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Necessary and sufficient conditions for the existence of $\alpha$-determinantal processes

Franck Maunoury *

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Abstract

We give necessary and sufficient conditions for existence and infinite divisibility of $\alpha$-determinantal processes. For that purpose we use results on negative binomial and ordinary binomial multivariate distributions.

Keywords: determinantal process, permanental process, $\alpha$-determinantal process, $\alpha$-permanental process, infinitely divisible, complete monotonicity, fermion process, boson process.

AMS 2010 subject classifications: 60G55, 60E07.

1 Introduction

Several authors have already established necessary and sufficient conditions for existence of $\alpha$-determinantal processes.


The same condition has also been established in a different way by Hough, Krishnapur, Peres and Virág in [7] in the case $\alpha = -1$. They have also given a sufficient condition of existence in the case $\alpha = 1$ and self-adjoint kernel.

In the special case when the configurations are on a finite space, the paper of Vere-Jones [12] provides necessary and sufficient conditions for any value of $\alpha$.

Finally, Shirai and Takahashi have given sufficient conditions for the existence of an $\alpha$-determinantal process for any values of $\alpha$. However, in the case $\alpha > 0$, their sufficient condition (Condition B) in [9] does not work for the following example: the space is reduced to a single point space and the reference measure $\lambda$ is a unit point mass. With their notations, the two kernels $K$ and $J_\alpha$ are respectively reduced to two real numbers $k$ and $j_\alpha$, with

$$j_\alpha = \frac{k}{1 + \alpha k}$$

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We can choose \( \alpha > 0 \) and \( k < 0 \) such that \( j_{\alpha} > 0 \). Under these assumptions, Condition B is fulfilled but the obtained point process has a negative correlation function \( (\rho_1(x) = k) \), which has to be excluded, since a correlation function is an almost everywhere non-negative function.

We are going to strengthen Condition B of Shirai and Takahashi and obtain a necessary and sufficient condition in the case \( \alpha > 0 \). This is presented in Theorem 1.
Besides, in the case \( \alpha < 0 \), we extend the result of Shirai and Takahashi to the case of non self-adjoint kernels and show that the obtained condition is also necessary (Theorems 4 and 5). Moreover, we show that \(-1/\alpha\) is necessarily an integer. This has been noticed by Vere-Jones in [13] in the case of configurations on a finite space.
We also give a necessary and sufficient condition for the infinite divisibility of an \( \alpha \)-determinantal process for all values of \( \alpha \).
The main results are presented in Section 3. Section 2 introduces the needed notation. In Section 4, we write a multivariate version of a Shirai and Takahashi formulae on Fredholm determinant expansion. Sections 5 and 6 present the proofs of the results concerning respectively the cases \( \alpha > 0 \) and \( \alpha < 0 \). The proofs concerning infinite divisibility are presented in Section 7.

2 Preliminaries

Let \( E \) be a locally compact Polish space. A locally finite configuration on \( E \) is an integer-valued positive Radon measure on \( E \). It can also be identified with a set \( \{(M, \alpha_M) : M \in F\} \), where \( F \) is a countable subset of \( E \) with no accumulation points (i.e. a discrete subset of \( E \)) and, for each point in \( F \), \( \alpha_M \) is a non-null integer that corresponds to the multiplicity of the point \( M \) (\( M \) is a multiple point if \( \alpha_M \geq 2 \)).
Let \( \lambda \) be a Radon measure on \( E \). Let \( \mathcal{X} \) be the space of the locally finite configurations of \( E \). The space \( \mathcal{X} \) is endowed with the vague topology of measures, i.e. the smallest topology such that, for every real continuous function \( f \) with compact support, defined on \( E \), the mapping

\[
\mathcal{X} \ni \xi \mapsto \langle f, \xi \rangle = \sum_{x \in \xi} f(x) = \int f d\xi
\]

is continuous. Details on the topology of the configuration space can be found in [1].
We denote by \( \mathcal{B}(\mathcal{X}) \) the corresponding \( \sigma \)-algebra. A point process on \( E \) is a random variable with values in \( \mathcal{X} \). We do not restrict ourselves to simple point processes, as the configurations in \( \mathcal{X} \) can have multiple points.

For a \( n \times n \) matrix \( A = (a_{ij})_{1 \leq i, j \leq n} \), set:

\[
\det_n A = \sum_{\sigma \in \Sigma_n} \alpha^{n-\nu(\sigma)} \prod_{i=1}^{n} a_{i\sigma(i)}
\]

where \( \Sigma_n \) is the set of all permutations on \( \{1, \ldots, n\} \) and \( \nu(\sigma) \) is the number of cycles of the permutation \( \sigma \).
For a relatively compact set \( \Lambda \subset E \), the Janossy densities of a point process \( \xi \) w.r.t. a Radon measure \( \lambda \) are functions (when they exist) \( j_n^\Lambda : E^n \to [0, \infty) \) for \( n \in N \), such that
\[
\begin{align*}
j_n^\Lambda(x_1, \ldots, x_n) &= n! \, \mathbb{P}(\xi(\Lambda) = n) \, \pi_n^\Lambda(x_1, \ldots, x_n) \\
j_0^\Lambda(\emptyset) &= \mathbb{P}(\xi(\Lambda) = 0),
\end{align*}
\]
where \( \pi_n^\Lambda \) is the density with respect to \( \lambda^{\otimes n} \) of the ordered set \( (x_1, \ldots, x_n) \), obtained by first sampling \( \xi \), given that there are \( n \) points in \( \Lambda \), then choosing uniformly an order between the points.

For \( \Lambda_1, \ldots, \Lambda_n \) disjoint subsets included in \( \Lambda \),
\[
f_{\Lambda_1 \times \ldots \times \Lambda_n} j_n^\Lambda(x_1, \ldots, x_n) \lambda(dx_1) \ldots \lambda(dx_n) \]
is the probability that there is exactly one point in each subset \( \Lambda_i \) (\( 1 \leq i \leq n \)), and no other point elsewhere.

We recall that we have the following formula, for a non-negative measurable function \( f \) with support in a relatively compact set \( \Lambda \subset E \):
\[
\mathbb{E}(f(\xi)) = f(\emptyset) \, j_0^\Lambda(\emptyset) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} f(x_1, \ldots, x_n) \, j_n^\Lambda(x_1, \ldots, x_n) \lambda(dx_1) \ldots \lambda(dx_n).
\]

For \( n \in \mathbb{N} \) and \( a \in \mathbb{R} \), we denote \( a^{(n)} = \prod_{i=0}^{n-1} (a - i) \).

The correlation functions (also called joint intensities) of a point process \( \xi \) w.r.t. a Radon measure \( \lambda \) are functions (when they exist) \( \rho_n : E^n \to [0, \infty) \) for \( n \geq 1 \), such that for any family of mutually disjoint relatively compact subsets \( \Lambda_1, \ldots, \Lambda_d \) of \( E \) and for any non-null integers \( n_1, \ldots, n_d \) such that \( n_1 + \cdots + n_d = n \), we have
\[
\mathbb{E} \left( \prod_{i=1}^d \xi(\Lambda_i)^{(n_i)} \right) = \int_{\Lambda_1^{n_1} \times \ldots \times \Lambda_d^{n_d}} \rho_n(x_1, \ldots, x_n) \lambda(dx_1) \ldots \lambda(dx_n).
\]

Intuitively, for a simple point process, \( \rho_n(x_1, \ldots, x_n) \lambda(dx_1) \ldots \lambda(dx_n) \) is the infinitesimal probability that there is at least one point in the vicinity of each \( x_i \) (each vicinity having an infinitesimal volume \( \lambda(dx_i) \) around \( x_i \)), \( 1 \leq i \leq n \).

Let \( \alpha \) be a real number and \( K \) a kernel from \( E^2 \) to \( \mathbb{R} \) or \( \mathbb{C} \). An \( \alpha \)-determinantal point process, with kernel \( K \) with respect to \( \lambda \) (also called \( \alpha \)-permanental point process) is defined, when it exists, as a point process with the following correlation functions \( \rho_n, n \in \mathbb{N} \) with respect to \( \lambda \):
\[
\rho_n(x_1, \ldots, x_n) = \det_{\alpha}(K(x_i, x_j))_{1 \leq i, j \leq n}.
\]

We denote by \( \mu_{\alpha,K,\lambda} \) the probability distribution of such a point process.

We exclude the case of a point process almost surely reduced to the empty configuration.

The case \( \alpha = -1 \) corresponds to a determinantal process and the case \( \alpha = 1 \) to a permanental process. The case \( \alpha = 0 \) corresponds to the Poisson point process. We suppose in the following that \( \alpha \neq 0 \).

