .

When  $\mathcal{I} + \alpha\mathcal{K}$  (resp.  $\mathcal{I} + \alpha\mathcal{K}_\Lambda$ ) is invertible,  $\mathcal{J}_\alpha$  (resp.  $\mathcal{J}_\alpha^\Lambda$ ) is the integral operator defined by:  $\mathcal{J}_\alpha = \mathcal{K}(\mathcal{I} + \alpha\mathcal{K})^{-1}$  (resp.  $\mathcal{J}_\alpha^\Lambda = \mathcal{K}_\Lambda(\mathcal{I} + \alpha\mathcal{K}_\Lambda)^{-1}$ ) and we denote by  $J_\alpha$  (resp.  $J_\alpha^\Lambda$ ) its kernel. Note that  $\mathcal{J}_\alpha^\Lambda$  is not the orthogonal projection of  $\mathcal{J}_\alpha$  on  $L^2(\Lambda, \lambda)$ .

### 3 Main results

**Theorem 1.** *For  $\alpha > 0$ , there exists an  $\alpha$ -permanental process with kernel  $K$  iff:*

- $\text{Det}(\mathcal{I} + \alpha\mathcal{K}_\Lambda) \geq 1$ , for any compact set  $\Lambda \subset E$
- $\det_\alpha(J_\alpha^\Lambda(x_i, x_j))_{1 \leq i, j \leq n} \geq 0$ , for any  $n \in \mathbb{N}$ , any compact set  $\Lambda \subset E$  and any  $\lambda^{\otimes n}$ -a.e.  $(x_1, \dots, x_n) \in \Lambda^n$ .

**Remark 2.** Even when  $E$  is a finite set, note that the second condition of Theorem 1 consists in an infinite number of computations. Finding a simpler condition, that could be checked in a finite number of steps is still an open problem.

**Theorem 3.** *For  $\alpha > 0$ , if an  $\alpha$ -permanental process with kernel  $K$  exists, then:*

$$\text{Spec } \mathcal{K}_\Lambda \subset \left\{ z \in \mathbb{C} : \text{Re } z > -\frac{1}{2\alpha} \right\}, \text{ for any compact set } \Lambda \subset E.$$

We remark that this condition is equivalent to

$$\text{Spec } \mathcal{J}_\alpha^\Lambda \subset \left\{ z \in \mathbb{C} : |z| < \frac{1}{\alpha} \right\}, \text{ for any compact set } \Lambda \subset E$$

**Theorem 4.** *For  $\alpha < 0$  and  $\mathcal{K}$  an integral operator such that  $\mathcal{I} + \alpha\mathcal{K}_\Lambda$  is invertible, for any compact set  $\Lambda \subset E$ , an  $\alpha$ -determinantal process with kernel  $K$  exists iff the two following conditions are fulfilled:*

- (i)  $-1/\alpha \in \mathbb{N}$
- (ii)  $\det(J_\alpha^\Lambda(x_i, x_j))_{1 \leq i, j \leq n} \geq 0$ , for any  $n \in \mathbb{N}$ , any compact set  $\Lambda \subset E$  and any  $\lambda^{\otimes n}$ -a.e.  $(x_1, \dots, x_n) \in \Lambda^n$ .

The arguments developed in the proof of Theorem 4 shows that actually (ii)  $\implies$  (i). Consequently, Condition (ii) is itself a necessary and sufficient condition. It also implies that  $\text{Det}(\mathcal{I} + \beta\mathcal{K}_\Lambda) > 0$  for any  $\beta \in [\alpha, 0]$  and any compact  $\Lambda \subset E$ .

**Theorem 5.** *For  $\alpha < 0$  and  $\mathcal{K}$  an integral operator such that for some compact set  $\Lambda_0 \subset E$ ,  $\mathcal{I} + \alpha\mathcal{K}_{\Lambda_0}$  is not invertible, an  $\alpha$ -determinantal process with kernel  $K$  exists iff:*

$$(i') \quad -1/\alpha \in \mathbb{N}$$

$$(ii') \quad \det(J_\beta^\Lambda(x_i, x_j))_{1 \leq i, j \leq n} \geq 0, \text{ for any } n \in \mathbb{N}, \text{ any } \beta \in (\alpha, 0), \text{ any compact set } \Lambda \subset E \\ \text{and any } \lambda^{\otimes n}\text{-a.e. } (x_1, \dots, x_n) \in \Lambda^n.$$

As in Theorem 4, we also have (ii')  $\implies$  (i') and Condition (ii') is itself a necessary and sufficient condition.

Note that  $\mathcal{I} + \alpha\mathcal{K}_{\Lambda_0}$  is not invertible if and only if there is almost surely at least one point in  $\Lambda_0$ .

**Corollary 6.** *For  $m$  a positive integer, the existence of a  $(-1/m)$ -determinantal process with kernel  $K$  is equivalent to the existence of a determinantal process with the kernel  $\frac{K}{m}$ .*

**Corollary 7.** *For  $\alpha < 0$  and  $\mathcal{K}$  a self-adjoint operator, an  $\alpha$ -determinantal process with kernel  $K$  exists iff:*

- $-1/\alpha \in \mathbb{N}$
- $\text{Spec } \mathcal{K} \subset [0, -1/\alpha]$

This result is well known in the case  $\alpha = -1$  (see for example Hough, Krishnapur, Peres and Virág in [7]).

The sufficient part of this necessary and sufficient condition corresponds to condition A in [9] of Shirai and Takahashi.

**Theorem 8.** *For  $\alpha < 0$ , an  $\alpha$ -determinantal process is never infinitely divisible.*

**Theorem 9.** *For  $\alpha > 0$ , an  $\alpha$ -determinantal process is infinitely divisible iff*

- $\text{Det}(\mathcal{I} + \alpha\mathcal{K}_\Lambda) \geq 1$ , for any compact set  $\Lambda \subset E$
- $\sum_{\sigma \in \Sigma_n: \nu(\sigma)=1} \prod_{i=1}^n J_\alpha^\Lambda(x_i, x_{\sigma(i)}) \geq 0$ , for any  $n \in \mathbb{N}$ , any compact set  $\Lambda \subset E$  and  $\lambda^{\otimes n}$ -a.e.  $(x_1, \dots, x_n) \in \Lambda^n$ .

This theorem gives a more general condition for infinite-divisibility of an  $\alpha$ -permanental process than the condition given by Shirai and Takahashi in [9].

**Theorem 10.** *For  $\mathcal{K}$  a real symmetric locally trace class operator and  $\alpha > 0$ , an  $\alpha$ -permanental process is infinitely divisible iff*

- $\text{Det}(\mathcal{I} + \alpha\mathcal{K}_\Lambda) \geq 1$ , for any compact set  $\Lambda \subset E$
- $J_\alpha^\Lambda(x_1, x_2) \dots J_\alpha^\Lambda(x_{n-1}, x_n) J_\alpha^\Lambda(x_n, x_1) \geq 0$ , for any  $n \in \mathbb{N}$ , any compact set  $\Lambda \subset E$  and  $\lambda^{\otimes n}$ -a.e.  $(x_1, \dots, x_n) \in \Lambda^n$ .

Following Griffith and Milne's remark in [6], when an  $\alpha$ -permanental process with kernel  $K$  exists and is infinitely divisible, we can replace  $J_\alpha^\Lambda$  by  $|J_\alpha^\Lambda|$  and obtain an  $\alpha$ -permanental process with the same probability distribution.

**Remark 11.** In Theorem 1, 9 and 10, the condition

$$\text{Det}(\mathcal{I} + \alpha\mathcal{K}_\Lambda) \geq 1, \text{ for any compact set } \Lambda \subset E$$

can be replaced by

$$\text{Det}(\mathcal{I} + \alpha\mathcal{K}_\Lambda) > 0, \text{ for any compact set } \Lambda \subset E.$$

## 4 Fredholm determinant expansion

In [9], Shirai and Takahashi have proved the following formula

$$\text{Det}(\mathcal{I} - \alpha z \mathcal{K})^{-1/\alpha} = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{E^n} \det_\alpha(K(x_i, x_j))_{1 \leq i, j \leq n} \lambda(dx_1) \dots \lambda(dx_n) \quad (2)$$

for a trace class integral operator  $\mathcal{K}$  with kernel  $K$  and for  $z \in \mathbb{C}$  such that  $\|\alpha z \mathcal{K}\| < 1$ . In the case where the space  $E$  is finite, this formula is also given by Shirai in [10].

