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How tempered representations of a semisimple Lie group contract to its Cartan motion group

Alexandre Afgoustidis[†]

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Abstract

George W. Mackey suggested in 1975 that there should be analogies between the irreducible unitary representations of a noncompact semisimple Lie group G and those of its Cartan motion group – the semidirect product G_0 of a maximal compact subgroup of G and a vector space. In these notes, I focus on the carrier spaces for these representations and try to give a precise meaning to some of Mackey’s remarks. I first describe a bijection, based on Mackey’s suggestions, between the tempered dual of G – the set of equivalence classes of irreducible unitary representations which are weakly contained in $\mathbb{L}^2(G)$ – and the unitary dual of G_0 . I then examine the relationship between the individual representations paired by this bijection : there is a natural continuous family of groups interpolating between G and G_0 , and starting from the Hilbert space \mathbf{H} for an irreducible representation of G , I prove that there is an essentially unique way of following a vector through the contraction from G to G_0 within a fixed Fréchet space that contains \mathbf{H} . It then turns out that there is a limit to this contraction process on vectors, and that the subspace of our Fréchet space thus obtained naturally carries an irreducible representation of G_0 whose equivalence class is that predicted by Mackey’s analogy.

1 Introduction

When G is a Lie group and K is a closed subgroup of G , one can use the linear action of K on the vector space $V = \text{Lie}(G)/\text{Lie}(K)$ to define a new Lie group G_0 – the semidirect product $K \ltimes V$. This group is known as the contraction of G with respect to K , and it is famous in mathematical physics : the Poincaré group of special relativity admits as a contraction the Galilei group of classical inertial changes, and it is itself a contraction of the de Sitter group which appears in general relativity¹.

Since the unitary irreducible representations of the Poincaré group are well-known to yield particle state spaces for quantum physics, it is quite natural to wonder whether there is a relationship between the representation theories of G and G_0 . For most Lie groups, including the Poincaré group, unitary representations do not behave well under the contraction : in general the parameters needed to identify representations of G and G_0 are rather different, and this is important for physics – a consequence of the bad behaviour in

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¹See the lecture by Freeman Dyson [29].

the case of the Poincaré group is that the notion of (rest) mass has different meanings in special and Galilean relativity.

In 1975 however, George Mackey - who had single-handedly developed the representation theory of semidirect products like G_0 in the 1950s - noticed that in the special case where G is a semisimple Lie group and K is a maximal compact subgroup of G , there is a coincidence between the parameters needed to describe rather large² subsets of the unitary duals of G and G_0 , and some analogies in the way to build these subsets. In this case, G_0 is often called the Cartan motion group of G : it acts through rigid motions on the flat symmetric space G_0/K , while G is the isometry group of the negatively curved G/K .

It is rather surprising that there should be a deep analogy between the representation theories of these two groups, and not only because the algebraic structures of G and G_0 are very different. When Γ is a Lie group, let us write $\widehat{\Gamma}$ for its reduced (!) unitary dual - gathering the equivalence classes of unitary irreducible representations which are weakly contained in the regular representation ; Mackey's theorems then make the description of $\widehat{G_0}$ quite accessible, while describing \widehat{G} is a formidable task which took all of Harish-Chandra's talent and energy. Mackey nevertheless went on to conjecture that there should be a natural one-to-one correspondence between large enough subsets of \widehat{G} and $\widehat{G_0}$:

In view of the facts outlined above for $SL_2(\mathbf{C})$ it is natural to wonder to what extent one can find a correspondence between "most" of the irreducible representations of G and those of the semidirect product G_0 .

The groups G and G_0 fit into a continuous one-parameter family (see Section 2), and in 1985 Dooley and Rice proved [15] that the operators for principal series representations of G do weakly converge, as the contraction is performed, to operators for a generic representation of G_0 (I discuss some of their results in section 4). Although the initial reactions to Mackey's ideas seem to have been rather skeptical, an interest in Mackey's suggestions later sprang from the deformation theory of C^* -algebras : as Baum, Connes and Higson pointed out in [7, 13], the Baum-Connes conjecture for G (in its "smooth" version due to Connes and Kasparov, proved by Wasserman since then) is a precise counterpart to Mackey's analogy at the level of cohomology.

But the interest for Mackey's proposal seems to have waned since then, and it is scarcely - if at all - mentioned in the recent representation-theoretic literature (see however [14]). One of the reasons for the subject not having been pursued further, even after the mentioned developments in operator algebras, may be the fact that at the level of representation spaces, for the deeper strata of the unitary duals (as one moves away from the principal series of G or the generic representations of G_0), the analogy seems doomed to be rather poor : for instance, there are unitary irreducible representations of G_0 whose carrier spaces are finite-dimensional, while all unitary irreducible representations of G are infinite-dimensional (the trivial one excepted). Thus, as Mackey says,

Above all [the analogy] is a mere coincidence of parametrizations, with no evident relationship between the constructions of corresponding representations.

²In Mackey's suggestions, the meaning of "large" here refers to the Plancherel measure on the unitary dual of G_0

However, Nigel Higson recently revived the subject [8, 9], starting back from the Connes-Kasparov conjecture and showing that a precise elaboration on Mackey's ideas leads to new proofs of the conjecture : the C^* -algebra point of view involves shifting the attention from the representation spaces to their matrix coefficients, and Higson noticed that when G is complex semisimple, there is a deep, though not obvious, analogy between the structure of reduced C^* -algebras of G and G_0 : they turn out to be assembled from identical building blocks, and fit into a continuous field³ which turns out to be assembled from constant fields through Morita equivalences, extensions and direct limits. He also made the important side observation that while Mackey's suggestions treated the unitary dual as a Borel space, the K -theory in the Connes-Kasparov phenomenon treats it as a topological space, and there is no natural way to relate these two points of view. This suggests that Mackey's analogy should extend to the full tempered dual of G , yielding an interesting bijection between \widehat{G}_0 and all of \widehat{G} ; this I will take up in Section 3 below.

In these notes, my aim is to consider Mackey's proposal from a rather naive perspective : I will start with spaces realizing elements of \widehat{G} and try to describe what happens to the (smooth, K -finite) vectors as one proceeds to the contraction. My hope is to give in this way a somewhat simpler picture than is usual for the relationship between \widehat{G} and \widehat{G}_0 , and to try to understand why the various strata in the unitary duals behave very differently.

As we shall see, the part of Mackey's analogy which relates spherical principal series representations of G to generic class-one representations of G_0 can be rephrased as transferring harmonic analysis on a symmetric space of the noncompact type to classical Fourier analysis on its (Euclidean) tangent space at a given point. This is a much-studied problem with beautiful ramifications [18, 21, 19, 20], and Higson's account of the Connes-Kasparov phenomenon shows that bringing Mackey's point of view into the picture is not at all devoid of interest in this case.

Now at the other end of the tempered spectrum, if G has discrete series representations, Mackey's proposal is that we should relate them to irreducible representations of K ; what makes this reasonable is the fact that a discrete series representation has a unique minimal K -type. If G_0 is to be brought into the picture here, my task is then to understand how its minimal K -type can emerge from a discrete series representation as the contraction from G to G_0 is performed.

The methods I will use here are not original in any way : on the contrary, I will try to take full advantage of the geometric realizations of unitary representations of G which were set forth in the years immediately following Mackey's proposal. These realizations provide natural topologies, defined on (dense subsets of) the Hilbert spaces, that are different from the Hilbert space norm : at least for principal series and discrete series representations, the smooth vectors in the Hilbert spaces for tempered representations of G can be seen as functions on, or sections of homogeneous bundles on, the symmetric space G/K . In these notes I will trace Mackey's analogy to phenomena which are invisible to the Hilbert space topology, but become obvious when the topology of uniform convergence on compact subsets of G/K comes in.

³The continuous field is defined using the deformation from G to G_0 which, from section 2 onwards, will play a key role in these notes.

Here is an outline of my notes. I will start with a description of the Cartan motion group, its unitary dual, and some aspects of the contraction from G to G_0 in section 2. After these preliminaries, I will write down a bijection between \widehat{G} and \widehat{G}_0 ; for this extension of Higson's analysis in the complex case, the main tool is Vogan's minimal K-type parametrization of the tempered irreducible representations of G which have real infinitesimal character. From section 3.3 onwards I focus on individual representation spaces, pursuing evidence for a phenomenon described in section 3.3 and summarized as Theorem 3.4 there. Sections 4 and 5 examine what happens in the case of the spherical principal series representations and the discrete series representations, respectively: both start with a presentation of the geometric realizations I will use, then watch a vector evolve as one proceeds to the contraction from G to G_0 . Since the full tempered dual of G can be in some sense assembled from the discrete series of reductive subgroups, I then use the results on the discrete series and the ideas (and lemmas) of the spherical principal case to work out the general case in sections 6 (real infinitesimal character) and 7 (general case).

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2 The Cartan motion group and its unitary dual

2.1 The contraction from G to G_0

Throughout these notes, I shall consider a real, connected, noncompact semisimple Lie group G , assume that it has finite center, and write \mathfrak{g} for its Lie algebra. Let's start from a maximal compact subgroup K of G , its Lie algebra \mathfrak{k} , and the orthogonal \mathfrak{p} of \mathfrak{k} with respect to the Killing form of \mathfrak{g} , so that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. The adjoint action of K leaves \mathfrak{p} invariant, so we can (re-)define G_0 as the set $K \times \mathfrak{p}$ with the group structure

$$(k_1, v_1) \cdot_0 (k_2, v_2) := (k_1 k_2, v_1 + Ad(k_1)v_2).$$

Let me start this section by saying more precisely how G_0 is a deformation of G . For each $t > 0$, we can use the diffeomorphism

$$\begin{aligned} \varphi_t : K \times \mathfrak{p} &\rightarrow G \\ (k, v) &\mapsto \exp_G(tv)k \end{aligned}$$

to endow the set $K \times \mathfrak{p}$ with a group structure which turns φ_t into an isomorphism.

So we will write G_t for be the set $K \times \mathfrak{p}$ with the composition

$$(k_1, v_1) \cdot_t (k_2, v_2) := \varphi_t^{-1}(\varphi_t(k_1, v_1) \cdot_G \varphi_t(k_2, v_2)).$$

Notice that we can use the Campbell-Hausdorff formula for G to introduce the map $Z : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ such that $\exp(X)\exp(Y) = \exp(X + Y + Z(X, Y))$ for all $X, Y \in \mathfrak{g}$; then

$$\begin{aligned} (k_1, v_1) \cdot_t (k_2, v_2) &= \varphi_t^{-1}(\exp[tv_1]\exp[tAd(k_1)v_2]k_1k_2) \\ &= \varphi_t^{-1}\left(\exp\left[t\left(v_1 + Ad(k_1)v_2 + \frac{1}{t}Z(tv_1, tAd(k_1)v_2)\right)\right]k_1k_2\right). \end{aligned}$$

It is now clear from the properties of Z that $(k_1, v_1) \cdot_t (k_2, v_2)$ goes to $(k_1, v_1) \cdot_0 (k_2, v_2)$ as t goes to zero.

Before we delve into representation theory, I will record here the effect of this contraction on the Riemannian symmetric space G/K . Recall that we can view the K -invariant Killing form of \mathfrak{g} as a quadratic form on the tangent space to G/K at the identity coset $\{K\}$, and then use the (left) action of G to build a G -invariant Riemannian metric on G/K ; this metric has constant scalar curvature -1 .

2.2 The G_t -actions on \mathfrak{p}

2.2.1. Now, the Cartan decomposition provides an explicit diffeomorphism between G/K and \mathfrak{p} , so we can use it to make \mathfrak{p} into a G -homogeneous space, and we can do this for each $t > 0$, using the natural maps

$$u : \mathfrak{p} \xrightarrow{\exp_G} G \twoheadrightarrow G/K.$$

$$u_t : \mathfrak{p} \xrightarrow{\exp_{G_t}} G_t \twoheadrightarrow G_t/K.$$

The fact that these maps are diffeomorphisms provides us with a transitive action of G on \mathfrak{p} , for which I will write $(g, x) \in G \times \mathfrak{p} \mapsto g \cdot x$ in these notes, as well as a transitive action of each G_t , for which my notation will be $(\gamma, x) \in G_t \times \mathfrak{p} \mapsto \gamma \cdot_t x$. Of course the stabilizer of the point 0 in \mathfrak{p} is K for each t . Let's also transfer the natural metric on G_t/K to \mathfrak{p} through u_t , but take into account the fact that the Killing forms of G_t and G are not quite the same: so let us start from the restriction to \mathfrak{p} of the Killing form of \mathfrak{g} , say B , and use it for each t to build a G_t -invariant metric η_t on \mathfrak{p} which coincides with B at zero. Let me also write η for the G -invariant metric on \mathfrak{p} built in this way and note that $\eta = \eta_1$.

If we do this, then the metric η_t has scalar curvature $-t$. On the other hand, we can build a G_0 -invariant Euclidean metric η_0 on \mathfrak{p} from B and the action of G_0 (which includes the translations of \mathfrak{p} , and for which my notation will be $(g_0, x) \in G_0 \times \mathfrak{p} \mapsto g_0 \cdot_0 x$), and the metrics η_t do tend to η_0 as t tends to zero (in the topology, say, of uniform convergence on compact sets for the metrics' coefficients in affine coordinates on \mathfrak{p}).

In the next three paragraphs, I will record simple facts on this geometrical setting.

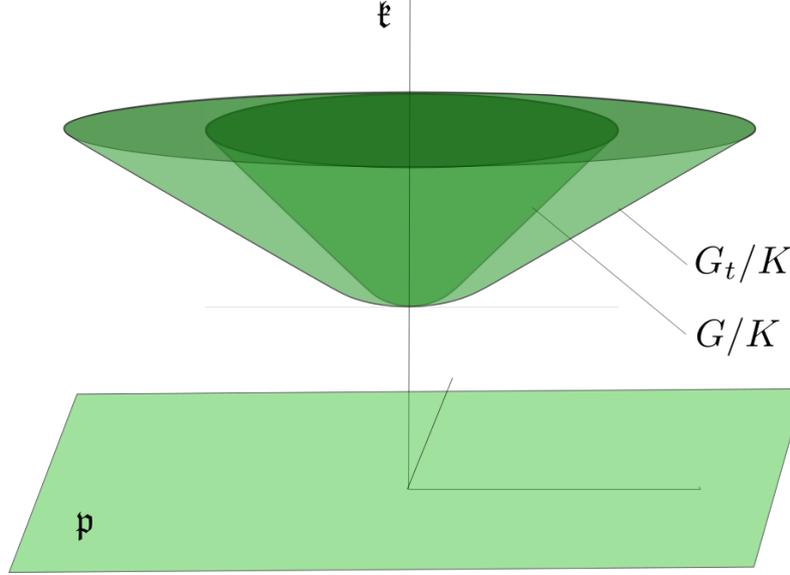


Figure 1: This is a picture of (co)adjoint orbits for G and G_t when G is $SL_2(\mathbb{R})$. The (green) horizontal plane is the space \mathfrak{p} of symmetric matrices with zero trace, the vertical axis is the line \mathfrak{k} of antisymmetric matrices ; the drawn orbits are the G - and G_t -coadjoint orbits of their common intersection with the vertical axis. Section 2.2.3, especially lemma 2.3, are a comment on this figure.

2.2.2. Let me consider the dilation

$$z_t : x \in \mathfrak{p} \mapsto \frac{x}{t}.$$

An important ingredient in these notes will be the fact that the relationship between G and G_t is simple enough that their actions on \mathfrak{p} are related through z_t :

Lemma 2.1. *For every x in \mathfrak{p} and every g in G , $\varphi_t^{-1}(g) \cdot_t z_t(x)$ is equal to $z_t(g \cdot x)$ for all x .*

Proof. Let us see where the diffeomorphism u_t sends the both of them. On the one hand,

$$u_t(\varphi_t^{-1}(g) \cdot_t z_t(x)) = \varphi_t^{-1}(g) \exp_{G_t}(z_t x) K = \varphi_t^{-1}(g \exp_G(x) K),$$

and on the other hand, using the definition of the G_t and G -actions,

$$u_t(z_t(g \cdot x)) = \exp_{G_t}[z_t(g \cdot x)] K = \varphi_t^{-1}(\exp_G(g \cdot x) K) = \varphi_t^{-1}(g \exp_G(x) K).$$

□

2.2.3. Here is a remark which says how the action of G_t on \mathfrak{p} admits the natural action of G_0 as a limit. Let me introduce the diffeomorphisms corresponding to the Cartan decomposition of each G_t , writing

$$\begin{aligned} \alpha_t : G_0 &\rightarrow G_t \\ (k, v) &\mapsto \exp_{G_t}(v)k. \end{aligned}$$

(of course none of the α_t is a group morphism), and $\alpha : G_0 \rightarrow G, (k, v) \mapsto \exp_G(v)k$. Note that φ_t sends $\alpha_t(k, v)$ to $\alpha(k, tv)$.

Lemma 2.2. For each g_0 in G_0 , $\alpha_t(g_0) \cdot_t x$ tends to $g_0 \cdot_0 x$ uniformly on \mathfrak{p} .

Proof. Recall that the G_t -action is defined by using

$$u_t : \mathfrak{p} \xrightarrow{\exp_{G_t}} G_t \twoheadrightarrow G_t/K.$$

to transfer the action (by left multiplication) of G_t on G_t/K .

Now if $g_0 = (k, v)$,

$$\begin{aligned} \alpha_t(g_0) \cdot_t x &= \varphi_t^{-1} [\alpha(k, tv)] \cdot_t x \\ &= \frac{1}{t} [\alpha(k, tv)] \cdot (tx) \quad \text{by Lemma 2.1.} \end{aligned}$$

What we need to show is that $\frac{1}{t} \alpha(k, tv) \cdot (tx)$ goes to $v + \text{Ad}(k)x$ as t goes to zero. But

$$\begin{aligned} u \left(\frac{1}{t} \alpha(k, tv) \cdot (tx) \right) &= \exp_G \left(\frac{1}{t} u^{-1} [\alpha(k, tv) \exp_G(tx)K] \right) K \\ &= \exp_G \left(\frac{1}{t} u^{-1} [\exp_G(tv)k \exp_G(tx)K] \right) K \\ &= \exp_G \left(\frac{1}{t} u^{-1} [\exp_G(tv) \exp_G(t\text{Ad}(k)[x])K] \right) K \\ &= \exp_G \left(\frac{1}{t} u^{-1} [\exp_G \{ (tv + t\text{Ad}(k)[x] + Z(tv, t\text{Ad}(k)[x])) \} K] \right) K \\ &= v + \text{Ad}(k)x + O(t) \end{aligned}$$

and the lemma follows . □

2.2.4. The simple facts I just described are well-displayed by Figure 1. Whenever G/K is a hermitian symmetric space, which is the case when G is $SL_2(\mathbb{R})$, there is an element in $\mathfrak{k} \subset \mathfrak{g}$ whose stabilizer under the adjoint action of G is K . Because \mathfrak{g} is the underlying vector space for the Lie algebra of G_t , too, we can look at the relationship between the orbits (although the adjoint and coadjoint actions are naturally equivalent for semisimple Lie groups, the usual vector space identification between \mathfrak{g}_t and \mathfrak{g}_t^* depends on the Killing form, which varies with t : while for some questions – like studying the distribution characters – it would be desirable to look at coadjoint orbits of course, I will stick with adjoint orbits here) .

