



HAL
open science

A note on the derivation of rigid-plastic models

Jean-François Babadjian, Gilles A. Francfort

► **To cite this version:**

Jean-François Babadjian, Gilles A. Francfort. A note on the derivation of rigid-plastic models. Non-linear Differential Equations and Applications, 2016, NoDEA Nonlinear Differential Equations Appl., 23 (3), pp.23-37. hal-01214821

HAL Id: hal-01214821

<https://hal.science/hal-01214821>

Submitted on 13 Oct 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A NOTE ON THE DERIVATION OF RIGID-PLASTIC MODELS

JEAN-FRANÇOIS BABADJIAN AND GILLES A. FRANCFORT

ABSTRACT. This note is devoted to a rigorous derivation of rigid-plasticity as the limit of elasto-plasticity when the elasticity tends to infinity.

1. INTRODUCTION

Small strain elasto-plasticity is formally modeled as follows. Consider a homogeneous elasto-plastic material occupying a volume $\Omega \subset \mathbb{R}^n$ with Hooke's law (elasticity tensor) \mathbb{C} . Assume that the body is subjected to a time-dependent loading process during a time interval $[0, T]$ with, say, $f(t)$ as body loads, $g(t)$ as surface loads on a part Γ_N of $\partial\Omega$, and $w(t)$ as displacement loads (hard device) on the complementary part Γ_D of $\partial\Omega$. Denoting by $Eu(t)$ the infinitesimal strain at t , that is, the symmetric part of the spatial gradient of the displacement field $u(t)$ at t , small strain elasto-plasticity requires that $Eu(t)$ decompose additively as

$$Eu(t) = e(t) + p(t) \text{ in } \Omega, \text{ with } u(t) = w(t) \text{ on } \Gamma_D$$

where $e(t)$ is the elastic strain and $p(t)$ the plastic strain. The elastic strain is related to the stress tensor $\sigma(t)$ through the constitutive law of linearized elasticity $\sigma(t) = \mathbb{C}e(t)$. In a quasi-static setting, the equilibrium equations read as

$$\operatorname{div} \sigma(t) + f(t) \text{ in } \Omega, \quad \sigma(t)\nu = g(t) \text{ on } \Gamma_N,$$

where ν denotes the outer unit normal to $\partial\Omega$. In plasticity, the stresses are constrained to remain below a yield stress at which permanent strains appear. Specifically, the deviatoric stress $\sigma_D(t)$ must belong to a fixed compact and convex subset K of the deviatoric (trace free) matrices

$$\sigma_D(t) \in K.$$

If $\sigma_D(t)$ lies inside the interior of K , the material behaves elastically ($p(t) = 0$). On the other hand, if $\sigma_D(t)$ reaches the boundary of K (called the yield surface), a plastic flow may develop, so that, after unloading, there will remain a non-trivial permanent plastic strain $p(t)$. Its evolution is described by the so-called flow rule

$$\dot{p}(t) \in N_K(\sigma_D(t))$$

where $N_K(\sigma_D(t))$ is the normal cone to K at $\sigma_D(t)$. By arguments of convex analysis, the flow rule can be equivalently written as Hill's principle of maximum plastic work

$$\sigma_D(t) : \dot{p}(t) = \max_{\tau_D \in K} \tau_D : \dot{p}(t) =: H(\dot{p}(t)),$$

where H is the support function of K , and $H(\dot{p}(t))$ identifies with the plastic dissipation.

In this self-contained note, we propose to show that rigid plasticity – that is the model where one formally sets $\mathbb{C} = \infty$ (and correspondingly $\dot{p}(t) = E\dot{u}(t)$, $\operatorname{div} \dot{u}(t) = 0$) in the system above – can be derived as an asymptotic limit of small strain elasto-plasticity as \mathbb{C} actually gets larger and larger. Rigid-plastic models are particularly useful in order to compute analytical solutions in a plane-strain setting. Indeed, inside the plastic zone, the stress equations can be formally written

Date: October 13, 2015.

Key words and phrases. Plasticity, Functions of bounded deformation, Calculus of variations.

as a non-linear hyperbolic system which is solved by the method of characteristics. The family of characteristics are the so-called *slip lines* along which some combinations of the stress remain constants, while the tangential velocities can jump. It thus seems appropriate to rigorously derive rigid-plasticity in order to investigate the hyperbolic structure of the equations. However, this later task falls outside the scope of the present work.

Notationwise, we denote by $\mathbb{M}_{sym}^{n \times n}$ the set of symmetric $n \times n$ matrices. If A and $B \in \mathbb{M}_{sym}^{n \times n}$, we use the Euclidean scalar product $A : B := \text{tr}(AB)$ and the associated Euclidean norm $|A| := \sqrt{A : A}$. The subset $\mathbb{M}_D^{n \times n}$ of $\mathbb{M}_{sym}^{n \times n}$ stands for trace free symmetric matrices. If $A \in \mathbb{M}_{sym}^{n \times n}$, it can be orthogonally decomposed as

$$A = A_D + \frac{\text{tr } A}{n} I,$$

where $A_D \in \mathbb{M}_D^{n \times n}$, and I is the identity matrix in \mathbb{R}^n . The notation \odot stands for the symmetrized tensor product between vectors in \mathbb{R}^n , *i.e.*, if a and $b \in \mathbb{R}^n$, $(a \odot b)_{ij} = (a_i b_j + a_j b_i)/2$ for all $1 \leq i, j \leq n$. Note in particular that $\frac{1}{\sqrt{2}}|a||b| \leq |a \odot b| \leq |a||b|$.

The Lebesgue measure in \mathbb{R}^n and the $(n-1)$ -dimensional Hausdorff measure are denoted by \mathcal{L}^n and \mathcal{H}^{n-1} , respectively. Given a locally compact set $E \subset \mathbb{R}^n$ and a Euclidean space X , we denote by $\mathcal{M}(E; X)$ (or simply $\mathcal{M}(E)$ if $X = \mathbb{R}$) the space of bounded Radon measures on E with values in X , endowed with the norm $\|\mu\|_{\mathcal{M}(E; X)} := |\mu|(E)$, where $|\mu| \in \mathcal{M}(E)$ is the variation of the measure μ . Moreover, if ν is a non-negative Radon measure over E , we denote by $d\mu/d\nu$ the Radon-Nikodym derivative of μ with respect to ν .

We use standard notation for Lebesgue and Sobolev spaces. In particular, for $1 \leq p \leq \infty$, the L^p -norms of the various quantities are denoted by $\|\cdot\|_p$. If $U \subset \mathbb{R}^n$ is an open set, the space $BD(U)$ of functions of bounded deformation in U is made of all functions $u \in L^1(U; \mathbb{R}^n)$ such that $Eu \in \mathcal{M}(U; \mathbb{M}_{sym}^{n \times n})$, where $Eu := (Du + Du^T)/2$ and Du is the distributional derivative of u . We refer to [14] for general properties of this space. Finally, $H(\text{div}, U)$ stands for the Hilbert space of all $\tau \in L^2(U; \mathbb{M}_{sym}^{n \times n})$ such that $\text{div } \tau \in L^2(U; \mathbb{R}^n)$.