As  $z \mapsto \text{Det}(\mathcal{I} - \alpha z \mathcal{K})$  is analytic on  $\mathbb{C}$  and  $z \mapsto z^{-1/\alpha}$  is analytic on  $\mathbb{C}^*$ , we obtain that  $z \mapsto \text{Det}(\mathcal{I} - \alpha z \mathcal{K}_{\Lambda, \alpha})^{-1/\alpha}$  is analytic on  $\{z \in \mathbb{C} : \mathcal{I} - \alpha z \mathcal{K}_{\Lambda, \alpha} \text{ invertible}\}$ .

Therefore, the formula can be extended to the open disc  $D$ , centered in 0 with radius  $R = \sup\{r \in \mathbb{R}_+ : \forall z \in \mathbb{C}, |z| < r \Rightarrow \mathcal{I} - \alpha z \mathcal{K} \text{ is invertible}\}$ .

$D$  is the open disc of center 0 and radius  $1/\|\alpha \mathcal{K}\|$ , if the operator  $\mathcal{K}$  is self-adjoint, but it can be larger if  $\mathcal{K}$  is not self-adjoint.

As remarked by Shirai and Takahashi, the formula (2) is valid for any  $z \in \mathbb{C}$  if  $-1/\alpha \in \mathbb{N}$ .

The following proposition extends (2) to a multivariate case.

**Proposition 12.** *Let  $\Lambda \subset E$  be a relatively compact set,  $\Lambda_1, \dots, \Lambda_d$  mutually disjoint subsets of  $\Lambda$  and  $\mathcal{K}$  a locally trace class integral operator with kernel  $K$ .*

*We have the following formula*

$$\begin{aligned} & \text{Det} \left( \mathcal{I} - \alpha \sum_{k=1}^d z_k \mathcal{K}_{\Lambda_k \Lambda} \right)^{-1/\alpha} \\ &= \sum_{n_1, \dots, n_d=0}^{\infty} \left( \prod_{k=1}^d \frac{z_k^{n_k}}{n_k!} \right) \int_{\Lambda_1^{n_1} \times \dots \times \Lambda_d^{n_d}} \det_\alpha(K(x_i, x_j))_{1 \leq i, j \leq n} \lambda(dx_1) \dots \lambda(dx_n) \quad (3) \end{aligned}$$

for any  $z_1, \dots, z_d \in \mathbb{C}$ , such that  $\mathcal{I} - \alpha \gamma \sum_{k=1}^d z_k \mathcal{K}_{\Lambda_k \Lambda}$  is invertible for any complex number  $\gamma$  satisfying  $|\gamma| < 1$  ( $n$  denotes  $n_1 + \dots + n_d$ ).

*Proof.* We apply the formula (2) to the class trace operator  $\sum_{k=1}^d z_k \mathcal{K}_{\Lambda_k \Lambda}$  and we use the multilinearity property of the  $\alpha$ -determinant of a matrix with respect to its rows.

We obtain

$$\begin{aligned}
& \text{Det} \left( \mathcal{I} - \alpha \sum_{k=1}^d z_k \mathcal{K}_{\Lambda_k \Lambda} \right)^{-1/\alpha} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{E^n} \det_{\alpha} \left( \sum_{k=1}^d z_k K_{\Lambda_k \Lambda}(x_i, x_j) \right)_{1 \leq i, j \leq n} \lambda(dx_1) \dots \lambda(dx_n) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{E^n} \sum_{k_1, \dots, k_n=1}^d \det_{\alpha} \left( z_{k_i} \mathbb{1}_{\Lambda_{k_i}}(x_i) \mathbb{1}_{\Lambda}(x_j) K(x_i, x_j) \right)_{1 \leq i, j \leq n} \lambda(dx_1) \dots \lambda(dx_n) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_1, \dots, k_n=1}^d \int_{\Lambda_{k_1} \times \dots \times \Lambda_{k_n}} \det_{\alpha} \left( z_{k_i} K(x_i, x_j) \right)_{1 \leq i, j \leq n} \lambda(dx_1) \dots \lambda(dx_n) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_1, \dots, k_n=1}^d \left( \prod_{i=1}^n z_{k_i} \right) \int_{\Lambda_{k_1} \times \dots \times \Lambda_{k_n}} \det_{\alpha} \left( K(x_i, x_j) \right)_{1 \leq i, j \leq n} \lambda(dx_1) \dots \lambda(dx_n)
\end{aligned}$$

where we have used the fact that  $K_{\Lambda_k \Lambda}(x_i, x_j) = \mathbb{1}_{\Lambda_k}(x_i) \mathbb{1}_{\Lambda}(x_j) K(x_i, x_j)$  for the equality between the first and the second line.

As the value of the  $\alpha$ -determinant of a matrix is unchanged by simultaneous interchange of its rows and its columns, the product  $z_1^{n_1} \dots z_d^{n_d}$  where  $n_1 + \dots + n_d = n$ , will be repeated  $\binom{n}{n_1 \dots n_d}$  times. This gives the desired formula.  $\square$

For a relatively compact set  $\Lambda \subset E$  and  $\Lambda_1, \dots, \Lambda_d$  mutually disjoint subsets of  $\Lambda$ , the computation of the Laplace functional of an  $\alpha$ -determinantal process for the function  $f : (z_1, \dots, z_d) \mapsto -\sum_{k=1}^d (\log z_k) \mathbb{1}_{\Lambda_k}$ , with  $z_1, \dots, z_d \in (0, 1]$  gives thanks to (1):

$$\mathbb{E}_{\mu_{\alpha, K, \lambda}} \left[ \prod_{k=1}^d z_k^{\xi(\Lambda_k)} \right] = \text{Det} \left( \mathcal{I} + \alpha \sum_{k=1}^d (1 - z_k) \mathcal{K}_{\Lambda_k \Lambda} \right)^{-1/\alpha} \quad (4)$$

which is the probability generating function (p.g.f.) of the finite-dimensional random vector  $(\xi(\Lambda_1), \dots, \xi(\Lambda_d))$ .

For  $\alpha < 0$ , the formula (4) reminds the multivariate binomial distribution p.g.f. and for  $\alpha > 0$ , the multivariate negative binomial distribution p.g.f., given by Vere-Jones in [12], in the special case where the space  $E$  is finite.

## 5 $\alpha$ - permanental process ( $\alpha > 0$ )

*Proof of Theorem 1.* We first prove that the conditions are necessary. We suppose that there exists an  $\alpha$ -permanental process with  $\alpha > 0$ , kernel  $K$  defining the locally trace class integral operator  $\mathcal{K}$ .



By taking  $d = 1$  in the formula (4), we have

$$\mathbb{E}_{\mu_{\alpha, K, \lambda}} \left( z^{\xi(\Lambda)} \right) = \text{Det}(\mathcal{I} + \alpha(1 - z)\mathcal{K}_\Lambda)^{-1/\alpha}$$

for any compact set  $\Lambda \subset E$  and  $z \in (0, 1]$ .

Thus,  $\text{Det}(\mathcal{I} + \alpha(1 - z)\mathcal{K}_\Lambda) \geq 1$  for  $z \in (0, 1]$ . By continuity (as  $z \mapsto \text{Det}(\mathcal{I} + (1 - z)\mathcal{K}_\Lambda)$  is indeed analytic on  $\mathbb{C}$ ), we obtain that  $\text{Det}(\mathcal{I} + \alpha\mathcal{K}_\Lambda) \geq 1$ , which is the first condition. This implies that for any compact set  $\Lambda \subset E$ ,  $\mathcal{I} + \alpha\mathcal{K}_\Lambda$  is invertible. Hence  $\mathcal{J}_\alpha^\Lambda$  exists and we have, for any non-negative function  $f$ , with compact support included in  $\Lambda$

$$\begin{aligned} \mathbb{E}_{\mu_{\alpha, K, \lambda}} \left( \prod_{x \in \xi} e^{-f(x)} \right) &= \text{Det}(\mathcal{I} + \alpha\mathcal{K}[1 - e^{-f}])^{-1/\alpha} \\ &= \text{Det}(\mathcal{I} + \alpha\mathcal{K}_\Lambda(1 - e^{-f}))^{-1/\alpha} \\ &= \text{Det}(\mathcal{I} + \alpha\mathcal{K}_\Lambda)^{-1/\alpha} \text{Det}(\mathcal{I} - \alpha\mathcal{J}_\alpha^\Lambda e^{-f})^{-1/\alpha} \\ &= \text{Det}(\mathcal{I} + \alpha\mathcal{K}_\Lambda)^{-1/\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \left( \prod_{i=1}^n e^{-f(x_i)} \right) \det_\alpha(J_\alpha^\Lambda(x_i, x_j))_{1 \leq i, j \leq n} \lambda(dx_1) \dots \lambda(dx_n) \end{aligned} \quad (5)$$

where we have used for the equality between the first and the second line the fact that  $\text{Det}(\mathcal{I} + \mathcal{A}\mathcal{B}) = \text{Det}(\mathcal{I} + \mathcal{B}\mathcal{A})$ , for any trace class operator  $\mathcal{A}$ , and any bounded operator  $\mathcal{B}$ .