Let me write ϕ_t for the derivative $X_{\mathfrak{k}} + X_{\mathfrak{p}} \in \mathfrak{k} \oplus \mathfrak{p} \mapsto X_{\mathfrak{k}} + tX_{\mathfrak{p}}$ of φ_t at the identity.

Lemma 2.3. The image under ϕ_t of a G_t -adjoint orbit, say Ω , is a G -adjoint orbit which has the same intersection with \mathfrak{k} as Ω .

Proof.

Start from the fact that φ_t is a group morphism, so $\varphi_t^{-1}(g)\varphi_t^{-1}(h)\varphi_t^{-1}(g) = \varphi_t^{-1}(ghg^{-1})$ for $g, h \in G$, and just take the derivative at $h = \mathbf{1}_G$; this yields

$$\text{Ad}_{G_t}(\varphi_t^{-1}g) [\phi_t^{-1}\lambda] = \phi_t^{-1} \text{Ad}_G(g)[\lambda]$$

so the G -adjoint orbit for λ is the image under ϕ_t of the G_t -adjoint orbit for $\phi_t^{-1}\lambda$, as announced. □

On the face of it Figure 1 seems to be a complete picture of the contents of this subsection when G is $SL_2(\mathbb{R})$, with the metrics η_t transferred from those of the upper sheets of various hyperboloids to \mathfrak{p} through the vertical projection. It would be a really complete picture if $u_t : \mathfrak{p} \rightarrow G_t/K$ were the vertical projection. This is not the case, but it is a near miss : an explicit calculation shows that u_t^{-1} is not quite the vertical projection on \mathfrak{p} , but that there is a very simple diffeomorphism of \mathfrak{p} , namely

$$\tau : x \mapsto \frac{\sinh(\|x\|_B)}{\|x\|_B}(\wedge x)$$

with \wedge a rotation of angle $\pi/2$, such that $u \circ \tau : \mathfrak{p} \rightarrow G/K$ coincides with the vertical projection.

The appearance of τ is not very surprising here⁴ : the geodesics of G/K which go through the identity coset are sent by both u_t and the vertical projection to straight lines through the origin, and the (nonconstant) dilation factor in τ compensates for the difference between the speeds at which geodesics spread in hyperbolic space and Euclidean space (see [41], Chap. 6).

2.2.5. In these notes, I shall try to trace the relationship between the representations of G and G_0 to the fact that the building blocks of the representation theory of G amount to studying spaces of functions on G/K , or sections of homogeneous bundles on it, which satisfy some invariant partial differential equation. The diffeomorphisms u_t , the relationship between the above actions of G and G_t on \mathfrak{p} and between the metrics η_t , will be simple enough to allow me to follow vectors through the contraction. But before I focus on individual representations from section 3.3 onwards, I will set to describe a common parametrization for \widehat{G} and \widehat{G}_0 .

2.3 The unitary dual of G_0

Let us start with some standard notation : suppose \mathfrak{a} is a maximal abelian subalgebra of \mathfrak{p} , and W is the Weyl group of the pair $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{a}_{\mathbf{C}})$ (here and throughout these notes, a subscript \mathbf{C} codes for the complexification of a Lie algebra). When χ is an element of \mathfrak{p}^* (the dual vector space), we write K_χ for its stabilizer (or "isotropy group") under the coadjoint action of K on \mathfrak{p} ; note that K_χ is usually not connected, though as we shall see it has finitely many components. One can use the Killing form of \mathfrak{g} to embed \mathfrak{a}^* in \mathfrak{p}^* as those linear forms on \mathfrak{p} which vanish on the orthocomplement of \mathfrak{a} , and then note that all χ s in \mathfrak{a}^* which are regular (that is, whose K -orbit in \mathfrak{p}^* has the largest possible dimension for a K -orbit in \mathfrak{p}^*) have the same stabilizer ; we will write M for it. We also write A for the abelian subgroup $\exp_G(\mathfrak{a})$.

I will now describe Mackey's results on the unitary dual of G_0 [11, 12]. For this, let's start with

Definition 2.1. A *Mackey datum* is a couple (χ, μ) in which χ is an element of \mathfrak{a}^* , and μ is an equivalence class of irreducible K_χ -modules.

⁴Thank you to Martin Puchol for pointing this out.

This non-standard vocabulary will be useful for us later ; but each K -orbit in \mathfrak{p}^* intersects \mathfrak{a}^* , so choosing a Mackey datum is the same as choosing first a K -orbit in \mathfrak{p}^* , then an irreducible representation of the isotropy group of one of its elements – these are the more usual parameters for \widehat{G}_0 .

From a Mackey datum $\delta = (\chi, \mu)$, one can produce a unitary representation of G_0 by unitary induction : set

$$\mathbf{M}_0(\delta) := \text{Ind}_{K_\chi \ltimes \mathfrak{p}}^{G_0} [\mu \otimes e^{i\chi}].$$

If we write W for an irreducible K_χ -module of class μ , a Hilbert space for $\mathbf{M}_0(\delta)$ is obtained by considering

$$\{f : K \rightarrow W \mid f(km) = \mu(m)f(k) \text{ for } k \in K, m \in K_\chi\}, \quad (1)$$

declaring that

$$g = (k, v) \in G_0 \text{ acts through } f \mapsto [u \mapsto e^{i(\chi, \text{Ad}(u^{-1})v)} f(k^{-1}u)], \quad (2)$$

and for the Hilbert space structure taking the \mathbf{L}^2 space associated to the Haar measure of K and a K_χ -invariant inner product on W .

Mackey proved (see [11, 12]) that each of the $\mathbf{M}_0(\delta)$ is irreducible.

Moreover, if we start from two Mackey data $\delta_1 = \mathbf{M}(\chi_1, \mu_1)$ and $\delta_2 = \mathbf{M}(\chi_2, \mu_2)$, the condition for $\mathbf{M}_0(\delta_1)$ and $\mathbf{M}_0(\delta_2)$ to be unitarily equivalent is that there be an element of the Weyl group of $(\mathfrak{g}_\mathbb{C}, \mathfrak{a}_\mathbb{C})$ which sends χ_1 to χ_2 and μ_1 to an irreducible K_{χ_2} -module which is unitarily equivalent with μ_2 . So we have an equivalence relation between Mackey data, and an injective map from the set of equivalence classes of Mackey data into \widehat{G}_0 . Mackey also proved that this map is surjective : the assignation

$$\delta \mapsto \mathbf{M}_0(\delta)$$

gives a bijection between the Mackey data, up to equivalence, and the unitary irreducible representations of G_0 , up to unitary equivalence.

Let us insist that this parametrization gives a stratification of \widehat{G}_0 into subsets recording the dimension of the orbit of the parameter χ in \mathfrak{p}^* ; parameters for the extreme strata are :

- Mackey data with $\chi = 0$, which correspond to the irreducible representations of K , with finite-dimensional carrier spaces ;

- Mackey data with regular χ ; the corresponding representations of G_0 are unitarily induced from $M \ltimes \mathfrak{p}$, so they have a realization as spaces of square-integrable vector-valued functions on K/M which transform according to (2) under the action of G_0 . There is a nice geometric picture for⁵ $\mathbf{M}_0(\chi, 1)$ which will be useful for us in section 4 : consider the tempered distributions on \mathfrak{p} whose Euclidean Fourier transform – a tempered distribution on \mathfrak{p}^* – is supported on $\text{Ad}^*(K) \cdot \chi$. These are automatically smooth functions on \mathfrak{p} , and when realized as functions on K/M through the Fourier transform, they do transform

⁵The 1 here means that I use the trivial representation of K_χ .

in the right way : (2) can be easily understood from the usual formula for the Fourier transform of $x \mapsto f(g_0 \cdot_0 x)$. By considering the smooth and square-integrable functions whose Fourier transform has the mentioned property, we get a realization of $\mathbf{M}_0(\chi, 1)$ for which members of the carrier space appear as functions on \mathfrak{p} which are combination of those plane waves with frequency vectors on $\text{Ad}^*(K) \cdot \chi$.

To close this subsection, let me note that every unitary irreducible representation of G_0 is weakly contained in the regular representation, so the reduced dual and the unitary dual of G_0 coincide. This is in sharp contrast with the situation for our semisimple group G , for which the unitary dual is quite larger than the reduced dual ; to give the simplest but significant example, the trivial representation of G is not in the reduced dual of G . In fact, the unitary irreducible representations which appear in the reduced dual of G are all *tempered* – this means that their matrix coefficients lie in $\mathbf{L}^{2+\epsilon}(G)$ for each positive ϵ , and although this definition leaves the terminology rather mysterious, it makes it quite clear that the trivial representation is not tempered.

When we write out a correspondence between \widehat{G}_0 and \widehat{G} in the next section, the trivial representation of G_0 will thus be associated with a quite non-trivial (and infinite-dimensional) representation of G .

3 Mackey's correspondence

3.1 Minimal K -type for discrete series, and a theorem of Vogan

In this subsection I will assume that G has a nonempty discrete series and write T for a maximal torus in K , so that T is also a (compact) Cartan subgroup of G .

Let us start with a unitary irreducible representation π of G in a Hilbert space \mathbf{H} ; the restriction $\pi|_K$ is a direct sum of irreducibles.

Given the choice of a system Δ_c^+ of positive roots for the pair $(\mathfrak{k}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})$, let's write ρ_c for the half-sum of the elements of Δ_c^+ ; it is an element of it^* . Recall that an element λ of \widehat{K} then has a *highest weight*, which is an element of it^* ; I shall also write λ for it. An element of \widehat{K} is a *minimal K -type* of π when, among the positive numbers $\|\lambda' + 2\rho_c\|$ in which λ' is the highest weight of a class occuring in $\pi|_K$, $\|\lambda + 2\rho_c\|$ is minimal (here $\|\cdot\|$ means the norm induced by the Killing form). We shall need only very simple instances of the deep problem of studying the minimal K -types in a unitary irreducible representation [5, 6].

The starting point for our common parametrization of \widehat{G} and \widehat{G}_0 is the fact that a *discrete series* representations π of G has a *unique* minimal K -type, and that non-equivalent discrete series representations have non-equivalent minimal K -types. This was noticed by Blattner in the wake of Harish-Chandra's formidable work on the discrete series ; see [2, 28] and the historical remarks in [23].

Let's record here that later in these notes, it will also be an important fact that its minimal K -type also occurs with multiplicity one in $\pi|_K$.

This theorem gives a very precise indication of which representations of G we should attach to the subset of \widehat{G}_0 gathering the representations of K . But not all K -types are to be obtained as minimal K -types of discrete series representations of G , and in the unitary dual of G_0 , there is no difference to be made between the elements of \widehat{K} ; to cover the remaining cases we will need a theorem of David Vogan which identifies tempered irreducible representations of G with a unique minimal K -type, in a way which treats all the elements in \widehat{K} on the same footing.

For the rest of this subsection, I no longer assume G to be connected nor semisimple, but only that it is a reductive Lie group in Harish-Chandra's class, as defined in [24] – the induction steps in the next subsection will make this technical detail necessary.

To state Vogan's theorem, let's first recall the notion of infinitesimal character⁶ (this presentation is taken from [4]).

When we consider the infinitesimal counterpart to an irreducible representation of G – a representation of the Lie algebra \mathfrak{g} on a space V , and its complexification, the elements in the center of the universal enveloping algebra $\mathfrak{Z}(\mathfrak{g}_{\mathbf{C}})$ act as scalar multiples of the identity on V ; we obtain an abelian character of the algebra $\mathfrak{Z}(\mathfrak{g}_{\mathbf{C}})$.

Now whenever $\mathfrak{h}_{\mathbf{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbf{C}}$ and $W_{\mathfrak{h}}$ is the corresponding Weyl group, there is a simple correspondence between characters of the commutative algebra $\mathfrak{Z}(\mathfrak{g}_{\mathbf{C}})$ on the one hand, and $\mathfrak{h}_{\mathbf{C}}^*/W_{\mathfrak{h}}$ on the other hand.

To define it, recall that Harish-Chandra defined an isomorphism $\xi_{\mathfrak{h}}$ from $\mathfrak{Z}(\mathfrak{g}_{\mathbf{C}})$ to the set $\mathbf{S}(\mathfrak{h}_{\mathbf{C}})^{W_{\mathfrak{h}}}$ of $W_{\mathfrak{h}}$ -invariant symmetric polynomials on $\mathfrak{h}_{\mathbf{C}}$ (for the definition of $\xi_{\mathfrak{h}}$, see the proof of lemma 6.5 below). Since evaluation at an element $\lambda \in \mathfrak{h}_{\mathbf{C}}^*$ yields a map from $\mathbf{S}(\mathfrak{h}_{\mathbf{C}})$ to \mathbf{C} , we can compose with the Harish-Chandra isomorphism to obtain an abelian character, say $\xi_{\mathfrak{h}}(\lambda)$, of $\mathfrak{Z}(\mathfrak{g}_{\mathbf{C}})$. Once the obvious equivalences are quotiented out, the map $\lambda \mapsto \xi_{\mathfrak{h}}(\lambda)$ provides us with the promised bijection between $\mathfrak{h}_{\mathbf{C}}^*/W_{\mathfrak{h}}$ and the set of abelian characters of $\mathfrak{Z}(\mathfrak{g}_{\mathbf{C}})$.

So if we start with an element λ in $\mathfrak{h}_{\mathbf{C}}^*$, we see what it means for an irreducible representation π of G to have infinitesimal character λ . To state Vogan's theorem we need to see what it means for π to have real infinitesimal character. Assume $\mathfrak{h}_{\mathbf{C}}$ is the complexification of the Lie algebra \mathfrak{h} of a Cartan subgroup of G which is stable under the Cartan involution associated to K . Splitting the subgroup into compact and vector subgroups, we get a decomposition $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$, and we can set $\mathfrak{R}(\mathfrak{h}) = i\mathfrak{t} \oplus \mathfrak{a}$. This is a subspace of $\mathfrak{h}_{\mathbf{C}}$, and of course it is a real form of $\mathfrak{h}_{\mathbf{C}}$; so our linear functional λ reads $\mathfrak{R}(\lambda) + i\mathfrak{I}(\lambda)$, where $\mathfrak{R}(\lambda)$ and $\mathfrak{I}(\lambda)$ are elements of $\mathfrak{h}_{\mathbf{C}}^*$ whose restriction to $\mathfrak{R}(\mathfrak{h})$ is real-valued.

The phrase " π has real infinitesimal character", i.e. $\mathfrak{I}(\lambda) = 0$, then turns out to be independent of the choice of H .

For $SL_2(R)$, there are but three of these representations which are not in the discrete series: the two "limits of discrete series" – which can be realized, like the discrete series

⁶It is not only for the reader's convenience that I recall the notion here: I will need it in Section 6.1 below.

representations, as spaces of holomorphic functions on the hyperbolic plane, but have a rather different Hilbert space structure – and the "principal series representations with spectral parameter zero" – the name and properties of this one are different, see section 6.3. Each of these three representations turns out to have a unique minimal $SO(2)$ -type as well ; the corresponding characters are the trivial one, the "identity" character corresponding to the usual embedding of $SO(2)$ in \mathbf{C} , and the "conjugation" character which is the complex-conjugate of the former, and they are precisely the characters that do not appear as minimal $SO(2)$ -type in any discrete series representation.

Returning to a reductive Lie group G in Harish-Chandra's class, Vogan proved that every irreducible tempered representation of G which has real infinitesimal character has a unique minimal K -type, that nonequivalent such representations have different minimal K -types, and that all K -types can be obtained in this way. This can be rephrased as follows.

Theorem (Vogan). *The minimal K -type map defines a bijection between the equivalence classes of irreducible tempered representations of G which have real infinitesimal character, and the equivalence classes of irreducible representations of K .*

Our results in sections 5 and 6 can be viewed as a way to use the contraction from G to G_0 to exhibit its minimal K -type from the carrier space of a tempered irreducible representation with real infinitesimal character. But now let us linger at the level of parameters ; it is time to give a precise meaning to Mackey's analogy.

3.2 Mackey and Higson's bijection

Let us come back to a connected semisimple Lie group G whose center is finite. I am now going to define a map from \widehat{G}_0 to \widehat{G} . Let us start with a Mackey datum $\delta = (\chi, \mu)$; out of $\chi \in \mathfrak{a}^*$, we will first build a cuspidal parabolic subgroup P_χ of G . For this, let us write T_χ for the intersection $K_\chi \cap T$; it is a maximal torus in K_χ . Consider now the centralizer \mathfrak{a}_χ of its Lie algebra \mathfrak{t}_χ in \mathfrak{a} , and the vector subgroup $A_\chi := \exp_G(\mathfrak{a}_\chi)$; now we have an abelian subgroup $H_\chi := T_\chi A_\chi$. It is a Cartan subgroup of G .

Note that in the case where G and K have equal ranks, H_0 is compact, while $H_\chi = MA$ whenever χ is regular.

Building a parabolic subgroup from the Cartan subalgebra $\mathfrak{t}_\chi \oplus \mathfrak{a}_\chi$ is now a standard matter. Let us write L_χ for the centralizer of A_χ in G ; it reads $L_\chi = M_\chi A_\chi$, where M_χ is the product of the centralizer of \mathfrak{a}_χ in K and the analytic subgroup of G whose Lie algebra is the orthocomplement \mathfrak{m}_χ of \mathfrak{a}_χ in \mathfrak{l}_χ .

The subgroup M_χ is no longer semisimple nor connected in general ; it is however a reductive group in Harish-Chandra's class, and given its definition, K_χ is a maximal compact subgroup in it. Note that T_χ is a compact Cartan subgroup of M_χ , so M_χ has a nonempty discrete series.

To complete the definition of a cuspidal parabolic subgroup P_χ , we need only obtain a subalgebra \mathfrak{n}_χ of \mathfrak{g} by choosing a system of positive roots, say Δ^+ , for the pair $(\mathfrak{g}_\mathbf{C}, \mathfrak{h}_\mathbf{C})$, defining a complex subalgebra of $\mathfrak{g}_\mathbf{C}$ as the the sum of root spaces for those positive roots

which do not vanish on \mathfrak{a}_χ , and choosing \mathfrak{n}_χ as its real part. Then we set $N_\chi := \exp_G(\mathfrak{n}_\chi)$, and finally $P_\chi := M_\chi A_\chi N_\chi$.