2. THE ELASTO-PLASTIC MODEL

We now consider a homogeneous elasto-plastic material with Hooke's law given by a fourth order tensor \mathbb{C} satisfying the usual symmetry properties

$$\mathbb{C}_{ijkl} = \mathbb{C}_{jikl} = \mathbb{C}_{klij}, \quad \text{for all } 1 \leq i, j, k, l \leq n, \quad (2.1)$$

and the growth and coercivity assumptions

$$\alpha|\xi|^2 \leq \mathbb{C}\xi : \xi \leq \beta|\xi|^2, \quad \text{for all } \xi \in \mathbb{M}_{sym}^{n \times n}, \quad (2.2)$$

where α and $\beta > 0$.

It occupies the domain Ω , a bounded and connected open subset of \mathbb{R}^n with at least Lipschitz boundary (see Definition 2.1) and outer normal ν . Its boundary $\partial\Omega$ is split into the union of a Dirichlet part Γ_D which is non empty and open in the relative topology of $\partial\Omega$, a Neumann part $\Gamma_N := \partial\Omega \setminus \overline{\Gamma_D}$, and their common relative boundary denoted by $\partial_{|\partial\Omega} \Gamma_D$.

Standard plasticity is characterized by the fact that the deviatoric stress is constrained to stay in a fixed compact and convex subset $K \subset \mathbb{M}_D^{n \times n}$ of deviatoric matrices. We further assume that

$$B(0, c_*) \subset K \subset B(0, c^*), \quad (2.3)$$

where $0 < c_* < c^* < \infty$, and denote by

$$\mathcal{K} := \{\sigma \in L^2(\Omega; \mathbb{M}_{sym}^{n \times n}) : \sigma_D(x) \in K \text{ for a.e. } x \in \Omega\}.$$

The support function of K , defined for any $p \in \mathbb{M}_D^{n \times n}$ by $H(p) := \sup_{\tau \in K} \tau : p$, satisfies, according to (2.3),

$$c_*|p| \leq H(p) \leq c^*|p|, \quad \text{for all } p \in \mathbb{M}_{sym}^{n \times n}.$$

On the Dirichlet part Γ_D of the boundary, the body is subjected to a hard device, *i.e.*, a boundary displacement which is the trace on Γ_D of a function $w \in AC([0, T]; H^1(\Omega; \mathbb{R}^n))$. In addition, the body is subjected to two types of forces: bulk forces $f \in AC([0, T]; L^n(\Omega; \mathbb{R}^n))$, and surface forces $g \in AC([0, T]; L^\infty(\Gamma_N; \mathbb{R}^n))$, the latter acting on the Neumann part Γ_N of the boundary. It is classical to assume a uniform safe load condition (see [12]) which ensures the existence of a plastically, as well as statically admissible state of stress π associated with the pair (f, g) . Specifically, there exists $\pi \in AC([0, T]; L^2(\Omega; \mathbb{M}_{sym}^{n \times n}))$ and some safety parameter $c > 0$ such that

$$\begin{cases} \pi_D(t, x) + B(0, c) \subset K \text{ for a.e. } x \in \Omega \text{ and all } t \in [0, T] \\ \operatorname{div} \pi(t) + f(t) = 0 \text{ in } \Omega, \quad \pi(t)\nu = g(t) \text{ on } \Gamma_N. \end{cases}$$

Given a boundary datum $\hat{w} \in H^1(\Omega; \mathbb{R}^n)$, we define the space of all kinematically admissible triples as

$$\begin{aligned} \mathcal{A}(\hat{w}) := \{ & (u, e, p) \in BD(\Omega) \times L^2(\Omega; \mathbb{M}_{sym}^{n \times n}) \times \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n}) : \\ & Eu = e + p \text{ in } \Omega, \quad p = (\hat{w} - u) \odot \nu \text{ on } \Gamma_D \}, \end{aligned}$$

where we still denote by u the trace of u on $\partial\Omega$ (see [2]). We also define the space of all statically admissible stresses as

$$\Sigma := \{ \sigma \in L^2(\Omega; \mathbb{M}_{sym}^{n \times n}) : \operatorname{div} \sigma \in L^n(\Omega; \mathbb{R}^n), \quad \sigma\nu \in L^\infty(\Gamma_N; \mathbb{R}^n), \quad \sigma_D \in L^\infty(\Omega; \mathbb{M}_D^{n \times n}) \},$$

where $\sigma\nu$ is the normal trace of $\sigma \in H(\operatorname{div}, \Omega)$ which is well defined as an element of $H^{-1/2}(\Gamma_N; \mathbb{R}^n)$, the dual space of $H_{00}^{1/2}(\Gamma_N; \mathbb{R}^n)$.

Following [7, Section 6], we introduce the following class of domains for which a meaningful duality pairing between stresses and strains can be defined. Note that the class contains in particular C^2 -domains [10], as well as hypercubes where Γ_D is one of its faces [7, Section 6].

Definition 2.1. We say that Ω is admissible if for any $\sigma \in \Sigma$, and any $p \in \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})$, with $(u, e, p) \in \mathcal{A}(\hat{w})$ for some $\hat{w} \in H^1(\Omega; \mathbb{R}^n)$, $u \in BD(\Omega)$ and $e \in L^2(\Omega; \mathbb{M}_{sym}^{n \times n})$, the distribution defined for all $\varphi \in C_c^\infty(\mathbb{R}^n)$ by

$$\begin{aligned} \langle [\sigma_D : p], \varphi \rangle := & \int_{\Omega} \varphi \sigma : (E\hat{w} - e) dx - \int_{\Omega} \varphi \operatorname{div} \sigma \cdot (u - \hat{w}) dx \\ & - \int_{\Omega} \sigma : [(u - \hat{w}) \odot \nabla \varphi] dx + \int_{\Gamma_N} \varphi \sigma \nu \cdot (u - \hat{w}) d\mathcal{H}^{n-1} \end{aligned}$$

extends to a bounded Radon measure in \mathbb{R}^n with $\|[\sigma_D : p]\| \leq \|\sigma_D\|_\infty |p|$. In this case, its mass is given by

$$\langle \sigma_D, p \rangle := \langle [\sigma_D : p], 1 \rangle = \int_{\Omega} \sigma : (E\hat{w} - e) dx - \int_{\Omega} \operatorname{div} \sigma \cdot (u - \hat{w}) dx + \int_{\Gamma_N} \sigma \nu \cdot (u - \hat{w}) d\mathcal{H}^{n-1}. \quad (2.4)$$

For any $e \in L^2(\Omega; \mathbb{M}_{sym}^{n \times n})$, the elastic energy is

$$\mathcal{Q}(e) = \frac{1}{2} \int_{\Omega} \mathbb{C}e : e dx,$$

while, for any $p \in \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})$, the dissipation energy is the convex functional of measure (see [9, 6])

$$\mathcal{H}(p) := \int_{\Omega \cup \Gamma_D} H \left(\frac{dp}{d|p|} \right) d|p|.$$