We will always assume that the kernel \( K \) defines a locally trace class integral operator \( K \) on \( L^2(E, \lambda) \). Under this assumption, one obtains an equivalent definition for the \( \alpha \)-determinantal process, using the following Laplace functional formula:
\[
\mathbb{E}_{\rho_n(K,\lambda)} \left[ \exp \left( - \int_E f d\xi \right) \right] = \text{Det} \left( \mathcal{I} + \alpha K[1 - e^{-f}] \right)^{-1/\alpha}
\] (1)
where $f$ is a compactly-supported non-negative function on $E$, $K[1 - e^{-f}]$ stands for
$\sqrt{1 - e^{-f}} K \sqrt{1 - e^{-f}}$, $I$ is the identity operator on $L^2(E, \lambda)$ and $\text{Det}$ is the Fredholm determinant. Details on the link between the correlation function and the Laplace functional of an $\alpha$-determinantal process can be found in the chapter 4 of [9]. Some explanations and useful formula on the Fredholm determinant are given in chapter 2.1 of [9].

For a subset $\Lambda \subset E$, set: $K_\Lambda = p_\Lambda K p_\Lambda$, where $p_\Lambda$ is the orthogonal projection operator from $L^2(E, \lambda)$ to the subspace $L^2(\Lambda, \lambda)$.

For two subsets $\Lambda, \Lambda' \subset E$, set: $K_{\Lambda \Lambda'} = p_\Lambda K p_{\Lambda'}$, and denote by $K_{\Lambda \Lambda'}$ its kernel. We have for any $x, y \in E$, $K_{\Lambda \Lambda'}(x, y) = 1_\Lambda(x) 1_{\Lambda'}(y) K(x, y)$.

When $I + \alpha K$ (resp. $I + \alpha K_\Lambda$) is invertible, $J_\alpha$ (resp. $J_\alpha^\Lambda$) is the integral operator defined by: $J_\alpha = K(I + \alpha K)^{-1}$ (resp. $J_\alpha^\Lambda = K_\Lambda (I + \alpha K_\Lambda)^{-1}$) and we denote by $J_\alpha$ (resp. $J_\alpha^\Lambda$) its kernel. Note that $J_\alpha^\Lambda$ is not the orthogonal projection of $J_\alpha$ on $L^2(\Lambda, \lambda)$.

3 Main results

**Theorem 1.** For $\alpha > 0$, there exists an $\alpha$-permanental process with kernel $K$ iff:

- $\text{Det}(I + \alpha K_\Lambda) \geq 1$, for any compact set $\Lambda \subset E$
- $\det_\alpha(J_\alpha^\Lambda(x_i, x_j))_{1 \leq i, j \leq n} \geq 0$, for any $n \in \mathbb{N}$, any compact set $\Lambda \subset E$ and any $\lambda^{\otimes n}$-a.e. $(x_1, \ldots, x_n) \in \Lambda^n$.

**Remark 2.** Even when $E$ is a finite set, note that the second condition of Theorem 1 consists in an infinite number of computations. Finding a simpler condition, that could be checked in a finite number of steps is still an open problem.

**Theorem 3.** For $\alpha > 0$, if an $\alpha$-permanental process with kernel $K$ exists, then:

$$\text{Spec} K_\Lambda \subset \{ z \in \mathbb{C} : \text{Re} z > -\frac{1}{2\alpha} \}, \text{ for any compact set } \Lambda \subset E.$$ 

We remark that this condition is equivalent to

$$\text{Spec} J_\alpha^\Lambda \subset \{ z \in \mathbb{C} : |z| < \frac{1}{\alpha} \}, \text{ for any compact set } \Lambda \subset E.$$ 

**Theorem 4.** For $\alpha < 0$ and $K$ an integral operator such that $I + \alpha K_\Lambda$ is invertible, for any compact set $\Lambda \subset E$, an $\alpha$-determinantal process with kernel $K$ exists iff the two following conditions are fulfilled:

- (i) $-1/\alpha \in \mathbb{N}$
- (ii) $\det(J_\alpha^\Lambda(x_i, x_j))_{1 \leq i, j \leq n} \geq 0$, for any $n \in \mathbb{N}$, any compact set $\Lambda \subset E$ and any $\lambda^{\otimes n}$-a.e. $(x_1, \ldots, x_n) \in \Lambda^n$. 
The arguments developed in the proof of Theorem 4 shows that actually $(ii) \implies (i)$. Consequently, Condition $(ii)$ is itself a necessary and sufficient condition. It also implies that $\text{Det}(\mathcal{I} + \beta K_\Lambda) > 0$ for any $\beta \in [\alpha, 0]$ and any compact $\Lambda \subset E$.

**Theorem 5.** For $\alpha < 0$ and $K$ an integral operator such that for some compact set $\Lambda_0 \subset E$, $\mathcal{I} + \alpha K_{\Lambda_0}$ is not invertible, an $\alpha$-determinantal process with kernel $K$ exists iff:

$(i') -1/\alpha \in \mathbb{N} \quad (ii') \quad \text{det}(J_{\beta}(x_i, x_j))_{1 \leq i, j \leq n} \geq 0,$ for any $n \in \mathbb{N}$, any $\beta \in (\alpha, 0)$, any compact set $\Lambda \subset E$ and any $\lambda^{\otimes n}$-a.e. $(x_1, \ldots, x_n) \in \Lambda^n$.

As in Theorem 4, we also have $(ii') \implies (i')$ and Condition $(ii')$ is itself a necessary and sufficient condition.

Note that $\mathcal{I} + \alpha K_{\Lambda_0}$ is not invertible if and only if there is almost surely at least one point in $\Lambda_0$.

**Corollary 6.** For $m$ a positive integer, the existence of a $(-1/m)$-determinantal process with kernel $K$ is equivalent to the existence of a determinantal process with the kernel $\frac{K}{m}$.

**Corollary 7.** For $\alpha < 0$ and $K$ a self-adjoint operator, an $\alpha$-determinantal process with kernel $K$ exists iff:

- $-1/\alpha \in \mathbb{N}$
- $\text{Spec} K \subset [0, -1/\alpha]$

This result is well known in the case $\alpha = -1$ (see for example Hough, Krishnapur, Peres and Virág in [7]).

The sufficient part of this necessary and sufficient condition corresponds to condition A in [9] of Shirai and Takahashi.

**Theorem 8.** For $\alpha < 0$, an $\alpha$-determinantal process is never infinitely divisible.

**Theorem 9.** For $\alpha > 0$, an $\alpha$-determinantal process is infinitely divisible iff

- $\text{Det}(\mathcal{I} + \alpha K_\Lambda) \geq 1$, for any compact set $\Lambda \subset E$
- $\sum_{\sigma \in \Sigma_n, \nu(\sigma) = 1} \prod_{i=1}^{n} J_{\alpha}(x_i, x_{\sigma(i)}) \geq 0$, for any $n \in \mathbb{N}$, any compact set $\Lambda \subset E$ and $\lambda^{\otimes n}$-a.e. $(x_1, \ldots, x_n) \in \Lambda^n$.

This theorem gives a more general condition for infinite-divisibility of an $\alpha$-permanental process than the condition given by Shirai and Takahashi in [9].

**Theorem 10.** For $K$ a a real symmetric locally trace class operator and $\alpha > 0$, an $\alpha$-permanental process is infinitely divisible iff
• \( \text{Det}(I + \alpha K_{\Lambda}) \geq 1 \), for any compact set \( \Lambda \subset E \)

• \( J_\alpha^\Lambda(x_1, x_2) \ldots J_\alpha^\Lambda(x_{n-1}, x_n) J_\alpha^\Lambda(x_n, x_1) \geq 0 \), for any \( n \in \mathbb{N} \), any compact set \( \Lambda \subset E \) and \( \lambda^{\otimes n} \)-a.e. \((x_1, \ldots, x_n) \in \Lambda^n\).

Following Griffith and Milne’s remark in [6], when an \( \alpha \)-permanental process with kernel \( K \) exists and is infinitely divisible, we can replace \( J_\alpha^\Lambda \) by \( |J_\alpha^\Lambda| \) and obtain an \( \alpha \)-permanental process with the same probability distribution.

**Remark 11.** In Theorem 1, 9 and 10, the condition

\[
\text{Det}(I + \alpha K_{\Lambda}) \geq 1, \text{ for any compact set } \Lambda \subset E
\]

can be replaced by

\[
\text{Det}(I + \alpha K_{\Lambda}) > 0, \text{ for any compact set } \Lambda \subset E.
\]

### 4 Fredholm determinant expansion

In [9], Shirai and Takahashi have proved the following formula

\[
\text{Det}(I - \alpha z K)^{-1/\alpha} = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{E^n} \text{det}_\alpha(K(x_i, x_j))_{1 \leq i, j \leq n} \lambda(dx_1) \ldots \lambda(dx_n)
\]

(2)

for a trace class integral operator \( K \) with kernel \( K \) and for \( z \in \mathbb{C} \) such that \( ||\alpha z K|| < 1 \).

In the case where the space \( E \) is finite, this formula is also given by Shirai in [10].

As \( z \mapsto \text{Det}(I - \alpha z K) \) is analytic on \( \mathbb{C} \) and \( z \mapsto z^{-1/\alpha} \) is analytic on \( \mathbb{C}^* \), we obtain that \( z \mapsto \text{Det}(I - \alpha z K_{\Lambda, \alpha})^{-1/\alpha} \) is analytic on \( \{ z \in \mathbb{C} : I - \alpha z K_{\Lambda, \alpha} \text{ invertible} \} \).