Let us pause on a few details on the Lie algebra level and record here that

$$\begin{aligned}\mathfrak{m}_\chi &= \bigoplus_{\alpha \in \Delta^+ | \alpha|_{\mathfrak{a}_\chi} = 0} \mathfrak{g}_\alpha; \\ \mathfrak{n}_\chi &= \bigoplus_{\alpha \in \Delta^+ | \alpha|_{\mathfrak{a}_\chi} \neq 0} \mathfrak{g}_\alpha.\end{aligned}$$

Now we shall perform parabolic induction from P_χ : our Mackey datum δ came with $\mu \in \widehat{K}_\chi$; let's consider the tempered representation $\mathbf{V}_{M_\chi}(\mu)$ of M_χ which is provided by Vogan's theorem. We extend χ to define a one-dimensional representation of $A_\chi N_\chi$, and then consider the unitarily induced representation

$$\mathbf{M}(\delta) := \text{Ind}_{P_\chi}^G [\mathbf{V}_{M_\chi}(\mu) \otimes e^{i\chi}].$$

There are important results of representation theory to be called upon here. They are simple consequences of deep work on irreducible tempered representations by Harish-Chandra on the one hand, Knapp and Zuckerman on the other, but since I will need to check a few things let me state them as three lemmas :

Lemma 3.1. *For each Mackey datum δ , this $\mathbf{M}(\delta)$ is irreducible and tempered.*

Lemma 3.2. *Suppose δ_1, δ_2 are Mackey data. Then the representations $\mathbf{M}(\delta_1)$ and $\mathbf{M}(\delta_2)$ are unitarily equivalent if and only if δ_1 and δ_2 are equivalent as Mackey data.*

So we get an injection from \widehat{G}_0 into \widehat{G} . Now, another consequence of Knapp and Zuckerman's results is that

Lemma 3.3. *Each irreducible tempered unitary representation is equivalent with one of the representations $\mathbf{M}(\delta)$.*

These three lemmas together prove

Theorem 3.1. *The map $\mathbf{M} \circ \mathbf{M}_0^{-1}$ induces a bijection between the unitary dual \widehat{G}_0 and the tempered dual \widehat{G} .*

To prove this theorem, we need only relate our three lemmas to an irreducibility theorem by Harish-Chandra on the one hand, and to the Knapp-Zuckerman classification of tempered irreducible representations on the other hand. Let's introduce a slight change of notation and write $\mathfrak{a}_\mathfrak{p}$, $A_\mathfrak{p}$, $M_\mathfrak{p}$, and so on, for the algebras and groups related to the minimal parabolic subgroup P_χ common to all regular χ s.

Proof of Lemma 3.1.

I will use a result of Harish-Chandra, cited as theorem 14.93 in [23] ; for a full proof and discussion see [33], theorem 4.11. Say that an element of $i\mathfrak{a}_\chi^*$ is \mathfrak{a}_χ -regular when its scalar product with each root of the pair $(\mathfrak{g}_\mathfrak{c}, \mathfrak{a}_{\chi, \mathfrak{C}})$ is nonzero ⁷. The result by Harish-Chandra

⁷NB : these roots do not give rise an abstract root system, but that will be no problem for us.

implies Lemma 3.1 if we can ensure that χ is \mathfrak{a}_χ -regular I will check this now.

Suppose a system of positive roots Σ is chosen so that the positive roots have a non-negative scalar product with χ . I now introduce four subsets of Σ :

$$\begin{aligned}\Sigma(\chi) &= \{\beta \in \Sigma \mid \langle \beta, \chi \rangle = 0\}; \\ \Sigma^{\max}(\chi) &= \{\beta \in \Sigma(\chi) \mid \forall \alpha \in \Sigma(\chi), \alpha + \beta \notin \Sigma\}; \\ \Phi(\chi) &= \{\beta \in \Sigma(\chi) \mid \beta|_{\mathfrak{a}_\chi} \text{ not identically } 0\}; \\ \Phi^{\max}(\chi) &= \Phi(\chi) \cap \Sigma^{\max}(\chi);\end{aligned}$$

and what is to be checked is that $\Phi(\chi)$ is empty. Note that $\Sigma(\chi)$ is nonempty exactly when χ is singular ; since there is nothing to be checked for regular χ s, I will assume this is the case. In this case, $\Sigma^{\max}(\chi)$ is nonempty as well.

Let's introduce the element H_χ of $\mathfrak{a}_\mathfrak{p}$ such that $\langle \beta, \chi \rangle = \beta(H_\chi)$ for each $\beta \in \Sigma$, and suppose a nonzero vector X_β is chosen in the root space \mathfrak{g}_β for each $\beta \in \Sigma(\chi)$. For each $\beta \in \Sigma(\chi)$ we know that $[H_\chi, X_\beta] = \beta(H_\chi)X_\beta$ is zero. Let's write $X_\beta = K_\beta + P_\beta$ with K_β and P_β in \mathfrak{k} and \mathfrak{p} , respectively. Then $[H_\chi, K_\beta] + [H_\chi, P_\beta]$ is zero, the first term is in \mathfrak{p} and the second in \mathfrak{k} , so they are both zero, and we see that K_β is in \mathfrak{k}_χ .

Since there was no other constraint on \mathfrak{k}_χ other than it be a maximal torus in K_χ , I can suppose that \mathfrak{k}_χ contains all of the K_β , $\beta \in \Sigma^{\max}(\chi)$. This is because they all commute : whenever γ and δ are in $\Sigma^{\max}(\chi)$, their bracket $[X_\gamma, X_\delta]$ is in the root space $\mathfrak{g}_{\gamma+\delta}$, which is zero since $\gamma + \delta$ is not a root. I will assume until the end of the ongoing proof that the choice of \mathfrak{k}_χ (which induces that of \mathfrak{a}_χ and the definition of \mathbf{M} and \mathbf{M}_0) is made accordingly.

I now claim that $\Phi^{\max}(\chi)$ is empty. If β is in it, then there is an element H_β in \mathfrak{a}_χ such that $c = \beta(H_\beta)$ is nonzero ; using $[H_\beta, X_\beta] = cX_\beta$ and the Cartan decomposition of \mathfrak{g} , we get

$$[H_\beta, K_\beta] = cP_\beta \quad , \quad [H_\beta, P_\beta] = cK_\beta$$

but the definition of \mathfrak{a}_χ means that the first bracket is zero, so P_β is zero, and the second equality proves K_β to be zero as well, so X_β is zero, a contradiction. Hence $\Phi^{\max}(\chi)$ is indeed empty.

To end the proof we need only remark that this could not happen if $\Phi(\chi)$ were nonempty. If γ is in $\Phi(\chi)$, then for each $\delta \in \Sigma(\chi)$, $\gamma + \delta$ is in $\Phi(\chi)$ as well : this is obvious when δ is not in $\Phi(\chi)$; when it is and when $\gamma + \delta$ is a root, both \mathfrak{g}_γ and \mathfrak{g}_δ lie in \mathfrak{n}_χ (recall that \mathfrak{n}_χ is the sum of the root spaces for the elements of $\Phi(\chi)$), so $\mathfrak{g}_{\gamma+\delta} = [\mathfrak{g}_\gamma, \mathfrak{g}_\delta]$ lies in it as well and $\gamma + \delta$ is in $\Phi(\chi)$. If $\Phi(\chi)$ were nonempty, we could then add elements of $\Sigma(\chi)$ to one of its elements and exhibit an element in $\Phi^{\max}(\chi)$, which is impossible.

We have proved at last that χ is \mathfrak{a}_χ -regular, and can conclude that for each Mackey datum δ , $\mathbf{M}(\delta)$ satisfies the hypotheses of [23], theorem 4.93, from which Lemma 3.1 follows. \square

Now that we know that the hypothesis in Harish-Chandra's theorem ([33], theorem 4.11) is satisfied, *Lemma 3.2* follows from it as well. \square

Proof of Lemma 3.3. We need Knapp and Zuckerman's results here. When M is a reductive group in Harish-Chandra's class, Knapp and Zuckerman work with a large-but-proper subset of those representations of M which are irreducible, tempered, and have real infinitesimal character : it consists of representations which are discrete series or "non-degenerate limits of discrete series". The definition of limits of discrete series is given in Section 6.1 below ; I will not need to say precisely what it means to be a "nondegenerate limit of discrete series" (see [30], section 1 and 8, for details), but it will be enough to use one of Knapp and Zuckerman's results, that σ is a tempered irreducible representation of M with real infinitesimal character if and only if there is a cuspidal parabolic subgroup $M_b A_b N_b$ of M and a discrete series or nondegenerate limit of discrete series representation σ^b of M_b such that $\sigma = \text{Ind}_{M_b A_b N_b}^M (\sigma^b \otimes \mathbf{1})$.

A central theorem by Knapp and Zuckerman is that every irreducible tempered representation π is "basic", that is, unitarily equivalent to

$$\text{Ind}_{MAN}^G (\tau \otimes e^{i\nu})$$

with $P = MAN$ a cuspidal parabolic subgroup of G , τ a discrete series or nondegenerate limit of discrete series representation of M , and ν a element of \mathfrak{a}^* . To prove Lemma 3.3 I need only check that this is in the image of my Mackey map \mathbf{M} . Of course I can assume P contains P_p , so let's extend ν to an element of \mathfrak{a}_p^* (deciding that it is zero on the orthocomplement of \mathfrak{a}) and consider the parabolic subgroup $P_\nu = M_\nu A_\nu N_\nu$ as before. Then P_ν contains P : a first remark is that the Lie algebra $\mathfrak{t}_P = T \cap M$ consists of elements which commute with H_ν , and so \mathfrak{t}_P contains \mathfrak{t}_ν . It follows that the centralizer $\mathfrak{z}_{\mathfrak{a}_p}(\mathfrak{t}_P)$ contains $\mathfrak{z}_{\mathfrak{a}_p}(\mathfrak{t}_\nu) = \mathfrak{a}_\nu$. Now, \mathfrak{a} is contained in $\mathfrak{z}_{\mathfrak{a}_p}(\mathfrak{t}_P)$, and $\mathfrak{t}_P \oplus \mathfrak{z}_{\mathfrak{a}_p}(\mathfrak{t}_P)$ is an abelian subalgebra of \mathfrak{g} . This subalgebra contains the Cartan subalgebra $\mathfrak{t}_P \oplus \mathfrak{a}$, so the centralizer $\mathfrak{z}_{\mathfrak{a}_p}(\mathfrak{t}_P)$ must be \mathfrak{a} . Hence \mathfrak{a}_ν is contained in \mathfrak{a} , and the inclusions $P_\nu \supset P$, $M_\nu \supset M$, $A_\nu \subset A$, $N_\nu \subset N$ follow from this and the definitions recalled above.

Let's introduce subgroups \tilde{A}, \tilde{N} such that $A = A_\nu \tilde{A}$ and $N = N_\nu \tilde{N}$, so that we can write

$$\text{Ind}_{MAN}^G (\tau \otimes e^{i\nu} \otimes \mathbf{1}) = \text{Ind}_{(M\tilde{A}\tilde{N})A_\nu N_\nu}^G ((\tau \otimes e^0) \otimes e^{i\nu}).$$

Here $\tilde{P} = M\tilde{A}\tilde{N}$ is a subgroup of M_ν , and $M\tilde{A}$ is the centralizer of \tilde{A} in M_ν : so \tilde{P} is in fact a parabolic subgroup of M_ν , and because M has a discrete series it is a cuspidal parabolic subgroup.

Now, $\sigma = \text{Ind}_{M\tilde{A}\tilde{N}}^{M_\nu} (\tau \otimes e^0)$ is a tempered representation of M_ν , it has real infinitesimal character, and it is irreducible (otherwise π would not be !). The double induction formula then yields

$$\text{Ind}_{\tilde{P}}^G (\sigma \otimes e^{i\nu}) = \text{Ind}_{M_\nu A_\nu N_\nu}^G (\text{Ind}_{\tilde{P}}^{M_\nu} (\tau \otimes e^0) \otimes e^{i\nu})$$

which proves that π is in the image of \mathbf{M} ; this is lemma 3.3. \square

Remark 3.2. In the case where G and K have different ranks, Vogan's theorem still holds, but Mackey's analogy relates representations of K to representations of G induced from

subgroups which, unlike G , have a discrete series. This is more awkward than the case of equal ranks ; Nigel Higson certainly understands the situation better than I do here !

3.3 What is coming next

If we start with a Mackey datum δ , the constructions above provide a Hilbert space \mathbf{H} and a morphism π from G to the unitary group of \mathbf{H} ; but we can also view δ as a Mackey datum for each of the G_t , getting a Hilbert space \mathbf{H}_t and a morphism $\pi_t : G_t \rightarrow \mathbf{U}(\mathbf{H}_t)$ for each $t > 0$. Now we have an explicit isomorphism φ_t from G_t to G , and the morphisms $\pi_t \circ \varphi_t^{-1}$ and π define irreducible representations of G . If we are careful about the interpretation of δ as a Mackey datum for G_t (see sections 4.2 and 7.2), they will be unitarily equivalent. In this case I will say that the equivalence class of π_t is $\mathbf{M}_t(\delta)$.

Definition 3.3. Suppose π is a unitary representation of G with Hilbert space \mathbf{H} and π^t is a unitary representation of G_t with Hilbert space \mathbf{H}_t . A linear map

$$\mathbf{C}_t : \mathbf{H} \rightarrow \mathbf{H}_t$$

will be called a *contraction map* when it intertwines π and $\pi_t \circ \varphi_t^{-1}$.

Notice that Schur's lemma says there cannot be many contraction maps ; when there is one it is unique up to to a scalar of modulus one, and as we shall see, upon introducing geometric realizations for \mathbf{H} and \mathbf{H}_t , it will be natural to add a finite number of small constraints to obtain a well-defined \mathbf{C}_t - "the" contraction operator.

Consider now a (smooth, K-finite) vector $f \in \mathbf{H}$, and set $f_t = \mathbf{C}_t f$.

Remember that the aim of these notes is to understand the relationship between \mathbf{H} and the Hilbert space \mathbf{H}_0 which carries the irreducible representation of G_0 attached to δ .

Is it possible that as t goes to zero, f_t should have a limit f_0 in some sense, and that f_0 should belong to \mathbf{H}_0 ? It is, but since f and f_t seem to live in different spaces, we have to be careful about what the "limit" means. In the rest of these notes, we shall embed the smooth, K-finite vectors of each \mathbf{H}_t in a fixed Fréchet space, and prove that for its Fréchet topology, f_t has a limit f_0 as t goes to zero. From the limits thus obtained we get a vector space which turns out to have a natural G_0 -module structure, and to be isomorphic with $\mathbf{M}_0(\delta)$. At this level of generality, my main result can be phrased as follows.

Theorem 3.4. *Suppose δ is a reasonable Mackey datum. Then there is a Fréchet space \mathbf{E} , a finite collection of continous linear functionals $\alpha_i \in \mathbf{E}'$, there is a vector subspace \mathbf{H} of \mathbf{E} , a map $\pi : G \rightarrow \text{End}(\mathbf{E})$, and for each $t > 0$ there is a vector subspace $\mathbf{H}_t \subset \mathbf{E}$ and a map $\pi_t : G_t \rightarrow \text{End}(\mathbf{E})$, which have the following properties.*

1. *The vector subspace \mathbf{H}_t is π_t -stable, and (\mathbf{H}_t, π_t) is a tempered irreducible representation of G_t with class $\mathbf{M}_t(\delta)$,*
2. *There is exactly one linear map from \mathbf{E} to itself which sends \mathbf{H} to \mathbf{H}_t and restricts to a contraction map between \mathbf{H} and \mathbf{H}_t , while satisfying $\alpha_i \circ \mathbf{C}_t = \alpha_i$ for all i . The family $(\mathbf{C}_t)_{t>0}$ is then weakly continuous.*

3. For each $f \in \mathbf{E}$, there is a limit (in \mathbf{E}) to $\mathbf{C}_t f$ as t goes to zero.
4. Define \mathbf{H}_0 as $\left\{ \lim_{t \rightarrow 0} \mathbf{C}_t f \mid f \in \mathbf{H} \right\}$, suppose f_0 is in \mathbf{H}_0 , consider an element f of \mathbf{H} such that $\lim_{t \rightarrow 0} \mathbf{C}_t f = f_0$, and set $f_t = \mathbf{C}_t f$. Then for each g_0 in G_0 , there is a limit to $\pi_t(\alpha_t(g_0))f_t$ as t goes to zero, this limit depends only on f_0 (and g_0), and it belongs to \mathbf{H}_0 . Call it $\pi_0(g_0)f_0$.
5. We thus obtain a vector subspace \mathbf{H}_0 of \mathbf{E} , and a representation π_0 of G_0 on \mathbf{H}_0 . This representation is then unitary irreducible, and its equivalence class is $\mathbf{M}_0(\delta)$.

"Reasonable" here means that if $\delta = (\chi, \mu)$, the irreducible-tempered-representation-with-real-infinitesimal-character $\mathbf{V}_{M_\chi}(\mu)$ is neither a discrete series nor a limit of discrete series representation of \mathbf{M}_χ , but is not trivial : we will of course see (in section 6.2) why I have not been able to remove this restriction (yet!). The space \mathbf{E} will roughly be a space of continuous functions with values in a finite-dimensional vector space, and the constraint enforced by the linear functionals will be that \mathbf{C}_t preserve the value of functions at a distinguished point. More detailed statements will be given along the way - see especially Theorem 7.2. below -, but the above statement might be helpful for orientation.

4 Spherical principal series representations

In this section, we choose a Mackey datum $\delta = (\lambda, \mu)$ with regular λ ; the representations of G with class $\mathbf{M}(\delta)$ are unitary principal series representations, and several existing results can be understood as giving flesh to Mackey's analogy at the level of carrier spaces. I will comment on some of them in section 4.3 below.

4.1 Two geometric realizations

There are several well-known function spaces carrying a representation of G with class $\mathbf{M}(\delta)$ – see for instance section VII.1. in [23]. I will use two of these function spaces here : in the first, the functions are defined on K/M – which has the same meaning in G and G_0 ; in the second, they are defined on G/K , or equivalently on \mathfrak{p} , and the geometrical setting in section 2.2 will prove helpful.

Before I proceed to the contraction, let me describe the corresponding realizations of $\mathbf{M}(\delta)$.

4.1.1 The compact picture

Since I will use this realization outside the principal series, until the end of this subsection I do not assume that $\delta = (\lambda, \mu)$ has λ regular. I write $P = MAN$ for the cuspidal parabolic subgroup we induce from.

Let me write V_σ for the space of a tempered irreducible M -module of class $\sigma = \mathbf{V}_M(\mu)$, and suppose that an M -invariant inner product is fixed on V_σ . A possible Hilbert space for $\mathbf{M}(\delta)$ is

$$\mathbf{H}_\sigma^{comp} = \{ f \in \mathbf{L}^2(K; V_\sigma) \mid f(km) = \sigma^{-1}(m)f(k), \forall (k, m) \in K \times M \}.$$

To say how G acts on \mathbf{H}_σ^{comp} I need the Iwasawa projections κ , \mathbf{m} , \mathbf{a} , ν sending an element of G to the unique quadruple

$$(\kappa(g), \mathbf{m}(g), \mathbf{a}(g), \nu(g)) \in K \times \exp_G(\mathfrak{m} \cap \mathfrak{p}) \times \mathfrak{a} \times N$$

such that $g = \kappa(g)\mathbf{m}(g)\exp_G(\mathbf{a}(g))\nu(g)$ (this quadruple is unique, see [23]). Note that if P is minimal, the map \mathbf{m} is trivial. The operator for the action of $g \in G$ on \mathbf{H}_σ^{comp} is then

$$\pi_{\lambda,\mu}^{comp}(g) = f \mapsto [k \mapsto \exp\langle -i\lambda - \rho, \mathbf{a}(g^{-1}k) \rangle \sigma(\mathbf{m}(g^{-1}k))^{-1} f(\kappa(g^{-1}k))].$$

Note that the Hilbert space does not depend on λ , but that the G -action does.