If $p : [0, T] \rightarrow \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})$, we define the total dissipation between times a and b by

$$\mathcal{V}_{\mathcal{H}}(p; [a, b]) := \sup \left\{ \sum_{i=1}^N \mathcal{H}(p(t_i) - p^\varepsilon(t_{i-1})) : N \in \mathbb{N}, a = t_0 < t_1 < \dots < t_N = b \right\}.$$

If additionally $p \in AC([0, T]; \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n}))$, then [4, Theorem 7.1] shows that

$$\mathcal{V}_{\mathcal{H}}(p; [a, b]) = \int_a^b \mathcal{H}(\dot{p}(s)) ds.$$

We finally impose the following initial condition on the evolution: $(u_0, e_0, p_0) \in \mathcal{A}(w(0))$ with $\sigma_0 := \mathbb{C}e_0$ such that

$$\operatorname{div} \sigma_0 + f(0) = 0 \text{ in } \Omega, \quad \sigma_0 \nu = g(0) \text{ on } \Gamma_N, \quad (\sigma_0)_D \in \mathcal{K}.$$

The following existence result has been established in [4, 7].

Theorem 2.2. *Under the previous assumptions, there exist a quasi-static evolution, i.e. a mapping $t \mapsto (u(t), e(t), p(t))$ with the following properties*

$$u \in AC([0, T]; BD(\Omega)), \quad \sigma, e \in AC([0, T]; L^2(\Omega; \mathbb{M}_{sym}^{n \times n})), \quad p \in AC([0, T]; \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})),$$

$$(u(0), e(0), p(0)) = (u_0, e_0, p_0),$$

and for all $t \in [0, T]$,

$$\begin{cases} Eu(t) = e(t) + p(t) \text{ in } \Omega, \\ p(t) = (w(t) - u(t)) \odot \nu \text{ on } \Gamma_D, \\ \sigma(t) = \mathbb{C}e(t) \text{ in } \Omega, \end{cases}$$

$$\begin{cases} \operatorname{div} \sigma(t) + f(t) = 0 \text{ in } \Omega, \\ \sigma(t) \nu = g(t) \text{ on } \Gamma_N, \\ \sigma_D(t) \in \mathcal{K}, \end{cases}$$

and for a.e. $t \in [0, T]$,

$$H(\dot{p}(t)) = [\sigma_D(t) : \dot{p}(t)] \text{ in } \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n}). \quad (2.5)$$

Remark 2.3. Equation (2.5) is a measure-theoretic formulation of the usual flow rule of perfect plasticity. Using the definition (2.4) of duality, it can be equivalently written as an energy balance

$$\begin{aligned} \mathcal{Q}(e(t)) + \int_0^t \mathcal{H}(\dot{p}(s)) ds &= \mathcal{Q}(e_0) + \int_0^t \int_{\Omega} \sigma(s) : E\dot{w}(s) dx ds \\ &\quad + \int_0^t \int_{\Omega} f(s) \cdot (\dot{u}(s) - \dot{w}(s)) dx ds + \int_0^t \int_{\Gamma_N} g(s) \cdot (\dot{u}(s) - \dot{w}(s)) d\mathcal{H}^{n-1} ds, \end{aligned}$$

or equivalently, according to the safe-load condition,

$$\begin{aligned} \mathcal{Q}(e(t)) + \int_0^t \mathcal{H}(\dot{p}(s)) ds &- \int_0^t \langle \pi_D(s), \dot{p}(s) \rangle ds + \int_{\Omega} \pi(t) : (Ew(t) - e(t)) dx \\ &= \mathcal{Q}(e_0) + \int_{\Omega} \pi(0) : (Ew(0) - e_0) dx + \int_0^t \int_{\Omega} \sigma(s) : E\dot{w}(s) dx ds \\ &\quad + \int_0^t \int_{\Omega} \dot{\pi}(s) : (Ew(s) - e(s)) dx ds. \end{aligned} \quad (2.6)$$

3. THE RIGID-PLASTIC MODEL

In order to derive the rigid-plastic model from elasto-plasticity, we assume that

$$\mathbb{C}^\varepsilon = \varepsilon^{-1}\mathbb{C}, \quad \text{where } \mathbb{C} \text{ satisfies (2.1) and (2.2),} \quad (3.1)$$

and $\varepsilon \rightarrow 0^+$. In addition, we suppose that the boundary data are compatible with rigid plasticity, that is

$$\operatorname{div} w(t) = 0 \text{ in } \Omega, \quad (3.2)$$

and, for simplicity, that the initial data satisfy

$$e_0 = \sigma_0 = 0. \quad (3.3)$$

Theorem 3.1. *Let u^ε , e^ε , p^ε and σ^ε be the solutions given by Theorem 2.2. There exist a subsequence (not relabeled), and functions $u \in AC([0, T]; BD(\Omega))$ and $\sigma \in L^2(0, T; L^2(\Omega; \mathbb{M}_{sym}^{n \times n}))$ such that*

$$\begin{aligned} u^\varepsilon(t) &\rightharpoonup u(t) \text{ weakly* in } BD(\Omega), \text{ for all } t \in [0, T], \\ \sigma^\varepsilon &\rightharpoonup \sigma \text{ weakly in } L^2(0, T; L^2(\Omega; \mathbb{M}_{sym}^{n \times n})). \end{aligned}$$

Denoting by $v := \dot{u} \in L_{w*}^\infty(0, T; BD(\Omega))$, then for a.e. $t \in [0, T]$, we have

$$\begin{cases} -\operatorname{div} \sigma(t) = f(t) \text{ in } \Omega, \\ \sigma(t)\nu = g(t) \text{ on } \Gamma_N, \\ \sigma(t) \in \mathcal{K}, \end{cases} \quad \begin{cases} \operatorname{div} v(t) = 0 \text{ in } \Omega, \\ (\dot{w}(t) - v(t)) \cdot \nu = 0 \text{ on } \Gamma_D, \\ H(Ev(t)) = [\sigma_D(t) : Ev(t)] \text{ in } \Omega \cup \Gamma_D. \end{cases} \quad (3.4)$$

The remaining of this paper is devoted to the proof of Theorem 3.1.

Remark 3.2. Although $Eu(t)$ is a measure *a priori* defined in Ω , we tacitly extend it by $(w(t) - u(t)) \odot \nu$ on Γ_D so that $Eu(t) \in \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})$.

Remark 3.3. In contrast with the framework of classical elasto-plasticity, that of rigid plasticity only involves the velocity field, and not the displacement field itself. As expressed above, time is merely a parameter, although the associated measurability properties of the various fields are obtained through the limit process $\varepsilon \searrow 0$ and would be difficult to obtain directly from the limit formulation.

3.1. A priori estimates. In this section all constants are independent of ε . We start with an estimate of the stress. Since $\sigma_D^\varepsilon(t) \in K$ in Ω , and K is bounded by (2.3), we first deduce that

$$\sup_{t \in [0, T]} \|\sigma_D^\varepsilon(t)\|_\infty \leq C. \quad (3.5)$$

The following result allows us to bound the hydrostatic stress.

