On the regularizing effect for unbounded solutions of first-order Hamilton-Jacobi equations

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ON THE REGULARIZING EFFECT FOR UNBOUNDED SOLUTIONS OF FIRST-ORDER HAMILTON-JACOBI EQUATIONS

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Abstract. We give a simplified proof of regularizing effects for first-order Hamilton-Jacobi Equations of the form $u_t + H(x,t,Du) = 0$ in $\mathbb{R}^N \times (0,+\infty)$ in the case where the idea is to first estimate u_t. As a consequence, we have a Lipschitz regularity in space and time for coercive Hamiltonians and, for hypo-elliptic Hamiltonians, we also have an Hölder regularizing effect in space following a result of L. C. Evans and M. R. James.

1. Introduction

In this short paper we give a new proof of regularizing effects for Hamilton-Jacobi Equations

$$u_t + H(x,t,Du) = 0 \quad \text{in } \mathbb{R}^N \times (0,\infty)\,, \tag{1.1}$$

in the case when the aim is to estimate u_t first. This new proof is inspired by ideas introduced in [3] and then simplified in [2]; a precise comparison between the results and ideas of [3, 2] and ours is provided just after the statement of the main results of this article, at the end of Section 2. More classical proofs can be found in [1] but with stronger assumptions and more tedious proof.

The model equations we have in mind are

$$u_t + |A(x,t)Du|^m = f(x,t) \quad \text{in } \mathbb{R}^N \times (0,\infty)\,, \tag{1.2}$$

where f is a continuous (typically bounded from below) function and A takes values, in the set of $N \times N$ symmetric matrices. For such equations, we consider two cases: the *coercive case* for which A is invertible and, as a consequence $|A(x,t)Du|^m \to +\infty$ as $|p| \to +\infty$, and the *non-coercive case* where A may be degenerate. In both cases, we provide regularizing effects for *bounded from below solutions*. The main improvement in the assumptions is easy to describe in the coercive case since we just require that A, f are continuous in x (no uniform continuity assumptions) and, in particular, f may have some growth at infinity. In the non-coercive case, analogous results hold except that we have to impose far more restrictive assumptions on the t-dependence of the equation.

Of course, for (1.2), the equation implies that $u_t \leq f(x,t)$ in $\mathbb{R}^N \times (0,\infty)$ and thefore we just need an estimate from below for u_t.

To do so, our approach consists in using the exponential transform, $v := -\exp(-u)$. Notice that provided u is bounded below (then we can always assume that u is nonnegative), we get that v is bounded since $-1 \leq v \leq 0$. Moreover, v solves a new Hamilton-Jacobi equation

$$v_t + G(x,t,v,Dv) = 0\,, \quad \text{with } G(x,t,v,p) := -vH\Big(x,t,-\frac{p}{v}\Big)\,. \tag{1.3}$$

In order to estimate v_t, a key property (as in all the regularizing effects proofs) is to have a large enough, positive G_v when $v_t = -G$ is large (but negative) i.e. when G is large and positive. This

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leads to an assumption on $(H_p \cdot p - H)(x,t,p)$ which is classical except that, here, this quantity has to be large when $H(x,t,p)$ is large, and not when $|p|$ is large as it is classical for the estimate on Du.

The paper is organized as follows : in Section 2, we state our main results, one in the case where H is assumed to be coercice, one in the degenerate case and we give and we deduce full regularizing effects, *i.e.* in space and time. The proofs of the main theorems are are given in Section 3. Then, in Section 4, we treat several explicit examples. We have put some technical results about Hamiltonian G in an appendix.

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2. Assumptions and main results

2.1. **Assumptions.** In order to state and prove our results, we use several structure conditions, which all rely on the following fundamental hypothesis

(**H0**) *The function H is continuous in $\mathbb{R}^N \times [0,T] \times \mathbb{R}^N$ and there exists $c_0 = c_0(H) \geq 0$ such that*

- *H is locally Lipschitz in the p-variable, in a neighborhood of the set $\{(x,t,p);\ H(x,t,p) \geq c_0\}$;*
- *there exists a continuous, increasing function $\phi : [c_0, \infty[\to [0, \infty[$ such that for some $A > c_0$,*

$$\int_A^{+\infty} \frac{1}{s\phi(s)}\, \mathrm{d}s < +\infty \tag{2.1}$$

and for almost all $x, p \in \mathbb{R}^N$, $t \in [0,T]$,

$$(H_p \cdot p - H)(x,t,p) \geq \phi\big(H(x,t,p)\big) \quad \text{a.e. in } \{(x,t,p);\ H(x,t,p) \geq c_0\}\,;$$

- *there exists $\kappa > 0$ such that*

$$|H_p(x,t,p)| \leq \kappa.(H_p \cdot p - H)(x,t,p) \quad \text{a.e. in } \{(x,t,p);\ H(x,t,p) \geq c_0\}\,.$$

We first point out that, for the model equation (1.2), (**H0**) is satisfied with $c_0 = 0$ and $\phi(s) = (m-1)s$ by homogeneity, both in the coercive and non-coercive cases.