As the Laplace functional defines a.e. uniquely the Janossy density of a point process, one obtains:

$$\det_\alpha(J_\alpha^\Lambda(x_i, x_j))_{1 \leq i, j \leq n} \geq 0 \quad \lambda^{\otimes n}\text{-a.e. } (x_1, \dots, x_n) \in E^n$$

$j_{\alpha, n}^\Lambda(x_1, \dots, x_n) = \text{Det}(\mathcal{I} + \alpha\mathcal{K}_\Lambda)^{-1/\alpha} \det_\alpha(J_\alpha^\Lambda(x_i, x_j))_{1 \leq i, j \leq n}$  is the Janossy density.

Conversely, if we assume  $\text{Det}(\mathcal{I} + \alpha\mathcal{K}_\Lambda)^{-1/\alpha} > 0$  and  $\det_\alpha(J_\alpha^\Lambda(x_i, x_j))_{1 \leq i, j \leq n} \geq 0$  for any  $n \in \mathbb{N}$ , any compact set  $\Lambda \subset E$  and any  $\lambda^{\otimes n}$ -a.e.  $(x_1, \dots, x_n) \in \Lambda^n$ , the Janossy density will be correctly defined and, on any compact set  $\Lambda$ , we get the existence of a point process  $\xi_\Lambda$  with kernel  $K_\Lambda$  (see Proposition 5.3.II. in [2] - here the normalization condition is automatic by choosing  $f = 0$  in (5)).

The restriction of a point process  $\eta$ , defined on  $\Lambda' \subset E$ , to a subspace  $\Lambda \subset \Lambda'$  is the point process denoted  $\eta|_\Lambda$ , obtained by keeping the points in  $\Lambda$  and deleting the points in  $\Lambda' \setminus \Lambda$ . For any compact sets  $\Lambda, \Lambda' \subset E$ , such that  $\Lambda \subset \Lambda'$ ,  $\xi_\Lambda$  and  $\xi_{\Lambda'}|_\Lambda$  have the same Laplace functional, because we have for any non-negative function  $f$ , with compact support included in  $\Lambda$ :

$$\begin{aligned} \mathbb{E} \left( \exp \left( - \int_\Lambda f d\xi_{\Lambda'}|_\Lambda \right) \right) &= \text{Det}(\mathcal{I} + \alpha\mathcal{K}_{\Lambda'}[1 - e^{-f}])^{-1/\alpha} \\ &= \text{Det}(\mathcal{I} + \alpha\mathcal{K}_\Lambda[1 - e^{-f}])^{-1/\alpha} \\ &= \mathbb{E} \left( \exp \left( - \int_\Lambda f d\xi_\Lambda \right) \right). \end{aligned}$$

Therefore,  $\xi_\Lambda$  and  $\xi_{\Lambda'}|_\Lambda$  have the same probability distribution. We say that the family  $(\mathcal{L}(\xi_\Lambda))$ ,  $\Lambda$  compact set included in  $E$ , is consistent.

Then we can obtain a point process on the complete space  $E$  by the Kolmogorov existence theorem for point processes (see Theorem 9.2.X in [3] with  $P_k(A_1, \dots, A_k; n_1, \dots, n_k) = \mathbb{P}(\xi_{\cup_{i=1}^k A_i}(A_1) = n_1, \dots, \xi_{\cup_{i=1}^k A_i}(A_k) = n_k)$ : as  $\xi_{\cup_{i=1}^k A_i}$  is a point process, it follows that the properties (i), (iii), (iv) are fulfilled ; (ii) is fulfilled because the family  $(\mathcal{L}(\xi_\Lambda))$ ,  $\Lambda$  compact set included in  $E$ , is consistent).

As we used, in this second part of the proof, only the fact that  $\text{Det}(\mathcal{I} + \alpha\mathcal{K}_\Lambda)^{-1/\alpha} > 0$  (instead of  $\text{Det}(\mathcal{I} + \alpha\mathcal{K}_\Lambda)^{-1/\alpha} \geq 1$ ), the assertion in remark 11 is also proved.  $\square$

*Proof of Theorem 3.* We suppose there exists an  $\alpha$ -permanental process with  $\alpha > 0$ , kernel  $K$  defining the locally trace class integral operator  $\mathcal{K}$ .

Then, following the proof of the preceding theorem, we get that, for all  $z \in [0, 1]$

$$\text{Det}(\mathcal{I} + \alpha(1 - z)\mathcal{K}_\Lambda) = \text{Det}(\mathcal{I} + \alpha\mathcal{K}_\Lambda) \text{Det}(\mathcal{I} - \alpha z\mathcal{J}_\alpha^\Lambda) > 0.$$

As the power series of  $\text{Det}(\mathcal{I} - \alpha z\mathcal{J}_\alpha^\Lambda)^{-1/\alpha}$  has all its terms non-negative,

$$|(\text{Det}(\mathcal{I} - \alpha z\mathcal{J}_\alpha^\Lambda)^{-1/\alpha}| \leq (\text{Det}(\mathcal{I} - \alpha|z|\mathcal{J}_\alpha^\Lambda)^{-1/\alpha}.$$

If  $z_0$  is a complex number with minimum modulus such that  $(\text{Det}(\mathcal{I} - \alpha z_0\mathcal{J}_\alpha^\Lambda) = 0$ , by analyticity of  $z \mapsto \text{Det}(\mathcal{I} - \alpha z\mathcal{J}_\alpha^\Lambda)$  on  $\mathbb{C}$  and  $z \mapsto z^{-1}$  on  $\mathbb{C}^*$ ,  $\text{Det}(\mathcal{I} - \alpha z\mathcal{J}_\alpha^\Lambda)^{-1/\alpha}$  converges for  $|z| < |z_0|$  and diverges for  $z = z_0$ . Thus the series diverges in  $z = |z_0|$  and  $|z_0| > 1$ . This means that the series converges for  $|z| \leq 1$  thus, in this case,  $\text{Det}(\mathcal{I} - \alpha z\mathcal{J}_\alpha^\Lambda) > 0$ .

This implies the necessary condition:  $\text{Spec } \mathcal{J}_\alpha^\Lambda \subset \{z \in \mathbb{C} : |z| < \frac{1}{\alpha}\}$ .

As  $\nu$  eigenvalue of  $\mathcal{K}$  is equivalent to  $\frac{\nu}{1 + \alpha\nu}$  eigenvalue of  $\mathcal{J}$ , and as,  $\mathcal{K}$  and  $\mathcal{J}$  being compact operators, their non-null spectral values are their eigenvalues, we get the other equivalent necessary condition:

$$\text{Spec } \mathcal{K}_\Lambda \subset \{z \in \mathbb{C} : \text{Re } z > -\frac{1}{2\alpha}\}.$$

$\square$

## 6 $\alpha$ -determinantal process ( $\alpha < 0$ )

We recall the following remark, already made for example in [7].

**Remark 13.** If we define kernels only  $\lambda^{\otimes 2}$ -almost everywhere, there can be problems when we consider only the diagonal terms, as  $\lambda^{\otimes 2}\{(x, x) : x \in \Lambda\} = 0$ . For example, in the formula

$$\text{tr } K_\Lambda = \int_\Lambda K(x, x)\lambda(dx),$$

$\text{tr } K_\Lambda$  is not uniquely defined. To avoid this problem, we write the kernel  $K_\Lambda$  as follows:

$$K_\Lambda(x, y) = \sum_{k=0}^{\infty} a_k \varphi_k(x) \overline{\psi_k}(y)$$

where  $(\varphi_k)_{k \in \mathbb{N}}$ ,  $(\psi_k)_{k \in \mathbb{N}}$  are orthonormal basis in  $L^2(\Lambda, \lambda)$  and  $(a_k)_{k \in \mathbb{N}}$  is a sequence of non-negative real number, which are the singular values of the operator  $\mathcal{K}_\Lambda$ .

The functions  $\varphi_k$  and  $\psi_k$ ,  $k \in \mathbb{N}$ , are defined  $\lambda$ -almost everywhere, but this gives then a unique value for the expression of type

$$\int_{\Lambda^n} F(K(x_i, x_j)_{1 \leq i, j \leq n}) G(x_1, \dots, x_n) \lambda(dx_1) \dots \lambda(dx_n)$$

where  $F$  is an arbitrary complex function from  $\mathbb{C}^{n^2}$  and  $G$  is an arbitrary complex function from  $\Lambda^n$ .