Lemma 3.4. *There exists a bounded sequence $(c^\varepsilon)_{\varepsilon > 0}$ in $L^2(0, T)$ such that for each $\varepsilon > 0$,*

$$\int_0^T \left\| \frac{\operatorname{tr} \sigma^\varepsilon(t)}{n} + c^\varepsilon(t) \right\|_2^2 dt \leq C.$$

Proof. Since the mapping $t \mapsto \sigma^\varepsilon(t)$ belongs to $L^2(0, T; H(\operatorname{div}, \Omega))$, there is a sequence $(\sigma_k^\varepsilon)_{k \in \mathbb{N}}$ of $H(\operatorname{div}, \Omega)$ -valued simple functions such that $\sigma_k^\varepsilon \rightarrow \sigma^\varepsilon$ strongly in $L^2(0, T; H(\operatorname{div}, \Omega))$ as $k \rightarrow +\infty$. For all $k \in \mathbb{N}$ and all $t \in [0, T]$, we have

$$\nabla \left(\frac{\operatorname{tr} \sigma_k^\varepsilon(t)}{n} \right) = \operatorname{div} \sigma_k^\varepsilon(t) - \operatorname{div}(\sigma_k^\varepsilon)_D(t) \text{ in } \Omega$$

which leads to

$$\int_0^T \left\| \nabla \left(\frac{\operatorname{tr} \sigma_k^\varepsilon(t)}{n} \right) \right\|_{H^{-1}(\Omega; \mathbb{R}^n)}^2 dt \leq \int_0^T \|\operatorname{div} \sigma_k^\varepsilon(t)\|_{H^{-1}(\Omega; \mathbb{R}^n)}^2 dt + \int_0^T \|(\sigma_k^\varepsilon)_D(t)\|_2^2 dt.$$

Since $\operatorname{div} \sigma_k^\varepsilon \rightarrow \operatorname{div} \sigma^\varepsilon$ in $L^2(0, T; L^2(\Omega; \mathbb{R}^n))$ and $-\operatorname{div} \sigma^\varepsilon = f \in L^2(0, T; L^2(\Omega; \mathbb{R}^n))$, we deduce that the first integral in the right-hand-side of the previous inequality is uniformly bounded with respect to ε and k . The second integral is bounded as well since $(\sigma_k^\varepsilon)_D \rightarrow \sigma_D^\varepsilon$ in $L^2(0, T; L^2(\Omega; \mathbb{M}_D^{n \times n}))$, and $(\sigma_D^\varepsilon)_{\varepsilon > 0}$ is uniformly bounded in that space in view of (3.5). Consequently, there exists a constant $C > 0$ (independent of k and ε) such that

$$\int_0^T \left\| \nabla \left(\frac{\operatorname{tr} \sigma_k^\varepsilon(t)}{n} \right) \right\|_{H^{-1}(\Omega; \mathbb{R}^n)}^2 dt \leq C.$$

Next, according to [8, Corollary 2.1] (see also [13, Lemma 9] in the case of smooth boundaries), for each $\varepsilon > 0$, $k \in \mathbb{N}$ and $t \in [0, T]$, there exists some $c_k^\varepsilon(t) \in \mathbb{R}$ such that

$$\left\| \frac{\operatorname{tr} \sigma_k^\varepsilon(t)}{n} + c_k^\varepsilon(t) \right\|_2 \leq C_\Omega \left\| \nabla \left(\frac{\operatorname{tr} \sigma_k^\varepsilon(t)}{n} \right) \right\|_{H^{-1}(\Omega; \mathbb{R}^n)},$$

for some constant $C_\Omega > 0$ only depending on Ω . Note that, since the mapping $t \mapsto \operatorname{tr} \sigma_k^\varepsilon(t)$ is a simple $L^2(\Omega)$ -valued function, $t \mapsto c_k^\varepsilon(t)$ is a simple real-valued measurable function as well. Additionally,

$$\int_0^T \left\| \frac{\operatorname{tr} \sigma_k^\varepsilon(t)}{n} + c_k^\varepsilon(t) \right\|_2^2 dt \leq C, \quad (3.6)$$

where $C > 0$ is again independent of k and ε . Setting $\hat{\sigma}_k^\varepsilon := \sigma_k^\varepsilon + c_k^\varepsilon I$ yields

$$\int_0^T \|\hat{\sigma}_k^\varepsilon(t)\|_{H(\operatorname{div}, \Omega)}^2 dt \leq C,$$

and thus,

$$\int_0^T \|\hat{\sigma}_k^\varepsilon(t)\nu\|_{H^{-1/2}(\Gamma_N; \mathbb{R}^n)}^2 dt \leq C.$$

Using that $\sigma_k^\varepsilon \nu \rightarrow \sigma^\varepsilon \nu = g$ in $L^2(0, T; H^{-1/2}(\Gamma_N; \mathbb{R}^n))$ and that $g \in L^2(0, T; L^2(\Gamma_N; \mathbb{R}^n))$, we obtain

$$\begin{aligned} & \int_0^T |c_k^\varepsilon(t)|^2 dt \|\nu\|_{H^{-1/2}(\Gamma_N; \mathbb{R}^n)}^2 \\ & \leq \int_0^T \|\hat{\sigma}_k^\varepsilon(t)\nu\|_{H^{-1/2}(\Gamma_N; \mathbb{R}^n)}^2 dt + \int_0^T \|\sigma_k^\varepsilon(t)\nu\|_{H^{-1/2}(\Gamma_N; \mathbb{R}^n)}^2 dt \leq C, \end{aligned} \quad (3.7)$$

for some constant $C > 0$, independent of k and ε . Therefore, the sequence $(c_k^\varepsilon)_{k \in \mathbb{N}}$ is bounded in $L^2(0, T)$ and a subsequence converges weakly in that space to some $c^\varepsilon \in L^2(0, T)$. Passing to the lower limit in (3.6) implies that

$$\int_0^T \left\| \frac{\operatorname{tr} \sigma^\varepsilon(t)}{n} + c^\varepsilon(t) \right\|_2^2 dt \leq C,$$

while (3.7) shows that $(c^\varepsilon)_{\varepsilon > 0}$ is bounded in $L^2(0, T)$. \square

As a consequence of the previous result and of (3.5), we deduce that

$$\int_0^T \|\sigma^\varepsilon(t)\|_2^2 dt \leq C. \quad (3.8)$$

Next, according to the energy balance (2.6), [4, Lemma 3.2], assumptions (3.2)–(3.3), and Cauchy-Schwarz inequality, we infer that

$$\begin{aligned} \frac{1}{2} \int_{\Omega} \mathbb{C}^\varepsilon e^\varepsilon(t) : e^\varepsilon(t) dx &\leq \int_{\Omega} \pi(t) : (e^\varepsilon(t) - Ew(t)) dx + \int_{\Omega} \pi(0) : Ew(0) dx \\ &\quad + \int_0^t \int_{\Omega} \sigma_D^\varepsilon(s) : E\dot{w}(s) dx ds + \int_0^t \int_{\Omega} \dot{\pi}(s) : (Ew(s) - e^\varepsilon(s)) dx ds \\ &\leq C \left(\sup_{t \in [0, T]} \|\pi(t)\|_2 + \int_0^T \|\dot{\pi}(s)\|_2 ds \right) \left(\sup_{t \in [0, T]} \|e^\varepsilon(t)\|_2 + \sup_{t \in [0, T]} \|Ew(t)\|_2 \right) \\ &\quad + \sup_{t \in [0, T]} \|\sigma_D^\varepsilon(t)\|_\infty \int_0^T \|E\dot{w}(s)\|_2 ds, \end{aligned}$$

which implies, according to the assumption (3.1) on \mathbb{C}^ε together with Young's inequality, that

$$\sup_{t \in [0, T]} \|e^\varepsilon(t)\|_2 \leq C\sqrt{\varepsilon}. \quad (3.9)$$