Remark 2.1. Roughly speaking, (**H0**) is a superlinearity condition on H either in the p-variable or a linear function of p (possibly depending on x, t and possibly degenerate). Indeed, it is easy to show that if $H(x,t,p)$ satisfies (**H0**) then $H(x,t,a(x,t)p)$ also satisfies (**H0**) for any bounded matrix-valued function a. We also refer to Section 4 for various examples. Even if we were not able to prove it in full generality, Condition (2.1) should be automatically satisfied for convex and superlinear Hamiltonians.

The first main consequence of (**H0**) is the following: if we set

$$\forall \tau \in (c_0, \infty)\,,\ F(\tau) := 2\int_\tau^\infty \frac{\mathrm{d}\sigma}{\sigma\phi(\sigma)} \quad \text{and} \quad \forall s \in (0, F(c_0))\,,\ \eta(s) = F^{-1}(s)\,,$$

then $\eta'(s) = -\eta(s)\phi(\eta(s))/2$, $s \mapsto \eta(s)$ is decreasing and $\eta(0^+) = +\infty$. Notice that if $c_0 = 0$, since $\phi(s) \sim \phi(0)s$ near $s = 0$, the integral blows-up and $F(0) = \infty$. Similarly, if $\phi(c_0) = 0$ we have $F(c_0) = +\infty$.

The next assumption concerns the dependence of H in t. In order to formulate it, we notice that $\eta \mapsto 2\eta + \phi(2\eta)$ is increasing from (c_0, ∞) to $(2c_0 + \phi(2c_0), \infty)$. Hence, its inverse, denoted by $\psi : (2c_0 + \phi(2c_0), \infty) \to (c_0, \infty)$ is well-defined and also increasing. Notice that ψ is sublinear but $\psi(\tau) \to \infty$ as $\tau \to \infty$.

(**H1**) *There exists $c_1 > 0$ such that the function H is locally Lipschitz in t in a neighborhood of the set $\{(x,t,p);\ H(x,t,p) \geq c_1\}$ and for any $x, p \in \mathbb{R}^N$, $t \in [0,T]$,*

$$|H_t(x,t,p)| \leq \psi\big(H(x,t,p)\big)(H_p \cdot p - H)(x,t,p) \quad \text{a.e. in } \{(x,t,p);\ H(x,t,p) \geq c_1\}\,.$$

For the model equation (1.2), (**H1**) is satisfied in the coercive case as soon as A is Lipschitz continuous in t, while, in the non-coercive case, it imposes rather restrictive (or at least non-natural assumptions) since A should satisfy $|A_t(x,t)p| \leq k|A(x,t)p|.(|A(x,t)p|^m)$ for any x, t, p such that $|A(x,t)p|^m \geq c_1$, a property which imposes particular types of degeneracies (see Section 4 for a study of the scalar case). However (**H1**) is satisfied with $c_1 = 0$ if A does not depend on t.

We then define precisely the coercive case: H is coercive if

(**H2**) *$H(x,t,p) \to +\infty$ when $|p| \to +\infty$, uniformly for (x,t) in a compact subset of $\mathbb{R}^N \times (0,T]$.*

And we have to impose some additional restrictions on the behavior of H in x in the non-coercive case *i.e.* when (**H2**) does not hold.

(**H3**) *There exists $c_2 > 0$ such that the function H is locally Lipschitz in x in a neighborhood of the set $\{(x,t,p);\ H(x,t,p) \geq c_2\}$ and there exists a constant $\gamma > 0$ such that, for any $x, p \in \mathbb{R}^N$, $t \in [0,T]$,*

$$|H_x(x,t,p)| \leq \gamma(|p|+1)(H_p \cdot p - H)(x,t,p) \quad \text{a.e. in } \{(x,t,p);\ H(x,t,p) \geq c_2\}\,.$$

For the model equation (1.2), (**H2**) is satisfied (in the non-coercive case) as soon as A is Lipschitz continuous in x.

2.2. **The main theorem.** In order to state our main results, we first introduce

$$c_* := \begin{cases} \max(c_0, c_1) & \text{if (\textbf{H0})-(\textbf{H1})-(\textbf{H2}) holds}\,, \\ \max(c_0, c_1, c_2) & \text{if (\textbf{H0})-(\textbf{H1})-(\textbf{H3}) holds}\,. \end{cases}$$

and we set

$$t_* = t_*(H) := \begin{cases} \eta^{-1}(c_*) = F(c_*) & \text{if } c_* > 0\,, \\ +\infty & \text{if } c_* = 0\,. \end{cases}$$

Notice that this is consistent with the fact that $F(0^+) = +\infty$.