With this remark, the quantities that appear with  $F = \det_\alpha$  are well defined.

**Lemma 14.** *Let  $K$  be a kernel defined as in Remark 13 and defining a trace class integral operator  $\mathcal{K}$  on  $L^2(\Lambda, \lambda)$ , where  $\Lambda$  is a non- $\lambda$ -null compact set included in the locally compact Polish space  $E$ ,  $\lambda$  be a Radon measure,  $n$  an integer and  $\alpha$  a real number. Let  $F$  be a continuous function from  $\mathbb{C}^{n^2}$  to  $\mathbb{C}$ . The three following assertions are equivalent*

- (i)  $F(K(x_i, x_j)_{1 \leq i, j \leq n}) \geq 0$   $\lambda^{\otimes n}$  - a.e.  $(x_1, \dots, x_n) \in \Lambda^n$
- (ii) there exists a set  $\Lambda' \subset \Lambda$  such that  $\lambda(\Lambda \setminus \Lambda') = 0$  and  $F((K(x_i, x_j))_{1 \leq i, j \leq n}) \geq 0$  for any  $(x_1, \dots, x_n) \in (\Lambda')^n$
- (iii) there exists a version of  $K$  such that  $F((K(x_i, x_j))_{1 \leq i, j \leq n}) \geq 0$  for any  $(x_1, \dots, x_n) \in \Lambda^n$

*Proof.* (i) is clearly a consequence of (ii). We assume now that (i) is satisfied and we denote by  $N$  the  $\lambda^{\otimes n}$ -null set of  $n$ -tuples  $(x_1, \dots, x_n) \in \Lambda^n$  such that  $F((K(x_i, x_j))_{1 \leq i, j \leq n}) < 0$ . As in remark 13, we write the kernel  $K$  as follows

$$K(x, y) = \sum_{k=0}^{\infty} a_k \varphi_k(x) \overline{\psi_k(y)} = \langle (\sqrt{a_k} \varphi_k)_{k \in \mathbb{N}}(x) | (\sqrt{a_k} \psi_k)_{k \in \mathbb{N}}(y) \rangle$$

where  $(\varphi_k)_{k \in \mathbb{N}}$ ,  $(\psi_k)_{k \in \mathbb{N}}$  are orthonormal basis in  $L^2(\Lambda, \lambda)$ ,  $(a_k)_{k \in \mathbb{N}}$  is a sequence of non-negative real number, which are the singular values of the operator  $\mathcal{K}$  and  $\langle \cdot | \cdot \rangle$  denote the inner product in the Hilbert space  $l_2(\mathbb{C})$ .

As  $\mathcal{K}$  is trace class, we have  $\sum_{k=0}^{\infty} a_k < \infty$ . Hence:

$$\sum_{k=0}^{\infty} a_k |\varphi_k(x)|^2 < \infty \text{ and } \sum_{k=0}^{\infty} a_k |\psi_k(x)|^2 < \infty \text{ } \lambda\text{-a.e. } x \in \Lambda$$

From Lusin's theorem, there exists an increasing sequence  $(A_p)_{p \in \mathbb{N}}$  of compact sets included in  $\Lambda$  such that, for any  $p \in \mathbb{N}$

$$(\sqrt{a_k} \varphi_k)_{k \in \mathbb{N}} \text{ and } (\sqrt{a_k} \psi_k)_{k \in \mathbb{N}} \text{ are continuous from } A_p \text{ to } l_2(\mathbb{C}) \text{ and } \lambda(\Lambda \setminus A_p) < \frac{1}{p}$$

Therefore the kernel  $K : (x, y) \mapsto \langle (\sqrt{a_k} \varphi_k)_{k \in \mathbb{N}}(x) | (\sqrt{a_k} \psi_k)_{k \in \mathbb{N}}(y) \rangle$  is continuous on  $A_p^2$ . As  $E$  is a Polish space, it can be endowed with a distance that we denote by  $d$ . We consider the sets

$$A'_p = \{x \in A_p : \forall r > 0, \lambda(B(x, r) \cap A_p) > 0\}$$

$$B_{p,n} = \{x \in A_p : \lambda(B(x, 1/n) \cap A_p) = 0\}$$

where  $B(x, r)$  is the open ball in  $E$  of radius  $r$  centered at  $x$  and  $n$  is an integer. Let  $(x_k)_{k \in \mathbb{N}}$  be a sequence in  $B_{p,n}$  converging to  $x \in A_p$ . Then we have, when  $d(x, x_k) < 1/n$ ,

$$\lambda(B(x, 1/n - d(x, x_k)) \cap A_p) \leq \lambda(B(x_k, 1/n) \cap A_p) = 0$$

Therefore  $\lambda(B(x, 1/n) \cap A_p) = 0$  and  $x \in B_{p,n} : B_{p,n}$  is closed, thus compact (as it is included in the compact set  $A_p$ ).

The set of open balls  $\{B(x, 1/n) : x \in B_{p,n}\}$  is a cover of  $B_{p,n}$ . Then, by compactness,  $B_{p,n}$  can be covered by a finite numbers of such balls. As the intersections of  $A_p$  and any such a ball is a  $\lambda$ -null set, we get  $\lambda(B_{p,n}) = 0$ .

Hence we have:  $\lambda(A'_p) = \lambda(A_p \setminus \cup_{n \in \mathbb{N}} B_{p,n}) = \lambda(A_p) > \lambda(\Lambda) - 1/p$ .

Let  $(x_1, \dots, x_n) \in (A'_p)^n$ . If  $(x_1, \dots, x_n) \notin N$ , then  $F((K(x_i, x_j))_{1 \leq i, j \leq n}) \geq 0$ .

Otherwise  $(x_1, \dots, x_n) \in N$ . For any  $i \in \llbracket 1, n \rrbracket$  and any  $r > 0$ , we have

$$\lambda(A_p \cap B(x_i, r)) > 0, \text{ then } \lambda^{\otimes n}(A_p^n \cap B((x_1, \dots, x_n), r)) = \lambda^{\otimes n}(\prod_{i=1}^n (A_p \cap B(x_i, r))) > 0.$$

where  $B((x_1, \dots, x_n), r)$  denotes the open ball of radius  $r$  centered at  $x$ , in  $E^n$  endowed with the distance  $d((x_1, \dots, x_n), (y_1, \dots, y_n)) = \max_{1 \leq i \leq n} d(x_i, y_i)$ .

Then, as  $\lambda^{\otimes n}(N) = 0$ , for any  $q \in \mathbb{N}$ , there exists  $(y_1^{(q)}, \dots, y_n^{(q)}) \in A_p^n \cap B((x_1, \dots, x_n), 1/q) \setminus N$  and thus  $(y_1^{(q)}, \dots, y_n^{(q)})$  converge to  $(x_1, \dots, x_n)$  when  $q \rightarrow \infty$ .

As  $(y_1^{(q)}, \dots, y_n^{(q)}) \notin N$ ,  $F((K(y_i^{(q)}, y_j^{(q)}))_{1 \leq i, j \leq n}) \geq 0$ .

As  $K$  is continuous on  $A_p^2$  and  $F$  is continuous on  $\mathbb{C}^{n^2}$ , we have that the function  $(x_1, \dots, x_n) \mapsto F((K(x_i, x_j))_{1 \leq i, j \leq n})$  is continuous on  $A_p^n$ . Hence we have:  $F((K(x_i, x_j))_{1 \leq i, j \leq n}) \geq 0$ .

Therefore, in all cases, if  $(x_1, \dots, x_n) \in (A'_p)^n$ ,  $F((K(x_i, x_j))_{1 \leq i, j \leq n}) \geq 0$ .

As  $(A_p)_{p \in \mathbb{N}}$  is an increasing sequence, it is the same for  $(A'_p)_{p \in \mathbb{N}}$ . Hence we have:  $\cup_{p \in \mathbb{N}} (A'_p)^n = (\cup_{p \in \mathbb{N}} A'_p)^n$ .

We obtain:

$$F((K(x_i, x_j))_{1 \leq i, j \leq n}) \geq 0 \text{ for any } (x_1, \dots, x_n) \in (\cup_{p \in \mathbb{N}} A'_p)^n$$

As  $\lambda(\Lambda \setminus (\cup_{p \in \mathbb{N}} A'_p)) = 0$ , we finally obtain (ii) with  $\Lambda' = \cup_{p \in \mathbb{N}} A'_p$ .

We obtained that (i) and (ii) are equivalent conditions.

