



**HAL**  
open science

# Velocity of the $L$ -branching Brownian motion

Michel Pain

► **To cite this version:**

| Michel Pain. Velocity of the  $L$ -branching Brownian motion. 2015. hal-01214605v1

**HAL Id: hal-01214605**

**<https://hal.science/hal-01214605v1>**

Preprint submitted on 12 Oct 2015 (v1), last revised 13 Nov 2023 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Velocity of the $L$ -branching Brownian motion

Michel PAIN\*

October 12, 2015

## Abstract

We consider a branching-selection system of particles on the real line that evolves according to the following rules: each particle moves according to a Brownian motion during an exponential lifetime and then splits into two new particles and, when a particle is at a distance  $L$  of the highest particle, it dies without splitting. This model has been introduced by Brunet, Derrida, Mueller and Munier [12] in the physics literature and is called the  $L$ -branching Brownian motion. We show that the position of the system grows linearly at a velocity  $v_L$  almost surely and we compute the asymptotic behavior of  $v_L$  as  $L$  tends to infinity:

$$v_L = \sqrt{2} - \frac{\pi^2}{2\sqrt{2}L^2} + o\left(\frac{1}{L^2}\right),$$

as conjectured in [12]. The proof makes use of results by Berestycki, Berestycki and Schweinsberg [5] concerning branching Brownian motion in a strip.

**MSC 2010:** 60J80, 60K35, 60J70.

**Keywords:** Branching Brownian motion, selection, F-KPP equation.

## 1 Introduction

The branching Brownian motion (or BBM) is a branching Markov process whose study dates back to [22]. It has been the subject of a large literature, especially for its connection with the F-KPP equation, highlighted by McKean [28]. It is defined as follows. Initially, there is a single particle at the origin. Each particle moves according to a Brownian motion, during an exponentially distributed time and then splits into two new particles, which start the same process from their place of birth. Every particle behaves independently of the others and the system goes on indefinitely.

We study here a branching Brownian motion with selection. A particle's position corresponds to its survival capacity and reproductive success (biologists call it fitness): it changes during the particle's life because of mutations and is then transmitted to the particle's children. The selection tends to eliminate the lowest particles, that have a too small fitness value by comparison with the best ones. Thus, we consider a system of particles evolving as before, but where in addition a particle dies as soon as it is at a distance  $L$  of the highest particle alive at the same time. This system is called the  $L$ -branching Brownian motion or  $L$ -BBM.

---

\*LPMA, UPMC and DMA, ENS.

## 1.1 Statement of the results

First of all, we need to define the branching Brownian motion for a more general initial condition. Let  $\mathcal{C} := \bigcup_{n \in \mathbb{N}^*} \mathbb{R}^n$  be the set of configurations. For each  $\xi \in \mathcal{C}$ ,  $\xi = (\xi_1, \dots, \xi_n)$ , we define the branching Brownian motion starting from this configuration as before, but with  $n$  particles at time 0 positioned at  $\xi_1, \dots, \xi_n$ . We denote by  $M(t)$  the number of particles in the BBM at time  $t$  and by  $X_1(t), \dots, X_{M(t)}(t)$  their positions. We will say that  $(X_k(t), 1 \leq k \leq M(t))_{t \geq 0}$  is a branching Brownian motion although this notation does not contain the genealogy of the process.

We work on a measure space  $(\Omega, \mathcal{F}, (\mathbb{P}_\xi)_{\xi \in \mathcal{C}})$  such that for each  $\xi \in \mathcal{C}$ , under  $\mathbb{P}_\xi$ ,  $(X_k(t), 1 \leq k \leq M(t))_{t \geq 0}$  is a branching Brownian motion starting from the configuration  $\xi$ . We equip this space with the canonical filtration associated to the branching Brownian motion, denoted by  $(\mathcal{F}_t)_{t \geq 0}$ .

For each  $t \geq 0$ , let  $X(t) := (X_1(t), \dots, X_{M(t)}(t)) \in \mathcal{C}$  be the configuration of the particles of the BBM living at time  $t$ . For  $\xi = (\xi_1, \dots, \xi_n)$  and  $\xi' = (\xi'_1, \dots, \xi'_m)$  configurations, we say that  $\xi' \subset \xi$  if  $\sum_{i=1}^m \delta_{\xi'_i} \leq \sum_{i=1}^n \delta_{\xi_i}$  as measures and we define  $\max \xi := \max_{1 \leq i \leq n} \xi_i$ . Thus,  $\max X(t)$  is the position of the highest particle of the BBM living at time  $t$ .

Then, we define the  $L$ -branching Brownian motion on the same space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (\mathbb{P}_\xi)_{\xi \in \mathcal{C}})$ , by coupling it canonically with the standard branching Brownian motion: for each realization of the BBM, we can define a realization of the  $L$ -BBM by killing every particle that is at a distance greater than  $L$  from the highest particle of the  $L$ -BBM. We denote by  $M^L(t)$  the number of particles in the  $L$ -BBM at time  $t$  and by  $X_1^L(t), \dots, X_{M^L(t)}^L(t)$  their position. Let  $X^L(t) := (X_1^L(t), \dots, X_{M^L(t)}^L(t))$ . By definition, we have the inclusion  $X^L(t) \subset X(t)$  for all  $t \geq 0$  and  $\omega \in \Omega$  and, for each  $\xi \in \mathcal{C}$ , under  $\mathbb{P}_\xi$ ,  $(X_k^L(t), 1 \leq k \leq M^L(t))_{t \geq 0}$  is an  $L$ -branching Brownian motion starting from the configuration  $\xi$ . See Figure 1.

Our aim here is to study the asymptotic behavior of the position of the highest particle of the  $L$ -branching Brownian motion at time  $t$ , that is  $\max X^L(t)$ . For the standard branching Brownian motion, it is known [13] that

$$\frac{1}{t} \max X(t) \xrightarrow[t \rightarrow \infty]{} \sqrt{2}$$

almost surely. So, the speed of the highest particle of the  $L$ -BBM, whenever it exists, has to be smaller than  $\sqrt{2}$ , because there are less particles in the  $L$ -BBM than in the BBM owing to selection. Moreover, it is clear that if this velocity exists, then all particles of the  $L$ -BBM moves at the same speed. Our first result ensures that the velocity of the  $L$ -branching Brownian motion is well-defined and does not depend on the initial configuration. The same result is proved in a similar way by Derrida and Shi in their article [14] in preparation, but, nonetheless, we present it here in order to work legitimately on  $v_L$  in what follows.

**Proposition 1.1.** *For each  $L > 0$ , there exists  $v_L \in \mathbb{R}$  such that for each  $\xi \in \mathcal{C}$  we have the following convergence*

$$\frac{1}{t} \max X^L(t) \xrightarrow[t \rightarrow \infty]{} v_L$$

$\mathbb{P}_\xi$ -almost surely.

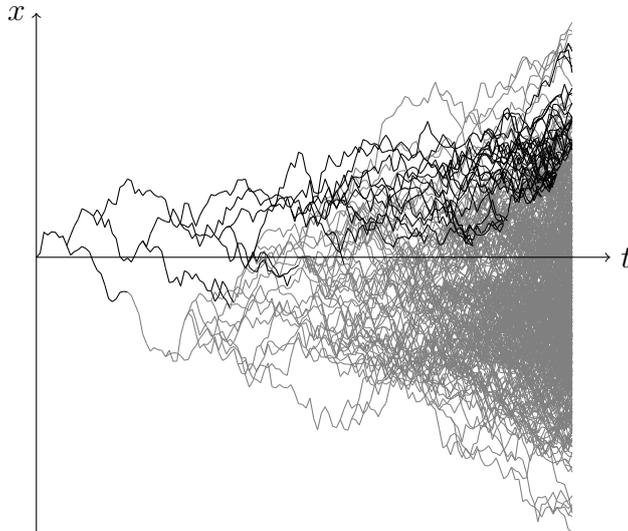


Figure 1: Simulation of a BBM starting with a single particle at 0 and of the coupled  $L$ -BBM with  $L = 3$  between times 0 and 7. Particles in black belong to the  $L$ -BBM and particles in grey belong only to the BBM without selection and not to the  $L$ -BBM. Note that quickly some particles killed by selection have descendants with a better fitness value than the particles of the  $L$ -BBM.

We focus in this paper on the behavior of  $v_L$  as  $L$  tends to infinity. It means that the selection effect vanishes, so we can expect that  $v_L$  tends to  $\sqrt{2}$ , which is the asymptotic velocity of the highest particle of the BBM without selection. Furthermore, we are interested in the asymptotic order of  $\sqrt{2} - v_L$ , which permits to estimate the slowdown due to selection. The main result of this work shows this second term in the asymptotic expansion of  $v_L$ , validating a conjecture of Brunet, Derrida, Mueller and Munier [12] (see Subsection 1.2 for more details).

**Theorem 1.2.** *We have the following asymptotic behavior:*

$$v_L = \sqrt{2} - \frac{\pi^2}{2\sqrt{2}L^2} + o\left(\frac{1}{L^2}\right),$$

as  $L$  tends to infinity.

## 1.2 Motivations

The  $L$ -branching Brownian motion has been introduced by Brunet, Derrida, Mueller and Munier [12] in order to describe the effect of a white noise on the F-KPP equation. The F-KPP equation (for Fisher [17] and Kolmogorov, Petrovsky, Piscounov [24])

$$\partial_t h = \frac{1}{2} \partial_x^2 h + h(1 - h) \tag{1.1}$$

is a traveling wave equation that describes how a stable phase ( $h = 1$ ) invades an unstable phase ( $h = 0$ ). Depending on the initial condition, the front between the two phases can travel at any velocity  $v$  larger than a minimal velocity  $v_{\min} = \sqrt{2}$  (which is as well the

asymptotic velocity of the highest particle of standard BBM). This equation often represents a large-scale limit or a mean-field description of some microscopic discrete stochastic processes. In order to understand the fluctuations that appear at the microscopic scale with a finite number of particles, one might consider instead the F-KPP equation with noise [10, 12, 11]

$$\partial_t h = \frac{1}{2} \partial_x^2 h + h(1-h) + \sqrt{\frac{h(1-h)}{N}} \dot{W}, \quad (1.2)$$

where  $N$  is the number of particles involved and  $\dot{W}$  is a normalized Gaussian white noise. Contrary to Equation (1.1), this equation with noise selects a single velocity  $v_N$  for the front propagation. A first approximation for Equation (1.2) consists in replacing the noise term by a deterministic cut-off, leading to the equation [8, 9]

$$\partial_t h = \frac{1}{2} \partial_x^2 h + h(1-h) \mathbb{1}_{h \geq 1/N}, \quad (1.3)$$

which selects also a single velocity  $v_N^{\text{cutoff}}$ . It has been conjectured in [8, 9] for the velocity  $v_N^{\text{cutoff}}$  and in [10] for the velocity  $v_N$  that, as  $N$  tends to infinity,

$$v_{\min} - v_N \sim v_{\min} - v_N^{\text{cutoff}} \sim \frac{\pi^2}{\sqrt{2}(\log N)^2},$$

which is an extremely slow convergence. These results have been given a rigorous mathematical proof in [15] for the F-KPP equation with cut-off and more recently in [29] for the F-KPP equation with white noise.

Brunet and Derrida in [8, 9, 10] support their conjecture by studying directly a particular microscopic stochastic processes involving  $N$  particles. In the same way, Brunet, Derrida, Mueller and Munier [12] introduce three different branching-selection particle systems, in order to describe more precisely the velocity and the diffusion constant of the front for Equation (1.2). Among these processes, they consider the  $L$ -BBM and the  $N$ -BBM. The latter is defined with a different selection rule: particles move according to Brownian motion, split after an exponential lifetime and, whenever the population size exceeds  $N$ , the lowest particle is killed. They conjecture, among other things, that the asymptotic velocity of the  $N$ -BBM is  $v_N$  and verifies the more precise asymptotic expansion as  $N \rightarrow \infty$ :

$$v_N = \sqrt{2} - \frac{\pi^2}{\sqrt{2}(\log N)^2} \left( 1 - \frac{6 \log \log N}{(\log N)^3} (1 + o(1)) \right).$$

This conjecture has not yet been mathematically proved, but some rigorous results have been obtained concerning the  $N$ -BBM or the  $N$ -branching random walk (its discrete time analog), in particular by Bérard and Gouéré [2], who showed the asymptotic behavior in  $\pi^2/\sqrt{2}(\log N)^2(1 + o(1))$  for  $\sqrt{2} - v_N$ , but also by Durrett and Remenik [16], Bérard and Maillard [3], Mallein [27] and Maillard [26]. For its part, the  $L$ -BBM has not yet been studied in the mathematical literature, but Brunet, Derrida, Mueller and Munier [12] conjectured that it behaves as the  $N$ -BBM by taking

$$L = \frac{\log N}{\sqrt{2}}, \quad (1.4)$$

which means that, with (1.4), the population size of the  $L$ -BBM is around  $N$  and the  $N$ -BBM has approximately a width  $L$ . It follows that the  $L$ -BBM must have an asymptotic velocity  $v_L$  that verifies, as  $L \rightarrow \infty$ ,

$$\sqrt{2} - v_L \sim \frac{\pi^2}{2\sqrt{2}L^2},$$

which is the result proved in this paper.

### 1.3 Proof overview and organization of the paper

One of the major difficulty in working with the  $L$ -BBM or the  $N$ -BBM is that they do not verify the branching property and, therefore, any form of many-to-one lemma (see Lemma 2.1): the offspring of a particle at a time  $t$  depends on the offspring of other particles alive at time  $t$ . For this reason, Bérard and Gouéré [2] compare the  $N$ -BRW with a branching random walk with absorption on a linear barrier in order to apply the precise results of Gantert, Hu and Shi [18]. In the same way, we come down here to results of Berestycki, Berestycki and Schweinsberg [5] concerning branching Brownian motion in a strip (see Subsection 2.2). In both cases, the study is reduced to another process that satisfies the branching property and, thus, on which the work is easier.

However, the arguments used here are quite different from those of Bérard and Gouéré [2]. The main reason is the absence of a monotonicity property for the  $L$ -BBM, like the one used by Bérard and Gouéré (see Lemma 1 of [2]): there is no obvious coupling such that, when one of the initial particles of the  $L$ -BBM is removed, its maximum becomes stochastically smaller.

The proof of Proposition 1.1 is based on the study of the return times to 1 for the population size. These times delimit i.i.d. pieces of the  $L$ -BBM, so, showing that they are sub-exponential, we can use the law of large numbers to prove the convergence of  $\max X^L(t)/t$ .

Although return times to 1 for the population size are sub-exponential, they are too large and, therefore, irrelevant for a more precise result. Thus, to prove Theorem 1.2, we work instead with stopping times  $(\tau_i)_{i \in \mathbb{N}}$  such that  $\tau_{i+1} - \tau_i$  is shorter than  $L^3$  and use on such a time interval a comparison with the BBM in a strip.

For the lower bound, we come down to a BBM in a strip that has exactly the same particles than the  $L$ -BBM at time  $\tau_i$  and is then included in the  $L$ -BBM until time  $\tau_{i+1}$ . Therefore, it is sufficient to show that this BBM in a strip goes up high enough between times  $\tau_i$  and  $\tau_{i+1}$ . It is done by using the monotonicity property of the BBM in a strip to consider the worst case with only a single initial particle at time  $\tau_i$  and, then, applying results of Berestycki, Berestycki and Schweinsberg [5].