Using again the energy balance (2.6), Cauchy-Schwarz inequality and (3.9), we find that

$$\begin{aligned} \int_0^t \mathcal{H}(\dot{p}^\varepsilon(s)) ds - \int_0^t \langle \pi_D(s), \dot{p}^\varepsilon(s) \rangle ds &\leq \int_{\Omega} \pi(t) : (e^\varepsilon(t) - Ew(t)) dx + \int_{\Omega} \pi(0) : Ew(0) dx \\ &\quad + \int_0^t \int_{\Omega} \sigma_D^\varepsilon(s) : E\dot{w}(s) dx ds + \int_0^t \int_{\Omega} \dot{\pi}(s) : (Ew(s) - e^\varepsilon(s)) dx ds \leq C. \end{aligned}$$

Applying [4, Lemma 3.2] again yields

$$\int_0^T \|\dot{p}^\varepsilon(s)\|_{\mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})} ds \leq C, \quad (3.10)$$

and thus

$$\sup_{t \in [0, T]} \|p^\varepsilon(t)\|_{\mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})} \leq C. \quad (3.11)$$

For the displacement, Poincaré-Korn's inequality (see [14, Chap. 2, Rmk. 2.5(ii)]) yields

$$\begin{aligned} \|u^\varepsilon(t)\|_{BD(\Omega)} &\leq c \left(\int_{\Gamma_D} |u^\varepsilon(t)| d\mathcal{H}^{n-1} + \|Eu^\varepsilon(t)\|_{\mathcal{M}(\Omega; \mathbb{M}_{sym}^{n \times n})} \right) \\ &\leq c \left(\int_{\Gamma_D} |w(t)| d\mathcal{H}^{n-1} + \int_{\Gamma_D} |u^\varepsilon(t) - w(t)| d\mathcal{H}^{n-1} + \|Eu^\varepsilon(t)\|_{\mathcal{M}(\Omega; \mathbb{M}_{sym}^{n \times n})} \right) \\ &\leq c \left(\|w(t)\|_{L^1(\Gamma_D; \mathbb{R}^n)} + \|p^\varepsilon(t)\|_{\mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})} + \|e^\varepsilon(t)\|_2 \right) \leq C, \end{aligned} \quad (3.12)$$

where we have used (3.9) and (3.11) in the last inequality.

3.2. Convergences. According to the stress estimate (3.8), there exist a subsequence (not relabeled) and $\sigma \in L^2(0, T; L^2(\Omega; \mathbb{M}_{sym}^{n \times n}))$ such that

$$\sigma^\varepsilon \rightharpoonup \sigma \text{ weakly in } L^2(0, T; L^2(\Omega; \mathbb{M}_{sym}^{n \times n})). \quad (3.13)$$

Consequently, since for all $t \in [0, T]$, we have $-\operatorname{div} \sigma^\varepsilon(t) = f(t)$ in Ω and $\sigma^\varepsilon(t)\nu = g(t)$ on Γ_N , we infer that for a.e. $t \in [0, T]$,

$$-\operatorname{div} \sigma(t) = f(t) \text{ in } \Omega, \quad \sigma(t)\nu = g(t) \text{ on } \Gamma_N.$$

In addition, since $\sigma_D^\varepsilon(t) \in \mathcal{K}$ for all $t \in [0, T]$, then

$$\sigma_D(t) \in \mathcal{K} \text{ for a.e. } t \in [0, T].$$

We then apply Helly's selection principle (see [11, Theorem 3.2]) which ensures, thanks to (3.10), the existence of a further subsequence (independent of time and still not relabeled) such that

$$p^\varepsilon(t) \rightharpoonup p(t) \text{ weakly* in } \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n}), \text{ for all } t \in [0, T], \quad (3.14)$$

for some $p \in BV([0, T]; \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n}))$.

Next according to (3.9), we have that

$$e^\varepsilon \rightarrow 0 \text{ strongly in } L^\infty(0, T; L^2(\Omega; \mathbb{M}_{sym}^{n \times n})). \quad (3.15)$$

Finally, as a consequence of the displacement estimate (3.12), for each $t \in [0, T]$, there exists a further subsequence $(u^{\varepsilon_j}(t))_{j \in \mathbb{N}}$ (now possibly depending on t) such that $u^{\varepsilon_j}(t) \rightharpoonup u(t)$ weakly* in $BD(\Omega)$, for some $u(t) \in BD(\Omega)$. Note that by (3.14)–(3.15), for a.e. $t \in [0, T]$, one has $Eu(t) = p(t)$ in Ω and, by [4, Lemma 2.1], $p(t) = (w(t) - u(t)) \odot \nu$ on Γ_D which shows that $u(t)$ is uniquely determined, and thus that the full sequence

$$u^\varepsilon(t) \rightharpoonup u(t) \text{ weakly* in } BD(\Omega), \text{ for all } t \in [0, T]. \quad (3.16)$$

In particular, since $Eu(t) = p(t) \in \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})$, we also deduce that

$$\operatorname{div} u(t) = 0 \text{ in } \Omega, \quad (w(t) - u(t)) \cdot \nu = 0 \text{ on } \Gamma_D. \quad (3.17)$$

3.3. Flow rule. According to the energy balance (2.6) and the fact that the plastic strain $p^\varepsilon \in AC([0, T]; \mathcal{M}(\bar{\Omega}; \mathbb{M}_D^{n \times n}))$, we can integrate by parts in time, so that for all $t \in [0, T]$,

$$\begin{aligned} \mathcal{V}_{\mathcal{H}}(p^\varepsilon; [0, t]) + \int_{\Omega} \pi(t) : (Ew(t) - e^\varepsilon(t)) \, dx - \langle \pi_D(t), p^\varepsilon(t) \rangle \\ \leq \int_{\Omega} \pi(0) : Ew(0) \, dx - \langle \pi_D(0), p_0 \rangle + \int_0^t \int_{\Omega} \sigma_D^\varepsilon(s) : E\dot{w}(s) \, dx \, ds \\ + \int_0^t \int_{\Omega} \dot{\pi}(s) : (Ew(s) - e^\varepsilon(s)) \, dx \, ds - \int_0^t \langle \dot{\pi}_D(s), p^\varepsilon(s) \rangle \, ds. \end{aligned}$$