It will be useful to recall how this is related to the usual ‘‘induced picture’’, for which the Hilbert space is

$$\mathbf{H}_\delta^{ind} = \left\{ f : G \rightarrow V_\sigma \mid f(gme^H n) = e^{\langle -i\lambda - \rho, H \rangle} \sigma(m)^{-1} f(g) \text{ for } (g, me^H n) \in G \times P, \text{ and } f|_K \in \mathbf{L}^2(K; V_\sigma) \right\},$$

the inner product is the \mathbb{L}^2 scalar product between restrictions to K , and the G -action is $\pi_\delta^{ind}(g) = f \mapsto [x \mapsto f(g^{-1}x)]$: because an element of \mathbf{H}_δ^{ind} is completely determined by its restriction to K thanks to its P -equivariance, restriction to K induces an isometry (say \mathbf{R}) between \mathbf{H}_δ^{ind} and \mathbf{H}_σ^{comp} ; the definition of $\pi_{\lambda,\mu}^{comp}$ is just what is needed to make \mathbf{R} an intertwining operator.

4.1.2 Helgason’s waves

Let me assume again that $\delta = (\lambda, \mu)$ has λ regular, and suppose in addition that μ is the trivial representation of M . Then there is a distinguished element in \mathbf{H}_σ^{comp} : the constant function on K with value one. Under the isometry \mathbf{R} , it corresponds to the function

$$\bar{e}_{\lambda,1} = ke^H n \mapsto e^{\langle -i\lambda - \rho, H \rangle}$$

in \mathbf{H}_δ^{ind} , which in turn defines a function on G/K if we set $e_{\lambda,1}(gK) = \bar{e}_{\lambda,1}(g^{-1})$, and a function on \mathfrak{p} if we set $e_{\lambda,1}(v) = e_{\lambda,1}(\exp_G(v)K)$.

Here is a plot of $e_{\lambda,1}$ when G is $SL_2(\mathbb{R})$:

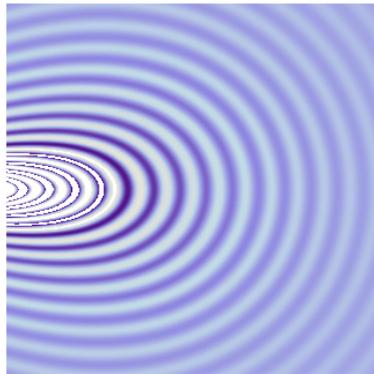


Figure 2: Plot of the real part of the Helgason wave $e_{30,1}$. I used the mapping from \mathbb{R}^2 to the unit disk provided by the Cartan decomposition, and the explicit formulae available on the unit disk: see [17], chapter 0. The x - and y - range is $[-1.5, 1.5]$ (this region is chosen so that the modulus varies clearly but within a displayable range, and the choice of λ is to have enough waviness in the region).

Now set $e_{\lambda,b}(v) = e_{\lambda,1}(b^{-1}v)$ for b in K/M and v in \mathfrak{p} . Then

$$\begin{aligned} \mathbf{L}^2(K/M) &= \mathbf{H}_\sigma^{comp} \rightarrow \mathbf{C}^\infty(\mathfrak{p}) \\ F &\mapsto \int_{K/M} e_{\lambda,b} F(b) db \end{aligned}$$

turns out to intertwine $\pi_{\lambda,1}^{comp}$ with the quasi-regular action of G on $\mathbf{C}^\infty(\mathfrak{p})$, and to be an injection (see [17], p. ??). I shall write $\mathbf{H}_\lambda^{H\text{elgason}}$ for the image of this map; of course it inherits a Hilbert space structure from that of $\mathbf{H}_\lambda^{comp}$.

4.2 The contraction operators

4.2.1 In the compact picture

Now, let me consider the spherical principal series representation $\pi_{\lambda,\sigma}^{t,comp}$ of G_t (here λ is regular and σ is any element of $\widehat{M}_\mathfrak{p}$) which acts on $\mathbf{H}_\sigma^{comp} = \mathbf{L}^2(K, V_\sigma)$. We can define a representation of G as the composition

$$\varpi_{\lambda,\sigma}^{t,comp} : G \xrightarrow{\varphi_t^{-1}} G_t \xrightarrow{\pi_{\lambda,\sigma}^{t,comp}} \text{End}(\mathbf{H}_\sigma^{comp}).$$

The next lemma indicates how (λ, σ) is to be interpreted as a Mackey datum for G_t :

Lemma 4.1. *For each $t > 0$, $\varpi_{\lambda,\sigma}^{t,comp}$ is equal to $\pi_{\lambda/t}^{comp}$.*

To prove this lemma, we need only write down the details for the definition of $\pi_{\lambda/t}^{t,comp}$. We have to understand what happens to the half-sum of positive roots when we go from G to G_t , and to make the relationship between the Iwasawa decompositions in both groups clear. Here is a first step :

Lemma 4.2. *If $\alpha \in \mathfrak{a}^*$ is a root of $(\mathfrak{g}, \mathfrak{a})$, then $t \cdot \alpha$ is a root of $(\mathfrak{g}_t, \mathfrak{a})$.*

Proof. When α is a root of $(\mathfrak{g}, \mathfrak{a})$, there is a nonzero $X \in \mathfrak{g}$ such that $[X, H] = \alpha(H)X$ for each $H \in \mathfrak{a}$. To keep track of X through the contraction, let's write $X = X_e + X_h$ with $X_e \in \mathfrak{k}$ and $X_h \in \mathfrak{p}$. Then

$$[X_e, H] - \alpha(H)X_h = [X_h, H] - \alpha(H)X_e. \quad (3)$$

The left-hand-side of (3) is in \mathfrak{p} and the right-hand-side is in \mathfrak{k} , so both are zero.

Now, the isomorphism ϕ_t^{-1} sends X to $X^t = \cdot X_e + \frac{1}{t} X_h \in \mathfrak{g}_t$, and for each $H \in \mathfrak{a}$,

$$[X^t, H]_{\mathfrak{g}_t} = [X_e, H]_{\mathfrak{g}_t} + \frac{1}{t} [X_h, H]_{\mathfrak{g}_t} = [X_e, H]_{\mathfrak{g}} + t [X_h, H]_{\mathfrak{g}}$$

But this is $\alpha(H)X_h + t \cdot \alpha(H)X_e = t \cdot \alpha(H)X^t$, so X^t is in the $(\mathfrak{g}_t, \mathfrak{a})$ root space for $t \cdot \alpha$, which proves lemma 4.2. \square

The proof shows that the root space for $t \cdot \alpha$ is the image of \mathfrak{g}_α under ϕ_t^{-1} ; a consequence of this is that the subgroups M_t, A_t and N_t of G_t provided by the constructions of section 3.2 are the images of M, A and N under φ_t^{-1} . If $g = k \exp_G(H)n$ is the Iwasawa decomposition of $g \in G$, the corresponding Iwasawa decomposition of $\varphi_t^{-1}g$ is then $\varphi_t^{-1}g = k \cdot \varphi_t^{-1} [\exp_G(H)] \cdot \varphi_t^{-1}(n)$. Thus

$$\kappa_t(\varphi_t^{-1}g) = \kappa(g);$$

$$\mathbf{a}_t(\varphi_t^{-1}g) = \frac{\mathbf{a}(g)}{t}.$$

The second equality uses the commutation relation between group exponentials and group morphisms. Now, because of Lemma 4.2, for each $\gamma \in G_t$ we know that

$$\pi_{\lambda,\sigma}^{t,comp}(\gamma) = f \mapsto [k \mapsto \exp\langle -i\lambda - t\rho, \mathbf{a}_t(\gamma^{-1}k) \rangle f(\kappa_t(\gamma^{-1}k))].$$

Hence

$$\pi_{\lambda,\sigma}^{t,comp}(\varphi_t^{-1}(g)) = f \mapsto [k \mapsto \exp\langle -i\lambda - t\rho, \mathbf{a}_t([\varphi_t^{-1}g]^{-1}k) \rangle f(\kappa_t([\varphi_t^{-1}g]^{-1}k))].$$

And rearranging,

$$\begin{aligned} \pi_{\lambda,\sigma}^{t,comp}(\varphi_t^{-1}(g)) &= f \mapsto [k \mapsto \exp\langle -i\frac{\lambda}{t} - \rho, t \cdot \mathbf{a}_t(\varphi_t^{-1}[g^{-1}k]) \rangle f(\kappa_t(\varphi_t^{-1}[g^{-1}k]))] \\ &= f \mapsto [k \mapsto \exp\langle -i\frac{\lambda}{t} - \rho, \mathbf{a}(g^{-1}k) \rangle f(\kappa(g^{-1}k))] \\ &= \pi_{\frac{\lambda}{t},\sigma}^{comp}(g), \end{aligned}$$

so the proof of lemma 4.1 is complete. \square

To discuss the contraction from G to G_0 the situation seems disappointingly trivial here : the Hilbert space is the same for each t , including $t = 0$, and because of Lemma 4.1 the natural "contraction" operator \mathbf{C}_t is the identity. However, this does not mean that Mackey's analogy is devoid of interest for the principal series, even from the point of view of Hilbert spaces ; the interplay with Helgason's picture will show this clearly, but let us linger in the compact picture for a moment.

I can use the diffeomorphisms $\alpha_t : G_0 \rightarrow G_t$ which realize the Cartan decomposition (see section 2.1) to define maps $\tilde{\pi}_t$ from G_0 to $\text{End}(\mathbf{H}_\sigma^{comp})$, setting

$$\tilde{\pi}_t = \pi_{\lambda,\sigma}^{t,comp} \circ \alpha_t.$$

Because of lemma 4.1, $\tilde{\pi}_t(g_0)$ is an operator for a principal series representation of G with infinitesimal character $\frac{i\lambda}{t}$; but as t goes to zero it gets closer and closer to an operator for the representation of G_0 with Mackey datum (λ, σ) :

Theorem 4.1. *For each g_0 in G_0 , there is a limit to $\tilde{\pi}_t(g_0)$ as t goes to zero ; it is the operator $\pi_0(g_0)$. The convergence holds both in the usual weak sense when the operators are viewed as unitary operators on $\mathbb{L}^2(K)$, and in the weak topology associated to that of uniform convergence on $\mathbf{C}(K)$.*

To prove this theorem, recall that

$$\pi_{\lambda,\sigma}^{t,comp}(k \exp_{G_t} v) = f \mapsto [u \mapsto \exp\langle -i\lambda - t\rho, \mathbf{a}_t((k^{-1} \cdot_t \exp_{G_t}(-v) \cdot_t u)) \rangle f(\kappa_t(k^{-1} \exp_{G_t}(-v)u))].$$

Here the products are products in G_t . On the other hand, recall from section 2.2 that

$$\pi_0(k, v) = f \mapsto [u \mapsto \exp\langle i\lambda, \text{Ad}(u^{-1})v \rangle f(k^{-1}u)].$$

To make the two look more similar, notice that

$$\begin{aligned} \left[\pi_{\lambda, \sigma}^{t, comp}(k \exp_{G_t} v) f \right] (u) &= e^{(-i\lambda - t\rho, \mathbf{a}_t((k^{-1}u)_t \exp_{G_t}(-Ad(u^{-1})v)))} f(\kappa_t(k^{-1}u \exp_{G_t}(-Ad(u^{-1})v))) \\ &= e^{(-i\lambda - t\rho, \mathbf{a}_t(\exp_{G_t}(-Ad(u^{-1})v)))} f((k^{-1}u) \cdot \kappa_t(\exp_{G_t}(-Ad(u^{-1})v))). \end{aligned}$$

We now need to see how the Iwasawa projection parts behave as t goes to zero. Let us write \mathfrak{K} and \mathfrak{J} for the maps from \mathfrak{p} to \mathfrak{a} sending $v \in \mathfrak{p}$ to the K - and \mathfrak{a} -Iwasawa components of $\exp_G(v)$, respectively (so \mathfrak{K} is $\kappa \circ \exp_G$ and \mathfrak{J} is $\mathbf{a} \circ \exp_G$); let us likewise set $\mathfrak{K}_t = \kappa_t \circ \exp_{G_t}$ and $\mathfrak{J}_t = \mathbf{a}_t \circ \exp_{G_t}$. The Iwasawa map \mathfrak{J}_t from \mathfrak{p} to \mathfrak{a} is a nonlinear map, but as t goes to zero it gets closer and closer to a linear projection :

Lemma 4.3. *As t tends to zero, \mathfrak{J}_t admits as a limit (in the sense of uniform convergence on compact subsets of \mathfrak{p}) the orthogonal projection from \mathfrak{p} to \mathfrak{a} , while \mathfrak{K}_t tends to the constant function on \mathfrak{p} with value $\mathbf{1}_K$.*

Proof. We will check now that \mathfrak{J}_t is none other than $v \mapsto \frac{1}{t} \mathfrak{J}(tv)$. Since φ_t is a group morphism from G_t to G , the definition of group exponentials does imply that $\exp_G(tv) = \exp_G(d\varphi_t(1)v) = \varphi_t(\exp_{G_t} v)$. Let us write $\exp_{G_t} v = ke^{\mathfrak{J}_t(v)} n_t$ with $k \in K$ and $n_t \in N_t$, then $\varphi_t(\exp_{G_t} v) = ke^{t\mathfrak{J}_t(v)} n$, with $n = \varphi_t(n_t)$ in N . So we know that

$$\exp_G(tv) = ke^{t\mathfrak{J}_t(v)} n \quad (4)$$

and thus that $\mathfrak{J}(tv) = t\mathfrak{J}_t(v)$, as announced.

But then as t goes to zero, the limit of $\mathfrak{J}_t(v)$ is the value at v of the derivative $d\mathfrak{J}(0)$. Now this does yield the orthogonal projection of v on \mathfrak{a} : although the Iwasawa decomposition of \mathfrak{g} is not an orthogonal direct sum because \mathfrak{k} and \mathfrak{n} are not orthogonal to each other, they are both orthogonal to \mathfrak{a} with respect to the Killing form of \mathfrak{g} , so the direct sum $\mathfrak{k} \oplus \mathfrak{n}$ is the orthogonal of \mathfrak{a} .

As for \mathfrak{K}_t , from 4 we see that $\mathfrak{K}_t(v) = \kappa_t(\exp_{G_t} v) = \kappa(\exp_G(tv))$, and this does go to the identity uniformly on compact subsets t goes to zero (here I measure distances on K with the bi-invariant metric on K whose volume form is the normalized Haar measure) \square

The “weak convergence with respect to the topology of uniform convergence” part of Theorem 4.1 follows immediately, and since we are dealing with continuous functions on a compact manifold here, uniform convergence implies \mathbb{L}^2 convergence. This concludes the proof of Theorem 4.1 ; for remarks, see 4.3.1 below. \square

4.2.2 In Helgason’s picture

For each $t > 0$, each $\lambda \in \mathfrak{a}^*$ and $b \in K/M$, define

$$\varepsilon_{\lambda, b}^t = v \in \mathfrak{p} \mapsto e^{(i\lambda + t\rho, \mathfrak{J}_t(Ad(b)v))}.$$

Let me simplify the notations a bit and write \mathbf{H}_λ^t for the Hilbert space $\mathbf{H}_\lambda^{t, Helgason}$ which one can associate to G_t as in section 4.1. Let me also set $B = K/M$. The next lemma gathers some simple consequences of Lemma 4.1, Lemma 4.3 and their proofs.

Lemma 4.4. *1. The Hilbert space \mathbf{H}_λ^t is exactly $\left\{ \int_B \varepsilon_{\lambda, b}^t F(b) db \mid F \in \mathbb{L}^2(B) \right\}$.*

2. For each $\lambda \in \mathfrak{a}^*$ and each $b \in K/M$, the Helgason waves $\varepsilon_{\lambda,b}^t$ converge uniformly on compact subsets of \mathfrak{p} to the Euclidean plane wave $v \mapsto \exp((i\lambda, \text{Ad}(b) \cdot v))$.

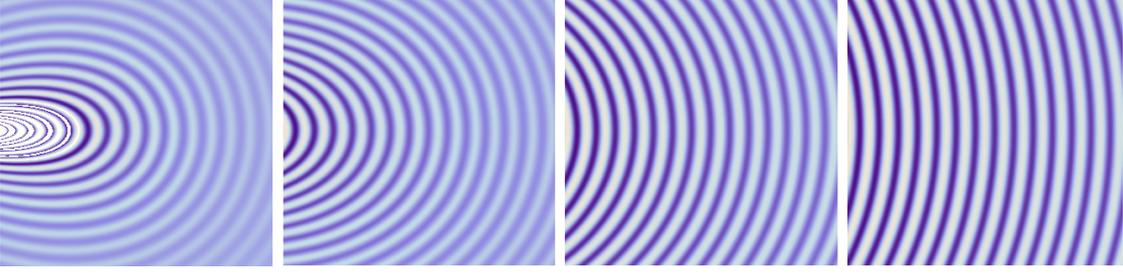


Figure 3: Illustration of lemma 4.4 : these are plots of $\varepsilon_{\lambda,1}^{1/2^k}$, $k = 0, 1, 2, 3$, in the same domain as in Figure 2. Each of these waves is a building block for a principal series representation of G whose continuous parameter is $2^k\lambda$, with $\lambda = 30$ here.

I think Lemma 4.4. is a nice way to understand how the principal series representation $\mathbf{M}(\lambda, 1)$ is related with the representation $\mathbf{M}_0(\lambda, 1)$ of G_0 , using the contraction from G_t to G_0 . In addition, the above reformulation of Helgason's realization as a space of functions on \mathfrak{p} makes it easy to write down a contraction operator in the spirit of what we are going to do for the discrete series below.

Recall that G_t acts on \mathbf{H}_λ^t via $\pi_t(\gamma) = f \mapsto [x \mapsto f(\gamma^{-1} \cdot_t x)]$. Define

$$\mathbf{C}_t^\lambda : \mathbf{H}_{\lambda/t} \rightarrow \mathbf{H}_\lambda^t$$

$$\int_B e_{\frac{\lambda}{t}, b} F(b) db \mapsto \int_B \varepsilon_{\lambda, b}^t F(b) db$$

Then \mathbf{C}_t^λ does intertwine $\pi_{\lambda/t}$ and $\varpi_{t, \lambda}$: to see this, notice that

$$\varepsilon_{\lambda, b}^t = v \in \mathfrak{p} \mapsto e \langle i \frac{\lambda}{t} + \rho, \mathfrak{I}(b \cdot (tv)) \rangle = \varepsilon_{\frac{\lambda}{t}, b}^1(tv) = \varepsilon_{\frac{\lambda}{t}, b}^1(z_t v)$$

so since $\varphi_t(g) \cdot z_t(v)$ is equal to $z_t(g \cdot_1 v)$ for all v (see 5.1),

$$\mathbf{C}_t^\lambda [v \mapsto \varepsilon_{\lambda, b}^t(\varphi_t(g^{-1}) \cdot v)] = [\mathbf{C}_t^\lambda \varepsilon_{\lambda, b}^t](g^{-1}v).$$

Note that strictly speaking $\varepsilon_{\lambda, b}^t$ is not in \mathbf{H}_λ^t , but the definition of \mathbf{C}_t^λ can be extended to $\{\int_B \varepsilon_{\lambda, b}^t F(b) db \mid F \text{ is a distribution on } B\}$.