In the same way, for the upper bound, we come down between times  $\tau_i$  and  $\tau_{i+1}$  to a BBM in a strip with more particles than the  $L$ -BBM and we show that it cannot rise too fast. But here, we need to control the population size at time  $\tau_i$ : the bad cases happen when there are too many particles. To this end, we use the following fact: a large population involves a quick increase of  $\max X^L$  but, when the maximum of the  $L$ -BBM rises fast, many particles are killed by selection (see Figure 2). This leads to the conclusion that  $\max X^L$  cannot grow quickly during a too long period.

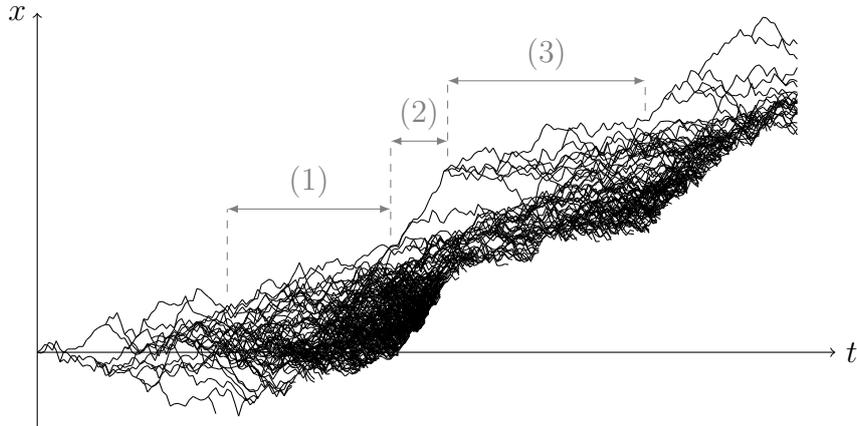


Figure 2: Simulation between times 0 and 10 of an  $L$ -BBM with  $L = 5$  starting with a single particle at 0. During the period (1),  $\max X^L$  grows slowly so the number of particles becomes very large. Among all these particles, there is one that goes up very fast during the period (2) and it reduces drastically the population. Therefore, after that, during period (3),  $\max X^L$  grows once again slowly, involving an increase of the population size.

The paper is organized as follows. Section 2 introduces some useful results concerning the standard BBM and the BBM in a strip. Then, Section 3 contains the proof of Proposition 1.1. Finally, the lower bound in Theorem 1.2 is proved in Section 4 and the upper bound in Section 5.

Throughout the paper,  $C$  denotes a positive constant that does not depend on the parameters and can change from line to line. For  $x$  and  $y$  real numbers, we set  $x \wedge y := \min(x, y)$  and  $x \vee y := \max(x, y)$ . For  $f: \mathbb{R} \rightarrow \mathbb{R}$  and  $g: \mathbb{R} \rightarrow \mathbb{R}_+^*$ , we say that  $f(x) \sim g(x)$  as  $x \rightarrow \infty$  if  $\lim_{x \rightarrow \infty} f(x)/g(x) = 1$  and that  $f(x) = o(g(x))$  as  $x \rightarrow \infty$  if  $\lim_{x \rightarrow \infty} f(x)/g(x) = 0$ . Lastly, we set  $\mathbb{N} := \{0, 1, 2, \dots\}$  and denote by  $\mathcal{C}([0, t], \mathbb{R})$  the set of continuous functions from  $[0, t]$  to  $\mathbb{R}$ .

## 2 Some useful results

### 2.1 Standard branching Brownian motion

In this section, we present some classical results concerning branching Brownian motion without selection. We assume here that there is initially one particle at 0, therefore we work under the probability  $\mathbb{P}_{(0)}$  and the associated expectation  $\mathbb{E}_{(0)}$ , where  $(0)$  denotes the configuration with a single particle at 0.

First of all,  $(M(t))_{t \geq 0}$  under  $\mathbb{P}_{(0)}$  is a Yule process (or pure birth process) with parameter 1, which means that it is a Markov process on  $\mathbb{N}$  with transitions  $i \rightarrow i + 1$  at rate  $i$  for all  $i \in \mathbb{N}$ . Thus,  $M(t)$  follows under  $\mathbb{P}_{(0)}$  a geometric distribution with parameter  $e^{-t}$ : for each  $k \geq 1$ ,  $\mathbb{P}_{(0)}(M(t) = k) = (1 - e^{-t})^{k-1} e^{-t}$ . In particular, we have  $\mathbb{E}_{(0)}[M(t)] = e^t$  and the next result, often called in the literature the many-to-one lemma, follows.

**Lemma 2.1** (Many-to-one lemma). *Let  $F: \mathcal{C}([0, t], \mathbb{R}) \rightarrow \mathbb{R}_+$  be a measurable function*

and  $(B_s)_{s \geq 0}$  denote a Brownian motion starting at 0. Then, we have

$$\mathbb{E}_{(0)} \left[ \sum_{k=1}^{M(t)} F\left((X_{k,t}(s))_{s \in [0,t]}\right) \right] = e^t \mathbb{E} \left[ F\left((B_s)_{s \in [0,t]}\right) \right],$$

where we denote by  $X_{k,t}(s)$  the position of the unique ancestor at time  $s$  of  $X_k(t)$ .

This lemma will be useful to compute the expectation of some functionals of the branching Brownian motion without selection. There is no similar result for the  $L$ -BBM because the number of particles of the  $L$ -BBM living at time  $t$  is not independent of their trajectories on  $[0, t]$ .

We saw before that the first order for the asymptotic behavior of the position of the highest particle of the BBM is  $\sqrt{2}t$ , but we will need more accurate information on the extremal particle of the BBM. The extremal particle has been a main topic in the study of the BBM, with in particular the seminal works of Bramson [6, 7] who shows the existence of a real random variable  $W$  such that

$$\max X(t) - m(t) \xrightarrow[t \rightarrow \infty]{\text{law}} W, \quad (2.1)$$

where

$$m(t) := \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t, \quad (2.2)$$

and of Lalley and Sellke [25] who describe the limit  $W$  as a random mixture of Gumbel distributions.

We will need some further information on the trajectory leading to the extremal particle at time  $t$ . The following proposition is an immediate consequence of convergence (2.1) and of Theorem 2.5 of Arguin, Bovier and Kistler [1]. It shows that there exists a high particle at time  $t$  (above  $m(t) - d$ ) whose trajectory does not go too far under the line  $s \mapsto \frac{s}{t}m(t)$ . Actually, Arguin, Bovier and Kistler [1] show that high particles at time  $t$  mainly satisfy this property.

**Proposition 2.2.** *For all  $\gamma > 0$  and  $\delta > 0$ , there exist  $d > 0$ ,  $r > 0$  and  $t_0 > 0$  large enough such that for all  $t \geq t_0$ ,*

$$\mathbb{P}_{(0)} \left( \exists k \in \llbracket 1, M(t) \rrbracket : X_k(t) \geq m(t) - d \right. \\ \left. \text{and } \forall s \in [0, t], X_{k,t}(s) \geq \frac{s}{t}m(t) - r \vee \left( s^{\frac{1}{2}+\gamma} \wedge (t-s)^{\frac{1}{2}+\gamma} \right) \right) \geq 1 - \delta,$$

where  $X_{k,t}(s)$  denotes the position of the unique ancestor at time  $s$  of  $X_k(t)$ .

## 2.2 Branching Brownian motion in a strip

The branching Brownian motion with absorption on a linear barrier has been introduced by Kesten [23]. In this process, each particle moves according to a Brownian motion with drift  $-\mu$ , splits into two new particles after an exponentially distributed time and is killed when it reaches a non-positive position. Kesten [23] showed that the process survives with positive probability if and only if  $\mu < \sqrt{2}$ . More recently, the branching Brownian motion

with absorption has been studied by Harris, Harris and Kyprianou [20], by Harris and Harris [19] and, in the near-critical case, that is when  $\mu \rightarrow \sqrt{2}$  while keeping  $\mu < \sqrt{2}$ , by Berestycki, Berestycki and Schweinsberg [5, 4]. For recent results concerning the branching Brownian motion in a strip, see Harris, Hesse and Kyprianou [21].

We focus here on results of Berestycki, Berestycki and Schweinsberg [5] concerning the branching Brownian motion in a strip, where particles move according to a Brownian motion with drift  $-\mu$ , split into two new particles after an exponentially distributed time and are killed when they come out of a fixed interval, that we will specify hereafter. We fix a positive real  $K$ , we set

$$\mu := \sqrt{2 - \frac{\pi^2}{K^2}} \quad (2.3)$$

and we choose the interval  $(0, K_A)$ , where we set

$$K_A := K - \frac{A}{\sqrt{2}},$$

in order to keep notation similar to those of Berestycki, Berestycki and Schweinsberg [5] (but with  $K$  and  $K_A$  instead of  $L$  and  $L_A$ ). We denote by  $(\tilde{X}_k^{K, K_A}(t), 1 \leq k \leq \tilde{M}^{K, K_A}(t))_{t \geq 0}$  the BBM in the strip  $(0, K_A)$  with drift  $-\mu$ , where  $\mu$  is given by (2.3). When  $A = 0$  (that is  $K = K_A$ ), we write  $\tilde{X}^K$  instead of  $\tilde{X}^{K, K_A}$ , in order to simplify notation.

Berestycki, Berestycki and Schweinsberg [5] introduce the following functional of the BBM in a strip, defined by

$$\tilde{Z}^{K, K_A}(t) := \sum_{k=1}^{\tilde{M}^{K, K_A}(t)} e^{\mu \tilde{X}_k^{K, K_A}(t)} \sin\left(\frac{\pi \tilde{X}_k^{K, K_A}(t)}{K_A}\right), \quad (2.4)$$

for each  $t \geq 0$ . The random variable  $\tilde{Z}^{K, K_A}(t)$  estimates the size of the process and the process  $(\tilde{Z}^{K, K_A}(t))_{t \geq 0}$  has the advantage of being a martingale in the particular case where  $K_A = K$ , as stated in the following proposition (see Lemma 7 of [5]).

**Proposition 2.3.** *If  $A = 0$ , then  $(\tilde{Z}^K(t))_{t \geq 0}$  is a martingale.*

So the drift  $-\mu$  given by (2.3) is exactly the right choice such that the size of the BBM in the strip  $(0, K)$  does not degenerate. When  $A > 0$  (that is  $K_A < K$ ), the population decreases and when  $A < 0$  (that is  $K_A > K$ ), the population explodes.

The following results concern the number of particles that are killed by hitting  $K_A$  between times 0 and  $\theta K^3$ . This corresponds to Subsection 3.2 of Berestycki, Berestycki and Schweinsberg [5] but with some modifications: here  $\theta$  is not assumed small (rather,  $\theta$  will be close to 1), we consider only the particular case  $A = 0$  and we do not make the same assumptions about the initial configuration.

This first proposition is a rewrite of Proposition 16 of Berestycki, Berestycki and Schweinsberg [5] in the case  $A = 0$  and estimates the mean number of particles that are killed by hitting  $K_A$  between times 0 and  $\theta K^3$  for any initial configuration.

**Proposition 2.4.** *Assume  $A = 0$ . For some fixed  $\theta > 0$ , let  $R$  be the number of particles that hit  $K$  between times 0 and  $\theta K^3$  and  $R'$  the number of particles that hit  $K$  between times  $K^{5/2}$  and  $\theta K^3$ . Then, for all  $\xi \in \mathcal{C}$ ,  $\mathbb{P}_\xi$ -a.s. we have the following inequalities as  $K \rightarrow \infty$ :*

$$\begin{aligned}\mathbb{E}_\xi[R] &\leq C \frac{e^{-\mu K}}{K} \tilde{V}^K(0) + 2\sqrt{2}\pi\theta K e^{-\mu K} \tilde{Z}^K(0)(1 + o(1)), \\ \mathbb{E}_\xi[R'] &\geq 2\sqrt{2}\pi\theta K e^{-\mu K} \tilde{Z}^K(0)(1 + o(1)),\end{aligned}$$

where  $C$  is a positive constant and  $\tilde{V}^K(0) := \sum_{i=1}^{\tilde{M}^K(0)} \tilde{X}_i^K(0) e^{\mu \tilde{X}_i^K(0)}$ .

*Proof.* This result follows directly from the proof of Proposition 16 of Berestycki, Berestycki and Schweinsberg [5], by noting that  $K = (\log N + 3 \log \log N)/\sqrt{2}$  and that the only place where they need  $\theta$  to be small enough is their inequality (69), whose left-hand side is simply zero in the case  $A = 0$ .  $\square$

The second proposition gives an upper bound for the second moment of  $R$  and follows from the proof of Proposition 18 of [5]. We still work under the assumption  $A = 0$  but moreover with a single initial particle, therefore the initial configuration is denoted by  $(x)$ , where  $x \in (0, K)$ . Here  $x$  can be close to  $K$ , so the initial configuration does not necessarily satisfy the assumption of Proposition 18 of [5].

**Proposition 2.5.** *Assume  $A = 0$  and suppose there is a single particle  $x$  at time 0, where  $x \in (0, K)$ . Let  $R$  be the number of particles that hit  $K$  between times 0 and  $K^3$ . Then, as  $K \rightarrow \infty$ ,*

$$\mathbb{E}_{(x)}[R^2] \leq C e^{-\mu K} e^{\mu x} \left(1 + K \sin\left(\frac{\pi x}{K}\right)\right) (1 + o(1)),$$

where  $C$  is a positive constant.

*Proof.* In the same way as Berestycki, Berestycki and Schweinsberg [5] in the proof of their Proposition 18, we write  $R^2 = R + Y$ , where  $Y$  is the number of distinct pairs of particles that hit  $K$  between times 0 and  $K^3$ . Here, as there is only a single initial particle, we have  $Y = Y_2 = Y_2^x$  with notation of [5], which means that all particles that hit  $K$  have the same ancestor at time 0. From their bounds (76) and (77), it follows that

$$\mathbb{E}_{(x)}[Y] \leq C e^{-\mu K} e^{\mu x} (1 + o(1)) + C e^{-\mu K} K e^{\mu x} \sin\left(\frac{\pi x}{K}\right) (1 + o(1)),$$

because they only use the assumption that  $\theta \leq 1$  and we chose here  $\theta = 1$ . Moreover, using Proposition 2.4, we get

$$\begin{aligned}\mathbb{E}_{(x)}[R] &\leq C \frac{e^{-\mu K}}{K} x e^{\mu x} + 2\sqrt{2}\pi K e^{-\mu K} e^{\mu x} \sin\left(\frac{\pi x}{K}\right) (1 + o(1)) \\ &\leq C e^{-\mu K} e^{\mu x} \left(1 + K \sin\left(\frac{\pi x}{K}\right)\right) (1 + o(1))\end{aligned}$$

and the result follows.  $\square$

### 3 Existence of the asymptotic velocity

In this section, we prove Proposition 1.1. We recall that the following proof is similar to the one presented by Derrida and Shi [14]. However, the methods carried out here are very rough and cannot be used to get more elaborate results as Theorem 1.2, but we want to be ensured that the velocity  $v_L$  of the  $L$ -branching Brownian motion is well defined.