Since by (3.14)–(3.16) $p^\varepsilon(t) \rightharpoonup Eu(t)$ weakly* in $\mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})$ for a.e. $t \in [0, T]$, Reshetnyak lower semicontinuity theorem, (3.13), (3.15), (3.16) and the definition (2.4) of duality ensures that

$$\begin{aligned} \mathcal{V}_{\mathcal{H}}(Eu; [0, t]) + \int_{\Omega} \pi(t) : Ew(t) \, dx - \langle \pi_D(t), Eu(t) \rangle \\ \leq \int_{\Omega} \pi(0) : Ew(0) \, dx - \langle \pi_D(0), Eu_0 \rangle + \int_0^t \int_{\Omega} \sigma_D(s) : E\dot{w}(s) \, dx \, ds \\ + \int_0^t \int_{\Omega} \dot{\pi}(s) : Ew(s) \, dx \, ds - \int_0^t \langle \dot{\pi}_D(s), Eu(s) \rangle \, ds. \quad (3.18) \end{aligned}$$

We now show the converse inequality. Since $\sigma_D \in L^1(0, T; L^2(\Omega; \mathbb{M}_D^{n \times n}))$, while $u - w \in L^1(0, T; L^{\frac{n}{n-1}}(\Omega; \mathbb{R}^n))$, and $u - w \in L^1(0, T; L^1(\Gamma_N; \mathbb{R}^n))$, [5, Lemma 7.5] implies the existence of a subdivision $0 = t_0 < t_1 < \dots < t_k = t$ of the time interval $[0, t]$ such that

$$\sum_{i=1}^k \chi_{[t_{i-1}, t_i]}(\sigma_D(t_i), u(t_i) - w(t_i), u(t_i) - w(t_i)) \rightarrow (\sigma_D, u - w, u - w)$$

and

$$\sum_{i=1}^k \chi_{[t_{i-1}, t_i]}(\sigma_D(t_{i-1}), u(t_{i-1}) - w(t_{i-1}), u(t_{i-1}) - w(t_{i-1})) \rightarrow (\sigma_D, u - w, u - w)$$

strongly in $L^1(0, T; L^2(\Omega; \mathbb{M}_D^{n \times n})) \times L^1(0, T; L^{\frac{n}{n-1}}(\Omega; \mathbb{R}^n)) \times L^1(0, T; L^1(\Gamma_N; \mathbb{R}^n))$, as $\max_{1 \leq i \leq k} (t_i - t_{i-1}) \rightarrow 0$. According to Proposition 3.9 in [7] and to the fact that Ω is admissible, we infer that for each $1 \leq i \leq k$,

$$\begin{aligned} \mathcal{H}(Eu(t_i) - Eu(t_{i-1})) &\geq \langle \sigma_D(t_i), Eu(t_i) - Eu(t_{i-1}) \rangle \\ &= \int_{\Omega} \sigma_D(t_i) : (Ew(t_i) - Ew(t_{i-1})) dx + \int_{\Omega} f(t_i) \cdot (u(t_i) - u(t_{i-1}) - w(t_i) + w(t_{i-1})) dx \\ &\quad + \int_{\Gamma_N} g(t_i) \cdot (u(t_i) - u(t_{i-1}) - w(t_i) + w(t_{i-1})) d\mathcal{H}^{n-1}. \end{aligned}$$

Summing up for $i = 1, \dots, k$, and performing discrete integration by parts yields

$$\begin{aligned} \mathcal{V}_{\mathcal{H}}(Eu, [0, t]) &\geq \sum_{i=1}^k \int_{t_{i-1}}^{t_i} \int_{\Omega} \sigma_D(t_i) : E\dot{w}(s) dx ds \\ &\quad - \sum_{i=1}^{k-1} \int_{t_i}^{t_{i+1}} \int_{\Omega} \dot{f}(s) \cdot (u(t_i) - w(t_i)) dx ds - \sum_{i=1}^{k-1} \int_{t_i}^{t_{i+1}} \int_{\Gamma_N} \dot{g}(s) \cdot (u(t_i) - w(t_i)) d\mathcal{H}^{n-1} ds \\ &\quad + \int_{\Omega} f(t) \cdot (u(t) - w(t)) dx + \int_{\Gamma_N} g(t) \cdot (u(t) - w(t)) d\mathcal{H}^{n-1} \\ &\quad - \int_{\Omega} f(t_1) \cdot (u_0 - w(0)) dx - \int_{\Gamma_N} g(t_1) \cdot (u_0 - w(0)) d\mathcal{H}^{n-1}. \end{aligned}$$

Passing to the limit as $\max_{1 \leq i \leq k} (t_i - t_{i-1}) \rightarrow 0$, and invoking the dominated convergence theorem yields

$$\begin{aligned} \mathcal{V}_{\mathcal{H}}(Eu, [0, t]) &\geq \int_0^t \int_{\Omega} \sigma_D(s) : E\dot{w}(s) dx ds \\ &\quad - \int_0^t \int_{\Omega} \dot{f}(s) \cdot (u(s) - w(s)) dx ds - \int_0^t \int_{\Gamma_N} \dot{g}(s) \cdot (u(s) - w(s)) d\mathcal{H}^{n-1} ds \\ &\quad + \int_{\Omega} f(t) \cdot (u(t) - w(t)) dx + \int_{\Gamma_N} g(t) \cdot (u(t) - w(t)) d\mathcal{H}^{n-1} \\ &\quad - \int_{\Omega} f(0) \cdot (u_0 - w(0)) dx - \int_{\Gamma_N} g(0) \cdot (u_0 - w(0)) d\mathcal{H}^{n-1}, \end{aligned}$$

and using the definition (2.4) of duality

$$\begin{aligned} \mathcal{V}_{\mathcal{H}}(Eu; [0, t]) &+ \int_{\Omega} \pi(t) : Ew(t) dx - \langle \pi_D(t), Eu(t) \rangle \\ &\geq \int_{\Omega} \pi(0) : Ew(0) dx - \langle \pi_D(0), Eu_0 \rangle + \int_0^t \int_{\Omega} \sigma_D(s) : E\dot{w}(s) dx ds \\ &\quad + \int_0^t \int_{\Omega} \dot{\pi}(s) : Ew(s) dx ds - \int_0^t \langle \dot{\pi}_D(s), Eu(s) \rangle ds. \end{aligned}$$

Thus, combining with (3.18) leads to the equality in the previous inequality, or still, integrating by parts with respect to time

$$\begin{aligned} \mathcal{V}_{\mathcal{H}}(Eu; [0, t]) &= \langle \pi_D(t), Eu(t) \rangle - \langle \pi_D(0), Eu_0 \rangle \\ &\quad + \int_0^t \int_{\Omega} (\sigma_D(s) - \pi_D(s)) : E\dot{w}(s) dx ds - \int_0^t \langle \dot{\pi}_D(s), Eu(s) \rangle ds. \quad (3.19) \end{aligned}$$