Theorem 2.2. *We assume that either* (**H0**)-(**H1**)-(**H2**) *or* (**H0**)-(**H1**)-(**H3**) *holds and let u be a continuous, bounded from below solution of* (1.1)*. Then for any $t \in (0, t_*)$ and $x \in \mathbb{R}^N$, the function $v := -\exp(-u)$ satisfies*

$$\inf_{0 \leq s \leq t} \big\{ v(x,t) - v(x,s) + (t-s)\eta(s) \big\} \geq 0\,. \tag{2.2}$$

Notice that since u and v are continuous functions, u_t and v_t are defined in the sense of distributions. A distributional formulation of (2.2) is the following: $v_t \geq -\eta(t)$ in $\mathbb{R}^N \times (0, t_*)$. Similarly, we have $u_t \geq -\eta(t)u$ on the same set.

2.3. **Consequences.** The general statement of this theorem may hide some non-trivial consequences concerning global estimates, at least in some particular cases. First, if $c_* > 0$ and we want to get an estimate for $t > t_*$, we can apply the estimate of the theorem to $w(x,s) := u(x, s + t - t_*/2)$ which is a solution of an analogous (just shifted in time) pde for $s \in (0, T - t + t_*/2)$. This interval corresponds to the interval $(t - t_*/2, T)$ for u. Hence the

Corollary 2.3. *If $c_* > 0$, setting $C = \eta(t_*/2)$ we have*

$$v_t \geq -\max(\eta(t), C) \quad \text{and} \quad u_t \geq -\max\big(\eta(t), C\big)u \quad \text{a.e. in } \mathbb{R}^N \times (0,T) \, .$$

If $T = +\infty$, the previous estimate gives the lower estimate $u_t \geq -Cu$ for t big enough. However, if $c_* = 0$ then $t_* = \infty$ and we get a much better asymptotic estimate

Corollary 2.4. *If $c_* = 0$ and $T = +\infty$,*

$$v_t \geq -\eta(t) \quad \text{and} \quad u_t \geq -\eta(t)u \quad \text{a.e. in } \mathbb{R}^N \times (0,\infty) \, .$$

In particular, $(u_t)^- \to 0$ locally uniformly in $\mathbb{R}^N$ as $t \to \infty$.

We first point out, as we already do it after the statement of (**H1**), that $c_* = 0$ implies in particular $c_1 = 0$, which is a rather restrictive assumption in the non-coercive case. We come back on this point in Section 4.

This asymptotic result leads us to the comparison with the results of [3, 2] which concern cases where H does not depend on t. The main issue in these papers was the asymptotic behavior of u which is expected to be of ergodic type, namely

$$u(x,t) = -\lambda t + \phi(x) + o_t(1) \quad \text{when } t \to +\infty \, ,$$

where λ is the ergodic constant and ϕ a solution of the ergodic problem

$$H(x, D\phi) = \lambda \quad \text{in } \mathbb{R}^N \, .$$

In [3, 2], the key idea is to reduce to the case when $\lambda = 0$ and then to show that $(u_t)^- \to 0$ (or $(u_t)^+ \to 0$) locally uniformly in $\mathbb{R}^N$ as $t \to \infty$. To do so, one has to take into account the fact that $\lambda = 0$ and ϕ plays a key role since it interferes in the analogue of (**H0**). With all these ingredients, the proofs of [3, 2] are slightly more sophisticated that ours. But we point out that, remarking that an analogous proof could provide regularizing effects is (to the best of our knowledge) completely new.

This analysis through the ergodic behavior of u also shows that Corollary 2.4 can only hold in particular situations.

2.4. **Gradient estimates and spatial regularity.** In the *coercive case*, we can deduce a gradient estimate from (2.2). Indeed, Theorem 2.2 implies

$$H(x,t,Du) \leq -u_t \leq \eta(t)u \, ,$$

and standard arguments in viscosity solutions' theory show that u is locally Lipschitz continuous. Moreover, a local estimate of Du follows from the above inequality.

In the non-coercive case we studied above (Example 3), the bound on u_t implies a local estimate on $|A(x,t)Du|$ and using a result of Evans and James [4], we can deduce a local $C^{0,\alpha}$ bound when A satisfies some "hypoelliptic" type assumptions.