We work here with a fixed  $L > 0$ . The strategy will be to show that the return time to 1 for the population size of the  $L$ -BBM is sub-exponential and then to apply the law of large numbers to the renewal structure obtained from the sequence of successive return times to 1 for the population size. Foremost, we need the following coupling result that will be useful to show that there cannot be too many particles in the  $L$ -BBM.

**Lemma 3.1.** *Let  $M$  be a positive integer and  $\xi = (\xi_1, \dots, \xi_N)$  be a configuration, with  $N \geq M$  and  $\xi_1 \geq \dots \geq \xi_N$ . Then, there exist independent Brownian motions  $B^1, \dots, B^M$  starting at  $\xi_1, \dots, \xi_M$  respectively such that, for all  $t \geq 0$ , on the event  $\{\forall s \in [0, t], M^L(s) \geq M\}$ , we have*

$$\max X^L(t) \geq \max_{1 \leq i \leq M} B_t^i$$

$\mathbb{P}_\xi$ -almost surely.

*Proof.* We choose for  $B^i$  the trajectory of the particle of the  $L$ -BBM that is at  $\xi_i$  at time 0. When this particle splits, we choose uniformly one of its children and  $B^i$  continues by following the trajectory of this child. We proceed like this until a particle followed by one of the Brownian motions  $B^1, \dots, B^M$ , say  $B^i$ , is killed by selection in the  $L$ -BBM. Then, we distinguish two cases:

- (i) If there are at least  $M$  particles in the  $L$ -BBM at this time denoted by  $t$ , then there is at least 1 particle that is not followed by one of the Brownian motions  $B^1, \dots, B^M$ , so we choose uniformly one of those particles and  $B^i$  continues by following, from the position  $B_t^i$ , the trajectory of the chosen particle. We proceed then as before.
- (ii) Otherwise, we lay the  $L$ -BBM aside and work only on the BBM without selection: each Brownian motion continues following its particle until it splits and then follows one of its child, and so on.

Thus,  $B^1, \dots, B^M$  are defined for all time and are independent Brownian motions. On the event  $\{\forall s \in [0, t], M^L(s) \geq M\}$ , we do not meet the case (ii) until time  $t$  and so we have for all  $s \in [0, t]$

$$\max X^L(s) \geq \max_{1 \leq i \leq M} B_s^i,$$

because  $B^i$  at time  $s$  is always lower than the particle it follows: when, in the case (i),  $B^i$  changes the particle it follows, the new particle is necessarily above  $B^i$ , because the new particle is in the  $L$ -BBM and  $B^i$  is at the low extremity of the  $L$ -BBM.  $\square$

We can now prove that the return time to 1 for the population size of the  $L$ -BBM is sub-exponential. For this purpose, we show with Lemma 3.1 that, if there is a very large number of particles in the population, then the maximum of the  $L$ -BBM can with high probability increase by  $2L$  in a short time and, thus, a large proportion of particles is killed by selection.

**Proposition 3.2.** *Let  $T := \inf\{t \geq 1 : M^L(t) = 1\}$ . There exist two positive constants  $C$  and  $c$  such that for all  $\xi \in \mathcal{C}$ ,  $\mathbb{P}_\xi(T \leq C) \geq c$ .*

*Proof.* The strategy will be to show first the same result for  $S_{M_0} := \inf\{t \geq 0 : M^L(t) \leq M_0\}$ , which is the return time under a fixed size  $M_0$  for the population of the  $L$ -BBM, where  $M_0$  will be chosen very large.

Let  $M$  be a positive integer and  $\xi = (\xi_1, \dots, \xi_N)$  a configuration, with  $N \geq M$  and  $\xi_1 \geq \dots \geq \xi_N$ . We can suppose with no loss of generality that  $\xi_1 = 0$ , because the law of  $(M^L(t))_{t \geq 0}$  is invariant under shift of the initial configuration, and that  $\xi_N > -L$ , otherwise the low particles will die instantly. Let  $a$  be a positive real number and  $t := aL^2/\log N$ . We have

$$\begin{aligned} \mathbb{P}_\xi(\forall s \in [0, t], M^L(s) \geq M) &\leq \mathbb{P}_\xi(\forall s \in [0, t], M^L(s) \geq M, \max X^L(t) < 2L) \\ &\quad + \mathbb{P}_\xi(M^L(t) \geq M, \max X^L(t) \geq 2L). \end{aligned} \quad (3.1)$$

The second term on the right-hand side of (3.1) can be bounded by noting that, on the event  $\{M^L(t) \geq M, \max X^L(t) \geq 2L\}$ , there are at least  $M$  particles of the BBM without selection that are above  $L$ . Thus,

$$\begin{aligned} \mathbb{P}_\xi(M^L(t) \geq M, \max X^L(t) \geq 2L) &\leq \mathbb{P}_\xi\left(\sum_{k=1}^{M(t)} \mathbb{1}_{X_k(t) \geq L} \geq M\right) \\ &\leq \frac{1}{M} \mathbb{E}_\xi \left[ \sum_{k=1}^{M(t)} \mathbb{1}_{X_k(t) \geq L} \right] \\ &= \frac{1}{M} \sum_{i=1}^N e^t \mathbb{P}(B_t + \xi_i \geq L), \end{aligned} \quad (3.2)$$

where we used successively the Markov inequality and the many-to-one lemma (Lemma 2.1) applied to the BBM starting at each  $\xi_i$ , with  $(B_t)_{t \geq 0}$  denoting a Brownian motion starting at 0. With our choice for  $t$  and recalling that  $\xi_i \leq 0$ , (3.2) is bounded from above by

$$\frac{N}{M} e^{aL^2/\log N} \mathbb{P}(\sqrt{t} \mathcal{N}(0, 1) \geq L) \leq e^{aL^2/\log N} \frac{N}{M} \frac{e^{-\log N/2a}}{2} = e^{aL^2/\log N} \frac{N^{1-1/2a}}{2M},$$

where  $\mathcal{N}(0, 1)$  denotes the standard normal distribution and we use that  $\mathbb{P}(\mathcal{N}(0, 1) \geq x) \leq e^{-x^2/2}/2$  for all  $x \geq 0$ . From now on, we set  $M := \lfloor N^{\lambda+1-1/2a} \rfloor$ , where  $0 < \lambda < 1/2a$ . Thus, since  $e^{aL^2/\log N} \rightarrow 1$  as  $N \rightarrow \infty$ , we proved that the second term on the right-hand side of (3.1) is smaller than  $N^{-\lambda}$  for  $N$  large enough.

We now deal with the first term on the right-hand side of (3.1). It can be bounded using Lemma 3.1: we have, with  $B^1, \dots, B^M$  independent Brownian motions starting at  $\xi_1, \dots, \xi_M$  under  $\mathbb{P}_\xi$ ,

$$\begin{aligned} \mathbb{P}_\xi(\forall s \in [0, t], M^L(s) \geq M, \max X^L(t) < 2L) &\leq \mathbb{P}_\xi\left(\max_{1 \leq i \leq M} B^i(t) < 2L\right) \\ &\leq \mathbb{P}(\sqrt{t} \mathcal{N}(0, 1) - L < 2L)^M, \end{aligned}$$

since  $\xi_i > -L$  for all  $i$ . Using that  $\mathbb{P}(\mathcal{N}(0, 1) \geq x) \geq e^{-x^2/2}/2x$  for  $x$  large enough and that  $\ln(1 - u) \leq -u$  for  $u < 1$ , with our choice for  $t$  and  $M$ , we get

$$\begin{aligned} \mathbb{P}\left(\sqrt{t}\mathcal{N}(0, 1) - L < 2L\right)^M &= \left(1 - \mathbb{P}\left(\mathcal{N}(0, 1) \geq \frac{3(\log N)^{1/2}}{a^{1/2}}\right)\right)^{\lfloor N^{\lambda+1-1/2a} \rfloor} \\ &\leq \exp\left(-\lfloor N^{\lambda+1-1/2a} \rfloor \frac{N^{-9/2a} a^{1/2}}{6(\log N)^{1/2}}\right), \end{aligned} \quad (3.3)$$

for  $N$  large enough. Thus, we want that  $\lambda + 1 - 10/2a > 0$ , but we need  $\lambda < 1/2a$  to have  $M < N$ . So that a such  $\lambda > 0$  exists, it is sufficient that  $10/2a - 1 < 1/2a$ , which means  $a > 9/2$ . Then, we get (3.3)  $\leq N^{-\lambda}$  for  $N$  large enough.

Going back to (3.1), we showed that there exist  $M_0$  large enough,  $a > 0$ ,  $\lambda > 0$  and  $0 < \mu < 1$  such that, for all  $\xi = (\xi_1, \dots, \xi_N)$  with  $N \geq M_0$ ,

$$\mathbb{P}_\xi\left(\exists s \in [0, aL^2/\log N] : M^L(s) \leq N^\mu\right) \geq 1 - N^{-\lambda}. \quad (3.4)$$

We fix  $N \geq M_0$  and we set  $k := \lceil \frac{\log \log N - \log \log M_0}{\log(1/\mu)} \rceil$ , so that  $k$  is the single integer that verifies  $N^{\mu^k} \leq M_0 < N^{\mu^{k-1}}$ . Then, by applying  $k$  times the inequality (3.4) and the strong Markov property, we get

$$\begin{aligned} \mathbb{P}_\xi\left(\exists s \in [0, akL^2/\log N] : M^L(s) \leq N^{\mu^k}\right) &\geq (1 - N^{-\lambda})(1 - (N^\mu)^{-\lambda}) \dots (1 - (N^{\mu^{k-1}})^{-\lambda}) \\ &= \prod_{i=0}^{k-1} (1 - N^{-\lambda\mu^i}) \geq \prod_{j=0}^{k-1} (1 - M_0^{-\lambda\mu^{-j}}), \end{aligned} \quad (3.5)$$

by using that  $N > M_0^{\mu^{-k+1}}$  and setting  $j = k - i - 1$ . It is easy to see that the product on the right-hand side of (3.5) converges to a positive limit as  $k$  tends to infinity, if  $M_0 > 1$ . Moreover, we can choose  $M_0$  large enough such that for all  $N \geq M_0$ ,  $akL^2/\log N \leq 1$ . Thus, we have proved that there exist  $c_1 > 0$  and  $M_0$  such that, for all  $\xi = (\xi_1, \dots, \xi_N)$  with  $N \geq M_0$ ,

$$\mathbb{P}_\xi\left(\exists s \in [0, 1] : M^L(s) \leq M_0\right) \geq c_1, \quad (3.6)$$

that is  $\mathbb{P}_\xi(S_{M_0} \leq 1) \geq c_1$  with  $S_{M_0} := \inf\{t \geq 0 : M^L(t) \leq M_0\}$ .

Now, we consider an initial configuration  $\xi = (\xi_1, \dots, \xi_N)$  but with  $N \leq M_0$ . We suppose  $\xi_1 \geq \dots \geq \xi_N$ . If no particle splits on the time interval  $[0, 1]$ , if the particle starting at  $\xi_1$  stays above  $\xi_1 - L/2$  on  $[0, 1]$  and reaches  $\xi_1 + 2L$  at time 1 and if all other particles stay strictly under  $\xi_1 + L/2$  on  $[0, 1]$ , then at time 1 only the particle starting at  $\xi_1$  is alive so  $T = 1$ . Therefore, if  $B$  denotes a Brownian motion starting at 0, we have

$$\mathbb{P}_\xi(T = 1) \geq (e^{-1})^N \mathbb{P}\left(\min_{s \in [0, 1]} B_s \geq -\frac{L}{2}, B_t \geq 2L\right) \mathbb{P}\left(\max_{s \in [0, 1]} B_s < \frac{L}{2}\right)^{N-1} \geq c_2,$$

where  $c_2 > 0$  is reached in the case  $N = M_0$ . So we can conclude that  $\forall \xi \in \mathcal{C}$ ,  $\mathbb{P}_\xi(T \leq 2) \geq c_1 c_2$ , by using (3.6) and the strong Markov property at time  $S_{M_0}$  in the case where there are more than  $M_0$  particles in the initial configuration.  $\square$

The controls performed in the previous proof are very loose: with slightly more computation, one can see that  $M_0$  needs to be larger than  $e^{bL^2}$  with some  $b > 0$ , whereas most of the time  $M^L(t)$  is of the order of  $e^{\sqrt{2}L}$ . Indeed, the return to 1 for the population size is a too infrequent event for a more accurate study of the  $L$ -BBM as in Sections 4 and 5. But it is sufficient here to prove the existence of the asymptotic velocity  $v_L$  for the  $L$ -BBM.

*Proof of Proposition 1.1.* Let  $\xi \in \mathcal{C}$  be a fixed configuration. We set  $T_0 := 0$  and  $T_{i+1} := \inf\{t \geq T_i + 1 : M^L(t) = 1\}$  for each  $i \in \mathbb{N}$ . Since for  $i \geq 1$  there is a single particle in the  $L$ -BBM at time  $T_i$ , by the strong Markov property,  $(T_{i+1} - T_i)_{i \geq 1}$  and  $(\max X^L(T_{i+1}) - \max X^L(T_i))_{i \geq 1}$  are sequences of i.i.d. random variables with the same laws as  $T$  and  $\max X^L(T)$  under  $\mathbb{P}_{(0)}$  respectively.

Using Proposition 3.2 and the strong Markov property, we have for all  $\xi \in \mathcal{C}$  and  $n \in \mathbb{N}$ ,

$$\mathbb{P}_\xi(T \geq nC) \leq (1 - c)^n,$$

which means that  $T$  has a sub-exponential distribution under  $\mathbb{P}_\xi$ , so  $T$  has finite moments and, in particular,  $\mathbb{P}_\xi$ -almost surely we have  $T_1 < \infty$ . Therefore, by the law of large numbers, we get that  $\mathbb{P}_\xi$ -a.s.  $T_n/n \rightarrow \mathbb{E}_{(0)}[T]$  as  $n \rightarrow \infty$ .