According to [4, Lemma 3.2], for all $0 \leq t_1 \leq t_2 \leq T$,

$$\begin{aligned} c\|Eu(t_2) - Eu(t_1)\|_{\mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})} &\leq \mathcal{H}(Eu(t_2) - Eu(t_1)) - \langle \pi_D(t_2), Eu(t_2) - Eu(t_1) \rangle \\ &\leq \mathcal{V}_{\mathcal{H}}(Eu, [t_1, t_2]) - \langle \pi_D(t_2), Eu(t_2) - Eu(t_1) \rangle. \end{aligned}$$

In view of (3.19), we get that

$$\begin{aligned} c\|Eu(t_2) - Eu(t_1)\|_{\mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})} &\leq \langle \pi_D(t_2) - \pi_D(t_1), Eu(t_1) \rangle \\ &\quad + \int_{t_1}^{t_2} \int_{\Omega} (\sigma_D(s) - \pi_D(s)) : E\dot{w}(s) \, dx \, ds - \int_{t_1}^{t_2} \langle \dot{\pi}_D(s), Eu(s) \rangle \, ds. \end{aligned}$$

Since $Eu = p$ and $p \in BV([0, T]; \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n}))$, we get that $Eu \in L_{w^*}^\infty(0, T; \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n}))$, and thus

$$\begin{aligned} c\|Eu(t_2) - Eu(t_1)\|_{\mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})} &\leq \int_{t_1}^{t_2} \left\{ \|Eu(t_1)\|_{\mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})} \|\dot{\pi}_D(s)\|_\infty \right. \\ &\quad \left. + (\|\pi_D(s)\|_2 + \|\sigma_D(s)\|_2) \|E\dot{w}(s)\|_2 + \|\dot{\pi}_D(s)\|_\infty \|Eu(s)\|_{\mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})} \right\} \, ds. \end{aligned}$$

The integrand being sommable, it ensures that the strain $Eu \in AC([0, T]; \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n}))$, and by the Poincaré-Korn inequality that $u \in AC([0, T]; BD(\Omega))$. Thus, integrating by part with respect to time and space in the energy equality (3.19),

$$\begin{aligned} \int_0^t \mathcal{H}(E\dot{u}(s)) \, ds &= \mathcal{V}_{\mathcal{H}}(Eu, [0, t]) = \int_0^t \int_{\Omega} \sigma_D(s) : E\dot{w}(s) \, dx \, ds \\ &\quad + \int_0^t \int_{\Omega} f(s) \cdot (\dot{u}(s) - \dot{w}(s)) \, dx \, ds + \int_0^t \int_{\Gamma_N} g(s) \cdot (\dot{u}(s) - \dot{w}(s)) \, d\mathcal{H}^{n-1} \, ds, \end{aligned}$$

and deriving this equality with respect to time yields, thanks to (2.4), for a.e. $t \in [0, T]$,

$$\mathcal{H}(E\dot{u}(t)) = \langle \sigma_D(t), E\dot{u}(t) \rangle.$$

Since, by [7, Proposition 3.9], $H(E\dot{u}(t)) \geq [\sigma_D(t) : E\dot{u}(t)]$ in $\mathcal{M}(\Omega \cup \Gamma_D)$, we finally deduce that $H(E\dot{u}(t)) = [\sigma_D(t) : E\dot{u}(t)]$ in $\mathcal{M}(\Omega \cup \Gamma_D)$.

Denoting by $v = \dot{u}$ the velocity, we proved that $v \in L_{w^*}^\infty(0, T; BD(\Omega))$, and recalling (3.17), we have for a.e. $t \in [0, T]$,

$$\operatorname{div} v(t) = 0 \text{ in } \Omega, \quad (\dot{w}(t) - v(t)) \cdot \nu = 0 \text{ on } \Gamma_D,$$

and

$$H(Ev(t)) = [\sigma_D(t) : Ev(t)] \text{ in } \Omega \cup \Gamma_D.$$

4. UNIQUENESS AND REGULARITY ISSUES FOR THE STRESS WITH A VON MISES YIELD CRITERION

We now specialize to the case where $K := \{\tau_D \in \mathbb{M}_D^{n \times n} : |\tau_D| \leq 1\}$. In such a setting, it is known (see [3]) when elasto-plasticity is considered the stress field is unique and belongs to $H_{\text{loc}}^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$. These properties fail in the case of rigid-plasticity as demonstrated below.

Example 4.1. Let us consider a two-dimensional body occupying the square $\Omega = (0, 1)^2$ in its reference configuration (the generalization to the n -dimensional case is obvious). We also assume that the boundary conditions are of pure Dirichlet type with a rigid body motion $\dot{w}(x) = Ax + b$ (where $A \in \mathbb{M}^{n \times n}$ is such that $A^T = -A^T$, and $b \in \mathbb{R}^n$) as boundary datum.

Then, defining $v(x) = Ax + b$ for all $x \in \Omega$ ensures that $Ev = 0$ in Ω . In particular, all equations on v are satisfied. Now define the stress as

$$\sigma(x) = \begin{pmatrix} f(x_2) & c \\ c & g(x_1) \end{pmatrix}$$

where $c \in \mathbb{R}$, $f, g \in L^\infty(0, 1)$ so that $\operatorname{div} \sigma = 0$ in Ω . In particular

$$\sigma_D(x) = \begin{pmatrix} \frac{f(x_2) - g(x_1)}{2} & c \\ c & \frac{g(x_1) - f(x_2)}{2} \end{pmatrix}$$

and $|\sigma_D(x)|^2 \leq 2c^2 + |f(x_2)|^2 + |g(x_1)|^2$ for a.e. $x \in \Omega$. Assuming that $\sqrt{2c^2 + \|f\|_\infty^2 + \|g\|_\infty^2} < 1/2$, we deduce that the one parameter family $\sigma^\lambda := \lambda\sigma$ still satisfies $\operatorname{div} \sigma^\lambda = 0$ and $|\sigma_D^\lambda| < 1$ in Ω provided that $|\lambda| \leq 2$.

In general, a certain amount of uniqueness holds true as shown below. It uses a notion of precise representative for the stress field first introduced in [1] (see also [4]).

Proposition 4.2. *Let $(\sigma^1, v^1), (\sigma^2, v^2) \in L^2(\Omega; \mathbb{M}_{sym}^{n \times n}) \times BD(\Omega)$ be two solutions of the rigid-plastic model (3.4) at a given time $t = t_0$. Then,*

- *There exist two $|Ev^1|$ -measurable functions $\hat{\sigma}_D^1$ and $\hat{\sigma}_D^2 \in L^\infty_{|Ev^1|}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})$ such that $\hat{\sigma}_D^1 = \sigma^1$ and $\hat{\sigma}_D^2 = \sigma_D^2$ \mathcal{L}^n -a.e. in $\Omega \cup \Gamma_D$, and*

$$\hat{\sigma}_D^1 = \hat{\sigma}_D^2 \quad |Ev^1| \text{-a.e. in } \Omega \cup \Gamma_D;$$