3. Proof of Theorem 2.2

In all the proof, we fix $0 < t < t_*$ where we recall that $t_* = \eta^{-1}(c_*)$. Assuming that for some $x_0 \in \mathbb{R}^N$, inequality (2.2) does not hold, we deduce that for $\alpha, \beta, \epsilon > 0$ small enough with $\epsilon \ll \alpha, \beta$, we have also

$$\min_{t \geq s \geq 0,\ x,y \in \mathbb{R}^N} \big\{ v(x,t) - v(y,s) + (t-s)\eta(s) + \frac{|x-y|^2}{\epsilon^2} + \alpha|y|^2 + \beta\eta(s) \big\} \leq -\delta\,, \tag{3.1}$$

for some $\delta > 0$ which does not depend on α, β, ϵ. Notice first that this min is attained because of the α-term, and it cannot be attained at $s = 0$ because of the β-term. For simplicity, we still denote by x, y, s, t the variables for which the min is attained. Since v is bounded, we have the usual estimates $|x-y| \leq O(\epsilon)$, $|y| = o(\alpha^{-1/2})$.

Next, the equation for v given in (1.3) involves a new Hamiltonian G whose properties follow from the assumptions on H (see the proof in Appendix): if $G(x,t,v,p) \geq c_*$,

($\mathbf{H0}$) implies ($\mathbf{G0}$) : $G_v(x,t,v,p) \geq \phi(G(x,t,v,p))$ and $|G_p(x,t,v,p)| \leq \kappa\, G_v(x,t,v,p)\,$,

($\mathbf{H1}$) implies ($\mathbf{G1}$) : $|G_t(x,t,v,p)| \leq \psi(G(x,t,v,p))\, G_v(x,t,v,p)\,$,

($\mathbf{H3}$) implies ($\mathbf{G3}$) : $|G_x(x,t,v,p)| \leq \gamma(|p|+1) G_v(x,t,v,p)\,$.

Using the supersolution inequality for $v(x,t)$ in the viscosity sense, and the subsolution for $v(y,s)$ we get, with $p_\epsilon = -2(x-y)/\epsilon^2$,

$$\begin{aligned} -\eta(s) + G\big(x,t,v(x,t),p_\epsilon\big) &\geq 0\,,\\ -\eta(s) + (t-s+\beta)\eta'(s) + G\big(y,s,v(y,s),p_\epsilon + 2\alpha y\big) &\leq 0\,. \end{aligned}$$

We begin with the *coercive* case, assuming ($\mathbf{H0}$)-($\mathbf{H1}$)-($\mathbf{H2}$). We remark that, if α and β remain fixed, then x, y stay in a compact subset of $\mathbb{R}^N$ and t, s in a compact subset of $(0, t_*)$ and we can let ϵ tend to zero. By the uniform continuity of H and v on such compact sets, we end up with

$$\begin{aligned} -\eta(s) + G\big(x,t,v(x,t),p\big) &\geq 0\,,\\ -\eta(s) + (t-s+\beta)\eta'(s) + G\big(x,s,v(x,s),p + 2\alpha x\big) &\leq 0\,, \end{aligned} \tag{3.2}$$

for some $p \in \mathbb{R}^N$. In order to estimate the difference of the hamiltonians, we denote by $v_1 = v(x,t)$, $v_2 = v(x,s)$, $q_\alpha = 2\alpha x$ and $\xi_\sigma := (x,t,v_1,p) + \sigma(0, s-t, v_2 - v_1, q_\alpha)$.

The following arguments rely on the next two lemmas, where s and t are fixed, and σ is the running variable.

Lemma 3.1. *Let $0 \leq \sigma_1 < \sigma_2 \leq 1$ such that*

$$\eta(s) \leq G(\xi_\sigma) \leq \eta(s) + \phi(\eta(s)) \quad \textit{for any } \sigma \in [\sigma_1, \sigma_2]\,. \tag{3.3}$$

Then, the function $\sigma \mapsto G(\xi_\sigma)$ is increasing on $[\sigma_1, \sigma_2]$.

Proof — We argue as if G were C^1 (otherwise a standard mollification argument allows to reduce to this case) and compute

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\big[G(\xi_\sigma)\big] = G_t(\xi_\sigma)(s-t) + G_v(\xi_\sigma)(v_2 - v_1) + G_p(\xi_\sigma)\cdot q_\alpha\,.$$

Inequality (3.1) yields (recall that we have let ϵ tend to 0)

$$v_2 - v_1 \geq (t-s+\beta)\eta(s) + \alpha|x|^2 + \delta\,, \tag{3.4}$$

and therefore, as long as $G(\xi_\sigma) \geq \eta(s)$, we have $G_v(\xi_\sigma) \geq 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\big[G(\xi_\sigma)\big] \geq G_t(\xi_\sigma)(s-t) + G_v(\xi_\sigma)\big((t-s+\beta)\eta(s) + \alpha|x|^2 + \delta\big) + G_p(\xi_\sigma)\cdot q_\alpha\,.$$