It is clear that  $\mathbb{P}_\xi$ -a.s.  $\max X^L(T_1) < \infty$  but in order to apply the law of large numbers to the sequence  $(\max X^L(T_{i+1}) - \max X^L(T_i))_{i \geq 1}$ , we need to check that  $\mathbb{E}_{(0)}[|\max X^L(T)|]$  is finite. We are going to prove something stronger that will be useful afterwards in the proof, which is  $\mathbb{E}_{(0)}[\zeta] < \infty$  where

$$\zeta := \max_{t \in [0, T]} |\max X^L(t)|.$$

For this, it is sufficient to prove that the function  $a \mapsto \mathbb{P}_{(0)}(\zeta \geq a)$  is integrable on  $\mathbb{R}_+$ . For all  $a > 0$ , we have

$$\mathbb{P}_{(0)}(\zeta \geq a) \leq \mathbb{P}_{(0)}(T > \sqrt{a}) + \mathbb{P}_{(0)}(\exists t \in [0, \sqrt{a}] : |\max X^L(t)| \geq a). \quad (3.7)$$

Since  $\mathbb{E}_{(0)}[T^2]$  is finite,  $a \mapsto \mathbb{P}_{(0)}(T > \sqrt{a})$  is integrable on  $\mathbb{R}_+$ . So we now have to deal with the second term on the right-hand side of (3.7). By the coupling with the BBM without selection, it is bounded by

$$\begin{aligned} \mathbb{P}_{(0)}(\exists t \in [0, \sqrt{a}] : |\max X(t)| \geq a) &\leq 2\mathbb{P}_{(0)}(\exists t \in [0, \sqrt{a}] : \max X(t) \geq a) \\ &\leq 2\mathbb{E}_{(0)} \left[ \sum_{k=1}^{M(\sqrt{a})} \mathbb{1}_{\exists t \in [0, \sqrt{a}] : X_{k, \sqrt{a}}(t) \geq a} \right], \end{aligned} \quad (3.8)$$

where we denote by  $X_{k, \sqrt{a}}(t)$  the position of the unique ancestor at time  $t$  of  $X_k(\sqrt{a})$ . Using the many-to-one lemma (Lemma 2.1), (3.8) is equal to

$$2e^{\sqrt{a}}\mathbb{P}(\exists t \in [0, \sqrt{a}] : B_t \geq a) = 4e^{\sqrt{a}}\mathbb{P}(a^{1/4}\mathcal{N}(0, 1) \geq a) \leq 2e^{\sqrt{a}}e^{-a^{3/2}/2},$$

which is an integrable function of  $a$ . This concludes the proof of the fact that  $\mathbb{E}_{(0)}[\zeta]$  is finite. In particular, we can apply the law of large numbers to the sequence  $(\max X^L(T_{i+1}) - \max X^L(T_i))_{i \geq 1}$

$\max X^L(T_i)_{i \geq 1}$  and get that  $\mathbb{P}_\xi$ -a.s.  $\max X^L(T_n)/n \rightarrow \mathbb{E}_{(0)}[\max X^L(T)]$  as  $n \rightarrow \infty$ . Thus, we have the convergence

$$\frac{\max X^L(T_n)}{T_n} \xrightarrow{n \rightarrow \infty} \frac{\mathbb{E}_{(0)}[\max X^L(T)]}{\mathbb{E}_{(0)}[T]} =: v_L \quad (3.9)$$

$\mathbb{P}_\xi$ -almost surely.

For each  $t \geq 0$ , let  $n_t$  be the integer such that  $T_{n_t} \leq t < T_{n_t+1}$ . It suffices now to show that  $\mathbb{P}_\xi$ -a.s.  $\max X^L(t)/t - \max X^L(T_{n_t})/T_{n_t} \rightarrow 0$  as  $t \rightarrow \infty$ . We have

$$\begin{aligned} \left| \frac{\max X^L(t)}{t} - \frac{\max X^L(T_{n_t})}{T_{n_t}} \right| &\leq \left| \frac{\max X^L(t) - \max X^L(T_{n_t})}{t} \right| + \left| \frac{\max X^L(T_{n_t})}{T_{n_t}} \right| \left( \frac{t - T_{n_t}}{t} \right) \\ &\leq \frac{\zeta_{n_t}}{t} + \left| \frac{\max X^L(T_{n_t})}{T_{n_t}} \right| \left( 1 - \frac{T_{n_t}}{t} \right), \end{aligned} \quad (3.10)$$

where, for  $n \in \mathbb{N}$ , we set

$$\zeta_n := \max_{t \in [T_n, T_{n+1}]} \left| \max X^L(t) - \max X^L(T_n) \right|.$$

Since  $T_{n_t}/n_t \leq t/n_t < T_{n_t+1}/n_t$ , we get that  $\mathbb{P}_\xi$ -a.s.  $t/n_t \rightarrow \mathbb{E}_{(0)}[T]$  as  $n \rightarrow \infty$ , so  $T_{n_t}/t \rightarrow 1$  and combining with (3.9), we deduce that  $\mathbb{P}_\xi$ -a.s. the second term on the right-hand side of (3.10) tends to 0. We now have to deal with the first term. Since  $(\zeta_n)_{n \geq 1}$  is a sequence of i.i.d. random variables with the same law as  $\zeta$  under  $\mathbb{P}_{(0)}$  and  $\mathbb{E}_{(0)}[\zeta]$  is finite, we have by the law of large numbers  $\mathbb{P}_\xi$ -a.s.  $\zeta_n/n \rightarrow 0$  as  $n \rightarrow \infty$  and so  $\mathbb{P}_\xi$ -a.s.  $\zeta_{n_t}/t \rightarrow 0$  as  $t \rightarrow \infty$ . It concludes the proof of Proposition 1.1.  $\square$

## 4 Lower bound for $v_L$

In this section, we fix  $0 < \varepsilon < 1$  and consider all processes with drift  $-\mu$ , where

$$\mu := \sqrt{2 - \frac{\pi^2}{(1-\varepsilon)^2 L^2}},$$

which means that, for each  $\xi \in \mathcal{C}$ , under  $\mathbb{P}_\xi$ ,  $(X_k(t), 1 \leq k \leq M(t))_{t \geq 0}$  is a BBM without selection, with drift  $-\mu$  and starting from the configuration  $\xi$  and  $(X_k^L(t), 1 \leq k \leq M^L(t))_{t \geq 0}$  is the associated  $L$ -BBM with drift  $-\mu$ . However,  $v_L$  still denotes the asymptotic velocity of  $L$ -BBM without drift, so Proposition 1.1 shows that, for all  $L > 0$  and all  $\xi \in \mathcal{C}$ ,

$$\frac{\max X^L(t)}{t} \xrightarrow{t \rightarrow \infty} v_L - \mu \quad (4.1)$$

$\mathbb{P}_\xi$ -almost surely. Actually, the aim is to show that for  $L$  large enough  $\lim \max X^L(t)/t \geq 0$   $\mathbb{P}_{(0)}$ -a.s. so that we can conclude that  $v_L \geq \mu$  and the lower bound follows by letting  $\varepsilon \rightarrow 0$ .

## 4.1 Proof of the lower bound

In this subsection, we prove the lower bound in Theorem 1.2, by postponing to the next subsection the proof of a proposition. The strategy is to study the  $L$ -BBM on time intervals of length at most  $L^3$  associated to a sequence of stopping times  $(\tau_i)_{i \in \mathbb{N}}$  defined by  $\tau_0 := 0$  and for each  $i \in \mathbb{N}$ ,

$$\tau_{i+1} := (\tau_i + L^3) \wedge \inf \left\{ t \geq \tau_i \mid \max X^L(t) - \max X^L(\tau_i) \notin (-(L-1), 1) \right\}.$$

We also define the event  $A_i := \{\max X^L(\tau_{i+1}) < \max X^L(\tau_i) + 1\}$  (see Figure 3 for an illustration of these definitions).

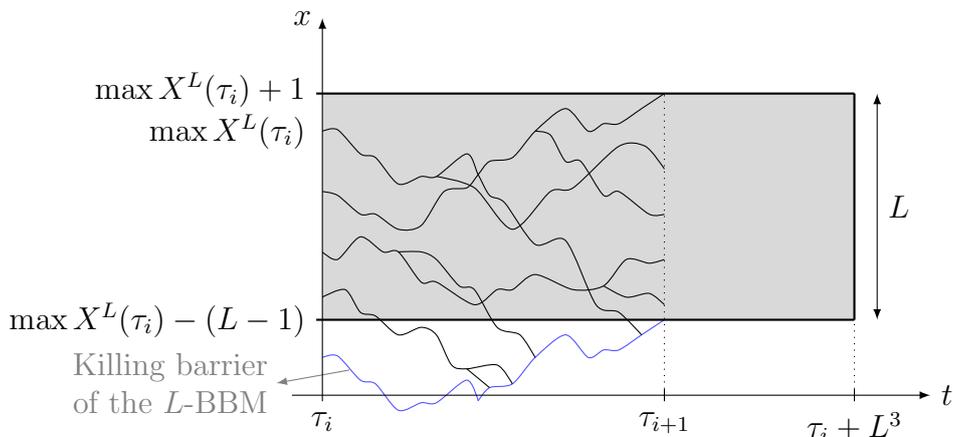


Figure 3: Representation of a  $L$ -BBM between times  $\tau_i$  and  $\tau_{i+1}$ . By definition,  $\tau_{i+1}$  is the time where  $t \mapsto \max X^L(t)$  leaves the gray area. We are here on the event  $A_i^c$  because  $t \mapsto \max X^L(t)$  leaves the area from above. Note that the killing barrier of  $X^L$  (drawn in blue) stays below  $\max X^L(\tau_i) - (L-1)$ .

The event  $A_i$  is a “bad” event, because on the event  $A_i^c$  the population of the  $L$ -BBM goes up between times  $\tau_i$  and  $\tau_{i+1}$ . The following proposition shows that the event  $A_i$  is very unlikely, regardless of the configuration of the  $L$ -BBM at time  $\tau_i$ . Its proof is postponed to Subsection 4.2.

**Proposition 4.1.** *Let  $h(L) := \sup_{\xi \in \mathcal{C}} \mathbb{P}_\xi(A_0)$ . Then, as  $L \rightarrow \infty$ , we have  $h(L) = o(1/L)$ .*

Using Proposition 4.1, we can now conclude the proof of the lower bound. Let  $K_n := \sum_{i=0}^{n-1} \mathbb{1}_{A_i}$  be the number of “bad” events  $A_i$  that happen before time  $\tau_n$ . On the event  $A_i^c$ , we have  $\max X^L(\tau_{i+1}) - \max X^L(\tau_i) = 1$  and, on the event  $A_i$ , we only have  $\max X^L(\tau_{i+1}) - \max X^L(\tau_i) \geq -(L-1)$  and, therefore, we get

$$\max X^L(\tau_n) - \max X^L(0) \geq (n - K_n) - (L-1)K_n = n - LK_n$$

and, thus,

$$\frac{\max X^L(\tau_n) - \max X^L(0)}{\tau_n} \geq \frac{n}{\tau_n} \left( 1 - L \frac{K_n}{n} \right). \quad (4.2)$$

We now want to prove that the right-hand side of (4.2) is non-negative as  $n$  tends to infinity. For this, we need to control  $K_n$ . Using Proposition 4.1 and the strong Markov

property, we get that, for each  $0 \leq k \leq n$ ,  $\mathbb{P}_{(0)}(K_n \geq k) \leq \binom{n}{k} h(L)^k$  and so

$$\mathbb{E}_{(0)}[e^{K_n}] \leq \sum_{k=0}^n \binom{n}{k} h(L)^k e^k = (1 + h(L)e)^n \leq e^{nh(L)e}.$$

Then, applying the Markov inequality, we get

$$\mathbb{P}_{(0)}(K_n \geq 3nh(L)) \leq e^{-3nh(L)} \mathbb{E}_{(0)}[e^{K_n}] \leq e^{-nh(L)(3-e)}$$

which is summable, so the Borel-Cantelli lemma implies that  $\mathbb{P}_{(0)}$ -a.s.  $\limsup_{n \rightarrow \infty} \frac{K_n}{n} \leq 3h(L)$ . Moreover, using Proposition 4.1, we get that for  $L$  large enough  $1 - 3h(L)L \geq 1/2$ , so we conclude that for  $L$  large enough,

$$\liminf_{n \rightarrow \infty} \frac{n}{\tau_n} \left(1 - L \frac{K_n}{n}\right) \geq 0. \quad (4.3)$$

Assume for now that  $\tau_n$  tends to infinity  $\mathbb{P}_{(0)}$ -almost surely as  $n \rightarrow \infty$ . Then, the left-hand side of (4.2) converges  $\mathbb{P}_{(0)}$ -almost surely to  $v_L - \mu$  as  $n \rightarrow \infty$ , so with (4.3) we get  $v_L - \mu \geq 0$  for  $L$  large enough, that is

$$v_L \geq \sqrt{2} - \frac{\pi^2}{2\sqrt{2}(1-\varepsilon)^2 L^2} + o\left(\frac{1}{L^2}\right)$$

and the lower bound in Theorem 1.2 follows by letting  $\varepsilon \rightarrow 0$ . It remains to show that  $\tau_n$  tends to infinity  $\mathbb{P}_{(0)}$ -almost surely. On the event  $\{(\tau_n)_{n \geq 0} \text{ is bounded}\}$ ,  $\tau_n$  tends to a finite limit  $\tau_\infty$  as  $n \rightarrow \infty$ , so it follows from (4.2) that  $\mathbb{P}_{(0)}$ -almost surely

$$\frac{\max X^L(\tau_\infty)}{\tau_\infty} = \infty,$$

for  $L$  large enough such that  $1 - 3h(L)L \geq 1/2$ . But  $\mathbb{P}_{(0)}$ -almost surely, for all  $t \geq 0$ ,  $\max X^L(t)/t < \infty$ , so  $\mathbb{P}_{(0)}((\tau_n)_{n \geq 0} \text{ is bounded}) = 0$ . It concludes the proof of the lower bound in Theorem 1.2.

## 4.2 Proof of Proposition 4.1

In this subsection, we prove Proposition 4.1, that is we show that the event  $A_0^\varepsilon$  is very likely. For this, we compare the  $L$ -BBM with a BBM in a strip that has less particles than the  $L$ -BBM between times 0 and  $\tau_1$ . Thus, if a particle of the BBM in a strip reaches  $\max X^L(0) + 1$ , then it is also the case for the  $L$ -BBM. To prove that there is a particle of the BBM in a strip reaching  $\max X^L(0) + 1$  with high probability, we will use Proposition 2.4, Proposition 2.5 and the following lemma that allows to have many independent BBM in a strip trying to reach  $\max X^L(0) + 1$  instead of only one.

**Lemma 4.2.** *Assume that  $X$  is a BBM with drift  $-\mu$  where  $\mu < \sqrt{2}$ . Then, for each  $\alpha > 0$ , for  $L$  large enough, we have*

$$\mathbb{P}_{(0)}(M(L^\alpha) < L \text{ or } \exists t \in [0, L^\alpha] : \min X(t) \leq -4L^\alpha) \leq 2Le^{-L^\alpha}.$$

Note that this lemma concerns branching Brownian motion without selection and is stated for a large choice of drift  $-\mu$  so that it can be applied here and also in Section 5.

*Proof.* Since  $M(L^\alpha)$  follows a geometric distribution with parameter  $e^{-L^\alpha}$ , we get

$$\mathbb{P}(M(L^\alpha) < L) = \sum_{k=1}^{L-1} e^{-L^\alpha} (1 - e^{-L^\alpha})^{k-1} = 1 - (1 - e^{-L^\alpha})^{L-1} \underset{L \rightarrow \infty}{\sim} L e^{-L^\alpha}. \quad (4.4)$$

Moreover, applying the many-to-one lemma (Lemma 2.1) without forgetting the drift  $-\mu$ , we have, with  $(B_t)_{t \geq 0}$  a Brownian motion,

$$\begin{aligned} \mathbb{P}(\exists t \in [0, L^\alpha] : \min X(t) \leq -4L^\alpha) &\leq e^{L^\alpha} \mathbb{P}(\exists t \in [0, L^\alpha] : B_t - \mu t \leq -4L^\alpha) \\ &\leq e^{L^\alpha} \mathbb{P}(\exists t \in [0, L^\alpha] : B_t \leq -(4 - \mu)L^\alpha) \\ &= e^{L^\alpha} 2\mathbb{P}(L^{\alpha/2} \mathcal{N}(0, 1) > (4 - \mu)L^\alpha) \\ &\leq e^{L^\alpha - (4 - \mu)^2 L^\alpha / 2}. \end{aligned} \quad (4.5)$$

The result follows from (4.4) and (4.5) with  $\mu \leq \sqrt{2}$ .  $\square$

We now have enough tools to prove Proposition 4.1. Actually, the control will be much more accurate than needed.