Because every element of $\mathbf{H}_{\lambda/t}$ is a combination of the $e_{\frac{\lambda}{t}, b}$ and the G -action commutes with the way the combinations are built, this does of course yield

$$\varpi_{t, \lambda}(\mathbf{C}_t^\lambda f) = \mathbf{C}_t^\lambda(\pi_{\lambda/t} f)$$

as announced.

Our contraction operator \mathbf{C}_t^λ is the only intertwining operator between $\pi_{\lambda/t}$ and $\varpi_{t, \lambda}$ which preserves the linear functional isolating the value of functions at zero. Because of

lemma 4.4 (ii), we see that $\mathbf{C}_t^\lambda \left(\int_B e_{\lambda,b} F(b) db \right) = \int_B \varepsilon_{\lambda,b}^t F(b) db$ converges to $\int_B e^{(i\lambda, \text{Ad}(b) \cdot v)} F(b) db$, a square-integrable, smooth function on \mathfrak{p} whose Fourier transform is concentrated on $\text{Ad}^*(K) \cdot \chi$.

As I recalled in section 2 (page 10), the vector space

$$\mathbf{H}_0 := \left\{ v \mapsto \int_B e^{(i\lambda, \text{Ad}(b) \cdot v)} F(b) db \mid F \in \mathbf{L}^2(B) \right\}$$

with the G_0 -action inherited from that of G_0 on \mathfrak{p} , is an irreducible G_0 -module with class $\mathbf{M}_0(\lambda, 1)$.

We can summarize the above discussion with the following statement.

Theorem 4.2. *For each $f \in \mathbf{H}_\lambda$, there is a limit f_0 to $\mathbf{C}_t^\lambda f$ for the topology of uniform convergence on compact subsets of \mathfrak{p} , and this limit belongs to \mathbf{H}_0 . In fact, $f \mapsto \lim_{t \rightarrow 0} \mathbf{C}_t^\lambda f$ defines a linear, K -invariant isometry between \mathbf{H}_λ and \mathbf{H}_0 .*

4.3 Some remarks

4.3.1. Theorem 4.1 can be viewed as a reformulation of Theorem 1 in Dooley and Rice's paper [15]. If I include it to these notes it is because I think the interplay with Helgason's picture throws some light on the phenomenon, because section 7 below will be a simple-but-technical adaptation of the strategy in section 4.2.1, and because all the ingredients in the proof of Lemma 4.1 and Theorem 4.1 will serve again in section 7.

4.3.2. The examples given here show that in principle, given a Mackey datum δ , there are several possible settings \mathbf{E} in which Theorem 3.4 is valid for δ . Let me give some precisions on the way Helgason's picture provides a setting for Theorem 3.4. Let \mathbf{E} be the space of smooth functions on \mathfrak{p} whose pushforward by u – smooth functions on G/K – are tempered as distributions on G/K (see [18]). Then \mathbf{E} is a closed subspace of the Fréchet space of smooth functions on \mathfrak{p} (with its usual Fréchet topology).

Furthermore, it is true that \mathbf{H}_λ^t is a vector subspace of \mathbf{E} for each $t > 0$ and each λ . In addition, there is a measure μ on \mathfrak{a}^* such that each element f of \mathbf{E} can be written as $\int_{\mathfrak{a}^*} f_\lambda d\mu(\lambda)$, with f_λ in \mathbf{H}_λ .

We defined a contraction map \mathbf{C}_t^λ in section 4.2.2, so we can define a linear operator on all of \mathbf{E} by setting

$$\mathbf{C}_t : \mathbf{E} \rightarrow \mathbf{E} \\ \int_{\mathfrak{a}^*} f_\lambda d\mu(\lambda) \mapsto \int_{\mathfrak{a}^*} \mathbf{C}_t^\lambda f_\lambda d\mu(\lambda).$$

This is obviously a contraction map, and it is the only contraction map which preserves the value of functions at zero (so that in the statement of Theorem 3.4, the only linear functional we need is $\alpha : f \mapsto f(0)$). Because of Lemma 4.4, it is true that for each f in \mathbf{E} , there is a limit $\mathbf{C}_0 f$ to $\mathbf{C}_t f$ as t goes to zero. Because of Lemma 2.1, we can easily work

with the actions of G_t , and point 4. in Theorem 3.4 follows readily.

4.3.3. Many existing studies compare harmonic analysis for functions on G/K with ordinary Fourier analysis for functions on \mathfrak{p} , with the hope of solving some apparently difficult problems on G/K , like the existence of fundamental solutions for G -invariant partial differential equations on G/K : see for instance [20, 19, 18], and especially Rouvière's book [21]. The contents of this section provide a way to turn a function on G/K into a function on \mathfrak{p} in a relatively natural manner which uses the fine structure of G . Is it possible that this transformation should be related to some of the issues in [21] ? I think it is likely to be, but still have to look closely into the matter.

5 The discrete series

In this section G will be connected, semisimple, with finite center, and I will assume that G and K have equal ranks, so that G has a nonempty discrete series. Let me again write T for a maximal torus in K .

5.1 Square-integrable solutions of the Dirac equation

Let us start with a class $\mu \in \widehat{K}$. If the highest weight of μ lies sufficiently far away from the root hyperplanes (I will make this precise immediately), the representation $\mathbf{V}_G(\mu)$ belongs to the discrete series of G . In this subsection I recall some results of Parthasarathy, Atiyah and Schmid [1, 22] which provide a Hilbert space for $\mathbf{V}_G(\mu)$.

I use standard terminology here and say that an element of it^* is in Λ if it is the derivative of a character of T ; let me write Δ_c for the set of roots of $(\mathfrak{k}_\mathbf{C}, \mathfrak{t}_\mathbf{C})$, and Δ for the set of roots of $(\mathfrak{g}_\mathbf{C}, \mathfrak{t}_\mathbf{C})$; of course $\Delta_c \subset \Delta$.

If P is any system of positive roots for Δ , we can consider the half-sum ρ_P of positive roots, and then set $\Lambda_\rho = \Lambda + \rho_P$; it is a lattice in it^* which does not depend on which positive system P we chose in Δ .

Suppose a positive root system Δ_c^+ has been chosen for Δ_c , and write $C \subset it^*$ for the corresponding Weyl chamber. I will write $\bar{\mu}^{\Delta_c^+}$ for the highest weight of μ : it is an element of $\Lambda \cap C \subset it^*$. Suppose a system of positive roots Δ^+ in Δ is chosen in a manner compatible with Δ_c^+ , write ρ^{Δ^+} for the corresponding half-sum of positive roots, $\rho_c^{\Delta^+}$ for the half-sum of positive, compact roots, and $\rho_n^{\Delta^+}$ for the half-sum $\rho^{\Delta^+} - \rho_c^{\Delta^+}$ of positive, noncompact roots. The "Harish-Chandra parameter" $\bar{\lambda}^{\Delta^+} := \bar{\mu}^{\Delta_c^+} + \rho_n^{\Delta^+} - \rho_c^{\Delta_c^+}$ is then an element of $\Lambda_\rho \cap C$.

Throughout section 5, I will suppose Δ^+ is chosen so as to make $\bar{\lambda}^{\Delta^+}$ dominant ⁸, I will remove the superscripts Δ^+ , and I will assume further that

$$\bar{\lambda} \text{ is a regular element of } it^* \tag{5}$$

(see the proof of lemma 3.1 for the definition) : (5) is the condition for $\mathbf{V}_G(\mu)$ to belong to the discrete series of G .

⁸This choice will allow me to use Atiyah and Schmid's work with the conventions in [1]. It can indeed be made !

Recall that the condition that G and K have equal ranks guarantees that $\dim(G/K)$ is an even integer, say $2q$. Let us write S for a 2^q -dimensional space on which $Spin(2q)$ acts through the spinor representation. The module S splits into two irreducible 2^{q-1} -dimensional $Spin(2q)$ -submodules S^+ and S^- , with ρ_n a weight of S^+ .

Suppose V_{μ^b} is the carrier space of an irreducible $\mathfrak{k}_{\mathbb{C}}$ -module with highest weight $\mu^b = \bar{\mu} - \rho_n$. Then we can consider the tensor product $V_{\mu^b} \otimes S^{\pm}$, and although neither V_{μ^b} nor S^{\pm} need be a K -module if G is not simply connected, it turns out that the action of \mathfrak{k} on $V_{\mu^b} \otimes S^{\pm}$ does lift to K – the half-integral ρ_n -shifts in the weights do compensate. So we can consider the equivariant bundle $\mathfrak{E} = G \otimes_K (V_{\mu} \otimes S)$ over G/K , as well as equivariant bundles $\mathfrak{E}^{\pm} = G \otimes_K (V_{\mu} \otimes S^{\pm})$.

Now, the natural G -invariant metric that G/K inherits from the Killing form of \mathfrak{g} and the built-in G -invariant spin structure of \mathfrak{E} make it possible to define a first-order differential operator D acting on smooth sections of \mathfrak{E} , the Dirac operator : since I will need a few immediate consequences of its definition the next subsection, let me give a quick definition, referring to [22] for details.

Suppose $(X_i)_{i=1..2q}$ is an orthonormal basis of \mathfrak{p} . Recall that the definition of spinors comes with a map \mathbf{c} from $\mathfrak{p}_{\mathbb{C}}$ to $\text{End}(S)$, called Clifford multiplication, such that $\mathbf{c}(X)$ sends S^{\pm} to S^{\mp} , and that every X in \mathfrak{p} defines a left-invariant vector field on G/K , which yields a first-order differential operator $X^{\mathfrak{E}}$ acting (componentwise in the natural trivialization associated to the action of G on G/K) on sections of \mathfrak{E} . The Dirac operator is then defined by

$$Ds = \sum_{i=1}^{2q} \mathbf{c}(X_i) X_i^{\mathfrak{E}} s$$

when s is a section of \mathfrak{E} . It splits as $D = D^+ + D^-$, with D^{\pm} sending sections of \mathfrak{E}^{\pm} to sections of \mathfrak{E}^{\mp} .

Let me now write \mathbf{H}_{μ} for the space of smooth, square integrable sections of \mathfrak{E} which are in the kernel of D . Since D is an elliptic operator, \mathbf{H}_{μ} is a closed subspace of the Hilbert space of square-integrable sections of \mathfrak{E} . And as D is G -invariant, \mathbf{H}_{μ} is invariant under the natural action of G on sections of E .

Theorem (Parthasarathy, Atiyah & Schmid). *If μ satisfies the hypothesis (5), then \mathbf{H} carries an irreducible unitary representation of G , whose equivalence class is $\mathbf{V}_G(\mu)$.*

But here something happens to which we must pay very special attention : the details in Atiyah and Schmid's proof show that solutions to the Dirac equation do not explore the whole fibers, but that they are actually sections of a sub-bundle whose fiber, a K -module, is irreducible and of class μ . In clearer words, let W denote the isotypical K -submodule of $V_{\mu^b} \otimes S^+$ for the highest weight $\bar{\mu} = \mu^b + \rho_n$; the K -module W turns out to be irreducible. Let us write p_W for the isotypical (orthogonal) projection to $V_{\mu^b} \otimes S^+$ to W . Let \mathbf{W} denote the equivariant bundle on G/K associated to W .

Proposition (Atiyah & Schmid). *If a section of \mathfrak{E} is a square-integrable solution of the Dirac equation, then it is in fact a section of \mathbf{W} .*

Although this is not isolated as a proposition in Atiyah and Schmid's paper [1], it is proved and stated there very clearly ; the statement contains a commentary which is quite interesting in the context of the present notes.⁹

We should remark that the arguments leading up to [the fact that the cokernel of the Dirac operator is zero] are really curvature estimates, in algebraic disguise. The curvature properties of the bundles and of the manifold G/K force all square-integrable, harmonic spinors to take values in a certain sub-bundle of $\mathbf{V}_\mu \otimes \mathbf{S}^+$, namely the one that corresponds to the K -submodule of highest weight $\mu + \rho_n$ in $V_\mu \otimes S^+$.

To be complete, I should mention here that the context of the above quotation is one in which another nondegeneracy condition is imposed on μ besides that which guarantees that is is the lowest K -type of a discrete series representation. Atiyah and Schmid's arguments to remove this nondegeneracy condition in their main theorem do imply also that the above remark holds without the proviso.

5.2 Contraction of a discrete series representation to its minimal K -type

It is time to set up the stage for the contraction of a discrete series representation (I'm afraid the notation has to be a bit pedantic here if I want to reduce the hand-waving to a minimum...).

Recall that in section 2.1, we used a diffeomorphism u_t between \mathfrak{p} and G_t/K to make \mathfrak{p} into a G_t -homogeneous space equipped with a metric η_t . We can then use the representation of K on $V_\mu \otimes S$ to build a G_t -invariant spinor bundle \mathfrak{E}_t over G_t/K , use u_t to turn it into a bundle over \mathfrak{p} , and use the action of G_t to make this bundle trivial : this yields a bundle map, say T_t , from the bundle $u_t^* \mathfrak{E}_t$ over \mathfrak{p} to the trivial bundle $\mathfrak{p} \times (V_{\mu^b} \otimes S)$.

The definition of the Dirac operator makes sense for the homogeneous bundle $u_t^* \mathfrak{E}$ over the Riemannian space (\mathfrak{p}, η_t) ; once we trivialize using T_t we end up with a Dirac operator D'_t , acting on $\mathbf{C}^\infty(\mathfrak{p}, V_{\mu^b} \otimes S)$ – and which is pushed forward by T_t -then- u_t to a constant multiple of the Dirac operator on G_t/K defined in the previous subsection. Motivated by the end of the previous subsection, we build from D'_t an operator which acts on $\mathbf{C}^\infty(\mathfrak{p}, W)$, setting

$$\Delta_t := P_W \circ (D'_t)^2 \Big|_{\mathbf{C}^\infty(\mathfrak{p}, W)}$$

where P_W is the orthogonal projection from $V_\mu \otimes S$ onto W .

Note that I need not assume that t is nonzero here : we get a G_0 -invariant operator Δ_0 on the Euclidean space (\mathfrak{p}, η_0) , as well as G_t -invariant operators $\Delta_t, t > 0$, on the negatively-curved spaces (\mathfrak{p}, η_t) . I introduced the clumsy notation in order to spell out the proof of the following simple fact.

Lemma 5.1. *For each $f \in \mathbf{C}^\infty(\mathfrak{p}, W)$, the family $(\Delta_t f)_{t \geq 0}$ is continuous with respect to the topology of uniform convergence on compact sets of \mathfrak{p} .*

Proof. The Dirac operator is a first-order differential operator, so if I introduce $2q$ cartesian coordinates on \mathfrak{p} using a linear basis, $D'_t \Big|_{\mathbf{C}^\infty(\mathfrak{p}, W)}$ will read

⁹Atiyah and Schmid's μ is our μ^b , their $\mathbf{V}_\mu \otimes \mathbf{S}^+$ is our \mathfrak{E} .

$$D'_t|_{\mathbf{C}^\infty(\mathfrak{p}, W)} = \sum_{i=1}^{2q} A_t^i \partial_i + K_t$$

where the A_t^i , $i = 1 \dots 2q$, as well as K_t , are continuous functions from \mathfrak{p} to $\text{Hom}(W, V_{\mu^b} \otimes S)$. I now claim that it is clear from the details given on the definition of the Dirac operator acting on sections on \mathfrak{E} , and from the properties of u_t and T_t , that the maps $(x, t) \mapsto A_t^i(x)$ and $(x, t) \mapsto K_t(x)$ are continuous $\text{Hom}(W, V_{\mu^b} \otimes S)$ -valued maps¹⁰ on $\mathfrak{p} \times \mathbb{R}$, which proves lemma 5.1. □

Now let us start bringing G_0 into the picture. We know from the previous subsection (and from the fact that D'_t and its square have the same \mathbb{L}^2 kernel) that the \mathbb{L}^2 kernel of each Δ_t , $t > 0$, carries a discrete series representation of G_t whose minimal K -type is μ . On the other hand, the \mathbb{L}^2 kernel of Δ_0 is zero ! To recover the representation of G_0 which we are interested in, we should consider an extended kernel in which the constants are allowed.

Definition 5.1. For each $t \geq 0$, the extended kernel of Δ_t is

$$\mathbf{H}_t = \{f \in \mathbf{C}^\infty(\mathfrak{p}, W) \mid \Delta_t f = 0, \text{ and there is a constant } c \in W \text{ such that } f + c \in \mathbb{L}^2(\eta_t, W)\}.$$

Note that when f is in \mathbf{H}_t , there can be only one constant c such that $f + c$ is square-integrable.

Lemma 5.2. For $t \neq 0$, the extended kernel \mathbf{H}_t coincides with the \mathbb{L}^2 kernel of Δ_t , whereas \mathbf{H}_0 is the space of constant W -valued functions on \mathfrak{p} .

Proof.

Let us come back to G/K and the Dirac operator D defined in subsection 5.1. Because of Parthasarathy's formula for its square, we know that there is a scalar σ such that

$$D^2 := D^- D^+ = -\Omega + \sigma$$

with Ω the Casimir operator acting on sections of \mathfrak{E} .

Suppose a G -invariant trivialization of \mathfrak{E} is chosen, so that D^2 is viewed as acting on functions from G/K to $V_{\mu^b} \otimes S$, and suppose $D^2 g = 0$, with $g = f + C$, $f \in \mathbb{L}^2(G/K, V_{\mu^b} \otimes S)$ and C a constant in $V_{\mu^b} \otimes S$. Then

$$\Omega f = \sigma f + \sigma C. \tag{6}$$

I claim that this cannot happen when C is nonzero. To see this, I use Helgason's Fourier transform for functions on G/K (see [18]). The Fourier transform of a smooth function with compact support on G/K is the function $(\lambda, b) \mapsto \int_{G/K} f(x) e_{\lambda, b}(x) dx$ on $\mathfrak{a}^* \times K/M$, and this extends to an isometry \mathbf{F} between $\mathbb{L}^2(G/K)$ and $\mathbb{L}^2(\mathfrak{a}^* \times K/M)$ for a suitable measure on $\mathfrak{a}^* \times K/M$. In addition, there is a notion of tempered distributions on G/K and $\mathfrak{a}^* \times K/M$, and when f is a smooth, square-integrable function both f and Ωf

¹⁰This is simply because the family $(T_t \circ u_t)^* X_i^{\mathfrak{e}_t}$ of vector fields on \mathfrak{p} is continuous with respect to the topology of uniform convergence on compact subsets.

are tempered distributions. Using this, the equality $\Omega f = \sigma f + \sigma C$ becomes an equality of tempered distributions on $\mathfrak{a}^* \times K/M$, namely

$$\mathbf{F}(\Omega f) - \sigma \mathbf{F}(f) = \sigma C \delta_{(0,1M)}$$

with $\delta_{(0,1M)}$ the Dirac distribution at the point $(0,1M)$. Of course, there are convenient transformation properties of \mathbf{F} with respect to the G -invariant differential operators, and $\mathbf{F}(\Omega f)$ is actually the product of $\mathbf{F}(f)$ – an element of $\mathbf{L}^2(\mathfrak{a}^* \times K/M)$ with a smooth function on \mathfrak{a}^* . So if f were a smooth, square-integrable solution of (6), $\sigma C \delta_0$ would be the product of an element in $\mathbf{L}^2(\mathfrak{a}^* \times K/M)$ with a smooth function on the same space. This can only happen if C is zero, and obviously Lemma 5.2 follows. \square

As a result of lemma 5.2, each \mathbf{H}_t carries an irreducible representation of G_t with minimal K-type μ , and \mathbf{H}_0 carries an irreducible representation of G_0 with the “right” equivalence class according to Mackey’s analogy.