Therefore, the formula can be extended to the open disc \( D \), centered in 0 with radius \( R = \sup \{ r \in \mathbb{R}_+ : \forall z \in \mathbb{C}, |z| < r \Rightarrow I - \alpha z K \text{ is invertible} \} \).

\( D \) is the open disc of center 0 and radius \( 1/||\alpha K|| \), if the operator \( K \) is self-adjoint, but it can be larger if \( K \) is not self-adjoint.

As remarked by Shirai and Takahashi, the formula (2) is valid for any \( z \in \mathbb{C} \) if \(-1/\alpha \in \mathbb{N}\).

The following proposition extends (2) to a multivariate case.

**Proposition 12.** Let \( \Lambda \subset E \) be a relatively compact set, \( \Lambda_1, \ldots, \Lambda_d \) mutually disjoint subsets of \( \Lambda \) and \( K \) a locally trace class integral operator with kernel \( K \).

We have the following formula

\[
\text{Det} \left( I - \alpha \sum_{k=1}^{d} z_k K_{\Lambda_k \Lambda} \right)^{-1/\alpha} = \sum_{n_1, \ldots, n_d = 0}^{\infty} \left( \prod_{k=1}^{d} \frac{z_k^{n_k}}{n_k!} \right) \int_{\Lambda_1^{n_1} \times \cdots \times \Lambda_d^{n_d}} \text{det}_\alpha(K(x_i, x_j))_{1 \leq i, j \leq n} \lambda(dx_1) \ldots \lambda(dx_n)
\]

(3)

for any \( z_1, \ldots, z_d \in \mathbb{C} \), such that \( I - \alpha \gamma \sum_{k=1}^{d} z_k K_{\Lambda_k \Lambda} \) is invertible for any complex number \( \gamma \) satisfying \( |\gamma| < 1 \) (\( n \) denotes \( n_1 + \cdots + n_d \)).
Proof. We apply the formula (2) to the class trace operator \( \sum_{k=1}^{d} z_k \mathbf{K}_{\Lambda k} \) and we use the multilinearity property of the \( \alpha \)-determinant of a matrix with respect to its rows.

We obtain

\[
\text{Det} \left( \mathbf{I} - \alpha \sum_{k=1}^{d} z_k \mathbf{K}_{\Lambda k} \right)^{-1/\alpha}
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{E^n} \det_{\alpha} \left( \sum_{k=1}^{d} z_k \mathbf{K}_{\Lambda k}(x_i, x_j) \right)_{1 \leq i,j \leq n} \lambda(dx_1) \ldots \lambda(dx_n)
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{E^n} \prod_{k=1}^{d} \mathbf{1}_{\Lambda k}(x_i) \mathbf{1}_{\Lambda}(x_j) K(x_i, x_j)_{1 \leq i,j \leq n} \lambda(dx_1) \ldots \lambda(dx_n)
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{E^n} \prod_{k=1}^{d} \mathbf{1}_{\Lambda k}(x_i) \mathbf{1}_{\Lambda}(x_j) \lambda(dx_1) \ldots \lambda(dx_n)
\]

where we have used the fact that \( \mathbf{K}_{\Lambda k}(x_i, x_j) = \mathbf{1}_{\Lambda k}(x_i) \mathbf{1}_{\Lambda}(x_j) K(x_i, x_j) \) for the equality between the first and the second line.

As the value of the \( \alpha \)-determinant of a matrix is unchanged by simultaneous interchange of its rows and its columns, the product \( z_1^{n_1} \ldots z_d^{n_d} \) where \( n_1 + \ldots + n_d = n \), will be repeated \( \binom{n}{n_1 \ldots n_d} \) times. This gives the desired formula.

For a relatively compact set \( \Lambda \subset E \) and \( \Lambda_1, \ldots, \Lambda_d \) mutually disjoint subsets of \( \Lambda \), the computation of the Laplace functional of an \( \alpha \)-determinantal process for the function \( f : (z_1, \ldots, z_d) \mapsto -\sum_{k=1}^{d} (\log z_k) \mathbf{1}_{\Lambda_k} \), with \( z_1, \ldots, z_d \in (0,1] \) gives thanks to (1):

\[
E_{\mu_{\alpha},K,\lambda} \left[ \prod_{k=1}^{d} z_k^{\xi(\Lambda_k)} \right] = \text{Det} \left( \mathbf{I} + \alpha \sum_{k=1}^{d} (1 - z_k) \mathbf{K}_{\Lambda k} \right)^{-1/\alpha}
\]

which is the probability generating function (p.g.f.) of the finite-dimensional random vector \( (\xi(\Lambda_1), \ldots, \xi(\Lambda_d)) \).

For \( \alpha < 0 \), the formula (4) reminds the multivariate binomial distribution p.g.f. and for \( \alpha > 0 \), the multivariate negative binomial distribution p.g.f., given by Vere-Jones in [12], in the special case where the space \( E \) is finite.

### 5 \( \alpha \)-permanental process (\( \alpha > 0 \))

Proof of Theorem 1. We first prove that the conditions are necessary. We suppose that there exists an \( \alpha \)-permanental process with \( \alpha > 0 \), kernel \( K \) defining the locally trace class integral operator \( \mathcal{K} \).
By taking $d = 1$ in the formula (4), we have

$$E_{\mu_\alpha,K,\lambda} \left(\frac{\lambda}{\xi}^{(\lambda)}\right) = \text{Det} \left( I + \alpha(1 - z)K_\lambda \right)^{-1/\alpha}$$

for any compact set $\Lambda \subset E$ and $z \in (0, 1]$.

Thus, $\text{Det}(I + \alpha(1 - z)K_\lambda) \geq 1$ for $z \in (0, 1]$. By continuity (as $z \mapsto \text{Det}(I + (1 - z)K_\lambda)$ is indeed analytic on $\mathbb{C}$), we obtain that $\text{Det}(I + \alpha K_\lambda) \geq 1$, which is the first condition. This implies that for any compact set $\Lambda \subset E$, $I + \alpha K_\lambda$ is invertible. Hence $J^\lambda_\alpha$ exists and we have, for any non-negative function $f$, with compact support included in $\Lambda$

$$E_{\mu_\alpha,K,\lambda} \left(\prod_{x \in \xi} e^{-f(x)}\right) = \text{Det}(I + \alpha K_\lambda \{1 - e^{-f}\})^{-1/\alpha}$$

$$= \text{Det}(I + \alpha K_\lambda(1 - e^{-f}))^{-1/\alpha}$$

$$= \text{Det}(I + \alpha K_\lambda)^{-1/\alpha} \text{Det}(I - \alpha J^\lambda_\alpha e^{-f})^{-1/\alpha}$$

$$= \text{Det}(I + \alpha K_\lambda)^{-1/\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \left(\prod_{i=1}^{n} e^{-f(x_i)}\right) \det_\alpha(J^\lambda_\alpha(x_i, x_j))_{1 \leq i,j \leq n} d(x_1) \ldots d(x_n)$$

(5)

where we have used for the equality between the first and the second line the fact that $\text{Det}(I + AB) = \text{Det}(I + BA)$, for any trace class operator $A$, and any bounded operator $B$.

As the Laplace functional defines a.e. uniquely the Janossy density of a point process, one obtains:

$$\det_\alpha(J^\lambda_\alpha(x_i, x_j))_{1 \leq i,j \leq n} \geq 0 \lambda^{\otimes n} \text{-a.e.} \ (x_1, \ldots, x_n) \in E^n$$

$$j^\lambda_{\alpha,n}(x_1, \ldots, x_n) = \text{Det}(I + \alpha K_\lambda)^{-1/\alpha} \det_\alpha(J^\lambda_\alpha(x_i, x_j))_{1 \leq i,j \leq n}$$

is the Janossy density.

Conversely, if we assume $\text{Det}(I + \alpha K_\lambda)^{-1/\alpha} > 0$ and $\det_\alpha(J^\lambda_\alpha(x_i, x_j))_{1 \leq i,j \leq n} \geq 0$ for any $n \in \mathbb{N}$, any compact set $\Lambda \subset E$ and any $\lambda^{\otimes n} \text{-a.e.} \ (x_1, \ldots, x_n) \in \Lambda^n$, the Janossy density will be correctly defined and, on any compact set $\Lambda$, we get the existence of a point process $\xi_\Lambda$ with kernel $K_\Lambda$ (see Proposition 5.3.II. in [2] - here the normalization condition is automatic by choosing $f = 0$ in (5)).