*Proof of Proposition 4.1.* Let  $\xi \in \mathcal{C}$  be a configuration. Note first that  $\mathbb{P}_\xi(A_0)$  is invariant under shift of  $\xi$ , so we can assume without loss of generality that  $\max \xi = L - 1$ . Then,  $\mathbb{P}_\xi$ -a.s.  $\tau_1 = L^3 \wedge \inf\{t \geq 0 : \max X^L(t) \notin (0, L)\}$  and  $A_0$  is  $\mathbb{P}_\xi$ -a.s. equal to the event “no particle of the  $L$ -BBM reaches  $L$  before reaching 0 on the time interval  $[0, L^3]$ ”. Moreover,  $\mathbb{P}_\xi$ -a.s. on the time interval  $[0, \tau_1]$  the killing barrier of the  $L$ -BBM stays under 0 (see Figure 3), so we have the inclusion

$$\forall t \in [0, \tau_1], \tilde{X}^{K,L}(t) \subset X^L(t), \quad (4.6)$$

where we set  $K := (1 - \varepsilon)L$  so that (2.3) is satisfied and  $A := -\varepsilon L$  so that  $K_A = L$ : thus, under  $\mathbb{P}_\xi$ ,  $(\tilde{X}_k^{K,L}(t), 1 \leq k \leq \tilde{M}^{K,L}(t))_{t \geq 0}$  is a BBM in the strip  $(0, L)$  with drift  $-\mu$  starting from the configuration  $\xi$ . Let  $C_0$  denote the event “no particle of  $\tilde{X}^{K,L}$  reaches  $L$  on the time interval  $[0, L^3]$ ”, then it follows from (4.6) that  $\mathbb{P}_\xi$ -a.s.  $A_0 \subset C_0$ . Moreover, the BBM in a strip verifies the branching property : the offspring of a single particle at  $x$  at time 0 is independent of the offspring of other initial particles and follows the law of a BBM in a strip under  $\mathbb{P}_x$ . So we have  $\mathbb{P}_\xi(C_0) \leq \mathbb{P}_{(L-1)}(C_0)$ , by keeping only the offspring of the highest initial particle. Thus, we get  $h(L) \leq \mathbb{P}_{(L-1)}(C_0)$  because it does not depend any more on the initial configuration.

Now, our aim is to give an upper bound for  $\mathbb{P}_{(L-1)}(C_0)$ . For this purpose, we will first use Lemma 4.2 to have at least  $L$  particles above  $L - 1 - 4L^\alpha$  after a short time  $L^\alpha$  and then apply Propositions 2.4 and 2.5 to show that each particle at time  $L^\alpha$  has a descendant that reaches  $L$  before time  $L^3$  with a positive probability that does not depend on  $L$ . We fix  $0 < \alpha < 1/2$ . Using Lemma 4.2, we get, for  $L$  large enough such that  $L - 1 - 4L^\alpha > 0$ ,

$$\begin{aligned} &\mathbb{P}_{(L-1)}(C_0) \\ &\leq 2Le^{-L^\alpha} + \mathbb{P}_{(L-1)}(\{M(L^\alpha) \geq L\} \cap \{\forall t \in [0, L^\alpha], \min X(t) > L - 1 - 4L^\alpha\} \cap C_0) \\ &\leq 2Le^{-L^\alpha} + \mathbb{P}_{(L-1)}(\{\tilde{M}^{K,L}(L^\alpha) \geq L\} \cap \{\min \tilde{X}^{K,L}(L^\alpha) > L - 1 - 4L^\alpha\} \cap C_0), \end{aligned} \quad (4.7)$$

because on the event  $\{\forall t \in [0, L^\alpha], \min X(t) \geq -4L^\alpha\} \cap C_0$ , no particle of the BBM in a strip is killed between times 0 and  $L^\alpha$ , so  $\tilde{M}(L^\alpha) = M(L^\alpha)$ . Applying the branching property at time  $L^\alpha$ , we bound from above the second term of (4.7) by

$$\mathbb{E}_{(L-1)} \left[ \mathbb{1}_{\tilde{M}^{K,L}(L^\alpha) \geq L} \mathbb{1}_{\min \tilde{X}^{K,L}(L^\alpha) > L-1-4L^\alpha} \prod_{k=1}^{\tilde{M}^{K,L}(L^\alpha)} \mathbb{P}_{(\tilde{X}_k^{K,L}(L^\alpha))} (C'_0) \right], \quad (4.8)$$

where  $C'_0$  denotes the event “no particle of  $\tilde{X}^{K,L}$  reaches  $L$  on the time interval  $[0, L^3 - L^\alpha]$ ”. Note that the function  $x \in (0, L) \mapsto \mathbb{P}_{(x)}(C'_0)$  is nondecreasing<sup>1</sup>. So it follows from (4.8) that

$$h(L) \leq 2Le^{-L^\alpha} + \mathbb{P}_{(L-1-4L^\alpha)}(C'_0)^L, \quad (4.9)$$

for  $L$  large enough.

Our aim is now to control  $\mathbb{P}_{(L-1-4L^\alpha)}(C'_0)$ . We set  $K' := L - 4L^\alpha$  and

$$\mu' := \sqrt{2 - \frac{\pi^2}{(L - 4L^\alpha)^2}},$$

the drift associated to  $K'$  (see equation (2.3)). Moreover, we define a new process  $\tilde{X}$  from the standard BBM  $X$  with drift  $-\mu$  by killing particles that go below  $t \mapsto (\mu' - \mu)t$  or above  $t \mapsto K' + (\mu' - \mu)t$  (see Figure 4). With this coupling, on the event  $C'_0$ , we have the

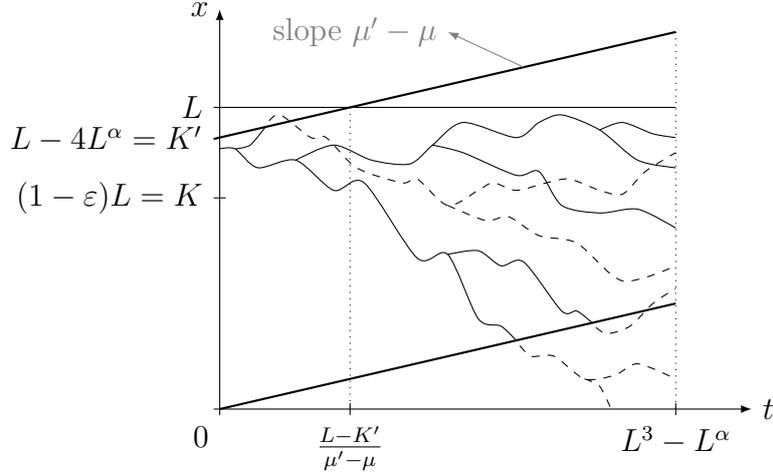


Figure 4: Representation of the coupled systems  $\tilde{X}$  (full line) and  $\tilde{X}^{K,L}$  (dashed line) starting with a single initial particle at  $K' - 1$ , on the event  $C'_0$ . The two thick straight lines of slope  $\mu' - \mu$  are the killing barriers that define  $\tilde{X}$ .

inclusion  $\forall t \in [0, L^3 - L^\alpha], \tilde{X}(t) \subset \tilde{X}^{K,L}(t)$  and it follows that

$$\begin{aligned} C'_0 &\subset \left\{ \text{no particle of } \tilde{X} \text{ reaches } L \text{ on time interval } \left[ \frac{L - K'}{\mu' - \mu}, L^3 - L^\alpha \right] \right\} \\ &\subset \left\{ \text{no particle of } \tilde{X} \text{ reaches } t \mapsto K' + (\mu' - \mu)t \text{ on } \left[ \frac{L - K'}{\mu' - \mu}, L^3 - L^\alpha \right] \right\}. \end{aligned}$$

<sup>1</sup>It follows from the fact that  $\mathbb{P}_{(x)}(C'_0) = \mathbb{P}_{(x)}(\text{no particle of } \hat{X} \text{ reaches } L \text{ on } [0, L^3 - L^\alpha])$ , where  $\hat{X}$  denotes the BBM with drift  $-\mu$ , with absorption at 0 and with a single initial particle at  $x$  under  $\mathbb{P}_{(x)}$ .

Then, recalling that  $\tilde{X}^{K'}$  denotes the BBM in the strip  $(0, K')$  with drift  $-\mu'$ , note that

$$\begin{aligned} & \mathbb{P}_{(L-1-4L^\alpha)} \left( \text{no particle of } \bar{X} \text{ reaches } t \mapsto K' + (\mu' - \mu)t \text{ on } \left[ \frac{L - K'}{\mu' - \mu}, L^3 - L^\alpha \right] \right) \\ & \leq \mathbb{P}_{(L-1-4L^\alpha)} \left( \text{no particle of } \tilde{X}^{K'} \text{ reaches } K' \text{ on } \left[ \frac{L - K'}{\mu' - \mu}, L^3 - L^\alpha \right] \right) \\ & \leq \mathbb{P}_{(L-1-4L^\alpha)} \left( \text{no particle of } \tilde{X}^{K'} \text{ reaches } K' \text{ on } [K'^{5/2}, K'^3] \right), \end{aligned}$$

using that  $L^3 - L^\alpha \geq K'^3$  and  $(L - K')/(\mu' - \mu) = C_\varepsilon L^{2+\alpha}(1 + o(1)) \leq K'^{5/2}$  for  $L$  large enough, with  $C_\varepsilon$  a positive constant depending only on  $\varepsilon$ . Thus, we get

$$\mathbb{P}_{(L-1-4L^\alpha)}(C'_0) \leq \mathbb{P}_{(L-1-4L^\alpha)}(R' = 0) = 1 - \mathbb{P}_{(K'-1)}(R' \geq 1) \leq 1 - \frac{\mathbb{E}_{(K'-1)}[R']^2}{\mathbb{E}_{(K'-1)}[R'^2]},$$

where  $R'$  is the number of particles of  $\tilde{X}^{K'}$  that hit  $K'$  between times  $K'^{5/2}$  and  $K'^3$ . We now want to give a lower bound for  $\mathbb{E}_{(K'-1)}[R']^2/\mathbb{E}_{(K'-1)}[R'^2]$ . Using first Proposition 2.4 with  $\theta = 1$ , we get

$$\mathbb{E}_{(K'-1)}[R'] \geq 2\sqrt{2}\pi K' e^{-\mu K'} e^{\mu(K'-1)} \sin\left(\frac{\pi(K'-1)}{K'}\right) (1 + o(1)) \xrightarrow{L \rightarrow \infty} 2\sqrt{2}\pi^2 e^{-\mu}.$$

Then, using Proposition 2.5 with  $R$  denoting the number of particles of  $\tilde{X}^{K'}$  that hit  $K'$  between times 0 and  $K'^3$ , we get

$$\begin{aligned} \mathbb{E}_{(K'-1)}[R'^2] & \leq \mathbb{E}_{(K'-1)}[R^2] \\ & \leq C e^{-\mu K'} e^{\mu(K'-1)} \left( 1 + K' \sin\left(\frac{\pi(K'-1)}{K'}\right) \right) (1 + o(1)) \\ & \xrightarrow{L \rightarrow \infty} C e^{-\mu} (1 + \pi). \end{aligned}$$

So, for  $L$  large enough, we have  $\mathbb{E}_{(K'-1)}[R']^2/\mathbb{E}_{(K'-1)}[R'^2] \geq c > 0$ . Coming back to (4.9), we get that, for  $L$  large enough,

$$h(L) \leq 2L e^{-L^\alpha} + (1 - c)^L,$$

and the result follows.  $\square$

## 5 Upper bound for $v_L$

As in Section 4, we fix  $0 < \varepsilon < 1/3$  and we consider all processes with drift  $-\mu$ , but we set here

$$\mu := \sqrt{2 - \frac{\pi^2}{(1 + 4\varepsilon)^2 L^2}}.$$

We want to show that for  $L$  large enough,  $\mathbb{P}_{(0)}$ -a.s.  $\limmax X^L(t)/t \leq C\varepsilon/L^2$ , that is  $v_L \leq \mu + C\varepsilon/L^2$ , and then the upper bound follows by letting  $\varepsilon \rightarrow 0$ . Moreover, we set for  $j \geq 1$ ,  $L_j := (1 + j\varepsilon)L$ . Thus,  $\mu$  is the drift corresponding to  $L_4$  according to the results of Subsection 2.2.

## 5.1 Proof of the upper bound

In this subsection, we prove the upper bound in Theorem 1.2, by postponing to the next subsections the proof of two propositions. As in Section 4, the  $L$ -BBM will be studied on time intervals of length at most  $L^3$  associated to a sequence of stopping times  $(\tau_i)_{i \in \mathbb{N}}$  defined by  $\tau_0 := 0$  and for each  $i \in \mathbb{N}$ ,

$$\tau_{i+1} := \inf \left\{ t \geq \tau_i : \max X^L(t) - \max X^L(\tau_i) \notin \left( -\varepsilon L + \frac{2\varepsilon}{L^2}(t - \tau_i), \varepsilon L \right) \right\}.$$

So we have  $\tau_{i+1} - \tau_i \leq L^3$  (see Figure 5). We also define the event  $A_i := \{\max X^L(\tau_{i+1}) = \max X^L(\tau_i) + \varepsilon L\}$ , as a “bad” event: on  $A_i$ ,  $\max X^L$  can increase quickly between times  $\tau_i$  and  $\tau_{i+1}$ .

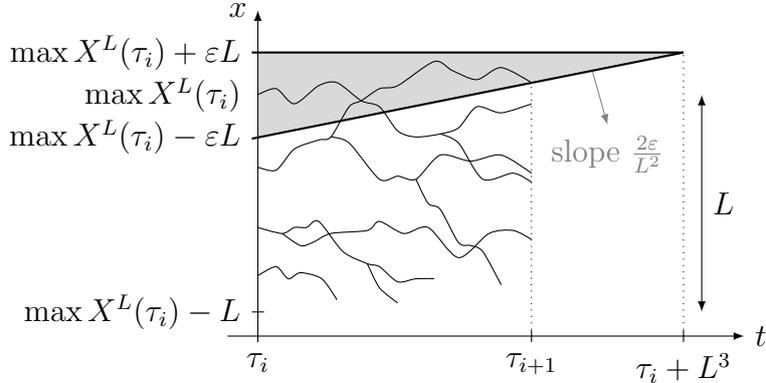


Figure 5: Representation of a  $L$ -BBM between times  $\tau_i$  and  $\tau_{i+1}$ . By definition,  $\tau_{i+1}$  is the time where  $t \mapsto \max X^L(t)$  leaves the gray area. The two thick straight lines that delimit the gray area intersect at time  $\tau_i + L^3$ . We are here on the event  $A_i^c$  because  $t \mapsto \max X^L(t)$  leaves the area from below.