Let us now follow a vector through the contraction. We can of course use the G -action on \mathfrak{p} to define a G -invariant Dirac operator on $C^\infty(\mathfrak{p}, W)$, and consider the Hilbert space \mathbf{H} of smooth, $\mathbf{L}^2(\eta)$ solutions of the corresponding Dirac equation. Recall from section 3.4 that we are looking for a contraction operator \mathbf{C}_t from \mathbf{H} to \mathbf{H}_t .

Definition 5.2. The *natural contraction* $\mathbf{C}_t : \mathbf{H} \rightarrow \mathbf{H}_t$ is the only contraction map (Definition 3.3) such that for each $f \in \mathbf{H}$, $(\mathbf{C}_t f)(0) = f(0)$.

Now, we set up the geometrical stage in a way which makes it very easy to identify \mathbf{C}_t .

Recall from lemma 2.2 that the dilation

$$z_t : x \mapsto \frac{x}{t}$$

intertwines the actions of G and G_t on \mathfrak{p} . As a consequence, $z_t^* \eta_t$ is a G -invariant metric on \mathfrak{p} ; but there are not many such metrics : since the derivative of z_t is multiplication by t and η_1 and η_t coincide at zero, we deduce that

$$z_t^* \eta_t = t^2 \cdot \eta_1. \tag{7}$$

(Note the coherence with the fact that η_t has curvature $-t$, while η_1 has curvature -1). Now we can use z_t to transform functions on \mathfrak{p} , setting

$$\mathbf{Z}_t f := x \mapsto f(t \cdot x).$$

As an immediate consequence of (7) and the definition of the Dirac operator, we get

$$\mathbf{Z}_t^{-1} \Delta_t \mathbf{Z}_t = t^4 \cdot \Delta_1.$$

Together with the fact that $\mathbf{Z}_t f$ is square-integrable with respect to η_t as soon as f is square-integrable with respect to η_1 , this means that \mathbf{Z}_t sends \mathbf{H}_1 to \mathbf{H}_t . Thus \mathbf{Z}_t satisfies the properties in Definition 5.2.

So \mathbf{C}_t is something very simple indeed :

Lemma 5.3. *The natural contraction \mathbf{C}_t is none other than the restriction to \mathbf{H}_t of the zooming-in operator \mathbf{Z}_t .*

Now we can get back to the program of section 3.3 and follow it to its end. Let's start with an element f of \mathbf{H} , and set $f_t = \mathbf{C}_t f$. We know that \mathbf{H} splits as a direct sum according to K -types, in other words, we can write f as a Fourier series

$$f = \sum_{\lambda \in \widehat{K}} f_\lambda$$

where f_λ belongs to a closed subspace \mathbf{H}^λ of \mathbf{H} on which $\pi|_K$ restricts as a direct sum of copies of λ ; from Harish-Chandra we know that each \mathbf{H}^λ is finite-dimensional (and from Blattner, whose conjecture was proved by Hecht and Schmid [28], we know that there is an explicit-but-computer-unfriendly formula for its dimension).

Of course a parallel decomposition holds for \mathbf{H}_t , $t \neq 0$, and f_t , too, has a Fourier series

$$f_t = \sum_{\lambda \in \widehat{K}} f_{t,\lambda}.$$

Naturally the dimension of \mathbf{H}_t^λ is independent of t , and the support of the above Fourier series does not depend on t .

The geometrical realization we chose is once more quite convenient here, because we can go a small step further and deal with each Fourier component separately :

Lemma 5.4. *The Fourier component $f_{t,\lambda}$ is actually $\mathbf{C}_t f_\lambda$.*

To prove this lemma, we need only notice that for each $\lambda \in \widehat{K}$ the map

$$f \mapsto P_\lambda f := \left[x \mapsto \int_K \xi_\lambda^*(k) \mu(k) \cdot f(k^{-1} \cdot x) dk \right]$$

has a meaning as a linear operator from $\mathbf{C}^\infty(\mathfrak{p}, W)$ to itself. In the above formula ξ_λ is the global character of λ – a continuous function from K to \mathbb{C} – and the star is complex conjugation. Now, if f is an element of \mathbf{H}_t , we can view K as a subgroup of G_t and since the adjoint action of K on \mathfrak{p} is the same as that inherited from the action of G_t , the formula for $P_\lambda f$ turns out to be exactly the formula for the isotypical projection from \mathbf{H}_t to \mathbf{H}_t^λ . Now we know \mathbf{C}_t from lemma 5.3, and according to it P_λ obviously commutes with \mathbf{C}_t . This proves lemma 5.4.

Now, on each compact subset of \mathfrak{p} , we know from lemma 5.3 that for the topology of uniform convergence on compact subsets of \mathfrak{p} , $\mathbf{C}_t f$ goes to $f(0)$ as t goes to zero. Lemma 5.4 adds the precision that each $\mathbf{C}_t f_\lambda$, $\lambda \in \widehat{K}$, goes to $f_\lambda(0)$.

Lemma 5.5. *If $\lambda \in \widehat{K}$ is different from the minimal K -type μ , then $f_\lambda(0) = 0$.*

Proof. The origin of \mathfrak{p} is a fixed point for the action of K on \mathfrak{p} ; so

$$f_\lambda(0) = (P_\lambda f)(0) = \int_K \xi_\lambda^*(k) \mu(k) \cdot f(0) dk. \quad (8)$$

Recall that $f(0)$ is in W , which is an irreducible K -module of class μ : now, (8) is the formula for the orthogonal projection of $f(0)$ onto the isotypical component of W corresponding to $\lambda \in \widehat{K}$, and this projection is zero whenever $\lambda \neq \mu$.

□

So each Fourier component of f , except that which corresponds to the minimal K-type, goes to zero as the contraction is performed. This is the end of the way :

Theorem 5.3. *For each $f \in \mathbf{E}$, there is a limit f_0 to $\mathbf{C}_t f$ for the topology of uniform convergence on compact subsets of \mathfrak{p} , and when f belongs to \mathbf{H} this limit belongs to \mathbf{H}_0 . Moreover, if f_{min} is the orthogonal projection of f onto the lowest K-type isotypical component of \mathbf{H} , then $\mathbf{C}_t(f - f_{min})$ tends to zero uniformly on compact sets of \mathfrak{p} .*

Remark 5.4. The limit f_0 is the constant function on \mathfrak{p} with value $f(0) \in W$.

Let me return to the statement of Theorem 3.4. The space $\mathbf{E} = C(\mathfrak{p}, W)$ of continuous functions from \mathfrak{p} to W is a Fréchet space when equipped with the topology of uniform convergence on compact subsets of \mathfrak{p} . What we just saw is that parts 1. to 3. of Theorem 3.4 hold as soon as $\mathbf{M}(\delta)$ is a discrete series representation. To prove parts 4. and 5., we just need the following simple observation.

Lemma 5.6. *Choose $g_0 \in G_0$. Then there is a distance on \mathbf{E} whose associated topology is that of uniform convergence on compact subsets of \mathfrak{p} , and with respect to which each of the $\pi_t(\alpha_t g_0)$ is 1-Lipschitz.*

Proof. Whenever $A \subset \mathfrak{p}$ is compact, the subset $\Pi(A) = \{(\alpha_t g_0) \cdot_t A \mid t \in [0, 1]\}$ is compact too. So there is an increasing family, say (A_i) , of compact subsets of \mathfrak{p} , such that $\Pi(A_i) \subset A_{i+1}$, and $\cup_n A_n = \mathfrak{p}$.

A consequence is that for each f and f' in \mathbf{E} , $\|\pi_t f - \pi_t f'\|_{A_n} \leq \|f - f'\|_{A_{n+1}}$. Recall that a distance whose associated topology is that of uniform convergence on compact subsets is $d(f, f') = \sum_n \frac{\|f - f'\|_{A_n}}{2^n (1 + \|f - f'\|_{A_n})}$. Then $d/2$ has the desired property. \square

The proof of Theorem 3.4 for discrete series representations is completed by the next

Corollary. *Suppose f is in \mathbf{E} , then if $g_0 = (k, v)$, $\pi_t(\alpha_t g_0) f_t$ goes to $\mu(k) f_0$ as t goes to zero.*

Proof. Note first that

$$\pi_t(\alpha_t g_0) f_t = \pi_t(\alpha_t g_0)(f_t - f_0) + \pi_t(\alpha_t g_0) f_0.$$

Because of Lemma 5.6 the first term goes to zero, and because f_0 is a constant function, $\pi_t(\alpha_t g_0) f_0$ is just $\mu(k) f_0$. \square

6 Other representations with real infinitesimal character

6.1 Limits of discrete series

If the highest weight $\bar{\mu}$ is integral and C -dominant but the corresponding Harish-Chandra parameter is singular (see (5) above), it is no longer true that $\mathbf{V}_G(\mu)$ belongs to the discrete series. But when $\bar{\mu}$ is "not too degenerate", we can build the carrier space for $\mathbf{V}_G(\mu)$ from that of a discrete series representation, following Zuckerman's translation principle : let us consider an element $\bar{\lambda}$ of \mathfrak{t}^* which is integral, C -dominant and nonsingular. Then we can start from the infinite-dimensional space¹¹ $\mathbf{H}_{\lambda+\mu}$, which carries a discrete series representation, and form the tensor product $E = \mathbf{H}_{\lambda+\mu} \otimes A(\mu)$, where $A(\mu)$ is the

¹¹For convenience I will be dropping the "vector" arrows for convenience from now on.

finite-dimensional carrier space of an irreducible representation of $\mathfrak{g}_{\mathbb{C}}$ with lowest weight $-\mu$. Then we can consider the isotypical component

$$E_{\lambda} := \{v \in E \mid \forall X \in \mathbf{Z}(\mathfrak{g}_{\mathbb{C}}), X \cdot v = \xi_{\mathfrak{b}}(\lambda)(X) \cdot v\}.$$

(here $\xi_{\mathfrak{b}}(\lambda)$ is the infinitesimal character of section 3.1).

Zuckerman and Knapp proved ([30], theorem 1.1) that E_{λ} is an irreducible $(\mathfrak{g}_{\mathbb{C}}, K)$ -module and that it has an invariant hermitian form. Depending on μ , this space is either zero or infinite-dimensional ; when it is nonzero, it is possible to complete it into an unitary irreducible representation of G , and when we do so the representation is of class $\mathbf{V}_G(\mu)$. If it is not in the discrete series, then it is called a limit of discrete series.

In general it is not easy to describe the unitary structure (think of the explicit, but not easily generalized, Hilbert space norm in the case $SL_2(\mathbf{R})$, see [23], II.5), but after all we shifted the attention away from the Hilbert space norm in these notes ; as we shall see the contraction maps \mathbf{C}_t and the weak convergence with respect to the Fréchet topology on E_{λ} inherited from that of $\mathbf{H}_{\lambda+\mu}$ are not difficult to describe.

We first need to understand how $\mathfrak{g}_{\mathbb{C}}$ acts on the finite-dimensional part $A(\mu)$, and how things evolve when we consider it as a $\mathfrak{g}_{t, \mathbb{C}}$ -module. For this, we need to recall a construction for $A(\mu)$. Instead of describing it through its lowest weight, I will write $\tilde{\mu}$ for its highest weight and recall a construction for $A(\mu)$ as the irreducible representation with highest weight $\tilde{\mu}$ (see [24], V.3).

Let us start with the subalgebra

$$\mathfrak{b} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{n} := \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{\alpha}$$

of $\mathfrak{g}_{\mathbb{C}}$. Setting $\chi(H + E) = (\tilde{\mu} - \rho)(H)$ when H is in $\mathfrak{t}_{\mathbb{C}}$ and E in \mathfrak{n} , we obtain an abelian character of \mathfrak{b} , and thus an abelian character of the enveloping algebra $U(\mathfrak{b})$. I will write \mathbb{C}_{χ} for \mathbb{C} with this $U(\mathfrak{b})$ -module structure.

The Verma module $B(\mu)$ is then defined as the induced module

$$B(\mu) = U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\chi}.$$

In our setting this means that as a vector space $B(\mu)$ is the quotient $U(\mathfrak{g}_{\mathbb{C}})/M$, with

$$M = \langle Y - \chi(Y), Y \in U(\mathfrak{b}) \rangle$$

(the ideal generated by the $Y - \chi(Y)$ s), and that the $U(\mathfrak{g}_{\mathbb{C}})$ -action is just the adjoint action passed through the quotient.

Note that $\mathfrak{g}_{\mathbb{C}} = \mathfrak{b} \oplus \mathfrak{n}^-$, with

$$\mathfrak{n}^- := \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha},$$

and that a consequence is that $U(\mathfrak{n}^-)$, viewed as a vector subspace of $U(\mathfrak{g}_{\mathbb{C}})$, is an algebraic complement to M ; the projection from $U(\mathfrak{g}_{\mathbb{C}})$ to $B(\mu)$ restricts to a vector space

isomorphism, obviously also a $U(\mathfrak{n}^-)$ -module isomorphism, between $U(\mathfrak{n}^-)$ and $B(\mu)$.

Now set

$$S = \text{Sum of all proper submodules of } B(\mu).$$

This is of course a submodule, and because the image of 1 (the unit of $U(\mathfrak{g}_{\mathbb{C}})$) in $B(\mu)$ can be contained in no submodule it is actually proper. The irreducible module $A(\mu)$ is the quotient $B(\mu)/S$, and an important step in the classification of finite-dimensional representations is proving that $\dim(A(\mu))$ is finite.

★

Now, we know that the isomorphism φ_t extends to an isomorphism $\tilde{\varphi}_t$ between $\mathbf{U}(\mathfrak{g}_{t,\mathbb{C}})$ and $\mathbf{U}(\mathfrak{g}_{\mathbb{C}})$. If $\rho : \mathfrak{g} \rightarrow \text{End}(U(\mathfrak{g}_{\mathbb{C}}))$ and $\rho_t : \mathfrak{g}_t \rightarrow \text{End}(U(\mathfrak{g}_{t,\mathbb{C}}))$ code for the canonical extensions of the adjoint actions in each of those Lie algebras, then of course $\tilde{\varphi}_t$ intertwines them :

$$\tilde{\varphi}_t \circ \rho_t = \rho \circ \varphi_t.$$

Naturally \mathfrak{g} and \mathfrak{g}_t are the same as vector spaces, so $U(\mathfrak{g}_{t,\mathbb{C}})$ and $U(\mathfrak{g}_{\mathbb{C}})$ are the same as vector spaces too. The construction above applies to \mathfrak{g}_t , yielding a Verma module $B_t(\mu) = U(\mathfrak{g}_{t,\mathbb{C}})/(\tilde{\varphi}_t M)$ and a finite-dimensional vector space $A_t(\mu) = B_t(\mu)/S_t$ with natural \mathfrak{g}_t -actions intertwined by $\tilde{\varphi}_t$.

Now let S be the sum of all proper $U(\mathfrak{g}_{\mathbb{C}})$ submodules of $B(\mu)$, and ψ_t the map between $U(\mathfrak{g}_{t,\mathbb{C}})/(\varphi_t M)$ and $U(\mathfrak{g}_{\mathbb{C}})/M$ induced by $\tilde{\varphi}_t$. The image $\psi_t S$ is the sum of all proper $U(\mathfrak{g}_{t,\mathbb{C}})$ -submodules of $B_t(\mu)$. To study the way vectors in $B(\mu)/S$ evolve as the contraction is performed, we need a way to relate $B_t(\mu)/(\psi_t S)$ with $B(\mu)/S$ inside a fixed space. For this it would be very nice if M and S were invariant, as vector spaces, under the contraction. While I have not been able to see whether it is true that neither M nor S move as the contraction is performed, the next lemma gives a way to view $B_t(\mu)/(\psi_t S)$ as a fixed subspace.

Lemma 6.1.

- a. *The vector subspace $U(\mathfrak{n}^-)$ of $U(\mathfrak{g}_{\mathbb{C}})$ is an algebraic complement to M which is $\tilde{\varphi}_t$ -invariant for all t , so each $\tilde{\varphi}_t$ induces an element, say $\bar{\varphi}_t$, of $GL(B(\mu))$.*
- b. *The maximal proper submodule S of $B(\mu)$ admits an algebraic complement which is $\bar{\varphi}_t$ -invariant for all $t > 0$.*

Proof.

- a. There is an important remark to be made here : because \mathfrak{t} is contained in \mathfrak{k} , the real parts of the root spaces for roots of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ are contained either in \mathfrak{k} (the corresponding roots are called compact roots) or in \mathfrak{p} (the corresponding roots are called noncompact roots). A consequence of this is that as vector subspaces of \mathfrak{g} , they will not move during the contraction.

This remark extends to $U(\mathfrak{g}_{\mathbb{C}})$ as follows. There is a natural basis for $\mathbf{U}(\mathfrak{g}_{\mathbb{C}})$ associated to any basis of \mathfrak{g} by the Poincaré-Birkhoff-Witt construction. Let's then choose a basis

$(K_1, \dots, K_n, P_1, \dots, P_{2q})$ with the K_i s in \mathfrak{k} and the P_j s in \mathfrak{p} , such that a subset of the K_i s, say K_1, \dots, K_{r_1} , spans $\mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta_c^+} \mathfrak{g}_\alpha$, a subset of the P_j s, say P_1, \dots, P_{r_2} , spans $\bigoplus_{\alpha \in \Delta^+ - \Delta_c^+} \mathfrak{g}_\alpha$, and the other elements span the root spaces for negative roots (so that $r_2 = q$, and n is $2r_1$ plus the rank of \mathfrak{g}).