(i) is clearly a consequence of (iii). Assume now (ii). We will define a version  $K_1$  of  $K$  satisfying the condition (iii).

As  $\lambda(\Lambda) \neq 0$ ,  $\Lambda' \neq \emptyset$ . We set an arbitrary  $x_0 \in \Lambda'$ .

For  $(x, x') \in \Lambda^2$ , we define,  $y = x$  if  $x \in \Lambda'$ ,  $y = x_0$  if  $x \in \Lambda \setminus \Lambda'$ ,  $y' = x'$  if  $x' \in \Lambda'$ ,  $y' = x_0$  if  $x' \in \Lambda \setminus \Lambda'$  and  $K_1(x, x') = K(y, y')$ .

For  $(x_1, \dots, x_n) \in \Lambda^n$ , we define, for  $1 \leq i \leq n$ ,  $y_i = x_i$  if  $x_i \in \Lambda'$  and  $y_i = x_0$  if  $x_i \in \Lambda \setminus \Lambda'$ . Then we have,  $F((K_1(x_i, x_j))_{1 \leq i, j \leq n}) = F((K(y_i, y_j))_{1 \leq i, j \leq n}) \geq 0$  and  $K_1$  is a version of  $K$  satisfying the condition (iii).  $\square$

**Remark 15.** Let  $F_n, n \in \mathbb{N}$ , be continuous functions from  $\mathbb{C}^{n^2}$  to  $\mathbb{C}$ . For any non- $\lambda$ -null compact set  $\Lambda$ , the condition:

$$(i) \quad F_n((J_\alpha^\Lambda(x_i, x_j))_{1 \leq i, j \leq n}) \geq 0, \text{ for any } n \in \mathbb{N} \text{ and } \lambda^{\otimes n}\text{-a.e. } (x_1, \dots, x_n) \in \Lambda^n$$

can always be replaced by the equivalent conditions:

$$(ii) \quad \text{there exists a set } \Lambda' \subset \Lambda \text{ such that } \lambda(\Lambda \setminus \Lambda') = 0 \text{ and } F_n((J_\alpha^\Lambda(x_i, x_j))_{1 \leq i, j \leq n}) \geq 0, \text{ for any } n \in \mathbb{N} \text{ and } (x_1, \dots, x_n) \in (\Lambda')^n.$$

or:

$$(iii) \quad \text{there exists a version of the kernel } J \text{ such that } F_n((J_\alpha^\Lambda(x_i, x_j))_{1 \leq i, j \leq n}) \geq 0, \text{ for any } n \in \mathbb{N} \text{ and } (x_1, \dots, x_n) \in \Lambda^n.$$

*Proof.* The proof of (ii)  $\implies$  (iii) is done in the same way as in Lemma 14. The other parts of the proof are a direct application of Lemma 14.  $\square$

*Proof that (i) is necessary in Theorem 4.* This has been mentioned by Vere-Jones in [12] for the multivariate binomial probability distribution, which corresponds to a determinantal process with  $E$  being finite. To our knowledge, this has not been proved in other cases.

We consider the  $n \times n$  matrix  $1_n$ , whose elements are all equal to one.

$$\text{We have: } \prod_{j=0}^{n-1} (1 + j\alpha) = 1 + \sum_{k=1}^{n-1} \sum_{1 \leq j_1 < \dots < j_k \leq n-1} j_1 \dots j_k \alpha^k$$

We will show by induction on  $n$  that the number of permutations in  $\Sigma_n$  having  $n-k$  cycles for  $k \neq 0$  is  $a_{nk} = \sum_{1 \leq j_1 < \dots < j_k \leq n-1} j_1 \dots j_k$ : this is true for  $n = 2$  and  $k = 1$ . Assume it is true for a given  $n \in \mathbb{N}^*$  and for any  $k \in \llbracket 1, n-1 \rrbracket$ . If we consider the permutations  $\sigma \in \Sigma_{n+1}$  having  $n+1-k$  cycles ( $0 \leq k \leq n$ ), we have 2 cases:

- either  $\sigma(n+1) = n+1$ : there is exactly  $a_{nk}$  permutations corresponding to this case (with the convention  $a_{nn} = 0$ , for the case  $k = n$ ),
- or  $\sigma(n+1) \neq n+1$ . Then, if we denote  $\tau_{n+1\sigma(n+1)}$  the transposition in  $\Sigma_{n+1}$  that exchange  $n+1$  and  $\sigma(n+1)$ ,  $\tau_{n+1\sigma(n+1)} \circ \sigma$  is a permutation having  $n+1$  as fixed point and  $n+1-k$  other cycles (with elements in  $\llbracket 1, n \rrbracket$ ): there is exactly  $na_{nk-1}$  permutations corresponding to this case.

Then we have

$$\begin{aligned} a_{n+1, n+1-k} &= a_{nk} + na_{nk-1} \\ &= \sum_{1 \leq j_1 < \dots < j_k \leq n-1} j_1 \dots j_k + \sum_{\substack{1 \leq j_1 < \dots < j_{k-1} \leq n-1 \\ j_k = n}} j_1 \dots j_k \\ &= \sum_{1 \leq j_1 < \dots < j_k \leq n} j_1 \dots j_k \end{aligned}$$

which is what we expected.

Thus:  $\det_\alpha 1_n = \prod_{j=0}^{n-1} (1 + j\alpha)$ .

If  $\alpha < 0$  but  $-1/\alpha \notin \mathbb{N}$ , there exists therefore  $n \in \mathbb{N}$  such that  $\det_\alpha 1_n < 0$ .

We suppose that there exists an  $\alpha$ -determinantal process with  $\alpha < 0$  but  $-1/\alpha \notin \mathbb{N}$  and kernel  $K$ . Then we have  $\det_\alpha (K(x_i, x_j))_{1 \leq i, j \leq n} \geq 0$   $\lambda^{\otimes n}$ -a.e.  $(x_1, \dots, x_n) \in E^n$ .

As we exclude the case of a point process having no point almost surely and there is a sequence of compact sets  $\Lambda_p$  such that  $\cup_{p \in \mathbb{N}} \Lambda_p = E$ , there exists a compact set  $\Lambda \in E$  such that

$$\mathbb{E}(\xi(\Lambda)) = \int_\Lambda K(x, x) \lambda(dx) > 0.$$

Applying Lemma 14, we get that there exist a version  $K_1$  of the kernel  $K$  such that  $\det_\alpha (K_1(x_i, x_j))_{1 \leq i, j \leq n} \geq 0$  for any  $(x_1, \dots, x_n) \in \Lambda^n$ . We also have:

$$\int_\Lambda K(x, x) \lambda(dx) = \int_\Lambda K_1(x, x) \lambda(dx) > 0.$$

Hence there exists  $x_0 \in \Lambda$  such that  $K_1(x_0, x_0) > 0$ .

For  $(x_1, \dots, x_n) = (x_0, \dots, x_0)$ , we get:

$$\det_\alpha (K_1(x_i, x_j))_{1 \leq i, j \leq n} = K(x_0, x_0)^n \det_\alpha 1_n < 0$$

which is a contradiction. Therefore if  $\alpha < 0$  and an  $\alpha$ -determinantal process exists, then  $\alpha$  must be in  $\{-1/m : m \in \mathbb{N}\}$ .

□

We consider a  $d \times d$  square matrix  $A$ . If  $n_1, \dots, n_d$  are  $d$  non-negative integers,  $A[n_1, \dots, n_d]$  is the  $(n_1 + \dots + n_d) \times (n_1 + \dots + n_d)$  square matrix composed of the block matrices  $A_{ij}$ :

$$A[n_1, \dots, n_d] = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1d} \\ A_{21} & A_{22} & \dots & A_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ A_{d1} & A_{d2} & \dots & A_{dd} \end{pmatrix},$$

where  $A_{ij}$  is the  $n_i \times n_j$  matrix whose elements are all equal to  $a_{ij}$  ( $1 \leq i, j \leq d$ ).

**Lemma 16.** *Given a  $d \times d$  square matrix  $A$ , the following assertions are equivalent*

- (i)  $\det_{-1/m} A[n_1, \dots, n_d] \geq 0, \forall n_1, \dots, n_d \in \mathbb{N}$
- (ii)  $\det_{-1/m} A[n_1, \dots, n_d] \geq 0, \forall n_1, \dots, n_d \in \{0, \dots, m\}$
- (iii)  $\det A[n_1, \dots, n_d] \geq 0, \forall n_1, \dots, n_d \in \mathbb{N}$
- (iv)  $\det A[n_1, \dots, n_d] \geq 0, \forall n_1, \dots, n_d \in \{0, 1\}$

*Proof.* If there exists  $k \in \llbracket 1, d \rrbracket$  such that  $n_k > 1$ , the matrix  $A[n_1, \dots, n_d]$  has at least two identical rows and its determinant is null. So it is clear that (iii) and (iv) are equivalent.