Denoting

$$I_1 := G_t(\xi_\sigma)(s-t) + \frac{1}{2}G_v(\xi_\sigma)(t-s)\eta(s)\,,$$
$$I_2 := \frac{1}{2}G_v(\xi_\sigma)\alpha|x|^2 - G_p(\xi_\sigma)\cdot q_\alpha\,,$$
$$I_3 := \frac{1}{2}G_v(\xi_\sigma)\delta\,,$$

we have to estimate these three terms, *as long as* $\eta(s) \leq G(\xi_\sigma) \leq \eta(s) + \phi(\eta(s))$. Recall that since $\eta(s) \geq \eta(t_*) = c_*$, as long as $G(\xi_\sigma) \geq \eta(s)$, we have $G(\xi_\sigma) \geq c*$ and we can apply the hypotheses on G.

First,

$$I_1 \geq \frac{1}{2}(t-s)G_v(\xi_\sigma)\left[\eta(s) - 2\frac{|G_t(\xi_\sigma)|}{G_v(\xi_\sigma)}\right]\,,$$

and by (**G1**), $|G_t(\xi_\sigma)|/G_v(\xi_\sigma) \leq \psi(G(\xi_\sigma)) \leq \psi\big(\eta(s) + \phi(\eta(s))\big) = \eta(s)/2$, the last equality coming from the definition of ψ. Hence $I_1 \geq 0$ on $[\sigma_1, \sigma_2]$.

Concerning I_2, we have

$$I_2 \geq \frac{1}{2}\alpha G_v(\xi_\sigma)\Big[\,|x|^2 - 2\frac{\big|G_p(\xi_\sigma)\big|}{G_v(\xi_\sigma)}|x|\Big] \geq \frac{1}{2}\alpha G_v(\xi_\sigma)\Big[\,|x|^2 - 2\kappa|x|\Big]\,.$$

Using (**G0**) we remark that the quantity in the bracket is positive if $|x| > 2\kappa$ and therefore we are left with considering the case when $|x| \leq 2\kappa$. But in this case, I_2 is estimated by $4\alpha\kappa^2 G_v(\xi_\sigma)$ and I_3 controls this term since $\delta \geq 4\alpha\kappa^2$ for α small enough, so that $I_3 + I_2 > 0$.

In any case, $I_1 + I_2 + I_3 > 0$ on $[\sigma_1, \sigma_2]$ and we deduce that, if t_* is chosen small enough and α is small enough, $\frac{\mathrm{d}}{\mathrm{d}\sigma}\big[G(\xi_\sigma)\big] > 0$ for $\sigma \in [\sigma_1, \sigma_2]$.

Q.E.D.

The next step shows that we can apply Lemma 3.1 on the whole interval $\sigma_1 = 0$ and $\sigma_2 = 1$:

Lemma 3.2. *For any* $\sigma \in [0,1]$

$$\eta(s) \leq G(\xi_\sigma) \leq \eta(s) + \phi(\eta(s))\,.$$

Proof — We recall that in this proof s is fixed. We first notice that viscosity inequalities (3.2) yield $G(\xi_0) \geq \eta(s)$ and

$$G(\xi_1) \leq \eta(s) - (t-s+\beta)\eta'(s) = \eta(s) + (t-s+\beta)\eta(s)\frac{\phi(\eta(s))}{2}\,.$$

From (3.4) we deduce that $(t-s+\beta)\eta(s) \leq v_2 - v_1 - \delta \leq 1-\delta$ so that $G(\xi_1) \leq \eta(s) + (1-\delta)\phi(\eta(s))$.

Now assume by contradiction that $G(\xi_\sigma) > \eta(s) + \phi(\eta(s))$ for some $\sigma \in [0,1)$ and denote by $\bar\sigma$ the supremum of such σ. Then, since $G(\xi_1) \leq \eta(s) + (1-\delta)\phi(\eta(s))$, necessarily $\bar\sigma < 1$ and $G(\xi_{\bar\sigma}) = \eta(s) + \phi(\eta(s))$. Since $G(\xi_1) < G(\xi_{\bar\sigma}) = \eta(s) + \phi(\eta(s))$, by continuity of $\sigma \mapsto G(\xi_\sigma)$, there exists a subinterval $[\sigma_1, \sigma_2] \subset [\bar\sigma, 1]$, such that $G(\xi_{\sigma_1}) > G(\xi_{\sigma_2})$ and for any $\sigma \in [\sigma_1, \sigma_2]$,

$$\eta(s) \leq G(\xi_\sigma) \leq \eta(s) + \phi(\eta(s))\,.$$

But this yields a contradiction with Lemma 3.1. Hence for any $\sigma \in [0,1]$, $G(\xi_\sigma) \leq \eta(s) + \phi(\eta(s))$.