Let  $K_n := \sum_{i=0}^{n-1} \mathbb{1}_{A_i}$  be the number of “bad” events  $A_i$  that happen before time  $\tau_n$ . On the event  $A_i^c$ , we have  $\max X^L(\tau_{i+1}) - \max X^L(\tau_i) = -\varepsilon L + 2\varepsilon(\tau_{i+1} - \tau_i)/L^2$  and, on the event  $A_i$ , we have  $\max X^L(\tau_{i+1}) - \max X^L(\tau_i) = \varepsilon L$ . Therefore, we get

$$\begin{aligned} \max X^L(\tau_n) - \max X^L(0) &= \sum_{i=0}^{n-1} \mathbb{1}_{A_i^c} \varepsilon L + \sum_{i=0}^{n-1} \mathbb{1}_{A_i} \left( -\varepsilon L + \frac{2\varepsilon}{L^2}(\tau_{i+1} - \tau_i) \right) \\ &\leq K_n \varepsilon L - (n - K_n) \varepsilon L + \frac{2\varepsilon}{L^2} \tau_n \end{aligned}$$

and so

$$\frac{\max X^L(\tau_n) - \max X^L(0)}{\tau_n} \leq \frac{2\varepsilon}{L^2} + \frac{n}{\tau_n} \left( -\varepsilon L + 2 \frac{K_n}{n} \varepsilon L \right). \quad (5.1)$$

We now need two propositions to conclude. The first one shows that events  $A_i$  happen only just more than half the time. Its proof is postponed to Subsections 5.2 and 5.3.

**Proposition 5.1.** *For  $L$  large enough,  $\mathbb{P}_{(0)}$ -a.s. we have*

$$\limsup_{n \rightarrow \infty} \frac{K_n}{n} \leq \frac{1}{2} + \frac{1}{\varepsilon L}.$$

The second proposition gives a lower bound for  $\tau_n/n$  as  $n \rightarrow \infty$  and shows that it is much larger than  $L^2$ . Its proof is postponed to Subsection 5.4.

**Proposition 5.2.** *There exists  $\gamma > 0$  such that, for  $L$  large enough,  $\mathbb{P}_{(0)}$ -a.s. we have*

$$\liminf_{n \rightarrow \infty} \frac{\tau_n}{n} \geq \frac{L^{2+\gamma}}{6}.$$

We can now conclude the proof of the upper bound. Using Proposition 5.1, we get that  $\mathbb{P}_{(0)}$ -a.s.  $\limsup_{n \rightarrow \infty} (-\varepsilon L + 2\frac{K_n}{n}\varepsilon L) \leq 2$ . Therefore, it follows from (5.1) that  $\mathbb{P}_{(0)}$ -a.s.

$$\lim_{n \rightarrow \infty} \frac{\max_{\tau_n} X^L(\tau_n)}{\tau_n} \leq \frac{2\varepsilon}{L^2} + 2 \limsup_{n \rightarrow \infty} \frac{n}{\tau_n} \leq \frac{2\varepsilon}{L^2} + \frac{12}{L^{2+\gamma}}, \quad (5.2)$$

for  $L$  large enough, applying Proposition 5.2. From Proposition 5.2, we get that  $\mathbb{P}_{(0)}$ -a.s.  $\tau_n \rightarrow \infty$  as  $n \rightarrow \infty$ , so the left-hand side of (5.2) is equal to  $v_L - \mu$ . Thus, we have

$$v_L \leq \sqrt{2} - \frac{\pi^2}{2\sqrt{2}(1+4\varepsilon)^2 L^2} + \frac{2\varepsilon}{L^2} + o\left(\frac{1}{L^2}\right)$$

and, letting  $\varepsilon \rightarrow 0$ , we get the upper bound in Theorem 1.2.

## 5.2 Comparison with the BBM in a strip

We use here results of Subsection 2.2 concerning BBM in a strip to show two lemmas that will be useful for the proof of Proposition 5.1 in the next subsection. To bound the probability of events  $A_i$ , we will need to control the size of the  $L$ -BBM at times  $\tau_i$ . For this, we introduce a functional of the  $L$ -BBM, analogous to the functional  $\tilde{Z}^{K,KA}$  of the BBM in a strip (see equation (2.4)): recalling that  $L_j = (1+j\varepsilon)L$ , we set, for  $i \in \mathbb{N}$ ,

$$U_i := \max X^L(\tau_i) - L_2$$

$$S_i^L := \sum_{k=1}^{M^L(\tau_i)} e^{\mu(X_k^L(\tau_i) - U_i)} \sin\left(\frac{\pi(X_k^L(\tau_i) - U_i)}{L_4}\right).$$

It amounts to shift the population at time  $\tau_i$  such that the highest particle is at  $L_2$  and then to take the value of the functional  $\tilde{Z}^{L_4}$  associated with the shifted population. We can now state the two lemmas of this subsection, used later for the proof of Proposition 5.1. The first one gives an upper bound for the conditional probability of event  $A_i$  given  $\mathcal{F}_{\tau_i}$ , in terms of  $S_i^L$ : if the size of the  $L$ -BBM at time  $\tau_i$  is small enough, then  $A_i$  is unlikely.

**Lemma 5.3.** *There exists  $C_\varepsilon > 0$  depending only on  $\varepsilon$  such that, for  $L$  large enough, for all  $i \in \mathbb{N}$ ,*

$$\mathbb{P}_{(0)}(A_i \mid \mathcal{F}_{\tau_i}) \leq C_\varepsilon L e^{-\mu L_3} S_i^L.$$

Therefore, according to Lemma 5.3, we need to control  $S_i^L$  in order to bound the probability of  $A_i$ . The second lemma controls the conditional expectation of  $S_{i+1}^L$  given  $\mathcal{F}_{\tau_i}$  in terms of  $S_i^L$ , with a poor bound in the general case but with a much more accurate

one on the event  $A_i$ . Indeed, even if  $A_i$  is a “bad” event because it involves a growth of  $\max X^L$ , it causes at the same time a large decrease of the population size: when  $\max X^L$  grows quickly, more particles are killed by selection. So each event  $A_i$  that happens makes the following events  $A_j$  less likely.

**Lemma 5.4.** *We have the following inequalities for  $L$  large enough and for all  $i \in \mathbb{N}$ :*

$$\mathbb{E}_{(0)}[S_{i+1}^L \mid \mathcal{F}_{\tau_i}] \leq 2e^{\mu\varepsilon L} S_i^L, \quad (5.3)$$

$$\mathbb{E}_{(0)}[S_{i+1}^L \mathbb{1}_{A_i} \mid \mathcal{F}_{\tau_i}] \leq 2e^{-\mu\varepsilon L} S_i^L. \quad (5.4)$$

*Proof of Lemma 5.3.* The strategy is as follows: we first come down to the study of  $\mathbb{P}_\xi(A_0)$ , where  $\max \xi = L_2$  and  $\min \xi \geq 2\varepsilon L$ , and then use Proposition 2.4 concerning BBM in a strip to bound from above the mean number of particles that hit  $L_3$  and thus the probability of event  $A_0$ . Applying the strong Markov property at the stopping time  $\tau_i$ , we get

$$\mathbb{P}_{(0)}(A_i \mid \mathcal{F}_{\tau_i}) = \mathbb{P}_{X^L(\tau_i)}(A_0) = \mathbb{P}_{X^L(\tau_i) - U_i}(A_0), \quad (5.5)$$

using for the second equality the fact that  $\mathbb{P}_\xi(A_0)$  is invariant under shift of the initial configuration  $\xi$ . We have  $\max X^L(\tau_i) - U_i = L_2$  by definition of  $U_i$  and therefore  $\min X^L(\tau_i) - U_i \geq 2\varepsilon L$ , so we have to bound  $\mathbb{P}_\xi(A_0)$  with an initial configuration  $\xi$  that verifies  $\max \xi = L_2$  and  $\min \xi \geq 2\varepsilon L$ .

We fix such a configuration  $\xi$ . Then,  $\mathbb{P}_\xi$ -a.s., for all  $t \in [0, \tau_1)$ , we have the inclusion  $X^L(t) \subset \tilde{X}^{L_4, L_3}(t)$ , because until time  $\tau_1$  the killing barrier of  $X^L$  stays above 0 (see Figure 6) and particles of  $X^L$  stay below  $L_3$  so no particle of  $\tilde{X}^{L_4, L_3}$  is killed by hitting  $L_3$ . Therefore, we get that  $\mathbb{P}_\xi$ -a.s.

$$A_0 \subset \left\{ \text{at least one particle of } \tilde{X}^{L_4, L_3} \text{ hits } L_3 \text{ on time interval } [0, L^3] \right\}. \quad (5.6)$$

Then, our aim is to come down to  $\tilde{X}^{L_3}$  instead of  $\tilde{X}^{L_4, L_3}$ , in order to apply Proposition 2.4. For this, let

$$\mu' := \sqrt{2 - \frac{\pi^2}{L_3^2}}$$

be the drift associated to  $L_3$  (see equation (2.3)). We have  $\mu' < \mu$ , so the right-hand side of (5.6) is obviously included in the event

$$\left\{ \text{at least one particle of } \tilde{X}^{L_4, L_3} \text{ hits } t \mapsto L_3 + (\mu' - \mu)t \text{ on time interval } [0, L^3] \right\}. \quad (5.7)$$

As in the proof of Proposition 4.1, we define now a new process  $\bar{X}$  from the standard BBM  $X$  with drift  $-\mu$  by killing particles that go below  $t \mapsto (\mu' - \mu)t$  or above  $t \mapsto L_3 + (\mu' - \mu)t$ . Thus, the event in (5.7) is included in the event

$$\left\{ \text{at least one particle of } \bar{X} \text{ hits } t \mapsto L_3 + (\mu' - \mu)t \text{ on time interval } [0, L^3] \right\},$$

because as long as no particle of  $\bar{X}$  reaches  $t \mapsto L_3 + (\mu' - \mu)t$ , the population of  $\tilde{X}^{L_4, L_3}$  is included in the population of  $\bar{X}$ . The probability of this last event under  $\mathbb{P}_\xi$  is equal to

$$\mathbb{P}_\xi \left( \text{at least one particle of } \tilde{X}^{L_3} \text{ hits } L_3 \text{ on time interval } [0, L^3] \right).$$

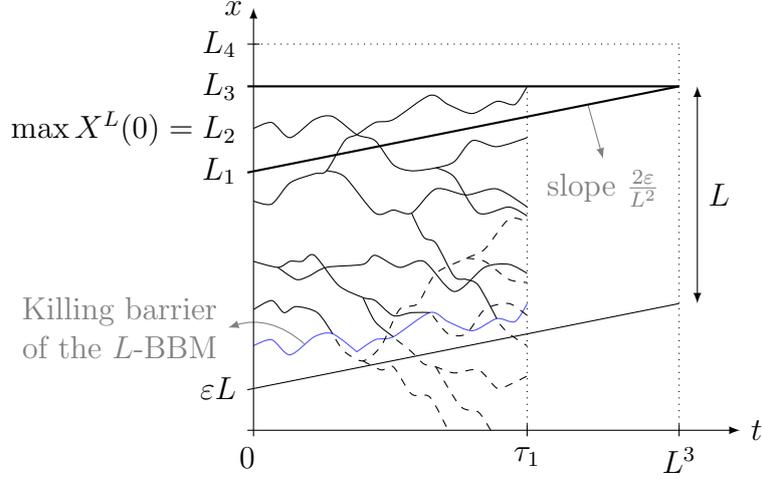


Figure 6: Representation of the coupled systems  $X^L$  (full line) and  $\tilde{X}^{L_4, L_3}$  (dashed line) between times 0 and  $\tau_1$ , starting with the highest particle at  $L_2$ , on the event  $A_0$ . Until time  $\tau_1$ , the killing barrier of  $X^L$  (drawn in blue) stays above the straight line  $t \mapsto \varepsilon L + \frac{2\varepsilon}{L^2}t$  so above 0.

Therefore, coming back to (5.6), we showed that  $\mathbb{P}_\xi(A_0) \leq \mathbb{E}_\xi[R]$ , where  $R$  is the number of particles of  $\tilde{X}^{L_3}$  that hit  $L_3$  between times 0 and  $L^3$ , and applying Proposition 2.4 with  $\theta = 1/(1 + 3\varepsilon)^3$ , we get that, as  $L \rightarrow \infty$ ,

$$\mathbb{E}_\xi[R] \leq C \frac{e^{-\mu L_3}}{L_3} \sum_{k=1}^n \xi_k e^{\mu \xi_k} + 2\sqrt{2}\pi\theta L_3 e^{-\mu L_3} \left( \sum_{k=1}^n e^{\mu \xi_k} \sin\left(\frac{\pi \xi_k}{L_3}\right) \right) (1 + o(1)) \quad (5.8)$$

where we wrote  $\xi = (\xi_1, \dots, \xi_n)$ . In order to make  $S_i^L$  appear, our upper bound has to depend on  $\sum_{k=1}^n e^{\mu \xi_k} \sin(\pi \xi_k / L_4)$ . Recalling that  $\xi_k \in [2\varepsilon L, L_2]$  for each  $1 \leq k \leq n$ , we get

$$\frac{\xi_k}{L_2} \leq \frac{\sin\left(\frac{\pi \xi_k}{L_3}\right)}{\sin\left(\frac{\pi L_2}{L_3}\right)} = \frac{\sin\left(\frac{\pi \xi_k}{L_3}\right)}{\sin\left(\frac{\pi \varepsilon}{(1+3\varepsilon)}\right)}$$

and it follows that

$$\sum_{k=1}^n \xi_k e^{\mu \xi_k} \leq \frac{L_2}{\sin\left(\frac{\pi \varepsilon}{(1+3\varepsilon)}\right)} \sum_{k=1}^n e^{\mu \xi_k} \sin\left(\frac{\pi \xi_k}{L_3}\right). \quad (5.9)$$

Moreover, using the inequality  $\sin(\pi \xi_k / L_3) / \sin(\pi \xi_k / L_4) \leq L_4 / L_3$ , we have

$$\sum_{k=1}^n e^{\mu \xi_k} \sin\left(\frac{\pi \xi_k}{L_3}\right) \leq \frac{1 + 4\varepsilon}{1 + 3\varepsilon} \sum_{k=1}^n e^{\mu \xi_k} \sin\left(\frac{\pi \xi_k}{L_4}\right). \quad (5.10)$$

Thus, bringing together (5.8), (5.9) and (5.10), we get that, for  $L$  large enough,

$$\mathbb{P}_\xi(A_0) \leq C_\varepsilon L e^{-\mu L_3} \sum_{k=1}^n e^{\mu \xi_k} \sin\left(\frac{\pi \xi_k}{L_4}\right)$$

and coming back to (5.5), the result follows with  $\xi = X^L(\tau_i) - U_i$ .  $\square$

*Proof of Lemma 5.4.* The reasoning is similar to the proof of Lemma 5.3: we first come down to the study of  $S_1^L$  under  $\mathbb{P}_\xi$  where  $\max \xi = L_2$ , then we compare  $S_1^L$  with  $\tilde{Z}^{L_4}(\tau_1)$  under  $\mathbb{P}_\xi$  in the general case and on the event  $A_0$  and finally we use Proposition 2.3 and the optional stopping theorem to get an upper bound for  $\mathbb{E}_\xi[\tilde{Z}^{L_4}(\tau_1)]$ .