The restriction of a point process $\eta$, defined on $\Lambda' \subset E$, to a subspace $\Lambda \subset \Lambda'$ is the point process denoted $\eta|_\Lambda$, obtained by keeping the points in $\Lambda$ and deleting the points in $\Lambda' \setminus \Lambda$. For any compact sets $\Lambda, \Lambda' \subset E$, such that $\Lambda \subset \Lambda'$, $\xi_\Lambda$ and $\xi_{\Lambda'}|_\Lambda$ have the same probability distribution. We say that the family $(\mathcal{L}(\xi_\Lambda))$, $\Lambda$ compact set included in $E$, is consistent.
Then we can obtain a point process on the complete space $E$ by the Kolmogorov existence theorem for point processes (see Theorem 9.2.X in [3] with $P_k(A_1,\ldots,A_k; n_1,\ldots,n_k) = P\left(\xi_{i=1}^k A_i(A_1) = n_1,\ldots,\xi_{i=1}^k A_i(A_k) = n_k\right)$: as $\xi_{i=1}^k A_i$ is a point process, it follows that the properties (i), (iii), (iv) are fulfilled; (ii) is fulfilled because the family $(\mathcal{L}(\xi_\Lambda), \Lambda$ compact set included in $E$, is consistent).

As we used, in this second part of the proof, only the fact that $\text{Det}(I + \alpha K_\Lambda)^{-1/\alpha} > 0$ (instead of $\text{Det}(I + \alpha K_\Lambda)^{-1/\alpha} \geq 1$), the assertion in remark 11 is also proved.

**Proof of Theorem 3.** We suppose there exists an $\alpha$-permanental process with $\alpha > 0$, kernel $K_\Lambda$ defining the locally trace class integral operator $\mathcal{K}$.

Then, following the proof of the preceding theorem, we get that, for all $z \in [0,1]$

$$\text{Det}(I + \alpha(1 - z)K_\Lambda) = \text{Det}(I + \alpha K_\Lambda) \text{Det}(I - \alpha z \mathcal{J}_\alpha^\lambda) > 0.$$  

As the power series of $\text{Det}(I - \alpha z \mathcal{J}_\alpha^\lambda)^{-1/\alpha}$ has all its terms non-negative,

$$|(\text{Det}(I - \alpha z \mathcal{J}_\alpha^\lambda)^{-1/\alpha}| \leq (\text{Det}(I - \alpha |z| \mathcal{J}_\alpha^\lambda)^{-1/\alpha}.$$  

If $z_0$ is a complex number with minimum modulus such that $(\text{Det}(I - \alpha z_0 \mathcal{J}_\alpha^\lambda) = 0$, by analyticity of $z \mapsto \text{Det}(I - \alpha z \mathcal{J}_\alpha^\lambda)$ on $\mathbb{C}$ and $z \mapsto z^{-1}$ on $\mathbb{C}^*$, $(\text{Det}(I - \alpha z \mathcal{J}_\alpha^\lambda)^{-1/\alpha}$ converges for $|z| < |z_0|$ and diverges for $z = z_0$. Thus the series diverges in $z = |z_0|$ and $|z_0| > 1$. This means that the series converges for $|z| \leq 1$ thus, in this case, $\text{Det}(I - \alpha z \mathcal{J}_\alpha^\lambda) > 0$.

This implies the necessary condition: $\text{Spec} \mathcal{J}_\alpha^\lambda \subset \{z \in \mathbb{C} : |z| < \frac{1}{\alpha}\}$.

As $\nu$ eigenvalue of $\mathcal{K}$ is equivalent to $\frac{\nu}{1 + \alpha \nu}$ eigenvalue of $\mathcal{J}$, and as, $\mathcal{K}$ and $\mathcal{J}$ being compact operators, their non-null spectral values are their eigenvalues, we get the other equivalent necessary condition:

$$\text{Spec} \mathcal{K}_\Lambda \subset \{z \in \mathbb{C} : \text{Re} z > -\frac{1}{2\alpha}\}.$$

6 $\alpha$- determinantal process ($\alpha < 0$)

We recall the following remark, already made for example in [7].

**Remark 13.** If we define kernels only $\lambda^{\otimes 2}$-almost everywhere, there can be problems when we consider only the diagonal terms, as $\lambda^{\otimes 2}\{(x,x) : x \in \Lambda\} = 0$. For example, in the formula

$$\text{tr} K_\Lambda = \int_\Lambda K(x,x) \lambda(dx),$$

$\text{tr} K_\Lambda$ is not uniquely defined. To avoid this problem, we write the kernel $K_\Lambda$ as follows:

$$K_\Lambda(x,y) = \sum_{k=0}^\infty a_k \varphi_k(x) \overline{\psi_k(y)}$$
where \((\varphi_k)_{k\in\mathbb{N}}, (\psi_k)_{k\in\mathbb{N}}\) are orthonormal basis in \(L^2(\Lambda, \lambda)\) and \((a_k)_{k\in\mathbb{N}}\) is a sequence of non-negative real number, which are the singular values of the operator \(K_{\Lambda}\).

The functions \(\varphi_k\) and \(\psi_k\), \(k \in \mathbb{N}\), are defined \(\lambda\)-almost everywhere, but this gives then a unique value for the expression of type

\[\int_{\Lambda^n} F(K(x_i, x_j)_{1\leq i,j\leq n}) G(x_1, \ldots, x_n) \lambda(dx_1) \ldots \lambda(dx_n)\]

where \(F\) is an arbitrary complex function from \(\mathbb{C}^{n^2}\) and \(G\) is an arbitrary complex function from \(\Lambda^n\).

With this remark, the quantities that appear with \(F = \det_{\alpha}\) are well defined.

**Lemma 14.** Let \(K\) be a kernel defined as in Remark 13 and defining a trace class integral operator \(K\) on \(L^2(\Lambda, \lambda)\), where \(\Lambda\) is a non-\(\lambda\)-null compact set included in the locally compact Polish space \(E\), \(\lambda\) be a Radon measure, \(n\) an integer and \(\alpha\) a real number. Let \(F\) be a continuous fonction from \(\mathbb{C}^{n^2}\) to \(\mathbb{C}\). The three following assertions are equivalent

(i) \(F(K(x_i, x_j)_{1\leq i,j\leq n}) \geq 0 \ \lambda^n - \text{a.e.}(x_1, \ldots, x_n) \in \Lambda^n\)

(ii) there exists a set \(\Lambda' \subset \Lambda\) such that \(\lambda(\Lambda \setminus \Lambda') = 0\) and \(F((K(x_i, x_j))_{1\leq i,j\leq n}) \geq 0\) for any \((x_1, \ldots, x_n) \in (\Lambda')^n\)

(iii) there exists a version of \(K\) such that \(F((K(x_i, x_j))_{1\leq i,j\leq n}) \geq 0\) for any \((x_1, \ldots, x_n) \in \Lambda^n\)

**Proof.** (i) is clearly a consequence of (ii). We assume now that (i) is satisfied and we denote by \(N\) the \(\lambda^n\)-null set of \(n\)-tuples \((x_1, \ldots, x_n) \in \Lambda^n\) such that \(F((K(x_i, x_j))_{1\leq i,j\leq n}) < 0\).

As in remark 13, we write the kernel \(K\) as follows

\[K(x, y) = \sum_{k=0}^{\infty} a_k \varphi_k(x) \overline{\psi_k(y)} = (\sqrt{a_k} \varphi_k)_{k\in\mathbb{N}}(x)|(\sqrt{a_k} \psi_k)_{k\in\mathbb{N}}(y)\]

where \((\varphi_k)_{k\in\mathbb{N}}, (\psi_k)_{k\in\mathbb{N}}\) are orthonormal basis in \(L^2(\Lambda, \lambda)\), \((a_k)_{k\in\mathbb{N}}\) is a sequence of non-negative real number, which are the singular values of the operator \(K\) and \(\langle \cdot, \cdot \rangle\) denote the inner product in the Hilbert space \(l_2(\mathbb{C})\).

As \(K\) is trace class, we have \(\sum_{k=0}^{\infty} a_k < \infty\). Hence:

\[\sum_{k=0}^{\infty} a_k |\varphi_k(x)|^2 < \infty \text{ and } \sum_{k=0}^{\infty} a_k |\psi_k(x)|^2 < \infty \text{ } \lambda\text{-a.e. } x \in \Lambda\]

From Lusin’s theorem, there exists an increasing sequence \((A_p)_{p\in\mathbb{N}}\) of compact sets included in \(\Lambda\) such that, for any \(p \in \mathbb{N}\)

\[\lambda(\Lambda \setminus A_p) < \frac{1}{p}\]

Therefore the kernel \(K : (x, y) \mapsto \langle (\sqrt{a_k} \varphi_k)_{k\in\mathbb{N}}(x)|(\sqrt{a_k} \psi_k)_{k\in\mathbb{N}}(y)\rangle\) is continuous on \(A_p^2\).