For the other inequality, the argument is the same: if there exists σ such that $G(\xi_\sigma) < \eta(s)$ (using this time $G(\xi_0)$ and the interval $[0, \bar\sigma]$ where $\bar\sigma$ is the infimum of the above set of σ), we get a similar contradiction with Lemma 3.1. Hence the proof is complete.

Q.E.D.

In order to conclude the proof of the first part of Theorem 2.2, we subtract the viscosity inequalities, we obtain

$$(t-s+\beta)\eta'(s) + G(\xi_1) - G(\xi_0) = (t-s+\beta)\eta'(s) + G\big(x,s,v_2,p+q_\alpha\big) - G(x,t,v_1,p) \le 0\,,$$

and we can perform almost the same computations to compute the left hand side, except that we write

$$G_v(\xi_\sigma) = \frac{1}{2}G_v(\xi_\sigma) + \frac{1}{2}G_v(\xi_\sigma) \ge \frac{1}{2}G_v(\xi_\sigma) + \frac{1}{2}\phi(\eta(s))\,,$$

and we obtain

$$\begin{aligned}&(t-s+\beta)\big(\eta'(s) + \frac{1}{2}\phi(\eta(s))\eta(s)\big)\\ &+\int_*^1 \Big\{G_t(\xi_\sigma)(s-t) + \frac{1}{2}G_v(\xi_\sigma)\big((t-s+\beta)\eta(s) + \alpha|x|^2 + \delta\big) + G_p(\xi_\sigma)\cdot q_\alpha\Big\}\,\mathrm{d}\sigma \le 0\,.\end{aligned}$$

By the choice of $\eta(\cdot)$, the first term is 0 and the integral is strictly positive for t_* and α small enough. Therefore we have a contradiction, proving the claim in the coercive case.

Now we turn to the *non-coercive case*, assuming (**H3**) instead of (**H2**), where the main difference is that we are not able to let ϵ tend to 0. We define $G(\xi_\sigma)$ in a similar way and prove that $G(\xi_\sigma) \ge \eta(s)$ (we omit the proof here since it follows, as in Lemma 3.1, from the same argument as below). Computing the difference of G in the same way, we can write the difference of the viscosity inequalities as

$$\int_*^1 \Big\{Q_1(\sigma) + Q_2(\sigma) + Q_3(\sigma) + Q_4(\sigma)\Big\}\,\mathrm{d}\sigma \le 0\,,$$

where

$$Q_1(\sigma) := (t-s+\beta)\eta'(s) + \frac{1}{2}G_v(\xi_\sigma)\big((t-s+\beta)\eta(s) + \delta\big)\,,$$

and by the same arguments as above $Q_1(\sigma) \ge \frac{1}{2}G_v(\xi_\sigma)\delta$.

$$Q_2(\sigma) := G_x(\xi_\sigma)\cdot(x-y) + \frac{1}{2}G_v(\xi_\sigma)\frac{|x-y|^2}{\epsilon^2}\,,$$

which can be rewritten as

$$Q_2(\sigma) \ge \frac{1}{2}\epsilon^2 G_v(\xi_\sigma)\Big[-\frac{|G_x(\xi_\sigma)|}{G_v(\xi_\sigma)}|p_\epsilon| + \frac{1}{4}|p_\epsilon|^2\Big] \ge \frac{1}{2}\epsilon^2 G_v(\xi_\sigma)\Big[-\gamma|p_\epsilon|(|p_\epsilon|+1) + \frac{1}{4}|p_\epsilon|^2\Big]\,.$$

But $\epsilon^2|p_\epsilon|^2 = \dfrac{|x-y|^2}{\epsilon^2} = o_\epsilon(1)$ if α, β are fixed and therefore $Q_2 \ge o_\epsilon(1)G_v(\xi_\sigma)$ is controlled by Q_1 for ϵ small enough.

$$Q_3(\sigma) := (t-s)\left[\frac{1}{2}G_v(\xi_\sigma)\eta(s) - G_t(\xi_\sigma)\right]$$

is as I_1 above and clearly the quantity in the bracket is positive if $\eta(s)$ is large enough, i.e. t_* small enough. Finally

$$Q_4(\sigma) := \frac{1}{2}\alpha G_v(\xi_\sigma)\Big[|y|^2 - 2\frac{|G_p(\xi_\sigma)|}{G_v(\xi_\sigma)}|y|\Big]\,,$$

is treated as I_2 above. And the proof is complete.