We have:

$$\det(I + ZA)^m = \sum_{n_1, \dots, n_d=0}^{\infty} m^{n_1 + \dots + n_d} \left( \prod_{k=1}^d \frac{z_k^{n_k}}{n_k!} \right) \det_{-1/m} A[n_1, \dots, n_d] \quad (6)$$

where  $Z = \text{diag}(z_1, \dots, z_d)$  and  $z_1, \dots, z_d$  are  $d$  complex numbers. It is a special case of the formula (3) with  $\alpha = -1/m$ , finite space  $E = \llbracket 1, d \rrbracket$  and reference measure  $\lambda$  atomic, where each point of  $E$  has measure 1,  $\Lambda_k = \{k\}$ , for  $k \in \llbracket 1, d \rrbracket$ ,  $\Lambda = E$ . Indeed,  $ZA = \sum_{k=1}^d z_k A_k$ , where  $A_k$  is the  $d \times d$  square matrix having the same  $k^{\text{th}}$  row as  $A$  and the other rows with all elements equal to 0. The matrix  $A$  corresponds to the operator  $\mathcal{K}$ , the matrix  $A_k$  corresponds to the operator  $\mathcal{K}_{\Lambda_k \Lambda}$ . Formula (6) also corresponds to the one given by Vere-Jones in [13].

We also have for  $m = 1$ :

$$\det(I + ZA) = \sum_{n_1, \dots, n_d=0}^1 \left( \prod_{k=1}^d \frac{z_k^{n_k}}{n_k!} \right) \det A[n_1, \dots, n_d]. \quad (7)$$

as  $\det A[n_1, \dots, n_d] = 0$  if there exists  $k \in \llbracket 1, d \rrbracket$  such that  $n_k > 1$ .

(i) is equivalent to the fact that the multivariate power series (6) has all its coefficients non-negative.

(iii) is equivalent to the fact that the multivariate power series (7) has all its coefficients non-negative.

The power series (6) being the  $m^{\text{th}}$  power of the power series (7), if there exists  $k \in \llbracket 1, d \rrbracket$  such that  $n_k > m$ , the coefficient of  $\prod_{k=1}^d z_k^{n_k}$  is null. Therefore, (i) is equivalent to (ii).

For the same reason, we also have that (i) is a consequence of (iii).

Conversely, following Vere-Jones in [12], we can show by induction on the order of the matrix  $A$ , that the fact that the power series (6) has all its coefficients non-negative implies that the power series (7) has all its coefficient non negative.

This proves the equivalence between (i) and (iii). □

**Proposition 17.** *Let  $\alpha < 0$  and  $\mathcal{K}$  be an integral operator such that  $\mathcal{I} + \alpha \mathcal{K}_\Lambda$  is invertible, for any compact set  $\Lambda \subset E$ . An  $\alpha$ -determinantal process with kernel  $K$  exists iff:*

$$\det_\alpha (J_\alpha^\Lambda(x_i, x_j))_{1 \leq i, j \leq n} \geq 0, \text{ for any } n \in \mathbb{N}, \text{ and any compact set } \Lambda \\ \lambda^{\otimes n}\text{-a.e. } (x_1, \dots, x_n) \in \Lambda^n \quad (8)$$

Condition (8) implies that  $-\frac{1}{\alpha} \in \mathbb{N}$  and  $\text{Det}(\mathcal{I} + \beta \mathcal{K}) > 0$  for any  $\beta \in [\alpha, 0]$ .

*Proof.* We assume that there exists an  $\alpha$ -determinantal process  $\xi$  with kernel  $K$ . We already proved that it is necessary to have  $-1/\alpha \in \mathbb{N}$ .

By taking  $d = 1$  in the formula (4), we have

$$\mathbb{E} \left( z^{\xi(\Lambda)} \right) = \text{Det} \left( \mathcal{I} + \alpha(1 - z) \mathcal{K}_\Lambda \right)^{-1/\alpha}$$

for any compact set  $\Lambda \subset E$  and  $z \in (0, 1]$ .

Then  $\text{Det} \left( \mathcal{I} + \alpha(1 - z) \mathcal{K}_\Lambda \right) > 0$  for  $z \in (0, 1]$ , and by continuity,  $\text{Det} \left( \mathcal{I} + \alpha \mathcal{K}_\Lambda \right) \geq 0$ . As we assumed that  $\mathcal{I} + \alpha \mathcal{K}_\Lambda$  is invertible, we have necessarily  $\text{Det} \left( \mathcal{I} + \alpha \mathcal{K}_\Lambda \right) > 0$ .

For any non-negative function  $f$ , with compact support included in  $\Lambda$

$$\begin{aligned} \mathbb{E} \left( \prod_{x \in \xi} e^{-f(x)} \right) &= \text{Det} \left( \mathcal{I} + \alpha \mathcal{K} [1 - e^{-f}] \right)^{-1/\alpha} \\ &= \text{Det} \left( \mathcal{I} + \alpha \mathcal{K}_\Lambda \right)^{-1/\alpha} \text{Det} \left( \mathcal{I} - \alpha \mathcal{J}_\alpha^\Lambda e^{-f} \right)^{-1/\alpha} \\ &= \text{Det} \left( \mathcal{I} + \alpha \mathcal{K}_\Lambda \right)^{-1/\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \left( \prod_{i=1}^n e^{-f(x_i)} \right) \det_\alpha \left( J_\alpha^\Lambda(x_i, x_j) \right)_{1 \leq i, j \leq n} \lambda(dx_1) \dots \lambda(dx_n) \end{aligned}$$

As the Laplace functional defines a.e. uniquely the Janossy density of a point process, one obtains:

$$\det_\alpha \left( J_\alpha^\Lambda(x_i, x_j) \right)_{1 \leq i, j \leq n} \geq 0 \quad \lambda^{\otimes n}\text{-a.e.} \quad (x_1, \dots, x_n) \in E^n$$

Conversely, we assume that the condition

$$\det_\alpha \left( J_\alpha^\Lambda(x_i, x_j) \right)_{1 \leq i, j \leq n} \geq 0, \text{ for any } n \in \mathbb{N}, \lambda^{\otimes n}\text{-a.e.} \quad (x_1, \dots, x_n) \in \Lambda^n \text{ and any compact set } \Lambda.$$

is fulfilled. We have

$$\text{Det} \left( \mathcal{I} - \alpha z \mathcal{J}_\alpha^\Lambda \right)^{-1/\alpha} = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{\Lambda^n} \det_\alpha \left( J_\alpha^\Lambda(x_i, x_j) \right)_{1 \leq i, j \leq n} \lambda(dx_1) \dots \lambda(dx_n)$$

As  $-1/\alpha \in \mathbb{N}$ , this formula is valid for any  $z \in \mathbb{C}$ . Then we obtain for  $z = 1$ ,  $\text{Det} \left( \mathcal{I} - \alpha \mathcal{J}_\alpha^\Lambda \right)^{-1/\alpha} \geq 0$ .

We also have  $(\mathcal{I} - \alpha \mathcal{J}_\alpha^\Lambda)(\mathcal{I} + \alpha \mathcal{K}_\Lambda) = (\mathcal{I} + \alpha \mathcal{K}_\Lambda)(\mathcal{I} - \alpha \mathcal{J}_\alpha^\Lambda) = \mathcal{I}$ .

Then  $\text{Det} \left( \mathcal{I} - \alpha \mathcal{J}_\alpha^\Lambda \right) > 0$  and  $\text{Det} \left( \mathcal{I} + \alpha \mathcal{K}_\Lambda \right) > 0$ .

Thus the Janossy density is correctly defined and, on any compact set  $\Lambda$  we get the existence of a point process with kernel  $K$  and reference measure  $\lambda$ .

Then it can be extended to the complete space  $E$  by the Kolmogorov existence theorem (see Theorem 9.2.X in [3]).