As \(E\) is a Polish space, it can be endowed with a distance that we denote by \(d\). We consider the sets

\[A'_p = \{ x \in A_p : \forall r > 0, \lambda(B(x, r) \cap A_p) > 0 \}\]

\[B_{p,n} = \{ x \in A_p : \lambda(B(x, 1/n) \cap A_p) = 0 \}\]
where $B(x, r)$ is the open ball in $E$ of radius $r$ centered at $x$ and $n$ is an integer. Let $(x_k)_{k \in \mathbb{N}}$ be a sequence in $B_{p,n}$ converging to $x \in A_p$. Then we have, when $d(x, x_k) < 1/n$,

$$\lambda(B(x, 1/n - d(x, x_k) \cap A_p) \leq \lambda(B(x_k, 1/n) \cap A_p) = 0$$

Therefore $\lambda(B(x, 1/n) \cap A_p) = 0$ and $x \in B_{p,n}$. Then, by compactness, $B_{p,n}$ can be covered by a finite numbers of such balls. As the intersections of $A_p$ and any such a ball is a $\lambda$-null set, we get $\lambda(B_{p,n}) = 0$.

Hence we have: $\lambda(A'_p) = \lambda(A_p \cup_{n \in \mathbb{N}} B_{p,n}) = \lambda(A_p) > \lambda(\Lambda) - 1/p$.

Let $(x_1, \ldots, x_n) \in (A'_p)^n$. If $(x_1, \ldots, x_n) \notin N$, then $F((K(x_i, x_j))_{1 \leq i, j \leq n}) \geq 0$.

Otherwise $(x_1, \ldots, x_n) \in N$. For any $i \in [1, n]$ and any $r > 0$, we have

$$\lambda(A_p \cap B(x_i, r)) > 0, \text{ then } \lambda^\otimes_n(A^n_p \cap B((x_1, \ldots, x_n), r)) = \lambda^\otimes_n(\prod_{i=1}^n (A_p \cap B(x_i, r))) > 0.$$  

where $B((x_1, \ldots, x_n), r)$ denotes the open ball of radius $r$ centered at $x$, in $E^n$ endowed with the distance $d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \max_{1 \leq i \leq n} d(x_i, y_i)$.

Then, as $\lambda^\otimes_n(N) = 0$, for any $q \in \mathbb{N}$, there exists $(y_1^{(q)}, \ldots, y_n^{(q)}) \in A^n_p \cap B((x_1, \ldots, x_n), 1/q) \setminus N$ and thus $(y_1^{(q)}, \ldots, y_n^{(q)})$ converge to $(x_1, \ldots, x_n)$ when $q \to \infty$.

As $(y_1^{(q)}, \ldots, y_n^{(q)}) \notin N$, $F((K(y_i^{(q)}, y_j^{(q)}))_{1 \leq i, j \leq n}) \geq 0$.

As $K$ is continuous on $A^2_p$ and $F$ is continuous on $\mathbb{C}^{n^2}$, we have that the function $(x_1, \ldots, x_n) \mapsto F((K(x_i, x_j))_{1 \leq i, j \leq n})$ is continuous on $A^n_p$. Hence we have: $F((K(x_i, x_j))_{1 \leq i, j \leq n}) \geq 0$.

Therefore, in all cases, if $(x_1, \ldots, x_n) \in (A'_p)^n$, $F((K(x_i, x_j))_{1 \leq i, j \leq n}) \geq 0$.

As $(A'_p)_{p \in \mathbb{N}}$ is an increasing sequence, it is the same for $(A'_p)_{p \in \mathbb{N}}$. Hence we have: $\cup_{p \in \mathbb{N}} (A'_p)^n = \left(\cup_{p \in \mathbb{N}} A'_p\right)^n$.

We obtain:

$$F((K(x_i, x_j))_{1 \leq i, j \leq n}) \geq 0 \text{ for any } (x_1, \ldots, x_n) \in \left(\cup_{p \in \mathbb{N}} A'_p\right)^n$$

As $\lambda(\Lambda \setminus \left(\cup_{p \in \mathbb{N}} A'_p\right)) = 0$, we finally obtain (ii) with $\Lambda' = \cup_{p \in \mathbb{N}} A'_p$.

We obtained that (i) and (ii) are equivalent conditions.

(i) is clearly a consequence of (iii). Assume now (ii). We will define a version $K_1$ of $K$ satisfying the condition (iii).

As $\lambda(\Lambda) \neq 0$, $\Lambda' \neq \emptyset$. We set an arbitrary $x_0 \in \Lambda'$.

For $(x, x') \in \Lambda^2$, we define, $y = x$ if $x \in \Lambda'$, $y = x_0$ if $x \in \Lambda \setminus \Lambda'$, $y' = x'$ if $x' \in \Lambda'$, $y' = x_0$ if $x' \in \Lambda \setminus \Lambda'$ and $K_1(x, x') = K(y, y')$. 

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For \((x_1, \ldots, x_n) \in \Lambda^n\), we define, for \(1 \leq i \leq n\), \(y_i = x_i\) if \(x_i \in \Lambda\) and \(y_i = x_0\) if \(x_i \in \Lambda^c\). Then we have, \(F((K_1(x_i, x_j))_{1 \leq i, j \leq n}) = F((K(y_i, y_j))_{1 \leq i, j \leq n}) \geq 0\) and \(K_1\) is a version of \(K\) satisfying the condition (iii).

**Remark 15.** Let \(F_n, n \in \mathbb{N}\), be continuous functions from \(\mathbb{C}^{n^2}\) to \(\mathbb{C}\). For any non-\(\lambda - null\) compact set \(\Lambda\), the condition:

1. \(F_n((J^1_{\alpha}(x_i, x_j))_{1 \leq i, j \leq n}) \geq 0\), for any \(n \in \mathbb{N}\) and \(\lambda \otimes^n\)-a.e. \((x_1, \ldots, x_n) \in \Lambda^n\)

can always be replaced by the equivalent conditions:

2. there exists a set \(\Lambda' \subset \Lambda\) such that \(\lambda(\Lambda \setminus \Lambda') = 0\) and \(F_n((J^1_{\alpha}(x_i, x_j))_{1 \leq i, j \leq n}) \geq 0\), for any \(n \in \mathbb{N}\) and \((x_1, \ldots, x_n) \in (\Lambda')^n\).

or:

3. there exists a version of the kernel \(J\) such that \(F_n((J^1_{\alpha}(x_i, x_j))_{1 \leq i, j \leq n}) \geq 0\), for any \(n \in \mathbb{N}\) and \((x_1, \ldots, x_n) \in \Lambda^n\).

**Proof.** The proof of (ii) \(\Rightarrow\) (iii) is done in the same way as in Lemma 14. The other parts of the proof are a direct application of Lemma 14.

**Proof that (i) is necessary in Theorem 4.** This has been mentioned by Vere-Jones in [12] for the multivariate binomial probability distribution, which corresponds to a determinantal process with \(E\) being finite. To our knowledge, this has not been proved in other cases.

We consider the \(n \times n\) matrix \(1_n\), whose elements are all equal to one.

We have: \(\prod_{j=0}^{n-1}(1 + j\alpha) = 1 + \sum_{i=1}^{n-1} \sum_{j_1 < \cdots < j_k \leq n-1} j_1 \cdots j_k \alpha^k\)

We will show by induction on \(n\) that the number of permutations in \(\Sigma_n\) having \(n-k\) cycles for \(k \neq 0\) is \(a_{nk} = \sum_{j_1 < \cdots < j_k \leq n-1} j_1 \cdots j_k\): this is true for \(n = 2\) and \(k = 1\). Assume it is true for a given \(n \in \mathbb{N}\) and for any \(k \in [1, n-1]\). If we consider the permutations \(\sigma \in \Sigma_{n+1}\) having \(n + 1 - k\) cycles \((0 \leq k \leq n)\), we have 2 cases:
- either \(\sigma(n + 1) = n + 1\): there is exactly \(a_{nk}\) permutations corresponding to this case (with the convention \(a_{n0} = 0\), for the case \(k = n\)),
- or \(\sigma(n + 1) \neq n + 1\). Then, if we denote \(\tau_{n+1} \sigma(n+1)\) the transposition in \(\Sigma_{n+1}\) that exchange \(n + 1\) and \(\sigma(n + 1)\), \(\tau_{n+1} \sigma(n+1) \circ \sigma\) is a permutation having \(n + 1\) as fixed point and \(n + 1 - k\) other cycles (with elements in \([1, n]\)) there is exactly \(na_{nk-1}\) permutations corresponding to this case.

Then we have:

\[
a_{n+1, n+1-k} = a_{nk} + na_{nk-1} = \sum_{1 \leq j_1 < \cdots < j_k \leq n-1} j_1 \cdots j_k + \sum_{1 \leq j_1 < \cdots < j_{k-1} \leq n-1} j_1 \cdots j_k
\]

\[
= \sum_{1 \leq j_1 < \cdots < j_k \leq n} j_1 \cdots j_k
\]
which is what we expected.
Thus: \( \det_{\alpha} 1_n = \prod_{j=0}^{n-1}(1 + j\alpha) \).
If \( \alpha < 0 \) but \(-1/\alpha \notin \mathbb{N} \), there exists therefore \( n \in \mathbb{N} \) such that \( \det_{\alpha} 1_n < 0 \).

We suppose that there exists an \( \alpha \)-determinantal process with \( \alpha < 0 \) but \(-1/\alpha \notin \mathbb{N} \) and kernel \( K \). Then we have \( \det_{\alpha}(K(x_i, x_j))_{1 \leq i,j \leq n} \geq 0 \) \( \lambda^\otimes n \)-a.e. \( (x_1, \ldots, x_n) \in E^n \).