4. Examples and applications

Example 1 — We first consider the model equation (1.2) where, for $m > 1$

$$H(x,t,p) = |A(x,t)p|^m - f(x,t)\,,$$

the function f being bounded from below. In this case

$$(H_p \cdot p - H)(x,t,p) = (m-1)|A(x,t)p|^m + f(x,t) = (m-1)H(x,t,p) + mf(x,t)\,.$$

We can assume without loss of generality that $f(x,t) \geq 1$ (which can be obtained by changing u into $u(x,t) + Ct$), so that (**H0**) is satisfied with $\phi(s) = (m-1)s$ for $s \geq 0$ and $c_0 = 0$. Moreover, $|H_p(x,t,p)| = m|A(x,t)p|^{m-1}$ and if $H(x,t,p) \geq 0$ then $|A(x,t)p| \geq f(x,t)^{1/m} \geq 1$. Therefore

$$|H_p(x,t,p)| \leq m|A(x,t)p|^m \leq \frac{m}{m-1}(H_p \cdot p - H)(x,t,p)\,.$$

A direct computation shows that $\eta(s) = \big[(m-1)s\big]^{-1}$ which yields the estimate $u_t \geq -u/\big[(m-1)t\big]$.

Example 2 — Let us now adress a non-homogeneous situation where, again for $m > 1$

$$H(x,t,p) = |A(x,t)p|^m - b(x,t)\cdot Du - f(x,t)\,. \tag{4.1}$$

Here, the function b is continuous and there exists $C_b > 0$ such that for any (x,t), $|b(x,t)| \leq C_b$. We are also considering the *coercive case* where A is an invertible matrix and we assume that there exists $C_A > 0$ such that for any (x,t), $\|A(x,t)\|\,, \|A^{-1}(x,t)\| \leq C_A$.

In this case, as for the homogeneous case we have

$$(H_p \cdot p - H)(x,t,p) = (m-1)|A(x,t)p|^m + f(x,t)\,.$$

By Young's inequality

$$|b(x,t)\cdot p| = |A^{-t}(x,t)b(x,t)\cdot A(x,t)p| \leq \frac{1}{m}|A(x,t)p|^m + \frac{(m-1)}{m}|A^{-t}(x,t)b(x,t)|^{m/(m-1)}$$

where A^{-t} is the inverse of the transpose matrix of A. Therefore, by our hypotheses on b and A there exists two constants $C_1, C_2 > 0$ such that

$$(1-\frac{1}{m})|A(x,t)p|^m - C_1 - f(x,t) \leq H(x,t,p) \leq (1+\frac{1}{m})|A(x,t)p|^m + C_2 - f(x,t)\,,$$

and from there we deduce that (**H0**) is satisfied with $\phi(s) = \big[m(m-1)/(m+1)\big]s - C$ for some constant $C > 0$ and for all $s > 0$.

Example 3 — In the non-coercive case for (4.1), *i.e.* if A is not invertible, the same computations for (**H0**) can be done if the Hamiltonian H has the form

$$H(x,t,p) = |A(x,t)p|^m - c(x,t)\cdot A(x,t)Du - f(x,t)\,,$$

where $c : \mathbb{R}^N \times [0,T] \to \mathbb{R}^N$. This means that b has to be of the form $b(x,t) = A^t(x,t)\cdot c(x,t)$. Notice that in this degenerate case, at least for large H, (**H3**) is automatically satisfied if A, c and f are Lipschitz continuous in x, uniformly in t.

Example 4 — In the non-coercive case it is also interesting to examine the simple example where

$$H(x,t,p) := a(x,t)|p|^m\,,$$

where $m > 1$ and $a : \mathbb{R}^N \times [0,T] \to [0,+\infty)$ is Lipschitz continuous in t. Taking into account that $\phi(s) = (m-1)s$ and $\psi(s) = s/(2m)$, assumption (**H1**) imposes

$$|a_t(x,t)| \leq \frac{m-1}{2m} a(x,t)\big(a(x,t)|p|^m\big)$$

for $a(x,t)|p|^m \geq c_1$, for some constant $c_1 \geq 0$. Clearly this implies $|a_t(x,t)| \leq c \cdot a(x,t)$ for some $c > 0$, which implies some particular degeneracy: for any x either $a(x,t) = 0$ for any t or $a(x,t) > 0$ for any t. An –admittedly– not very natural hypothesis. Moreover if we want $c_1 = 0$ then necessarily the above constant c has to be 0 and therefore a cannot depend on t.

Remarks — (i) Concerning (**H1**), in all the previous examples above the function $\psi(H)$ behaves like $c.H$ for some constant $c > 0$. And in the case of (4.1), at least for large H, this is automatically satisfied if A, b and f are Lipschitz continuous in t, and uniformly in x in the coercive case. In the non-coercive case, we are in a similar situation as for Example 4: we have to impose that $|A_t(x,t)p| \leq C|A(x,t)p|^{m+1}$ for some constant $C > 0$, at least for $|A(x,t)p|$ large enough. And again, this is not a very natural assumptions. The conclusion is that in non-coercive cases, the t-dependance is a problem in general.