Note first that each element of the associated basis of $U(\mathfrak{g}_\mathbb{C})$ is a product $K_1^{u_1} \dots K_n^{u_n} P_1^{v_1} \dots P_{2q}^{v_{2q}}$, with $(u_1, \dots, u_n, v_1, \dots, v_{2q}) \in \mathbb{N}^{n+2q}$, and that the definition of $\tilde{\varphi}_t$ is equivalent with the fact that

$$\tilde{\varphi}_t [K_1^{u_1} \dots K_n^{u_n} P_1^{v_1} \dots P_{2q}^{v_{2q}}] = t^{v_1 + \dots + v_{2q}} [K_1^{u_1} \dots K_n^{u_n} P_1^{v_1} \dots P_{2q}^{v_{2q}}].$$

The elements of $U(\mathfrak{n}^-)$, viewed as elements of $U(\mathfrak{g}_\mathbb{C})$, are just the combinations of those basis elements which have $u_1 = \dots = u_{r_1} = v_1 = \dots = v_{r_2} = 0$. So the subspace $U(\mathfrak{n}^-)$ of $U(\mathfrak{g}_\mathbb{C})$ is indeed $\tilde{\varphi}_t$ -invariant for all t .

b. The second part is a consequence of the following simple observation :

Lemma 6.2. *Suppose $V = \bigoplus_{k \geq 0} V^k$ is a graded vector space, and S is a linear subspace with finite codimension. Then there is an algebraic complement to S for which a basis consists of homogeneous elements.*

Proof. I will write v^{\max} for the highest-degree homogeneous component of a vector v in V here.

Let's use induction on the codimension of S .

If $\text{codim}(S)$ is one, and $V = \mathbb{C}e_1 \oplus S$, it is not possible that every homogeneous component of e_1 be in S . Any homogeneous component that is not in S then yields a homogeneous algebraic complement to S .

Suppose now $\text{codim}(S)$ is higher. When E is a finite-dimensional subspace of V , let me write $d_E = \max\{d \in \mathbb{N} \mid V^d \cap E \neq \{0\}\}$. Choose d as the smallest integer such that there is an algebraic complement E to S with $d_E = d$, and let me start with E_0 such that $E_0 \oplus S = V$ and $d_{E_0} = d$. Choose a basis (e_1, \dots, e_n) of E_0 , and order it so that the e_i s have decreasing degrees, and (e_1, \dots, e_k) are the ones with maximal degree. Then there are two possible cases :

Case 1 : $e_1^{\max} \notin \text{Span}[e_2, \dots, e_n] \oplus S$. Then $V = \text{Span}[e_2, \dots, e_n] \oplus (\mathbb{C}e_1 \oplus S)$, and the conclusion for S follows from the induction hypothesis.

Case 2 : $e_1^{\max} \in \text{Span}[e_2, \dots, e_n] \oplus S$. Then $E_1 = \text{Span}[e_2, \dots, e_k, e_1 - e_1^{\max}, e_{k+1}, \dots, e_n]$ is an algebraic complement to S . Check whether e_2^{\max} is in $\text{Span}[e_3, \dots, e_k, e_1 - e_1^{\max}, e_{k+1}, \dots, e_n] \oplus S$, and if it is, define $E_2 = \text{Span}[e_2, \dots, e_k, e_2 - e_2^{\max}, e_1 - e_1^{\max}, e_{k+1}, \dots, e_n]$ and start again. This algorithm cannot fail to produce a situation in which Case 1 appears for one E_i , $i \leq k$, since if that were the case d would not be minimal.

Lemma 6.2 follows. \square

To prove lemma 6.1.b, we use the grading on $B(\mu)$ provided by the isomorphism between $U(\mathfrak{n}^-)$ and $B(\mu)$, deciding that the image of $[K_{r_1+1}^{u_{r_1+1}} \dots K_n^{u_n} P_{r_2+1}^{v_{r_2+1}} \dots P_{2q}^{v_{2q}}]$ in $B(\mu)$ has degree $v_{q+1} + \dots + v_{2q}$. The linear map $\bar{\varphi}_t$ then acts as multiplication by t^v on the subspace consisting of homogeneous elements with degree v , so that a subspace generated by homogeneous elements is $\bar{\varphi}_t$ -stable. We can then use lemma 6.2 to conclude the proof of lemma 6.1. \square

Because of lemma 6.1, we know that there is a $\tilde{\varphi}_t$ -invariant, finite-dimensional subspace $F(\mu)$ of $U(\mathfrak{n}^-)$ on which for each t , the composition of the two projections from $U(\mathfrak{g}_{t,\mathbb{C}})$ to $B_t(\mu)$ and from $B_t(\mu)$ to $A_t(\mu)$ restricts to a linear isomorphism. We know that φ_t induces a linear map which intertwines the actions of \mathfrak{g} and \mathfrak{g}_t on $A(\mu)$ and $A_t(\mu)$, so using our linear isomorphisms to lift these actions to $F(\mu)$, we end up with maps ρ' and ρ'_t from \mathfrak{g} and \mathfrak{g}_t to $\text{End}(F_\mu)$, which turn F_μ into a finite-dimensional irreducible $\mathfrak{g}_{\mathbb{C}}$ -module with lowest weight $-\mu$ and a finite-dimensional irreducible $\mathfrak{g}_{t,\mathbb{C}}$ -module with lowest weight $-\mu$, and which satisfy in addition

$$\tilde{\varphi}_t \circ \rho'_t = \rho' \circ \varphi_t.$$

We have thus exhibited our linear map $\tilde{\varphi}_t$ as a contraction map from $F(\mu)$ to itself. We now rename it as \mathbf{C}_t^{fd} .

But we explicitly know how $\tilde{\varphi}_t$ acts on $\mathbf{U}(\mathfrak{g}_C)$, so we can use this to see whether there is a limit to this contraction operator as t goes to zero. In the proof of lemma 6.1.b, we saw that a linear basis for $U(\mathfrak{g}_C)$ consists of monomials for which

$$\tilde{\varphi}_t [K_1^{u_1} \dots K_n^{u_n} P_1^{v_1} \dots P_{2q}^{v_{2q}}] = t^{v_1 + \dots + v_{2q}} [K_1^{u_1} \dots K_n^{u_n} P_1^{v_1} \dots P_{2q}^{v_{2q}}].$$

But of course these formulae make sense in the limit $t = 0$. Here is the conclusion :

Lemma 6.3. *For each $v \in F(\mu)$, there is a limit to $\mathbf{C}_t^{fd}v$ as t goes to zero.*

Here the convergence is in the sense of any norm-induced topology on $A(\mu)$, and the limit is naturally an element of $U(\mathfrak{k}_C)$.

It is time to return to limits of discrete series. Suppose $\mathbf{E}_{\lambda+\mu}^{ds}$ is the Fréchet space we associated to the representation $\mathbf{H}_{\lambda+\mu}$ in section 5. Consider now a vector F in $\mathbf{E} = \mathbf{E}_{\lambda+\mu}^{ds} \otimes F(\mu)$. It can be written as a finite sum $F = \sum_i f_i \otimes v_i$, with the f_i s in $\mathbf{E}_{\lambda+\mu}^{ds}$ and the v_i s in $A(\mu)$. We now set

$$\mathbf{Z}_t f = f_t := \sum_i (\mathbf{C}_t^{ds} f_i) \otimes (\mathbf{C}_t^{fd} v_i)$$

where $\mathbf{C}_t^{ds} \in \text{End}(\mathbf{E}_{\lambda+\mu}^{ds})$ is the contraction operator defined in section 5.2.

Lemma 6.3, together with the results of section 5, yields :

Lemma 6.4. *For each vector $F \in \mathbf{E}$, there is a limit F_0 to $\mathbf{Z}_t F$ as t goes to zero.*

Let us now see what remains if we start from the carrier Hilbert space \mathbf{E}_λ of our limit of discrete series, viewed as a vector subspace of \mathbf{E} .

For the moment the map \mathbf{Z}_t is defined on all of \mathbf{E} , which is much larger than the space we are actually interested in. If this \mathbf{Z}_t is to be our contraction map between representation spaces, we need the following fact. Let me use the notations of section 5.2 and write \mathbf{E}_λ for the vector subspace $\mathbf{H}_{\lambda+\mu} \otimes F(\mu)$ of \mathbf{E} , which carries our limit of discrete series representation as recalled above, and $\mathbf{E}_{t,\lambda}$ for the vector subspace $\mathbf{H}_{\lambda+\mu}^t \otimes F(\mu)$.

Lemma 6.5. *For each $F \in \mathbf{E}_\lambda$, $\mathbf{Z}_t F$ belongs to $\mathbf{E}_{t,\lambda}$*

To prove this, we need to start with an element $X \in \mathbf{Z}(\mathfrak{g}_{t,\mathbb{C}})$ and to see how it acts on $\mathbf{Z}_t F$. What we know is the infinitesimal character of the action of $U(\mathfrak{g}_{\mathbb{C}})$ on \mathbf{E}_λ , so writing π and π_t for the actions of $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g}_{t,\mathbb{C}}$ on \mathbf{E} naturally defined from those in Section 5, we know that $\pi(\tilde{\varphi}_t X)F$ is $\xi_t(\lambda)(\tilde{\varphi}_t X)F$. Because \mathbf{Z}_t intertwines the actions on \mathbf{E} by definition, this means that $\pi_t(X)(\mathbf{Z}_t F) = \xi_t(\lambda)(\tilde{\varphi}_t X)F$.

Does this mean that $X \mapsto \xi_t(\lambda)(\tilde{\varphi}_t X)$ is the abelian character of $\mathbf{Z}(\mathfrak{g}_{t,\mathbb{C}})$ which, through the Harish-Chandra isomorphism associated to the pair $(\mathfrak{g}_{t,\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, has parameter λ ? Yes, it does.

For this I recall the definition of the Harish-Chandra isomorphism γ between $\mathbf{Z}(\mathfrak{g}_{\mathbb{C}})$ and $\mathfrak{t}^*/W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ (see [24], V.7) : one starts with the decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{n} \oplus \mathfrak{n}^-$, and this yields a direct sum decomposition

$$U(\mathfrak{g}_{\mathbb{C}}) = U(\mathfrak{t}_{\mathbb{C}}) \oplus [U(\mathfrak{g}_{\mathbb{C}})\mathfrak{n} \oplus \mathfrak{n}^-U(\mathfrak{g}_{\mathbb{C}})].$$

Write $p_{U(\mathfrak{t}_{\mathbb{C}})}$ for the associated projection $U(\mathfrak{g}_{\mathbb{C}}) \rightarrow U(\mathfrak{t}_{\mathbb{C}})$, and recall that ρ is the half-sum of positive roots of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ with respect to the ordering we have been working with in this section. The linear map

$$H \in \mathfrak{t}_{\mathbb{C}} \mapsto H - \rho(H)1 \in U(\mathfrak{t}_{\mathbb{C}})$$

extends to an algebra automorphism, say τ , of $U(\mathfrak{t}_{\mathbb{C}})$, and the Harish-Chandra isomorphism is

$$\gamma := \tau \circ p_{U(\mathfrak{t}_{\mathbb{C}})}.$$

Of course this construction also yields an algebra isomorphism γ_t between $\mathbf{Z}(\mathfrak{g}_{t,\mathbb{C}})$ and $\mathfrak{t}^*/W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, and I claim that $\gamma_t = \gamma \circ \tilde{\varphi}_t$. The reason is that

If α is a root of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, then it is also a root of $(\mathfrak{g}_{t,\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, and the root spaces correspond under ϕ_t .

Indeed, if X is an element of $\mathfrak{g}_{\mathbb{C}}$ such that $[H, X]_{\mathfrak{g}_{\mathbb{C}}} = \alpha(H)X$ for all H in $\mathfrak{t}_{\mathbb{C}}$, then $[\phi_t^{-1}H, \phi_t^{-1}X]_{\mathfrak{g}_{t,\mathbb{C}}} = \alpha(H)\phi_t^{-1}X$, and the statement in italics follows because ϕ induces the identity on $\mathfrak{t}_{\mathbb{C}}$ (this is the difference with Lemma 4.2).

An immediate consequence is that γ_t is defined from the decomposition $\mathfrak{g}_{t,\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus (\phi_t^{-1}\mathfrak{n}) \oplus (\phi_t^{-1}\mathfrak{n}^-)$, hence that the projection $p_{t,U(\mathfrak{t}_{\mathbb{C}})}$ defined from \mathfrak{g}_t is just $p_{U(\mathfrak{t}_{\mathbb{C}})} \circ \tilde{\varphi}_t$. Another immediate consequence is that the half-sum of positive roots of $(\mathfrak{g}_{t,\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ is also ρ , so that the shift between $p_{t,U(\mathfrak{t}_{\mathbb{C}})}$ and γ_t is still τ . This proves lemma 6.5. \square

Because I defined \mathbf{Z}_t in a manner compatible with the definition of tensor products of $\mathfrak{g}_{\mathbb{C}}$ -modules, and because of Lemma 6.5, we now know that \mathbf{Z}_t intertwines the actions of $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g}_{t,\mathbb{C}}$ on \mathbf{E}_{λ} and $\mathbf{E}_{\lambda,t}$. When these infinitesimal versions are integrated and $\mathbf{E}_{\lambda,t}$ is viewed as the space of smooth, K -finite vectors in the carrier space of a unitary irreducible representation of G_t , \mathbf{Z}_t becomes a well-defined contraction operator in the sense of section 3.3 ; we now rename \mathbf{Z}_t as \mathbf{C}_t .

Lemma 6.6. *The vector space $\mathbf{H}_0 := \{F_0 \mid F \in E_{\lambda}\}$ carries an irreducible K -module of class μ .*

Proof. Let me write $\mathbf{H}_{\lambda+\mu}^0$ for the finite-dimensional vector space gathering the limits of the $\mathbf{C}_t^{ds} f$, $f \in \mathbf{H}_{\lambda+\mu}$, and $F(\mu)^0$ for the subspace of $F(\mu)$ gathering the limits of the $\mathbf{C}_t^{fd} v$, $v \in F(\mu)$. Our \mathbf{H}_0 is then the image under p_{λ} of the tensor product $\mathbf{H}_{\lambda+\mu}^0 \otimes F(\mu)^0$, viewed as a subspace of \mathbf{E} . But suppose we start with the tensor product, say $\mathbf{A}_{\lambda+\mu}^0 \otimes G(\mu)^0$, of the carrier space for an irreducible K -module with highest weight $\lambda + \mu$, with the carrier space for an irreducible $\mathfrak{k}_{\mathbb{C}}$ -module with lowest weight $-\mu$, then look at the isotypical component corresponding to the infinitesimal character which the Harish-Chandra isomorphism for the pair $(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ associates to λ . Then, because K is a reductive Lie group and an irreducible K -module with highest weight $\lambda + \mu$ can be viewed as a discrete series representation, a trivial case of the result described above for limits of discrete series says that this isotypical component is the carrier space for an irreducible representation with highest weight λ . Now, it is true that $F(\mu)^0$ is the carrier space for an irreducible $\mathfrak{k}_{\mathbb{C}}$ -module with lowest weight $-\mu$: the definition of $F(\mu)$ means that

$$F(\mu)^0 = F(\mu) \cap U(\mathfrak{k}_{\mathbb{C}}) \cap U(\mathfrak{n}^-) = F(\mu) \cap \sum_{\alpha \in \Delta_{\mathfrak{k}}^+} \mathfrak{k}_{-\alpha};$$

and except at zero $F(\mu)^0$ does not intersect M , especially not

$$M^0 := M \cap U(\mathfrak{k}_{\mathbb{C}}) = \left\langle Y - \chi(Y), Y \in U(\mathfrak{t}_{\mathbb{C}} \oplus \sum_{\alpha \in \Delta_{\mathfrak{k}}^+} \mathfrak{k}_{\alpha}) \right\rangle.$$

In addition, the image of $F(\mu)^0$ in $U(\mathfrak{k}_{\mathbb{C}})/M^0$ is an algebraic complement to

$$S^0 := S \cap U(\mathfrak{k}_{\mathbb{C}}) = \text{Sum of all proper } U(\mathfrak{k}_{\mathbb{C}}) \text{ submodules of } U(\mathfrak{k}_{\mathbb{C}})/M^0$$

(the last equality is because the proper $U(\mathfrak{g}_{\mathbb{C}})$ -submodules of $B(\mu)$ are those that do not contain the highest weight $\tilde{\mu} - \rho$ upon restriction to \mathfrak{t} , and each of those decomposes under $U(\mathfrak{k}_{\mathbb{C}})$ as a sum of $U(\mathfrak{k}_{\mathbb{C}})$ -submodules which do not contain the highest weight $\tilde{\mu} - \rho_{\mathbb{C}}$ upon restriction to \mathfrak{t} , so that they project in $U(\mathfrak{k}_{\mathbb{C}})/M^0$ as proper $U(\mathfrak{k}_{\mathbb{C}})$ -modules).

Hence the double projection from $F(\mu)^0$ to $(U(\mathfrak{k}_{\mathbb{C}})/M^0)/S^0$ is a vector space isomorphism which commutes with the action of \mathfrak{k} on both spaces, as announced.

This proves lemma 6.6. □

Let me summarize the situation for limits of discrete series representations : because of lemma 6.6 and the interpretation of the "limit subspace" for discrete series as a space of constant functions, it is also true that our Fréchet space \mathbf{E}_{λ} can be viewed as the space of continuous functions with values in a fixed vector space carrying the minimal K -type of our representation π . The convergence of vectors in the subspace of \mathbf{E}_{λ} which carries the limit of discrete series is summarized in the following statement.

Theorem 6.1. *For each $f \in \mathbf{E}$, there is a limit f_0 to $\mathbf{C}_t f$ for the Fréchet topology of uniform convergence on compact subsets of \mathfrak{p} when one views \mathbf{E} as a space of $W_{\lambda+\mu} \otimes F(\mu)$ -valued functions on \mathfrak{p} ; the space of limits obtained from elements of \mathbf{E}_λ carries an irreducible K -module whose equivalence class is the lowest K -type of π . Moreover, if f_{min} is the orthogonal projection of f onto the lowest K -type isotypical component of \mathbf{H} , then $\mathbf{C}_t(f - f_{min})$ tends to zero uniformly on compact sets of \mathfrak{p} .*

Note that because $F(\mu)$ is a priori larger than $F(\mu)^0$, when we interpret \mathbf{E}_λ as a space of functions on \mathfrak{p} the value of functions at zero is modified by the contraction process: in the limit it is projected on $W \otimes F(\mu)^0$. This did not happen in the case of discrete series. To come back to the statement of Theorem 3.4, it is no longer quite true that \mathbf{C}_t is the only contraction map which preserves the value of functions at zero, but it is the only contraction map which preserves the projection on $W \otimes F(\mu)^0$ of their value at zero.

Because the action of \mathfrak{g} on the finite-dimensional part is through bounded operators, the end of point 4. in Theorem 3.4 follows immediately from the analogous statement for the discrete series (lemma 5.6 and the corollary), and this completes the proof of Theorem 3.4 when $\mathbf{M}(\delta)$ is a nonzero limit of discrete series.

6.2 The remaining representations

We will now consider the representations of G which are irreducible tempered and have real infinitesimal character, and hence a minimal K -type, but which are neither in the discrete series nor limits of discrete series.

Example 6.2. When G is $SL_2(\mathbf{R})$, there is only one such representation: the irreducible principal series representation with continuous parameter zero, whose minimal K -type is the trivial representation.

For the representation $\mathbf{V}_G(1)$ whose minimal K -type is the trivial representation, we gave two geometric realizations in section 4: in the compact picture, the Hilbert space is $\mathbf{L}^2(K)$ and $g \in G$ acts as $f \mapsto [k \mapsto \exp\{-\rho, \mathbf{a}(g^{-1}k)\}\sigma(\mathbf{m}(g^{-1}k))f(\kappa(g^{-1}k))]$, and in the second the Hilbert space is a space of functions on \mathfrak{p} .