As we exclude the case of a point process having no point almost surely and there is a sequence of compact sets \( \Lambda_p \) such that \( \cup_{p \in \mathbb{N}} \Lambda_p = E \), there exists a compact set \( \Lambda \in E \) such that

\[
\mathbb{E}(\xi(\Lambda)) = \int_{\Lambda} K(x, x)\lambda(dx) > 0.
\]

Applying Lemma 14, we get that there exist a version \( K_1 \) of the kernel \( K \) such that \( \det_{\alpha}(K_1(x_i, x_j))_{1 \leq i,j \leq n} \geq 0 \) for any \( (x_1, \ldots, x_n) \in \Lambda^n \). We also have:

\[
\int_{\Lambda} K(x, x)\lambda(dx) = \int_{\Lambda} K_1(x, x)\lambda(dx) > 0.
\]

Hence there exists \( x_0 \in \Lambda \) such that \( K_1(x_0, x_0) > 0 \).
For \( (x_1, \ldots, x_n) = (x_0, \ldots, x_0) \), we get:

\[
\det_{\alpha}(K_1(x_i, x_j))_{1 \leq i,j \leq n} = K(x_0, x_0)^n \det_{\alpha} 1_n < 0
\]

which is a contradiction. Therefore if \( \alpha < 0 \) and an \( \alpha \)-determinantal process exists, then \( \alpha \) must be in \( \{-1/m : m \in \mathbb{N}\} \).

We consider a \( d \times d \) square matrix \( A \). If \( n_1, \ldots, n_d \) are \( d \) non-negative integers, \( A[n_1, \ldots, n_d] \) is the \( (n_1 + \cdots + n_d) \times (n_1 + \cdots + n_d) \) square matrix composed of the block matrices \( A_{ij} \):

\[
A[n_1, \ldots, n_d] = \begin{pmatrix}
A_{11} & A_{12} & \cdots & A_{1d} \\
A_{21} & A_{22} & \cdots & A_{2d} \\
\vdots & \vdots & \ddots & \vdots \\
A_{d1} & A_{d2} & \cdots & A_{dd}
\end{pmatrix},
\]

where \( A_{ij} \) is the \( n_i \times n_j \) matrix whose elements are all equal to \( a_{ij} \) \( (1 \leq i, j \leq d) \).

**Lemma 16.** Given a \( d \times d \) square matrix \( A \), the following assertions are equivalent

\begin{align*}
(i) \quad & \det_{-1/m} A[n_1, \ldots, n_d] \geq 0, \forall n_1, \ldots, n_d \in \mathbb{N} \\
(ii) \quad & \det_{-1/m} A[n_1, \ldots, n_d] \geq 0, \forall n_1, \ldots, n_d \in \{0, \ldots, m\} \\
(iii) \quad & \det A[n_1, \ldots, n_d] \geq 0, \forall n_1, \ldots, n_d \in \mathbb{N} \\
(iv) \quad & \det A[n_1, \ldots, n_d] \geq 0, \forall n_1, \ldots, n_d \in \{0, 1\}
\end{align*}

**Proof.** If there exists \( k \in [1, d] \) such that \( n_k > 1 \), the matrix \( A[n_1, \ldots, n_d] \) has at least two identical rows and its determinant is null. So it is clear that (iii) and (iv) are equivalent.

\[\square\]
We have:

$$\det(I + ZA)^m = \sum_{n_1, \ldots, n_d=0}^{\infty} m^{n_1+\ldots+n_d} \left( \prod_{k=1}^{d} \frac{z_k^{n_k}}{n_k!} \right) \det_{-1/m} A[n_1, \ldots, n_d]$$

(6)

where $Z = \text{diag}(z_1, \ldots, z_d)$ and $z_1, \ldots, z_d$ are $d$ complex numbers. It is a special case of the formula (3) with $\alpha = -1/m$, finite space $E = [1, d]$ and reference measure $\lambda$ atomic, where each point of $E$ has measure 1, $\Lambda_k = \{ k \}$, for $k \in [1, d]$, $\Lambda = E$. Indeed, $ZA = \sum_{k=1}^{d} z_k A_k$, where $A_k$ is the $d \times d$ square matrix having the same $k^{th}$ row as $A$ and the other rows with all elements equal to 0. The matrix $A$ corresponds to the operator $K$, the matrix $A_k$ corresponds to the operator $K_{\Lambda_k \Lambda}$. Formula (6) also corresponds to the one given by Vere-Jones in [13].

We also have for $m = 1$:

$$\det(I + ZA) = \sum_{n_1, \ldots, n_d=0}^{1} \left( \prod_{k=1}^{d} \frac{z_k^{n_k}}{n_k!} \right) \det A[n_1, \ldots, n_d].$$

(7)

as $\det A[n_1, \ldots, n_d] = 0$ if there exists $k \in [1, d]$ such that $n_k > 1$.

(i) is equivalent to the fact that the multivariate power series (6) has all its coefficients non-negative.

(iii) is equivalent to the fact that the multivariate power series (7) has all its coefficients non-negative.

The power series (6) being the $m^{th}$ power of the power serie (7), if there exists $k \in [1, d]$ such that $n_k > m$, the coefficient of $\prod_{k=1}^{d} z^{n_k}$ is null. Therefore, (i) is equivalent to (ii).

For the same reason, we also have that (i) is a consequence of (iii).

Conversely, following Vere-Jones in [12], we can show by induction on the order of the matrix $A$, that the fact that the power series (6) has all its coefficients non-negative implies that the power series (7) has all its coefficient non negative.

This proves the equivalence between (i) and (iii).

Proposition 17. Let $\alpha < 0$ and $K$ be an integral operator such that $I+\alpha K_\Lambda$ is invertible, for any compact set $\Lambda \subset E$. An $\alpha$-determinantal process with kernel $K$ exists iff:

$$\det_{\alpha}(J_{\alpha}(x_1, x_j))_{1 \leq i, j \leq n} \geq 0, \text{ for any } n \in \mathbb{N}, \text{ and any compact set } \Lambda \lambda^{\otimes n} \text{-a.e. } (x_1, \ldots, x_n) \in \Lambda^n$$

(8)

Condition (8) implies that $-\frac{1}{\alpha} \in \mathbb{N}$ and $\text{Det}(I + \beta K) > 0$ for any $\beta \in [\alpha, 0]$.

Proof. We assume that there exists an $\alpha$-determinantal process $\xi$ with kernel $K$. We already proved that it is necessary to have $-1/\alpha \in \mathbb{N}$.
By taking \( d = 1 \) in the formula (4), we have

\[
E \left( z^{\xi(A)} \right) = \text{Det} \left( I + \alpha (1 - z) K_\Lambda \right)^{-1/\alpha}
\]

for any compact set \( \Lambda \subseteq E \) and \( z \in (0, 1] \).

Then \( \text{Det} \left( I + \alpha (1 - z) K_\Lambda \right) > 0 \) for \( z \in (0, 1] \), and by continuity, \( \text{Det} \left( I + \alpha K_\Lambda \right) \geq 0 \). As we assumed that \( I + \alpha K_\Lambda \) is invertible, we have necessarily \( \text{Det} \left( I + \alpha K_\Lambda \right) > 0 \).

For any non-negative function \( f \), with compact support included in \( \Lambda \)

\[
E \left( \prod_{x \in \xi} e^{-f(x)} \right) = \text{Det} \left( I + \alpha \mathcal{J}[1 - e^{-f}] \right)^{-1/\alpha} = \text{Det} \left( I + \alpha K_\Lambda \right)^{-1/\alpha} \text{Det} \left( I - \alpha \mathcal{J}_\alpha^\Lambda e^{-f} \right)^{-1/\alpha} = \text{Det} \left( I + \alpha K_\Lambda \right)^{-1/\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \left( \prod_{i=1}^{n} e^{-f(x_i)} \right) \text{det}_\alpha(J_\alpha^\Lambda(x_i, x_j))_{1 \leq i, j \leq n} \lambda(dx_1) \ldots \lambda(dx_n)
\]

As the Laplace functional defines a.e. uniquely the Janossy density of a point process, one obtains:

\[
\text{det}_\alpha(J_\alpha^\Lambda(x_i, x_j))_{1 \leq i, j \leq n} \geq 0 \lambda^{\otimes n}\text{-a.e. } (x_1, \ldots, x_n) \in E^n
\]

Conversely, we assume that the condition

\[
\text{det}_\alpha(J_\alpha^\Lambda(x_i, x_j))_{1 \leq i, j \leq n} \geq 0, \text{ for any } n \in \mathbb{N}, \lambda^{\otimes n}\text{-a.e. } (x_1, \ldots, x_n) \in \Lambda^n \text{ and any compact set } \Lambda
\]

is fulfilled. We have

\[
\text{Det} \left( I - \alpha z \mathcal{J}_\alpha^\Lambda \right)^{-1/\alpha} = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{\Lambda^n} \text{det}_\alpha(J_\alpha^\Lambda(x_i, x_j))_{1 \leq i, j \leq n} \lambda(dx_1) \ldots \lambda(dx_n)
\]

As \(-1/\alpha \in \mathbb{N}\), this formula is valid for any \( z \in \mathbb{C} \). Then we obtain for \( z = 1 \), \( \text{Det} \left( I - \alpha \mathcal{J}_\alpha^\Lambda \right)^{-1/\alpha} \geq 0 \).