(ii) We can consider other types of growths: if for instance $H(x,t,p) = e^{|p|}$ we have $\phi(s) = s\ln(s-1)$, defined for $s > 2 = c_0$. Hypotheses (**H0**)-(**H1**)-(**H2**) are satisfied with $c_1 = 2$ and in this case, $\eta(s) \gg s^{-1}$ as $s \to 0$.

(iii) If on most examples the function ϕ grows at least linearly, this is not always true. For instance, the case $H(p) = |p|\ln(1+|p|)$ leads to $H_p \cdot p - H \simeq |p|$ for $|p|$ big, which leads to a function ϕ which is not superlinear, only asymptotically linear at infinity. Indeed, for any $\epsilon > 0$, one can check that for $\sigma > 0$ big enough, $\sigma \leq \phi(\sigma) \leq \sigma^{1-\epsilon}$ so that the function $[\sigma\phi(\sigma)]^{-1}$ is still integrable near infinity. Hence, (**H0**) is satisfied and our results apply.

Appendix

Lemma 4.1. *Assumptions* (**H0**),(**H1**),(**H3**) *imply that for any* (x,t,v,p) *such that* $G(x,t,v,p) \geq c_*$,

(**G0**) $G_v(x,t,v,p) \geq \phi(G(x,t,v,p))$ *and* $|G_p(x,t,v,p)| \leq \kappa\, G_v(x,t,v,p)$,

(**G1**) $|G_t(x,t,v,p)| \leq \psi(G(x,t,v,p))\, G_v(x,t,v,p)$,

(**G3**) $|G_x(x,t,v,p)| \leq \gamma(|p|+1) G_v(x,t,v,p)$.

Proof — We first recall that $G(x,t,v,p) = -vH(x,t,-p/v) \leq H(x,t,-p/v)$ since $v \in [-1,0)$. So, if $G(x,t,v,p) \geq c_*$, necessarily $H(x,t,-p/v) \geq c_*$ also.

For (**G0**), straightforward computations show that $G_v(x,t,v,p) = (H_p \cdot p - H)(x,t,v,p)$ and $G_p(x,t,v,p) = H_p(x,t,-p/v)$. Thus, (**H0**) implies that if $G(x,t,v,p) \geq c_*$,

$$G_v(x,t,v,p) \geq \phi(H(x,t,-p/v)) \geq \phi(G(x,t,v,p))$$

since ϕ is increasing. The second part of (**G0**) is obvious.

To get (**G1**) we start from (**H1**):

$$|G_t(x,t,v,p)| = |-vH_t(x,t,-p/v)| \leq (-v)\psi(H(x,t,-p/v))G_v(x,t,v,p)\,.$$

Then we notice that ψ satisfies the following sublinear property: for any $\lambda \in [0,1]$, $\lambda\psi(H) \leq \psi(\lambda H)$. Indeed, since ϕ is increasin, for any $\lambda \in [0,1]$ and $\eta > c_*$, $(\lambda 2\eta)\phi(\lambda 2\eta) \leq \lambda(2\eta\phi(2\eta))$. Taking the inverse yields the result. Thus, taking $\lambda = -v \in [0,1]$ we get $|G_t(x,t,v,p)| \leq \psi(G(x,t,v,p))G_v(x,t,v,p)$.

The proof of (**G3**) is similar: $|G_x(x,t,v,p)| \leq |v||H_x(x,t,-p/v)|$. Then, using (**H3**), $|v| \leq 1$, we get

$$|G_x(x,t,v,p)| \leq \gamma\left(\left|\frac{p}{v}\right| + 1\right)|v|G_v(x,t,v,p) \leq \gamma(|p|+1)G_v(x,t,v,p)\,,$$

which gives the result.

Q.E.D.

References

[1] Barles, Guy, *Regularity results for first-order Hamilton-Jacobi Equations*. J. Differential and Integral Equations, vol 3, n° 1,(1990), pp 103-125.

[2] Barles, Guy; Ishii, Hitoshi; Mitake, Hiroyoshi, *A new PDE approach to the large time asymptotics of solutions of Hamilton-Jacobi equations*. Bull. Math. Sci. 3 (2013), no. 3, 363–388.

[3] Barles, Guy; Souganidis, Panagiotis E, On the Large Time Behavior of Solutions of Hamilton-Jacobi Equations. SIAM J. Math. Anal. 31, No.4, 925-939 (2000).

[4] Evans, L. C.; James, M. R., The Hamilton-Jacobi-Bellman equation for time-optimal control, SIAM J. Control Optim., 27, (1989), 6, 1477–1489.

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