□

*Proof of Theorem 4.* For any  $m \in \mathbb{N}$ , applying Lemma 16, we have for any compact set  $\Lambda$

$$\det_{-1/m} \left( J_{-1/m}^\Lambda(x_i, x_j) \right)_{1 \leq i, j \leq n} \geq 0, \text{ for any } n \in \mathbb{N}, \text{ and any } (x_1, \dots, x_n) \in \Lambda^n$$

is equivalent to

$$\det \left( J_{-1/m}^\Lambda(x_i, x_j) \right)_{1 \leq i, j \leq n} \geq 0, \text{ for any } n \in \mathbb{N}, \text{ and any } (x_1, \dots, x_n) \in \Lambda^n$$



Now, assume we only have

$$\det_{-1/m}(J_{-1/m}^\Lambda(x_i, x_j))_{1 \leq i, j \leq n} \geq 0, \text{ for any } n \in \mathbb{N}, \lambda^{\otimes n}\text{-a.e. } (x_1, \dots, x_n) \in \Lambda^n.$$

By lemma 14, for each  $n \in \mathbb{N}$ , there exists a set  $\Lambda'_n \subset \Lambda$  such that  $\lambda(\Lambda \setminus \Lambda'_n) = 0$  and  $\det_{-1/m}(J_{-1/m}^\Lambda(x_i, x_j))_{1 \leq i, j \leq n} \geq 0$  for any  $(x_1, \dots, x_n) \in (\Lambda'_n)^n$ .

If  $\Lambda' = \bigcap_{n \in \mathbb{N}} \Lambda'_n$ , we have  $\lambda(\Lambda \setminus \Lambda') = 0$  and  $\det_{-1/m}(J_{-1/m}^\Lambda(x_i, x_j))_{1 \leq i, j \leq n} \geq 0$  for any  $n \in \mathbb{N}$  and  $(x_1, \dots, x_n) \in (\Lambda')^n$ .

Then, by Lemma 16, we have:  $\det(J_{-1/m}^\Lambda(x_i, x_j))_{1 \leq i, j \leq n} \geq 0$ , for any  $n \in \mathbb{N}$  and  $(x_1, \dots, x_n) \in (\Lambda')^n$ .

Therefore, we have

$$\det(J_{-1/m}^\Lambda(x_i, x_j))_{1 \leq i, j \leq n} \geq 0, \text{ for any } n \in \mathbb{N}, \lambda^{\otimes n}\text{-a.e. } (x_1, \dots, x_n) \in \Lambda^n.$$

The converse is done through a similar proof, using Lemma 14 and 16.

Thus, we obtain:

$$\det_\alpha(J_\alpha^\Lambda(x_i, x_j))_{1 \leq i, j \leq n} \geq 0, \text{ for any } n \in \mathbb{N}, \lambda^{\otimes n}\text{-a.e. } (x_1, \dots, x_n) \in \Lambda^n$$

is equivalent to

$$\det(J_\alpha^\Lambda(x_i, x_j))_{1 \leq i, j \leq n} \geq 0, \text{ for any } n \in \mathbb{N}, \lambda^{\otimes n}\text{-a.e. } (x_1, \dots, x_n) \in \Lambda^n$$

Theorem 4 is then a consequence of Proposition 17. □

*Proof of Theorem 5.* We assume that there exists  $\xi$  an  $\alpha$ -determinantal process with kernel  $K$ .

For  $p \in (0, 1)$ , let  $\xi_p$  be the process obtained by first sampling  $\xi$ , then independently deleting each point of  $\xi$  with probability  $1 - p$ .

Computing the correlation functions, we obtain that  $\xi_p$  is an  $\alpha$ -determinantal process with kernel  $pK$ .

Thus we get from Theorem 4 that the conditions of the theorem must be fulfilled.

Conversely, we assume that these conditions are fulfilled. We obtain from Theorem 4 that an  $\alpha$ -determinantal process  $\xi_p$  with kernel  $pK$  exists, for any  $p \in (0, 1)$ .

We consider a sequence  $(p_k) \in (0, 1)^\mathbb{N}$  converging to 1 and a compact  $\Lambda$ .

$$\mathbb{E}(\exp(-t\xi_{p_k}(\Lambda))) = \text{Det}(\mathcal{I} + \alpha p_k K_\Lambda(1 - e^{-t}))^{-1/\alpha} \xrightarrow[k \rightarrow \infty]{} \text{Det}(\mathcal{I} + \alpha K_\Lambda(1 - e^{-t}))^{-1/\alpha}$$

As  $t \mapsto \text{Det}(\mathcal{I} + \alpha K_\Lambda(1 - e^{-t}))^{-1/\alpha}$  is continuous in 0,  $(\mathcal{L}(\xi_{p_k}(\Lambda)))_{k \in \mathbb{N}}$  converge weakly. Thus  $(\mathcal{L}(\xi_{p_k}(\Lambda)))_{k \in \mathbb{N}}$  is tight.

$\Gamma \subset \mathcal{X}$  is relatively compact if and only if, for any compact set  $\Lambda \subset E$ ,  $\{\xi(\Lambda) : \xi \in \Gamma\}$  is bounded.

Let  $(\Lambda_n)_{n \in \mathbb{N}}$  be an increasing sequence of compact sets such that  $\bigcup_{n \in \mathbb{N}} \Lambda_n = E$ .

As, for any  $n \in \mathbb{N}$ ,  $(\mathcal{L}(\xi_{p_k}(\Lambda_n)))_{k \in \mathbb{N}}$  is tight, we have that, for any  $\epsilon > 0$  and  $n \in \mathbb{N}$ , there exists  $M_n > 0$  such that for any  $k \in \mathbb{N}$ ,  $\mathbb{P}(\xi_{p_k}(\Lambda_n) > M_n) < \epsilon 2^{-n-1}$

Let  $\Gamma = \{\gamma \in \mathcal{X} : \forall n \in \mathbb{N}, \gamma(\Lambda_n) \leq M_n\}$ . It is a compact set and for any  $k \in \mathbb{N}$ ,  $\mathbb{P}(\xi_{p_k} \in \Gamma^c) < \epsilon$ .

Therefore,  $(\mathcal{L}(\xi_{p_k}))_{k \in \mathbb{N}}$  is tight. As  $E$  is Polish,  $\mathcal{X}$  is also Polish (endowed with the Prokhorov metric). Thus there is a subsequence of  $(\mathcal{L}(\xi_{p_k}))_{k \in \mathbb{N}}$  converging weakly to the probability distribution of a point process  $\xi$ . By unicity of the distribution of an  $\alpha$ -determinantal process for given kernel and reference measure,  $\xi$  must be an  $\alpha$ -determinantal process with kernel  $K$ , which gives the existence.  $\square$

**Lemma 18.** *Let  $\mathcal{J}$  be a trace class self-adjoint integral operator with kernel  $J$ . We have*

$$\det(J(x_i, x_j))_{1 \leq i, j \leq n} \geq 0 \text{ for any } n \in \mathbb{N}, \lambda^{\otimes n}\text{-a.e. } (x_1, \dots, x_n) \in \Lambda^n$$

*if and only if*

$$\text{Spec } \mathcal{J} \subset [0, \infty)$$

*Proof.* If we assume that the operator  $\mathcal{J}$  is positive, the kernel can be written as follows:

$$J(x, y) = \sum_{k=0}^{\infty} a_k \varphi_k(x) \overline{\varphi_k(y)}$$

where  $a_k \geq 0$  for  $k \in \mathbb{N}$ .

Hence:

$$\det(J(x_i, x_j))_{1 \leq i, j \leq n} \geq 0 \text{ for any } n \in \mathbb{N}, \text{ and any } (x_1, \dots, x_n) \in \Lambda^n$$

Conversely, assume that

$$\det(J(x_i, x_j))_{1 \leq i, j \leq n} \geq 0 \text{ for any } n \in \mathbb{N}, \lambda^{\otimes n}\text{-a.e. } (x_1, \dots, x_n) \in \Lambda^n.$$

From formula (2) with  $\alpha = -1$ , we have then for any  $z \in \mathbb{C}$

$$\text{Det}(\mathcal{I} + z\mathcal{J}) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{E^n} \det(J(x_i, x_j))_{1 \leq i, j \leq n} \lambda(dx_1) \dots \lambda(dx_n). \quad (9)$$

As  $\mathcal{J}$  is assumed to be self-adjoint, its spectrum is included in  $\mathbb{R}$ . Thanks to (9), it is impossible to have an eigenvalue in  $\mathbb{R}_-^*$ , as the power series has all its coefficients real non-negative and the first coefficient ( $n = 0$ ) is real positive. Hence  $\text{Spec } \mathcal{J} \subset [0, \infty)$ .  $\square$

*Proof of Corollary 7.* We assume:  $-1/\alpha \in \mathbb{N}$  and  $\text{Spec } \mathcal{K} \subset [0, -1/\alpha]$ . Then we have, as  $\mathcal{K}$  is self-adjoint, that for any compact set  $\Lambda$ ,  $\text{Spec } \mathcal{K}_\Lambda \subset [0, -1/\alpha]$ . Then  $\text{Det}(\mathcal{I} + \beta \mathcal{K}_\Lambda) > 0$  for any  $\beta \in (\alpha, 0]$ .