We also have \( (I - \alpha \mathcal{J}_\alpha^\Lambda) (I + \alpha K_\Lambda) = (I + \alpha K_\Lambda) (I - \alpha \mathcal{J}_\alpha^\Lambda) = I \).

Thus the Janossy density is correctly defined and, on any compact set \( \Lambda \) we get the existence of a point process with kernel \( K \) and reference measure \( \lambda \).

Then it can be extended to the complete space \( E \) by the Kolmogorov existence theorem (see Theorem 9.2.X in [3]).

\( \square \)

**Proof of Theorem 4.** For any \( m \in \mathbb{N} \), applying Lemma 16, we have for any compact set \( \Lambda \)

\[
\text{det}_{-1/m}(J_{-1/m}^\Lambda(x_i, x_j))_{1 \leq i, j \leq n} \geq 0, \text{ for any } n \in \mathbb{N}, \text{ and any } (x_1, \ldots, x_n) \in \Lambda^n
\]

is equivalent to

\[
\text{det}(J_{-1/m}^\Lambda(x_i, x_j))_{1 \leq i, j \leq n} \geq 0, \text{ for any } n \in \mathbb{N}, \text{ and any } (x_1, \ldots, x_n) \in \Lambda^n
\]

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Now, assume we only have
\[ \det_{-1/m}(J^\Lambda_{-1/m}(x_i, x_j))_{1 \leq i, j \leq n} \geq 0, \] for any \( n \in \mathbb{N} \), \( \lambda^{\otimes n} \)-a.e. \( (x_1, \ldots, x_n) \in \Lambda^n \).

By lemma 14, for each \( n \in \mathbb{N} \), there exists a set \( \Lambda_n' \subset \Lambda \) such that \( \lambda(\Lambda \setminus \Lambda_n') = 0 \) and
\[ \det_{-1/m}(J^\Lambda_{-1/m}(x_i, x_j))_{1 \leq i, j \leq n} \geq 0, \] for any \( (x_1, \ldots, x_n) \in (\Lambda_n')^n \).

If \( \Lambda' = \cap_{n \in \mathbb{N}} \Lambda_n' \), we have \( \lambda(\Lambda \setminus \Lambda') = 0 \) and \( \det_{-1/m}(J^\Lambda_{-1/m}(x_i, x_j))_{1 \leq i, j \leq n} \geq 0, \) for any \( n \in \mathbb{N} \) and \( (x_1 \ldots, x_n) \in (\Lambda')^n \).

Then, by Lemma 16, we have: \( \det(J^\Lambda_{-1/m}(x_i, x_j))_{1 \leq i, j \leq n} \geq 0, \) for any \( n \in \mathbb{N} \) and \( (x_1 \ldots, x_n) \in (\Lambda')^n \).

Therefore, we have
\[ \det(J^\Lambda_{-1/m}(x_i, x_j))_{1 \leq i, j \leq n} \geq 0, \] for any \( n \in \mathbb{N} \), \( \lambda^{\otimes n} \)-a.e. \( (x_1, \ldots, x_n) \in \Lambda^n \).

The converse is done through a similar proof, using Lemma 14 and 16.

Thus, we obtain:
\[ \det_{\alpha}(J^\Lambda_{\alpha}(x_i, x_j))_{1 \leq i, j \leq n} \geq 0, \] for any \( n \in \mathbb{N} \), \( \lambda^{\otimes n} \)-a.e. \( (x_1, \ldots, x_n) \in \Lambda^n \)

is equivalent to
\[ \det(J^\Lambda_{\alpha}(x_i, x_j))_{1 \leq i, j \leq n} \geq 0, \] for any \( n \in \mathbb{N} \), \( \lambda^{\otimes n} \)-a.e. \( (x_1, \ldots, x_n) \in \Lambda^n \).

Theorem 4 is then a consequence of Proposition 17.

\[ \square \]

**Proof of Theorem 5.** We assume that there exists \( \xi \) an \( \alpha \)-determinantal process with kernel \( K \).

For \( p \in (0, 1) \), let \( \xi_p \) be the process obtained by first sampling \( \xi \), then independently deleting each point of \( \xi \) with probability \( 1-p \).

Computing the correlation functions, we obtain that \( \xi_p \) is an \( \alpha \)-determinantal process with kernel \( pK \).

Thus we get from Theorem 4 that the conditions of the theorem must be fulfilled.

Conversely, we assume that these conditions are fulfilled. We obtain from Theorem 4 that an \( \alpha \)-determinantal process \( \xi_p \) with kernel \( pK \) exists, for any \( p \in (0, 1) \).

We consider a sequence \( (p_k) \in (0, 1)^N \) converging to 1 and a compact \( \Lambda \).

\[ \mathbb{E}(\exp(-t\xi_{p_k}(\Lambda))) = \text{Det}(\mathcal{I} + \alpha p_k K\Lambda(1 - e^{-t}))^{-1/\alpha} \xrightarrow{k \to \infty} \text{Det}(\mathcal{I} + \alpha K\Lambda(1 - e^{-t}))^{-1/\alpha} \]

As \( t \mapsto \text{Det}(\mathcal{I} + \alpha K\Lambda(1 - e^{-t}))^{-1/\alpha} \) is continuous in 0, \( (\mathcal{L}(\xi_{p_k}(\Lambda)))_{k \in \mathbb{N}} \) converge weakly.

Thus \( (\mathcal{L}(\xi_{p_k}(\Lambda)))_{k \in \mathbb{N}} \) is tight.

\( \Gamma \subset \mathcal{X} \) is relatively compact if and only if, for any compact set \( \Lambda \subset E \), \( \{\xi(\Lambda) : \xi \in \Gamma\} \) is bounded.

Let \( (\Lambda_n)_{n \in \mathbb{N}} \) be an increasing sequence of compact sets such that \( \cup_{n \in \mathbb{N}} \Lambda_n = E \).

As, for any \( n \in \mathbb{N} \), \( (\mathcal{L}(\xi_{p_k}(\Lambda_n)))_{k \in \mathbb{N}} \) is tight, we have that, for any \( \epsilon > 0 \) and \( n \in \mathbb{N} \), there exists \( M_n > 0 \) such that for any \( k \in \mathbb{N} \), \( \mathbb{P}(\xi_{p_k}(\Lambda_n) > M_n) < 2^{-n-1} \).

Let \( \Gamma = \{\gamma \in \mathcal{X} : \forall n \in \mathbb{N}, \gamma(\Lambda_n) \leq M_n\} \). It is a compact set and for any \( k \in \mathbb{N} \), \( \mathbb{P}(\xi_{p_k} \in \Gamma^c) < \epsilon \).
Therefore, \((L(\xi_{p_k}))_{k \in \mathbb{N}}\) is tight. As \(E\) is Polish, \(\mathcal{X}\) is also Polish (endowed with the Prokhorov metric). Thus there is a subsequence of \((L(\xi_{p_k}))_{k \in \mathbb{N}}\) converging weakly to the probability distribution of a point process \(\xi\). By unicity of the distribution of an \(\alpha\)-determinantal process for given kernel and reference measure, \(\xi\) must be an \(\alpha\)-determinantal process with kernel \(K\), which gives the existence.

**Lemma 18.** Let \(\mathcal{J}\) be a trace class self-adjoint integral operator with kernel \(J\). We have
\[
\det(J(x_i, x_j))_{1 \leq i, j \leq n} \geq 0 \text{ for any } n \in \mathbb{N}, \lambda^{\otimes n}\text{-a.e.}(x_1, \ldots, x_n) \in \Lambda^n
\]
if and only if
\[
\text{Spec } \mathcal{J} \subset [0, \infty)
\]

**Proof.** If we assume that the operator \(\mathcal{J}\) is positive, the kernel can be written as follows:
\[
J(x, y) = \sum_{k=0}^{\infty} a_k \varphi_k(x) \bar{\varphi}_k(y)
\]
where \(a_k \geq 0\) for \(k \in \mathbb{N}\).

Hence:
\[
\det(J(x_i, x_j))_{1 \leq i, j \leq n} \geq 0 \text{ for any } n \in \mathbb{N}, \text{ and any } (x_1, \ldots, x_n) \in \Lambda^n
\]

Conversely, assume that
\[
\det(J(x_i, x_j))_{1 \leq i, j \leq n} \geq 0 \text{ for any } n \in \mathbb{N}, \lambda^{\otimes n}\text{-a.e.}(x_1, \ldots, x_n) \in \Lambda^n
\]

From formula (2) with \(\alpha = -1\), we have then for any \(z \in \mathbb{C}\)
\[
\text{Det}(I + z\mathcal{J}) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{E^n} \det(J(x_i, x_j))_{1 \leq i, j \leq n} \lambda(dx_1) \ldots \lambda(dx_n).
\]

As \(\mathcal{J}\) is assumed to be self-adjoint, its spectrum is included in \(\mathbb{R}\). Thanks to (9), it is impossible to have an eigenvalue in \(\mathbb{R}^*\), as the power series has all its coefficients real non-negative and the first coefficient \((n = 0)\) is real positive. Hence \(\text{Spec } \mathcal{J} \subset [0, \infty)\).