Applying the strong Markov property at the stopping time  $\tau_i$  and using that the laws of  $S_1^L$  and  $S_1^L \mathbb{1}_{A_0}$  are invariant under shift of the initial configuration, we get

$$\mathbb{E}_{(0)}[S_i^L \mid \mathcal{F}_{\tau_i}] = \mathbb{E}_{X^L(\tau_i)}[S_1^L] = \mathbb{E}_{X^L(\tau_i)-U_i}[S_1^L] \quad (5.11)$$

and, in the same way,  $\mathbb{E}_{(0)}[S_i^L \mathbb{1}_{A_0} \mid \mathcal{F}_{\tau_i}] = \mathbb{E}_{X^L(\tau_i)-U_i}[S_1^L \mathbb{1}_{A_0}]$ . Thus, we now have to bound  $\mathbb{E}_\xi[S_1^L]$  and  $\mathbb{E}_\xi[S_1^L \mathbb{1}_{A_0}]$ , for an initial configuration  $\xi$  that verifies  $\max \xi = L_2$ .

We fix such a configuration  $\xi$ . As in the proof of Lemma 5.3, we note that  $\mathbb{P}_\xi$ -a.s., for all  $t \in [0, \tau_1]^2$ , we have the inclusion  $X^L(t) \subset \tilde{X}^{L_4}(t)$  (see Figure 6). Thus, we have  $\mathbb{P}_\xi$ -a.s.

$$\sum_{k=1}^{M^L(\tau_1)} e^{\mu X_k^L(\tau_1)} \sin\left(\frac{\pi X_k^L(\tau_1)}{L_4}\right) \leq \tilde{Z}^{L_4}(\tau_1). \quad (5.12)$$

Therefore, we want to compare  $S_1^L$  with the left-hand side of (5.12), first in the general case and then on the event  $A_0$ . Note that  $U_1 \in [-\varepsilon L, \varepsilon L]$  and for  $1 \leq k \leq M^L(\tau_1)$ ,  $X_k^L(\tau_1) \in [\max X^L(\tau_1) - L, \max X^L(\tau_1)] = [U_1 + 2\varepsilon L, U_1 + L_2]$ . On the one hand, if  $u \leq 0$ , then the function  $x \in (0, \pi + u] \mapsto \sin(x - u) / \sin(x)$  is nonincreasing, so if  $U_1 \leq 0$ , we get the upper bound

$$\frac{\sin\left(\frac{\pi(X_k^L(\tau_1)-U_1)}{L_4}\right)}{\sin\left(\frac{\pi X_k^L(\tau_1)}{L_4}\right)} \leq \frac{\sin\left(\frac{\pi 2\varepsilon L}{L_4}\right)}{\sin\left(\frac{\pi(U_1+2\varepsilon L)}{L_4}\right)} \leq \frac{\sin\left(\frac{\pi 2\varepsilon}{1+4\varepsilon}\right)}{\sin\left(\frac{\pi\varepsilon}{1+4\varepsilon}\right)} \leq 2.$$

On the other hand, if  $u \geq 0$ , then the function  $x \in (u, \pi] \mapsto \sin(x - u) / \sin(x)$  is nondecreasing, so if  $U_1 \geq 0$ , we get the upper bound

$$\frac{\sin\left(\frac{\pi(X_k^L(\tau_1)-U_1)}{L_4}\right)}{\sin\left(\frac{\pi X_k^L(\tau_1)}{L_4}\right)} \leq \frac{\sin\left(\frac{\pi L_2}{L_4}\right)}{\sin\left(\frac{\pi(U_1+L_2)}{L_4}\right)} \leq \frac{\sin\left(\frac{\pi 2\varepsilon}{1+4\varepsilon}\right)}{\sin\left(\frac{\pi\varepsilon}{1+4\varepsilon}\right)} \leq 2.$$

It follows that

$$S_1^L \leq 2e^{-\mu U_1} \sum_{k=1}^{M^L(\tau_1)} e^{\mu X_k^L(\tau_1)} \sin\left(\frac{\pi X_k^L(\tau_1)}{L_4}\right).$$

Coming back to (5.12) and using that  $U_1 \geq -\varepsilon L$  and, on the event  $A_0$ ,  $U_1 = \varepsilon L$ , we get

$$\begin{aligned} \mathbb{E}_\xi[S_1^L] &\leq 2e^{\mu\varepsilon L} \mathbb{E}_\xi[\tilde{Z}^{L_4}(\tau_1)], \\ \mathbb{E}_\xi[S_1^L \mathbb{1}_{A_0}] &\leq 2e^{-\mu\varepsilon L} \mathbb{E}_\xi[\tilde{Z}^{L_4}(\tau_1)]. \end{aligned} \quad (5.13)$$

---

<sup>2</sup>Note that here the inclusion is still true at time  $\tau_1$ , because a particle of  $\tilde{X}^{L_4}$  that hits  $L_3$  is not killed, unlike a particle of  $\tilde{X}^{L_4, L_3}$ .

Note that the bound for  $S_1^L \mathbb{1}_{A_0}$  is better not because the event  $A_0$  is unlikely, but because it involves a large increase of  $\max X^L$  while most of the particles stay at the same height (because  $\mu$  is chosen such that  $\tilde{Z}^{L_4}$  is a martingale) and so all these particles have a much smaller weight in  $S_1^L$ .

Finally, applying the optional stopping theorem to  $(\tilde{Z}^{L_4}(t))_{t \geq 0}$ , which is a martingale by Proposition 2.3, and to  $\tau_1$ , which is a bounded stopping time, we get

$$\mathbb{E}_\xi \left[ \tilde{Z}^{L_4}(\tau_1) \right] \leq \sum_{k=1}^n e^{\mu \xi_k} \sin\left(\frac{\pi \xi_k}{L_4}\right). \quad (5.14)$$

The result follows from (5.11), (5.13) and (5.14) with  $\xi = X^L(\tau_i) - U_i$ .  $\square$

### 5.3 Proof of Proposition 5.1

We prove here Proposition 5.1 that states that  $K_n$ , the number of events  $A_i$  that happen before time  $\tau_n$ , cannot be much larger than  $n/2$ .

*Proof of Proposition 5.1.* For  $1 \leq k \leq n$ , we first give an upper bound for  $\mathbb{P}_{(0)}(K_n \geq k)$ . We have

$$\mathbb{P}_{(0)}(K_n \geq k) \leq \sum_{1 \leq i_1 < \dots < i_k \leq n} \mathbb{P}_{(0)}(A_{i_1} \cap \dots \cap A_{i_k}).$$

So we fix  $1 \leq i_1 < \dots < i_k \leq n$  and deal with  $\mathbb{P}_{(0)}(A_{i_1} \cap \dots \cap A_{i_k})$ . The strategy is to control  $S_i^L$  for  $1 \leq i \leq i_k$  using Lemma 5.4 and then to bound  $\mathbb{P}_{(0)}(A_{i_k} | \mathcal{F}_{\tau_{i_k}})$  with Lemma 5.3 and our control of  $S_{i_k}^L$ : if  $k$  is large, then  $S_{i_k}^L$  is small and  $A_{i_k}$  is unlikely. First conditioning on  $\mathcal{F}_{\tau_{i_k}}$ , using that for all  $i \geq 0$ ,  $A_i \in \mathcal{F}_{\tau_{i+1}}$  and then applying Lemma 5.3, we get

$$\begin{aligned} \mathbb{P}_{(0)}(A_{i_1} \cap \dots \cap A_{i_k}) &= \mathbb{E}_{(0)} \left[ \mathbb{1}_{A_{i_1}} \dots \mathbb{1}_{A_{i_{k-1}}} \mathbb{P}_{(0)}(A_{i_k} | \mathcal{F}_{\tau_{i_k}}) \right] \\ &\leq C_\varepsilon L e^{-\mu L_3} \mathbb{E}_{(0)} \left[ \mathbb{1}_{A_{i_1}} \dots \mathbb{1}_{A_{i_{k-1}}} S_{i_k}^L \right]. \end{aligned} \quad (5.15)$$

Then, conditioning successively on all  $\mathcal{F}_i$  for  $i$  from  $i_k - 1$  to 0 and applying Lemma 5.4 (we use (5.3) if  $i \notin \{i_1, \dots, i_{k-1}\}$  and (5.4) otherwise), we bound (5.15) by

$$C_\varepsilon L e^{-\mu L_3} \left( 2e^{\mu \varepsilon L} \right)^{i_k - (k-1)} \left( 2e^{-\mu \varepsilon L} \right)^{k-1} \mathbb{E}_{(0)}[S_0^L]$$

and it follows that

$$\mathbb{P}_{(0)}(K_n \geq k) \leq \binom{n}{k} C_\varepsilon L \left( 2e^{\mu \varepsilon L} \right)^{n - (k-1)} \left( 2e^{-\mu \varepsilon L} \right)^{k-1}, \quad (5.16)$$

using that  $\mathbb{E}_{(0)}[S_0^L] = e^{\mu L_2} \sin(\pi L_2 / L_4) \leq e^{\mu L_3}$ .

We take now

$$k := \left\lceil n \left( \frac{1}{2} + \frac{1}{\varepsilon L} \right) \right\rceil.$$

Using  $\binom{n}{k} \leq 2^n$ , (5.16) becomes

$$\mathbb{P}_{(0)}(K_n \geq k) \leq C_\varepsilon L 2^{2n} \exp(\mu \varepsilon L (n - 2(k-1))) \leq CL 2^{2n} \exp(-2\mu n),$$

which is summable because  $2 < e^\mu$  for  $L$  large enough, so the Borel-Cantelli lemma implies that  $\mathbb{P}_{(0)}$ -almost surely for  $n$  large enough we have  $K_n \leq k - 1$  and the result follows.  $\square$

## 5.4 Proof of Proposition 5.2

In this section, we fix  $0 < \gamma < 1/7$  and we will show that, on the event  $A_i^c$ , with high probability we have  $\tau_{i+1} - \tau_i \geq L^{2+\gamma}$ : the standard BBM starting at  $\max X^L(\tau_i)$  at time  $\tau_i$  has with high probability a particle that stays above  $t \mapsto \max X^L(\tau_i) - \varepsilon L + 2\varepsilon(t - \tau_i)/L^2$  between times  $\tau_i$  and  $\tau_i + L^{2+\gamma}$ . Then, by Proposition 5.1, we know that events  $A_i^c$  happen about half time so we have often enough  $\tau_{i+1} - \tau_i \geq L^{2+\gamma}$  and Proposition 5.2 will follow. Therefore, we first show a lemma concerning the standard BBM starting with a single particle (we still work with drift  $-\mu$ ).

**Lemma 5.5.** *We define  $T := L^\gamma + L^{2-5\gamma}$  and the event*

$$C := \left\{ \exists k \in \llbracket 1, M(T) \rrbracket : X_k(T) \geq -5L^\gamma \text{ and } \forall t \in [0, T], X_{k,T}(t) \geq -\frac{\varepsilon L}{2} \right\},$$

where  $(X_{k,T}(t))_{0 \leq t \leq T}$  denotes the trajectory between times 0 and  $T$  of the particle that is at  $X_k(T)$  at time  $T$ . Then, for  $L$  large enough, we have  $\mathbb{P}_{(0)}(C) \geq 1 - 3Le^{-L^\gamma}$ .

*Proof.* The strategy is to use first Lemma 4.2 in order to get more than  $L$  particles after a short time  $L^\gamma$  and then Proposition 2.2 to see that each of these particles is likely to stay high between times  $L^\gamma$  and  $T$ . Applying Lemma 4.2 (we still have  $\mu \leq \sqrt{2}$ ), we get

$$\mathbb{P}_{(0)}(C^c) \leq 2Le^{-L^\gamma} + \mathbb{P}(\{M(L^\gamma) \geq L\} \cap \{\forall t \in [0, L^\gamma], \min X(t) > -4L^\gamma\} \cap C^c). \quad (5.17)$$

Then, using the branching property at time  $L^\gamma$ , the probability on the right-hand side of (5.17) is equal to

$$\mathbb{E}_{(0)} \left[ \mathbb{1}_{M(L^\gamma) \geq L} \mathbb{1}_{\forall t \in [0, L^\gamma], \min X(t) > -4L^\gamma} \prod_{i=1}^{M(L^\gamma)} \mathbb{P}_{(X_i(L^\gamma))}(C_1^c) \right], \quad (5.18)$$

for  $L$  large enough, where we set

$$C_1 := \left\{ \exists k \in \llbracket 1, M(L^{2-5\gamma}) \rrbracket : X_k(L^{2-5\gamma}) \geq -5L^\gamma \text{ and } \min X_{k, L^{2-5\gamma}} \geq -\frac{\varepsilon L}{2} \right\}.$$

We now want to bound  $\mathbb{P}_{(x)}(C_1^c)$  for  $x \geq -4L^\gamma$ . First, as  $x \mapsto \mathbb{P}_{(x)}(C_1^c)$  is nonincreasing, it is clear that  $\mathbb{P}_{(x)}(C_1^c) \leq \mathbb{P}_{(-4L^\gamma)}(C_1^c) = \mathbb{P}_{(0)}(C_2^c)$ , where we set

$$C_2 := \left\{ \exists k \in \llbracket 1, M(L^{2-5\gamma}) \rrbracket : X_k(L^{2-5\gamma}) \geq -L^\gamma \text{ and } \min X_{k, L^{2-5\gamma}} \geq -\frac{\varepsilon L}{2} + 4L^\gamma \right\}.$$

Then, we apply Proposition 2.2 to  $\delta = 1/2$ : there exist  $d > 0$ ,  $r > 0$  and  $t_0 > 0$  large enough such that for all  $t \geq t_0$ ,  $\mathbb{P}_{(0)}(D_t) \geq 1/2$ , where we set for all  $t \geq 0$

$$D_t := \left\{ \exists k \in \llbracket 1, M(t) \rrbracket : X_k(t) \geq m(t) - \mu t - d \right. \\ \left. \text{and } \forall s \in [0, t], X_{k,t}(s) \geq \frac{s}{t} m(t) - \mu s - r \vee \left( s^{\frac{1}{2}+\gamma} \wedge (t-s)^{\frac{1}{2}+\gamma} \right) \right\}.$$

Note that  $D_{L^{2-5\gamma}} \subset C_2$  for  $L$  large enough<sup>3</sup>, so we have showed that for  $x \geq -4L^\gamma$ ,  $\mathbb{P}_{(x)}(C_1^c) \leq \mathbb{P}_{(0)}(D_{L^{2-5\gamma}}^c) \leq 1/2$  for  $L$  large enough such that  $L^{2-5\gamma} \geq t_0$  (because  $2-5\gamma > 0$ ). Thus, we get (5.18)  $\leq 1/2^L$  and, combining with (5.17), the result follows.  $\square$

We now state a corollary that will be used in the proof of Proposition 5.2: it says that a standard BBM with drift  $-\mu$  starting with a single particle at 0 has with high probability a particle that stays above  $t \mapsto -\varepsilon L + 2\varepsilon t/L^2$  between times 0 and  $L^{2+\gamma}$ .

**Corollary 5.6.** *We define the event*

$$E := \left\{ \exists k \in \llbracket 1, M(L^{2+\gamma}) \rrbracket : \forall t \in [0, L^{2+\gamma}], X_{k, L^{2+\gamma}}(t) > -\varepsilon L + \frac{2\varepsilon}{L^2} t \right\},$$

where  $(X_{k, L^{2+\gamma}}(t))_{0 \leq t \leq L^{2+\gamma}}$  denotes the trajectory between times 0 and  $T$  of the particle that is at  $X_k(L^{2+\gamma})$  at time  $L^{2+\gamma}$ . Then,  $\mathbb{P}_{(0)}(E)$  tends to 1 as  $L \rightarrow \infty$ .