- *There exist two $|Ev^2|$ -measurable functions $\tilde{\sigma}_D^1$ and $\tilde{\sigma}_D^2 \in L^\infty_{|Ev^2|}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})$ such that $\tilde{\sigma}_D^1 = \sigma^1$ and $\tilde{\sigma}_D^2 = \sigma_D^2$ \mathcal{L}^n -a.e. in $\Omega \cup \Gamma_D$, and*

$$\tilde{\sigma}_D^1 = \tilde{\sigma}_D^2 \quad |Ev^2| \text{-a.e. in } \Omega \cup \Gamma_D.$$

Proof. Since $(\sigma^1, v^1), (\sigma^2, v^2)$ are two solutions of the rigid-plastic model (3.4), the following inequalities in $\mathcal{M}(\Omega \cup \Gamma_D)$ hold true

$$[\sigma_D^1 : Ev^1] = |Ev^1| \geq [\sigma_D^2 : Ev^1], \quad [\sigma_D^2 : Ev^2] = |Ev^2| \geq [\sigma_D^1 : Ev^2].$$

As a consequence,

$$[(\sigma_D^1 - \sigma_D^2) : Ev^1] \geq 0, \quad [(\sigma_D^2 - \sigma_D^1) : Ev^2] \geq 0,$$

and thus,

$$[(\sigma_D^1 - \sigma_D^2) : (Ev^1 - Ev^2)] \geq 0.$$

In addition, by definition (2.4) of the duality pairing, the total mass of the measure on the left-hand side of the previous inequality is given by

$$\langle \sigma_D^1 - \sigma_D^2, Ev^1 - Ev^2 \rangle = 0.$$

It thus follows that

$$[(\sigma_D^1 - \sigma_D^2) : Ev^1] = 0, \quad [(\sigma_D^2 - \sigma_D^1) : Ev^2] = 0,$$

or still that

$$[\sigma_D^1 : Ev^1] = |Ev^1| = [\sigma_D^2 : Ev^1], \quad [\sigma_D^2 : Ev^2] = |Ev^2| = [\sigma_D^1 : Ev^2]. \quad (4.1)$$

Arguing as in [4], since \mathcal{L}^n and $E^s v^1$ are mutually singular Borel measures, it is possible to find two disjoint Borel sets A and $B \subset \Omega \cup \Gamma_D$ such that $A \cup B = \Omega \cup \Gamma_D$, and $\mathcal{L}^n(B) = |E^s v^1|(A) = 0$. Then, defining (for $i = 1, 2$)

$$\hat{\sigma}_D^i := \begin{cases} \sigma_D^i & \mathcal{L}^n \text{-a.e. in } A, \\ \frac{dEv^1}{d|Ev^1|} & |E^s v^1| \text{-a.e. in } B, \end{cases}$$

it follows that $\hat{\sigma}_D^1$ and $\hat{\sigma}_D^2 \in L^\infty_{|Ev^1|}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})$, and

$$\hat{\sigma}_D^1 : \frac{dEv^1}{d|Ev^1|} |Ev^1| = [\sigma_D^1 : Ev^1] = |Ev^1| = [\sigma_D^2 : Ev^1] = \hat{\sigma}_D^2 : \frac{dEv^1}{d|Ev^1|} |Ev^1|.$$

By definition, we have that $\hat{\sigma}_D^1 = \hat{\sigma}_D^2 |E^s v^1|$ -a.e. in $\Omega \cup \Gamma_D$. In addition, taking the absolutely continuous part in (4.1) yields (see [4, 7]),

$$\sigma_D^1 : E^a v^1 = [\sigma_D^1 : Ev^1]^a = |E^a v^1| = [\sigma_D^2 : Ev^1]^a = \sigma_D^2 : E^a v^1.$$

Thus $\sigma_D^1 = \sigma_D^2 \mathcal{L}^n$ -a.e. in $\{|E^a v^1| > 0\}$ and finally $\hat{\sigma}_D^1 = \hat{\sigma}_D^2 |Ev^1|$ -a.e. in $\Omega \cup \Gamma_D$ as requested. \square

ACKNOWLEDGEMENTS

The authors wish to thank the hospitality of the Courant Institute of Mathematical Sciences at New York University where a large part of this work has been carried out.

REFERENCES

- [1] G. ANZELLOTTI: On the extremal stress and displacement in Hencky plasticity, *Duke Math. J.* **51**(1) (1984) 133–147.
- [2] J.-F. BABADJIAN: Traces of functions of bounded deformation, *Indiana Univ. Math. J.* **64** (2015) 1271–1290.
- [3] A. BENSOUSSAN, J. FREHSE: Asymptotic behaviour of the time dependent Norton-Hoff law in plasticity theory and H^1 regularity, *Comment. Math. Univ. Carolin.* **37** (1996) 285–304.
- [4] G. DAL MASO, A. DE SIMONE, M. G. MORA: Quasistatic evolution problems for linearly elastic–perfectly plastic materials, *Arch. Rational Mech. Anal.* **180** (2006) 237–291.
- [5] G. DAL MASO, A. DE SIMONE, F. SOLOMBRINO: Quasistatic evolution for Cam-Clay plasticity: a weak formulation via viscoplastic regularization and time rescaling, *Calc. Var. Partial Diff. Eq.* **40** (2011) 125–181.
- [6] F. DEMENGEL, R. TEMAM: Convex function of a measure, *Indiana Univ. Math. J.* **33** (1984) 673–709.
- [7] G.A. FRANCFORT, A. GIACOMINI: Small strain heterogeneous elasto-plasticity revisited, *Comm. Pure Appl. Math.* **65** (2012) 1185–1241.
- [8] V. GIRAULT, P.-A. RAVIART: *Finite element method for Navier-Stokes equations. Theory and algorithm*, Springer-Verlag (1986).
- [9] C. GOFFMAN, J. SERRIN: Sublinear functions of measures and variational integrals, *Duke Math. J.* **31** (1964) 159–178.
- [10] R.V. KOHN, R. TEMAM: Dual spaces of stresses and strains, with applications to Hencky plasticity, *Appl. Math. Optim.* **10** (1983) 1–35.
- [11] A. MAINIK, A. MIELKE: Existence results for energetic models for rate-independent systems, *Calc. Var. PDEs* **22** (2005) 73–99.
- [12] P. SUQUET: Sur les équations de la plasticité: existence et régularité des solutions, *J. Mécanique* **20** (1981) 3–39.
- [13] L. TARTAR: *Topics in nonlinear analysis*, publications mathématiques d’Orsay (1982).
- [14] R. TEMAM: *Mathematical problems in plasticity*, Gauthier-Villars, Paris (1985). Translation of *Problèmes mathématiques en plasticité*, Gauthier-Villars, Paris (1983).

(J.-F. Babadjian) SORBONNE UNIVERSITÉS, UPMC UNIV PARIS 06, CNRS, UMR 7598, LABORATOIRE JACQUES-LOUIS LIONS, F-75005, PARIS, FRANCE

E-mail address: jean-francois.babadjian@upmc.fr

(G.A. Francfort) UNIVERSITÉ PARIS-NORD, LAGA, AVENUE J.-B. CLÉMENT, 93430 - VILLETANEUSE, FRANCE & INSTITUT UNIVERSITAIRE DE FRANCE

E-mail address: gilles.francfort@univ-paris13.fr