If  $\mathcal{I} + \alpha \mathcal{K}_\Lambda$  is invertible for any compact set  $\Lambda \subset E$ , we have  $\text{Spec } J_\alpha^\Lambda \subset [0, \infty)$  and  $J_\alpha^\Lambda$  is a trace class self adjoint operator for any compact set  $\Lambda$ .

Then, applying Lemma 18, we get that

$$\det(J(x_i, x_j))_{1 \leq i, j \leq n} \geq 0 \text{ for any } n \in \mathbb{N}, \text{ compact set } \Lambda \text{ and } \lambda^{\otimes n}\text{-a.e. } (x_1, \dots, x_n) \in \Lambda^n$$

Using Theorem 4, we get the existence of an  $\alpha$ -determinantal process with kernel  $K$ .

When there exists a compact set  $\Lambda_0$  such that  $\mathcal{I} + \alpha K_{\Lambda_0}$  is not invertible, by the same line of proof, we obtain the announced result, using Theorem 5.

Conversely, we assume that there exists an  $\alpha$ -determinantal process with kernel  $K$ . Then, from Theorem 4 or 5, we get that  $-1/\alpha \in \mathbb{N}$ .

If  $\mathcal{I} + \alpha K_\Lambda$  is invertible for any compact set  $\Lambda \subset E$ , we have  $\text{Spec } J_\alpha^\Lambda \subset [0, \infty)$ , using Theorem 4 and lemma 18. Then  $\text{Spec } K_\Lambda \subset [0, -1/\alpha) \subset [0, -1/\alpha]$ , for any compact set  $\Lambda$ .

If there exists a compact set  $\Lambda_0$  such that  $\mathcal{I} + \alpha K_{\Lambda_0}$  is not invertible, we have  $\text{Spec } J_\beta^\Lambda \subset [0, \infty)$  for any compact set  $\Lambda$  and any  $\beta \in (\alpha, 0)$ , using Theorem 5 and lemma 18. Then  $\text{Spec } K_\Lambda \subset [0, -1/\beta)$  for any  $\beta \in (\alpha, 0)$ . Therefore  $\text{Spec } K_\Lambda \subset [0, -1/\alpha]$  for any compact set  $\Lambda$ .

As  $K$  is self-adjoint, this implies in both cases that  $\text{Spec } K \subset [0, -1/\alpha]$ . □

**Remark 19.** Using the known result in the case  $\alpha = -1$  (see for example Hough, Krishnapur, Peres and Virág in [7]) and corollary 6, one obtains a direct proof of Corollary 7.

## 7 Infinite divisibility

*Proof of Theorem 8.* For  $\alpha < 0$ , we have proved that it is necessary to have  $-1/\alpha \in \mathbb{N}$ . If an  $\alpha$ -determinantal process was infinitely divisible, with  $\alpha < 0$ , it would be the sum of  $N$  i.i.d  $\alpha N$ -determinantal processes for any  $N \in \mathbb{N}^*$ , as it can be seen for the Laplace functional formula (1). This would imply that  $-1/(N\alpha) \in \mathbb{N}$ , for any  $N \in \mathbb{N}^*$ , which is not possible. Therefore, an  $\alpha$ -determinantal process with  $\alpha < 0$  is never infinitely divisible. □

Some characterization on infinite divisibility have also been given in [4] in the case  $\alpha > 0$ .

*Proof of Theorem 9.* For  $\alpha > 0$ , assume that  $\text{Det}(\mathcal{I} + \alpha K_\Lambda) \geq 1$  and

$$\sum_{\sigma \in \Sigma_n: \nu(\sigma)=1} \prod_{i=1}^n J_\alpha^\Lambda(x_i, x_{\sigma(i)}) \geq 0,$$

for any compact set  $\Lambda \subset E$ ,  $n \in \mathbb{N}$  and  $\lambda^{\otimes n}$ -a.e.  $(x_1, \dots, x_n) \in \Lambda^n$ . Then we have:

$$\begin{aligned} \sum_{\sigma \in \Sigma_n: \nu(\sigma)=k} \prod_{i=1}^n J_\alpha^\Lambda(x_i, x_{\sigma(i)}) &= \sum_{\substack{\{I_1, \dots, I_k\} \\ \text{partition of } \llbracket 1, n \rrbracket}} \sum_{\substack{\sigma_1 \in \Sigma(I_1), \dots, \sigma_k \in \Sigma(I_k): \\ \nu(\sigma_1) = \dots = \nu(\sigma_k) = 1}} \prod_{q=1}^k \prod_{i \in I_q} J_\alpha^\Lambda(x_i, x_{\sigma_q(i)}) \\ &= \sum_{\substack{\{I_1, \dots, I_k\} \\ \text{partition of } \llbracket 1, n \rrbracket}} \prod_{q=1}^k \left( \sum_{\substack{\sigma \in \Sigma(I_q): \\ \nu(\sigma)=1}} \prod_{i \in I_q} J_\alpha^\Lambda(x_i, x_{\sigma(i)}) \right) \geq 0, \end{aligned}$$

for any compact set  $\Lambda \subset E$ ,  $n \in \mathbb{N}$ ,  $k \in \llbracket 1, n \rrbracket$  and  $\lambda^{\otimes n}$ -a.e.  $(x_1, \dots, x_n) \in \Lambda^n$ , where, for a finite set  $I$ ,  $\Sigma(I)$  denotes the set of all permutations on  $I$ .

Then, for any  $N \in \mathbb{N}^*$  and any compact set  $\Lambda \in E$ ,  $\det_{N\alpha}(J_\alpha^\Lambda(x_i, x_j)/N)_{1 \leq i, j \leq n} \geq 0$ . From Theorem 1, we get that there exists a  $(N\alpha)$ -permanental process with kernel  $K/N$ . This

means that an  $\alpha$ -permanental process with kernel  $K$  is infinitely divisible.

Conversely, if we assume an  $\alpha$ -permanental process with kernel  $K$  is infinitely divisible, we get the existence of a  $N\alpha$ -permanental process with kernel  $K/N$ , for any  $N \in \mathbb{N}^*$ .

From Theorem 1, we have that  $\text{Det}(\mathcal{I} + \alpha K_\Lambda) \geq 1$  for any compact set  $\Lambda \in E$ .

We also have

$$\frac{1}{(N\alpha)^{n-1}} \det_{N\alpha}(J_\alpha^\Lambda(x_i, x_j))_{1 \leq i, j \leq n} \geq 0,$$

for any  $N \in \mathbb{N}^*$ , any  $n \in \mathbb{N}$ , any compact set  $\Lambda \in E$  and  $\lambda^{\otimes n}$ -a.e.  $(x_1, \dots, x_n) \in \Lambda^n$ .  
When  $N$  tends to  $\infty$ , we obtain:

$$\sum_{\sigma \in \Sigma_n: \nu(\sigma)=1} \prod_{i=1}^n J_\alpha^\Lambda(x_i, x_{\sigma(i)}) \geq 0,$$

which is the desired result. □

*Proof of Theorem 10.* We use the argument of Griffiths in [5] and Griffiths and Milne in [6]. Assume

$$\sum_{\sigma \in \Sigma_n: \nu(\sigma)=1} \prod_{i=1}^n J_\alpha^\Lambda(x_i, x_{\sigma(i)}) \geq 0,$$

for any  $n \in \mathbb{N}$  and any  $(x_1, \dots, x_n) \in \Lambda^n$ .

The condition  $J_\alpha^\Lambda(x_1, x_2) \dots J_\alpha^\Lambda(x_{n-1}, x_n) J_\alpha^\Lambda(x_n, x_1) \geq 0$  is satisfied for the elementary cycles, i.e. cycles such that  $J_\alpha^\Lambda(x_i, x_j) = 0$  if  $i < j + 1$  and  $(i \neq 1 \text{ or } j \neq n)$ . Then it can be extended to any cycle by induction, using  $J_\alpha^\Lambda(x_i, x_j) = J_\alpha^\Lambda(x_j, x_i)$ .

With Lemma 14, we can then extend the proof to the case when

$$\sum_{\sigma \in \Sigma_n: \nu(\sigma)=1} \prod_{i=1}^n J_\alpha^\Lambda(x_i, x_{\sigma(i)}) \geq 0,$$

for any  $n \in \mathbb{N}$  and  $\lambda^{\otimes n}$ -a.e.  $(x_1, \dots, x_n) \in \Lambda^n$ . □

**Remark 20.** Note that the argument from Griffiths and Milne in [5] and [6] is only valid for real symmetric matrices.

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