*Proof.* First note that the slope  $2\varepsilon/L^2$  plays only a negligible role on a time period of length  $L^{2+\gamma}$  and that we can replace  $L^{2+\gamma}$  by  $NT$  where  $N := \lceil L^{6\gamma} \rceil$ : we have  $\bar{E} \subset E$  with

$$\bar{E} := \{ \exists k \in \llbracket 1, M(NT) \rrbracket : \forall t \in [0, NT], X_{k, NT}(t) \geq -\varepsilon L + L^\gamma \}.$$

We now want to apply Lemma 5.5 to  $N$  consecutive time intervals of length  $T$ . Formally, we introduce for  $0 \leq j \leq N$  the event

$$\bar{E}_j := \{ \exists k \in \llbracket 1, M(jT) \rrbracket : \min X_{k, jT} \geq -\varepsilon L + (5(N-j)+1)L^\gamma \}$$

and we have then, for all  $0 \leq j \leq N-1$  and for  $L$  large enough such that  $(5N+1)L^\gamma \leq \varepsilon L/2$  (which is possible since  $\gamma < 1/7$ ),

$$\begin{aligned} \bar{E}_{j+1} \supset \left\{ \exists i \in \llbracket 1, M(T) \rrbracket : X_i(T) \geq -5L^\gamma \text{ and } \forall t \in [0, T], X_{i, T}(t) \geq -\varepsilon L/2 \text{ and} \right. \\ \left. \exists k \in \llbracket 1, M^i(jT) \rrbracket : X_i(T) + \min X_{k, jT}^i \geq -\varepsilon L + (5(N-j-1)+1)L^\gamma \right\}, \end{aligned} \quad (5.19)$$

where  $X^i$  denotes the BBM emanating from the particle at  $X_i(T)$  at time  $T$  in the BBM  $X$ , that is then shifted so that it starts from 0 at time 0. According to the branching property,  $(X^i, 1 \leq i \leq M^L(T))$  is a family of independent BBM, which is moreover independent of  $\mathcal{F}_T$ , so it follows from (5.19) that  $\mathbb{P}_{(0)}(\bar{E}_{j+1}) \geq \mathbb{P}_{(0)}(C)\mathbb{P}_{(0)}(\bar{E}_j)$ , where  $C$  is defined in Lemma 5.5. As  $E_0 = \Omega$  and  $\bar{E}_N = \bar{E}$ , we get

$$\mathbb{P}_{(0)}(E) \geq \mathbb{P}_{(0)}(C)^N \geq (1 - 2Le^{-L^\gamma})^{\lceil L^{4\gamma} \rceil} = \exp\left(-2Le^{-L^\gamma} L^{4\gamma}(1 + o(1))\right) \xrightarrow{L \rightarrow \infty} 1,$$

for  $L$  large enough, using Lemma 5.5.  $\square$

<sup>3</sup>On the one hand, with  $t := L^{2-5\gamma}$  and  $0 \leq s \leq t$ , we have  $t^{\frac{1}{2}+\gamma} = L^{1-\frac{\gamma}{2}-5\gamma^2} = o(L)$  and  $\frac{\varepsilon}{t}m(t) - \mu s \geq (\sqrt{2} - \mu)s \geq 0$  so  $\frac{\varepsilon}{t}m(t) - \mu s - r \vee (s^{\frac{1}{2}+\gamma} \wedge (t-s)^{\frac{1}{2}+\gamma}) \geq -\frac{\varepsilon L}{2} + 4L^\gamma$  for  $L$  large enough. On the other hand,  $m(t) - \mu t - d \geq -\frac{3(2-5\gamma)}{2\sqrt{2}} \log L - d \geq -L^\gamma$  for  $L$  large enough.

*Proof of Proposition 5.2.* We introduce for  $i \geq 0$  the event  $E_i$  defined by “in the BBM  $X$ , the particle at  $\max X^L(\tau_i)$  at time  $\tau_i$  has a descendant at time  $\tau_i + L^{2+\gamma}$  whose trajectory between times  $\tau_i$  and  $\tau_i + L^{2+\gamma}$  stays above  $t \mapsto \max X^L(\tau_i) - \varepsilon L + \frac{2\varepsilon}{L^2}(t - \tau_i)$ ”. It is clear that  $\mathbb{P}_{(0)}(E_i | \mathcal{F}_{\tau_i}) = \mathbb{P}_{(0)}(E)$ , where  $E$  is defined in Corollary 5.6. Moreover, we have  $A_i^c \cap E_i \subset \{\tau_{i+1} - \tau_i \geq L^{2+\gamma}\}$ : on the event  $A_i^c$ ,  $\tau_{i+1}$  is the first time after  $\tau_i$  when  $\max X^L$  goes below  $t \mapsto \max X^L(\tau_i) - \varepsilon L + \frac{2\varepsilon}{L^2}(t - \tau_i)$ , so on the event  $A_i^c \cap E_i$  the descendant at time  $\tau_i + L^{2+\gamma}$  in the definition of  $E_i$  cannot be killed by selection<sup>4</sup> and so belongs to the  $L$ -BBM and guarantees that  $\tau_{i+1} \geq L^{2+\gamma} + \tau_i$ . Thus, we have

$$\mathbb{P}_{(0)}\left(A_i^c \cap \{\tau_{i+1} - \tau_i < L^{2+\gamma}\} \mid \mathcal{F}_{\tau_i}\right) \leq \mathbb{P}_{(0)}(E_i^c \mid \mathcal{F}_{\tau_i}) = 1 - \mathbb{P}_{(0)}(E) \quad (5.20)$$

for all  $i \geq 0$ .

Now the reasoning is as follows: by (5.20) and Corollary 5.6, on each event  $A_i^c$  we have  $\tau_{i+1} - \tau_i \geq L^{2+\gamma}$  with high probability and, by Proposition 5.1, we know that events  $A_i^c$  happen about half time, thus  $\tau_n/n$  must be larger than  $L^{2+\gamma}/6$  for  $n$  large enough. For  $n \in \mathbb{N}^*$ , we have

$$\mathbb{P}_{(0)}\left(\frac{\tau_n}{n} < \frac{L^{2+\gamma}}{6}\right) \leq \mathbb{P}_{(0)}\left(\frac{\tau_n}{n} < \frac{L^{2+\gamma}}{6} \text{ and } \frac{K_n}{n} \leq \frac{2}{3}\right) + \mathbb{P}_{(0)}\left(\frac{K_n}{n} > \frac{2}{3}\right). \quad (5.21)$$

By the proof of Proposition 5.1, we know that  $\mathbb{P}_{(0)}(K_n/n > 2/3)$  is summable in  $n$  for  $L$  large enough. Moreover, denoting by  $\mathcal{P}_k(S)$  the set of the subsets with  $k$  elements of a set  $S$ , the first term on the right-hand side of (5.21) is equal to

$$\begin{aligned} & \sum_{k=0}^{\lfloor 2n/3 \rfloor} \mathbb{P}_{(0)}\left(\frac{\tau_n}{n} < \frac{L^{2+\gamma}}{6} \text{ and } K_n = k\right) \\ &= \sum_{k=0}^{\lfloor 2n/3 \rfloor} \sum_{I \in \mathcal{P}_{n-k}(\llbracket 1, n \rrbracket)} \mathbb{P}_{(0)}\left(\left\{\frac{\tau_n}{n} < \frac{L^{2+\gamma}}{6}\right\} \cap \bigcap_{i \in I} A_i^c \cap \bigcap_{i \notin I} A_i\right). \end{aligned} \quad (5.22)$$

But, on the event  $\{\tau_n/n < L^{2+\gamma}/6\}$ , the events  $\{\tau_{i+1} - \tau_i \geq L^{2+\gamma}\}$  happen for at most  $\lfloor n/6 \rfloor$  indices  $i$ . Therefore, (5.22) is bounded by

$$\begin{aligned} & \sum_{k=0}^{\lfloor 2n/3 \rfloor} \sum_{I \in \mathcal{P}_{n-k}(\llbracket 1, n \rrbracket)} \sum_{J \in \mathcal{P}_{n-k-\lfloor n/6 \rfloor}(I)} \mathbb{P}_{(0)}\left(\bigcap_{i \in J} (A_i^c \cap \{\tau_{i+1} - \tau_i < L^{2+\gamma}\})\right) \\ & \leq \sum_{k=0}^{\lfloor 2n/3 \rfloor} \binom{n}{n-k} \binom{n-k}{n-k-\lfloor n/6 \rfloor} (1 - \mathbb{P}_{(0)}(E))^{n-k-\lfloor n/6 \rfloor}, \end{aligned} \quad (5.23)$$

by conditioning successively on  $\mathcal{F}_{\tau_i}$  for all  $i \in J$  (in descending order) and using repeatedly (5.20). Then, we bound (5.23) from above by

$$\left(\left\lfloor \frac{2n}{3} \right\rfloor + 1\right) 2^n 2^n (1 - \mathbb{P}_{(0)}(E))^{n-\lfloor 2n/3 \rfloor - \lfloor n/6 \rfloor} \leq n \left(4(1 - \mathbb{P}_{(0)}(E))^{1/6}\right)^n,$$

which is summable for  $L$  large enough according to Corollary 5.6. Coming back to (5.21),  $\mathbb{P}_{(0)}(\tau_n/n < L^{2+\gamma}/6)$  is summable and the result follows by the Borel-Cantelli lemma.  $\square$

<sup>4</sup>The killing barrier of the  $L$ -BBM stays below  $\max X^L(\tau_i) + \varepsilon L - L$  and, thus, below  $t \mapsto \max X^L(\tau_i) - \varepsilon L + \frac{2\varepsilon}{L^2}(t - \tau_i)$

## References

- [1] L.-P. Arguin, A. Bovier, and N. Kistler. The genealogy of extremal particles of branching Brownian motion. *Comm. Pure Appl. Math.*, 64(12):1647–1676, 2011.
- [2] J. Bérard and J.-B. Gouéré. Brunet-Derrida behavior of branching-selection particle systems on the line. *Comm. Math. Phys.*, 298(2):323–342, 2010.
- [3] J. Bérard and P. Maillard. The limiting process of  $N$ -particle branching random walk with polynomial tails. *Electron. J. Probab.*, 19:no. 22, 17, 2014.
- [4] J. Berestycki, N. Berestycki, and J. Schweinsberg. Survival of near-critical branching Brownian motion. *J. Phys. Stat.*, 143(5):833–854, 2011.
- [5] J. Berestycki, N. Berestycki, and J. Schweinsberg. The genealogy of branching Brownian motion with absorption. *Ann. Probab.*, 41(2):527–618, 03 2013.
- [6] M. D. Bramson. Maximal displacement of branching Brownian motion. *Comm. Pure Appl. Math.*, 31(5):531–581, 1978.
- [7] M. D. Bramson. *Convergence of solutions to the Kolmogorov equation to travelling waves*, volume 44 of *Mem. Amer. Math. Soc.* American Mathematical Society, 1983.
- [8] É. Brunet and B. Derrida. Shift in the velocity of a front due to a cutoff. *Phys. Rev. E (3)*, 56(3, part A):2597–2604, 1997.
- [9] É. Brunet and B. Derrida. Microscopic models of traveling wave equations. *Comput. Phys. Commun.*, 121-122:376–381, 1999. Proceedings of the Europhysics Conference on Computational Physics CCP 1998.
- [10] É. Brunet and B. Derrida. Effect of microscopic noise on front propagation. *J. Statist. Phys.*, 103(1-2):269–282, 2001.
- [11] É. Brunet, B. Derrida, A. H. Mueller, and S. Munier. Noisy traveling waves: effect of selection on genealogies. *Europhys. Lett.*, 76(1):1–7, 2006.
- [12] É. Brunet, B. Derrida, A. H. Mueller, and S. Munier. Phenomenological theory giving the full statistics of the position of fluctuating pulled fronts. *Phys. Rev. E*, 73:056126, 2006.
- [13] A. Champneys, S. Harris, J. Toland, J. Warren, and D. Williams. Algebra, analysis and probability for a coupled system of reaction-diffusion equations. *Philos. Trans. Roy. Soc. London Ser. A*, 350(1692):69–112, 1995.
- [14] B. Derrida and Z. Shi. Large deviations for the branching Brownian motion in presence of selection or coalescence. 2015. (in preparation).
- [15] F. Dumortier, N. Popović, and T. J. Kaper. The critical wave speed for the Fisher-Kolmogorov-Petrovskii-Piscounov equation with cut-off. *Nonlinearity*, 20(4):855–877, 2007.

- [16] R. Durrett and D. Remenik. Brunet-Derrida particle systems, free boundary problems and Wiener-Hopf equations. *Ann. Probab.*, 39(6):2043–2078, 2011.
- [17] R. A. Fisher. The wave of advance of advantageous genes. *Annals of Eugenics*, 7(4):355–369, 1937.
- [18] N. Gantert, Y. Hu, and Z. Shi. Asymptotics for the survival probability in a killed branching random walk. *Ann. Inst. H. Poincaré Probab. Statist.*, 47(1):111–129, 02 2011.
- [19] J. W. Harris and S. C. Harris. Survival probabilities for branching brownian motion with absorption. *Electron. Commun. Probab.*, 12:no. 10, 81–92, 2007.
- [20] J. W. Harris, S. C. Harris, and A. E. Kyprianou. Further probabilistic analysis of the Fisher-Kolmogorov-Petrovskii-Piscounov equation: one sided travelling-waves. *Ann. Inst. H. Poincaré Probab. Statist.*, 42(1):125–145, 2006.
- [21] S. C. Harris, M. Hesse, and A. E. Kyprianou. Branching Brownian motion in strip: survival near criticality. 2014. [arXiv:1212.1444v1 \[math.PR\]](#).
- [22] N. Ikeda, M. Nagasawa, and S. Watanabe. Branching Markov processes II. *J. Math. Kyoto Univ.*, 8(3):365–410, 1968.
- [23] H. Kesten. Branching Brownian motion with absorption. *Stochastic Processes Appl.*, 7(1):9–47, 1978.
- [24] A. Kolmogorov, I. Petrovsky, and N. Piskunov. Etude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique. *Bulletin Université d'Etat à Moscou*, 1:1–25, 1937.
- [25] S. P. Lalley and T. Sellke. A conditional limit theorem for the frontier of a branching Brownian motion. *Ann. Probab.*, 15(3):1052–1061, 1987.
- [26] P. Maillard. Speed and fluctuations of N-particle branching Brownian motion with spatial selection. 2015. [arXiv:1304.0562v3 \[math.PR\]](#).
- [27] B. Mallein. N-branching random walk with  $\alpha$ -stable spine. 2015. [arXiv:1503.03762v1 \[math.PR\]](#).
- [28] H. P. McKean. Application of Brownian motion to the equation of Kolmogorov-Petrovskii-Piskunov. *Comm. Pure Appl. Math.*, 28(3):323–331, 1975.
- [29] C. Mueller, L. Mytnik, and J. Quastel. Effect of noise on front propagation in reaction-diffusion equations of KPP type. *Invent. Math.*, 184(2):405–453, 2011.