**Proof of Corollary 7.** We assume: \(-1/\alpha \in \mathbb{N}\) and \(\text{Spec } \mathcal{K} \subset [0, -1/\alpha]\). Then we have, as \(\mathcal{K}\) is self-adjoint, that for any compact set \(\Lambda\), \(\text{Spec } \mathcal{K}_\Lambda \subset [0, -1/\alpha]\). Then \(\text{Det}(I + \beta \mathcal{K}_\Lambda) > 0\) for any \(\beta \in (\alpha, 0]\).

If \(I + \alpha \mathcal{K}_\Lambda\) is invertible for any compact set \(\Lambda \subset E\), we have \(\text{Spec } J^\Lambda_\alpha \subset [0, \infty)\) and \(J^\Lambda_\alpha\) is a trace class self adjoint operator for any compact set \(\Lambda\).

Then, applying Lemma 18, we get that
\[
\det(J(x_i, x_j))_{1 \leq i, j \leq n} \geq 0 \text{ for any } n \in \mathbb{N}, \text{ compact set } \Lambda \text{ and } \lambda^{\otimes n}\text{-a.e.}(x_1, \ldots, x_n) \in \Lambda^n
\]

Using Theorem 4, we get the existence of an \(\alpha\)-determinantal process with kernel \(K\).
When there exists a compact set \( \Lambda_0 \) such that \( I + \alpha K_{\Lambda_0} \) is not invertible, by the same line of proof, we obtain the announced result, using Theorem \( 5 \).

Conversely, we assume that there exists an \( \alpha \)-determinantal process with kernel \( K \). Then, from Theorem \( 4 \) or \( 5 \), we get that \( -1/\alpha \in \mathbb{N} \). If \( I + \alpha K_{\Lambda} \) is invertible for any compact set \( \Lambda \subset E \), we have \( \text{Spec} J_\alpha \subset [0, \infty) \), using Theorem \( 4 \) and lemma \( 18 \). Then \( \text{Spec} K_{\Lambda} \subset [0, -1/\alpha) \subset [0, -1/\alpha] \), for any compact set \( \Lambda \).

If there exists a compact set \( \Lambda_0 \) such that \( I + \alpha K_{\Lambda_0} \) is not invertible, we have \( \text{Spec} J_\beta \subset [0, \infty) \) for any compact set \( \Lambda \) and any \( \beta \in (\alpha, 0) \), using Theorem \( 5 \) and lemma \( 18 \). Then \( \text{Spec} K_{\Lambda} \subset [0, -1/\beta) \) for any \( \beta \in (\alpha, 0) \). Therefore \( \text{Spec} K_{\Lambda} \subset [0, -1/\alpha] \) for any compact set \( \Lambda \).

As \( K \) is self-adjoint, this implies in both cases that \( \text{Spec} K \subset [0, -1/\alpha] \).

\[ \square \]

\textbf{Remark} 19. Using the known result in the case \( \alpha = -1 \) (see for example Hough, Krishnapur, Peres and Virág in [7]) and corollary \( 6 \), one obtains a direct proof of Corollary \( 7 \).

### 7 Infinite divisibility

\textbf{Proof of Theorem} \( 8 \). For \( \alpha < 0 \), we have proved that it is necessary to have \(-1/\alpha \in \mathbb{N}\). If an \( \alpha \)-determinantal process was infinitely divisible, with \( \alpha < 0 \), it would be the sum of \( N \) i.i.d \( \alpha N \)-determinantal processes for any \( N \in \mathbb{N}^* \), as it can be seen for the Laplace functional formula (1). This would imply that \(-1/(N\alpha) \in \mathbb{N} \), for any \( N \in \mathbb{N}^* \), which is not possible. Therefore, an \( \alpha \)-determinantal process with \( \alpha < 0 \) is never infinitely divisible.

Some characterization on infinite divisibility have also been given in [4] in the case \( \alpha > 0 \).

\textbf{Proof of Theorem} \( 9 \). For \( \alpha > 0 \), assume that \( \text{Det}(I + \alpha K_{\Lambda}) \geq 1 \) and

\[ \sum_{\sigma \in \Sigma_n, \nu(\sigma) = 1} \prod_{i=1}^{n} J_\alpha(x_i, x_{\sigma(i)}) \geq 0, \]

for any compact set \( \Lambda \subset E \), \( n \in \mathbb{N} \) and \( \lambda^{\otimes n} \)-a.e. \( (x_1, \ldots, x_n) \in \Lambda^n \). Then we have:

\[ \sum_{\sigma \in \Sigma_n, \nu(\sigma) = k} \prod_{i=1}^{n} J_\alpha(x_i, x_{\sigma(i)}) = \sum_{\{I_1, \ldots, I_k\}, \nu(I) = \nu(\sigma) = 1} \prod_{q=1}^{k} \prod_{i \in I_q} J_\alpha(x_i, x_{\sigma(i)}) \]

\[ = \sum_{\{I_1, \ldots, I_k\}, \nu(\sigma) = 1} \prod_{q=1}^{k} \left( \sum_{\nu(\sigma) = 1} \prod_{i \in I_q} J_\alpha(x_i, x_{\sigma(i)}) \right) \geq 0, \]

for any compact set \( \Lambda \subset E \), \( n \in \mathbb{N} \), \( k \in [1, n] \) and \( \lambda^{\otimes n} \)-a.e. \( (x_1, \ldots, x_n) \in \Lambda^n \), where, for a finite set \( I \), \( \Sigma(I) \) denotes the set of all permutations on \( I \).

Then, for any \( N \in \mathbb{N}^* \) and any compact set \( \Lambda \subset E \), \( \det_{N\alpha}(J_\alpha(x_i, x_j)/N)_{1 \leq i, j \leq n} \geq 0 \). From Theorem \( 1 \), we get that there exists a \( (N\alpha) \)-permanental process with kernel \( K/N \). This
means that an $\alpha$-permanental process with kernel $K$ is infinitely divisible.

Conversely, if we assume an $\alpha$-permanental process with kernel $K$ is infinitely divisible, we get the existence of a $\alpha$-permanental process with kernel $K/N$, for any $N \in \mathbb{N}^*$. From Theorem 1, we have that $\text{Det}(I + \alpha K_{\Lambda}) \geq 1$ for any compact set $\Lambda \in E$. We also have

$$\frac{1}{(N\alpha)^{n-1}} \det_{N\alpha}(J_{\alpha}^n(x_i, x_j))_{1 \leq i, j \leq n} \geq 0,$$

for any $N \in \mathbb{N}^*$, any $n \in \mathbb{N}$, any compact set $\Lambda \in E$ and $\lambda^{\otimes n}$-a.e. $(x_1, \ldots, x_n) \in \Lambda^n$.

When $N$ tends to $\infty$, we obtain:

$$\sum_{\sigma \in \Sigma_n: \nu(\sigma) = 1} n \prod_{i=1}^n J_{\alpha}^n(x_i, x_{\sigma(i)}) \geq 0,$$

which is the desired result. \hfill \Box

**Proof of Theorem 10.** We use the argument of Griffiths in [5] and Griffiths and Milne in [6]. Assume

$$\sum_{\sigma \in \Sigma_n: \nu(\sigma) = 1} n \prod_{i=1}^n J_{\alpha}^n(x_i, x_{\sigma(i)}) \geq 0,$$

for any $n \in \mathbb{N}$ and any $(x_1, \ldots, x_n) \in \Lambda^n$. The condition $J_{\alpha}^n(x_1, x_2) \cdots J_{\alpha}^n(x_{n-1}, x_n) J_{\alpha}^n(x_n, x_1) \geq 0$ is satisfied for the elementary cycles, i.e. cycles such that $J_{\alpha}^n(x_i, x_j) = 0$ if $i < j + 1$ and $(i \neq 1$ or $j \neq n)$. Then it can be extended to any cycle by induction, using $J_{\alpha}^n(x_i, x_j) = J_{\alpha}^n(x_j, x_i)$.

With Lemma 14, we can then extend the proof to the case when

$$\sum_{\sigma \in \Sigma_n: \nu(\sigma) = 1} n \prod_{i=1}^n J_{\alpha}^n(x_i, x_{\sigma(i)}) \geq 0,$$

for any $n \in \mathbb{N}$ and $\lambda^{\otimes n}$-a.e. $(x_1, \ldots, x_n) \in \Lambda^n$. \hfill \Box

**Remark 20.** Note that the argument from Griffiths and Milne in [5] and [6] is only valid for real symmetric matrices.

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References