6.2.1. The trivial representation of G_0 in Helgason's picture.

Let us start with Helgason's picture, and recall that in section 4.2. I used functions $e_{\lambda,b}$ on G/K . The definition makes sense with $\lambda = 0$, and it is true that

$$\mathbf{H}^{Helgason} = \left\{ \int_{K/M} e_{0,b} F(b) db \mid F \in \mathbf{L}^2(K/M) \right\}$$

carries an irreducible representation of G whose equivalence class is $\mathbf{V}_G(1)$. Because of the results in section 4, which do hold when $\lambda = 0$, we know that the contracted waves $\varepsilon_{0,b}^t$ give rise to the corresponding representation of G_t with Hilbert space

$$\mathbf{H}_t^{Helgason} = \left\{ \int_{K/M} \varepsilon_{0,b}^t F(b) db \mid F \in \mathbf{L}^2(K/M) \right\}$$

But now when t goes to zero, all of the $\varepsilon_{0,b}^t$ converge to the constant function with value 1! Because of the results on page 23, the map $\mathbf{C}_t = f \mapsto [x \mapsto f(tx)]$ turns out to be a contraction map between the various $\mathbf{H}_t^{Helgason}$ s, and the conclusion is that if \mathbf{E} is the space of continuous, complex-valued functions on \mathfrak{p} equipped with the topology of uniform convergence on compact subsets,

Theorem 6.3. *For each $f \in \mathbf{E}$, there is a limit to $\mathbf{C}_t f$ as t goes to zero ; the limit is the constant function with value $f(0)$. For each g_0 in G_0 and each f in $\mathbf{H}^{\text{Helgason}}$, $[x \mapsto (\mathbf{C}_t f)((\alpha_t g_0)^{-1} \cdot_t x)]$ converges (in \mathbf{E}) to the constant function with value $f(0)$.*

This proves Theorem 3.4 when $\mathbf{M}_0(\delta)$ is the trivial representation.

6.2.2. The trivial representation of G_0 in the compact picture.

If instead of Helgason's picture we take up the compact picture and try to perform the contraction, the situation is less promising. Here \mathbf{H}^{comp} is just $\mathbf{L}^2(K/M)$, and as we saw earlier, the only contraction map between $\pi_{0,1}^{\text{comp}}$ and $\pi_{0,1}^{t,\text{comp}}$ which preserves the value of functions at zero is the identity ! So in the limit, we will certainly not get the carrier space for the trivial representation. Instead, the proof of theorem 4.1 shows that if $g_0 = (k, v)$, $\pi_{\lambda,\sigma}^{t,\text{comp}}(\alpha_t(k, v))$ weakly converges to $f \mapsto [u \mapsto f(k^{-1}u)]$, so that in the limit we get the quasi-regular representation of K on $\mathbf{L}^2(K/M)$ instead of the trivial representation of K !

6.2.3. The remaining cases.

It would be very nice if the other real-infinitesimal-character cases could be treated in the same way. I do not see how, though. The only description I know for tempered irreducible representations which have real infinitesimal character, but are not in the discrete series or limits of discrete series, is the following simple consequence of the Knapp-Zuckerman classification theorem (I already used it in the proof of lemma 3.3) :

Fact (Knapp-Zuckerman). If σ is a irreducible tempered (unitary) representation of G which has real infinitesimal character, then there is a cuspidal parabolic subgroup MAN of G , and there is a discrete series or nondegenerate limit of discrete series representation σ^b of M , such that

$$\sigma = \text{Ind}_{MAN}^G (\sigma^b \otimes \mathbf{1}).$$

Given what I said in sections 6.2.1 and 6.2.2, it is pretty clear that the compact picture for σ that this provides will not be enough. The other usual pictures do not seem to lead to a setting in which the contraction can easily be described. In view of what precedes, the following question seems natural : is there a realization for this which would be analogous to Helgason's picture, and would allow for the Hilbert space for σ to be viewed as a space of functions on \mathfrak{p} , or perhaps on a vector subspace or quotient of \mathfrak{p} ? To my knowledge none has been set forth yet. It is likely that Camporesi's paper [42] might be helpful in that direction, but I have not looked deep enough into the matter at present.

7 General tempered representations

7.1 Discrete series for disconnected groups

To cover the general case, we need to describe the discrete series representations of M . I will follow [30] here and refer to [23], XII.8.

Let us first consider the identity component M_0 of M . It is a non-semisimple, connected Lie group and can be decomposed as $M_0 = M_{ss}(Z_M)_0$, with M_{ss} a connected semisimple Lie group with finite center. The abelian group $(Z_M)_0$ is compact and central in M_0 ([23], section V.5).

Suppose we start with a discrete series representation of M_0 . Then the elements in $(Z_M)_0$ will act as scalars, and we will get an abelian character of $(Z_M)_0$. It is easy to check that the restriction to M_{ss} of our representation will then be irreducible and belong to the discrete series of M_{ss} , because its matrix elements will be square-integrable.

A discrete series representation π_0 of M_0 is thus specified by a discrete series representation π_{ss} of M_{ss} and an abelian character ξ of $(Z_M)_0$ whose restriction to $M_{ss} \cap (Z_M)_0$ coincides with $(\pi_{ss})|_{M_{ss} \cap (Z_M)_0}$. The Hilbert space for π_0 is that of π_{ss} , and the formula for π_0 is $g = g_{ss}g_{(Z_M)_0} \mapsto \xi(g_{(Z_M)_0})\pi_{ss}(g_{ss})$.

Now that we know how to describe the discrete series of M_0 , let us write M^\sharp for the subgroup M_0Z_M of G ; because of [23], lemma 12.30, M_0 has finite index in M^\sharp , and in addition there is a finite, abelian subgroup F of K (it is the subgroup called $F(B^-)$ in [23]) such that

$$M^\sharp = M_0F$$

and F is in the center of M (hence of M^\sharp).

Of course the arguments we recalled for M_0 go through here, and a discrete series representation π^\sharp of M^\sharp is thus specified by a discrete series representation π_0 of M_0 and an abelian character χ of F whose restriction to $M_0 \cap F$ coincides with $(\pi_0)|_{M_0 \cap F}$. The Hilbert space for π^\sharp is that of π_0 , and the formula for π^\sharp is $g = g_0f \mapsto \chi(f)\pi_0(g_0)$.

To obtain a unitary representation of M , we can start from a discrete series representation π^\sharp of M^\sharp and set

$$\pi = \text{Ind}_{G^\sharp}^G(\pi^\sharp).$$

It turns out ([23], Proposition 12.32) that π is irreducible, is in the discrete series of M , and that $\pi^\sharp \mapsto \pi$ maps the discrete series of M^\sharp onto the discrete series of M . In addition, the restriction of π to M^\sharp decomposes as

$$\pi|_{M^\sharp} = \sum_{w \in M/M^\sharp} w\pi^\sharp$$

where $w\pi^\sharp$ is $m \mapsto \pi^\sharp(w^{-1}mw)$. Notice that M^\sharp has finite index in M (see (12.74) in [23]), so the sum is finite here.

★

Now, the above description makes it easy to describe the contraction maps between the carrier spaces for discrete series representations of M and M_t .

A first remark is that both $(Z_M)_0$ and F are contained in K (see [23], sections V.5 and XII.8). So we can consider the discrete series representations of M and M_t assembled from χ , ξ and discrete series representations of M_{ss} and $(M_t)_{ss}$ with the same minimal $(K \cap M_{ss})$ -type, and these will have the same minimal $(K \cap M)$ -type.

Suppose now \mathbf{H} and \mathbf{H}_t are the carrier spaces for discrete series representations of M_{ss} and $(M_{ss})_t$ with the same minimal K_{ss} -type, and suppose \mathbf{C}_t is the contraction map between \mathbf{H} and \mathbf{H}_t defined in Section 5.

Then for each choice of ξ and χ , \mathbf{C}_t intertwines the discrete series representations representations of M^\sharp and $(M_t)^\sharp$. This is because the elements of M_0 and F act through multiples of the identity on \mathbf{H} , so of course

$$\pi^\sharp(g_{ss}g_{(Z_M)_0}f) \circ C_t = \xi(g_{(Z_M)_0})\chi(f) (\pi^{ss} \circ C_t).$$

Let us now start with discrete series representations of M and M_t which have the same minimal $K \cap M$ -type. Let's write the decomposition of their restrictions to M^\sharp and $(M_t)^\sharp$ as

$$\begin{aligned} \mathbf{H} &= \sum_{\omega \in M/M^\sharp} \mathbf{H}_\omega \\ \mathbf{H}_t &= \sum_{\omega \in M/M^\sharp} \mathbf{H}_{t,\omega} \end{aligned}$$

and suppose we intertwine each of the summands with a geometric realization as a space of solutions of a Dirac equation, so that for each ω , we can view \mathbf{H}_ω and the various $\mathbf{H}_{t,\omega}$ s ($t > 0$) as subspaces of a fixed Fréchet space \mathbf{E}_ω as described in section 4. We can then view \mathbf{H} and the \mathbf{H}_t s, $t > 0$, as subspaces of a fixed Fréchet space \mathbf{E} (the finite direct sum of the \mathbf{E}_ω s). In section 4 we defined maps \mathbf{C}_t^ω from \mathbf{E}_ω to itself which send \mathbf{H}_ω to $\mathbf{H}_{t,\omega}$. Let us define

$$\mathbf{C}_t \left(\sum_{\omega \in M/M^\sharp} f_\omega \right) = \sum_{\omega \in M/M^\sharp} \mathbf{C}_t^\omega f_\omega.$$

This is a linear map from \mathbf{E} to itself sending \mathbf{H} to \mathbf{H}_t .

Lemma 7.1. *This operator is a contraction map.*

Proof. Of course this map is defined in such a way that it commutes with the restrictions to M^\sharp and $(M_t)^\sharp$. What we need to check is just that it commutes with the M and M_t -actions. But this is clear from the definition of induced representations when one induces from a subgroup with finite index : suppose m is in M and $(m_\omega)_{\omega \in M/M^\sharp}$ is a section of the projection $M \rightarrow M/M^\sharp$ (so each m_ω is in M), then there is a collection $(m_{[m\omega]}^\sharp)_{\omega \in M/M^\sharp}$ of elements of M^\sharp such that $mm_\omega = m_{[m\omega]}^\sharp m_{[m\omega]}$, and the action of m on \mathbf{H} is

$$\sum_{\omega} x_\omega \mapsto \sum_{\omega} \pi_\omega^\sharp(m_{[m\omega]}^\sharp) x_{[m\omega]}.$$

Then M_t/M_t^\sharp and M/M^\sharp coincide, $(\varphi_t^{-1}m_\omega)_{\omega \in M/M^\sharp}$ is a full set of representatives, $\varphi_t^{-1}m$ will satisfy $(\varphi_t^{-1}m)(\varphi_t^{-1}m_\omega) = (\varphi_t^{-1}m_{[m\omega]})(\varphi_t^{-1}m_{[m\omega]}^\sharp)$, and will act on \mathbf{H}_t through

$$\sum_{\omega} x_\omega \mapsto \sum_{\omega} \pi_{t,\omega}^\sharp(\varphi_t^{-1}m_{[m\omega]}^\sharp) x_{[m\omega]}.$$

Then of course

$$\begin{aligned}
\mathbf{C}_t \left(\pi(m) \sum_{\omega} f_{\omega} \right) &= \sum_{\omega} \mathbf{C}_t^{\omega} \pi_{\omega}^{\sharp}(m_{[m\omega]}) f_{[m\omega]}. \\
&= \sum_{\omega} \pi_{t,\omega}^{\sharp}(\varphi_t^{-1} m_{[m\omega]}) \mathbf{C}_t^{\omega} f_{[m\omega]}. \\
&= \pi_t(\varphi_t^{-1} m) \left(\sum_{\omega} \mathbf{C}_t^{\omega} f_{[m\omega]} \right) \\
&= \pi_t(\varphi_t^{-1} m) \left(\mathbf{C}_t \left[\sum_{\omega} f_{\omega} \right] \right)
\end{aligned}$$

and the lemma follows. \square

Lemma 7.2. *For each vector $F \in \mathbf{E}$, there is a limit F_0 to $\mathbf{C}_t F$ as t goes to zero.*

The limit in the statement is with respect to the Fréchet topology of \mathbf{E} , and the lemma is obvious from Theorem 5.1. Now our aim was to describe the contraction of a discrete series representation onto a space carrying a K -module whose equivalence class is the minimal K -type μ of the discrete series we started from, so the following result is the end of the way :

Lemma 7.3. *The vector space $\mathbf{H}_0 := \{F_0 \mid F \in \mathbf{E}\}$ carries an irreducible K -module of class μ .*

Proof. Let me write K^{\sharp} , K_0 , K_{ss} for the intersections of K with M^{\sharp} , M_0 , M_{ss} . For each $\omega \in M/M^{\sharp}$, write

$$\mathbf{V}_{\omega} := \{F_0 \mid F \in \mathbf{H}_{\omega}\}$$

This is an irreducible K_{ss} -module whose equivalence class is the minimal K_{ss} -type, say μ_{ω}^b , of \mathbf{H}_{ω} . One can use the characters ξ and χ to turn \mathbf{V}_{ω} into a K^{\sharp} -module as above ; I will write μ_{ω}^{\sharp} for its equivalence class, which is also the minimal K^{\sharp} -type of \mathbf{H}_{ω} .

Now, we know that the inclusion from K to M induces an isomorphism between K/K^{\sharp} and M/M^{\sharp} (see (12.74) in [23]), so the outcome of the contraction can be rewritten as

$$\mathbf{H}_0 = \sum_{K/K^{\sharp}} \mathbf{V}_{\omega}.$$

The fact that each \mathbf{C}_t^{ω} is K -equivariant and induces an intertwining map between the restriction of π^{ω} to the minimal K -type component of \mathbf{H}_{ω} on the one hand, and the action μ^{ω} on \mathbf{V}_{ω} on the other hand, means that the action of K on \mathbf{H} will induce an action of K on \mathbf{H}_0 . In this way an element k in K will act as

$$\sum_{\omega} x_{\omega} \mapsto \sum_{\omega} \mu_{\omega}^{\sharp}(k_{[k\omega]}^{\sharp}) x_{[k\omega]}$$

where the $k_{[k\omega]}^{\sharp}$ s are the elements defined in the proof of lemma 7.1 if we take care to ask that the representatives m_{ω} , $\omega \in M/M^{\sharp}$, belong to K .

Of course the description of induced representations given in the proof of lemma 7.1 means that for any ω_0 in K/K^{\sharp} ,

$$\mathbf{H}_0 \simeq \text{Ind}_{K^{\sharp}}^K (\mu_{\omega_0}^{\sharp}).$$

But as a particular case of the description of the discrete series of a reductive group M from the discrete series of M^{ss} recalled above, we do know that

$$\mu \simeq \text{Ind}_{K^\sharp}^K (\mu_{\omega_0}^\sharp).$$

So the equivalence class of \mathbf{H}_0 as a K -module is really that of μ . □

I worked with discrete series representations of M_χ for convenience here, but it is clear from the constructions recalled above that the remarks in this subsection yield a description of both the limits of discrete series representations of M_χ and their contraction onto their minimal K -type.

7.2 Contraction of a basic representation

Let me finally consider a general Mackey datum $\delta = (\chi, \mu)$ and the cuspidal parabolic subgroup $P_\chi = M_\chi A_\chi N_\chi$ from section 3.2. Since I have not been able to write down what happens for the contraction of $\mathbf{V}_{M_\chi}(\mu)$ if it is neither a limit of discrete series (or discrete series) representation nor the one with trivial minimal K -type, I will assume that δ is a reasonable Mackey datum in the sense of Section 3.3¹².

Let me consider a carrier Hilbert space \mathbf{S}^μ for the tempered-irreducible-representation-with-real-infinitesimal-character $\mathbf{V}_{M_\chi}(\mu)$ of M_χ , σ for the morphism from M_χ to $\text{End}(\mathbf{S}^\mu)$, and let me introduce Hilbert spaces \mathbf{S}_t^μ for the corresponding representations σ_t of $M_{\chi,t}$. As I explained in subsection 7.1, we can view all those carrier Hilbert spaces as subspaces of a fixed Fréchet space \mathbf{E}_μ , and we identified a distinguished linear map

$$\mathbf{C}_t^\mu : \mathbf{E}^\mu \rightarrow \mathbf{E}^\mu$$

which restricts to a contraction map between \mathbf{S}^μ and \mathbf{S}_t^μ .

Consider now the vector space $\mathbf{E} = C(K, \mathbf{E}^\mu)$ of continuous functions from K to \mathbf{E}^μ , and endow it with the Fréchet topology of uniform convergence.

Pointwise composition with \mathbf{C}_t^μ defines a linear map

$$\mathbf{C}_t : \mathbf{E} \rightarrow \mathbf{E}$$

which sends the subspace $\mathbf{H} \subset \mathbf{E}$ of \mathbf{S}^μ -valued, continuous functions on K which satisfy $f(ku) = \sigma(u)^{-1}f(k)$ for each (k, u) in $K \times (K \cap M_\chi)$, to the subspace \mathbf{H}_t of \mathbf{S}_t^μ -valued, continuous functions on K which satisfy $f(ku) = \sigma_t(u)^{-1}f(k)$ for each (k, u) in $K \times (K \cap M_{t,\chi})$.

Recall from section 4.1 that the representation of G on \mathbf{H} defined by

$$\pi_{\lambda,\mu}^{comp}(g) = f \mapsto [k \mapsto \exp \langle i\lambda + \rho, \mathbf{a}(g^{-1}k) \rangle \sigma(\mathbf{m}(g^{-1}k)) f(\kappa(g^{-1}k))].$$

is the compact picture for $\mathbf{M}(\delta)$.

¹²Once the extension to all tempered-irreducible-with-real-infinitesimal-character representation is obtained, I will be able to just drop this sentence and the results will apply to the full tempered dual.

Let me define a $\pi_{\lambda,\mu}^{t,comp}$ of G_t which acts on \mathbf{H}_t , and

$$\varpi_{\lambda,\mu}^{t,comp} : G \xrightarrow{\varphi_t^{-1}} G_t \xrightarrow{\pi_{\lambda,\mu}^{t,comp}} \text{End}(\mathbf{H}_t).$$

Lemma 7.4. *The linear map \mathbf{C}_t intertwines $\varpi_{\lambda,\mu}^{t,comp}$ and $\pi_{\lambda/t,\mu}^{comp}$.*

Proof. This is but an adaptation of Lemma 4.1. A trivial adaptation of its proof shows that the subgroups $M_{t,\chi}$, $A_{t,\chi}$, $N_{t,\chi}$ used in the definition of $\pi_{\lambda,\mu}^{t,comp}$ are sent by φ_t to the subgroups M_χ , A_χ , N_χ used to define $\pi_{\lambda,\mu}^{comp}$, and that the projections κ_t , \mathbf{m}_t , \mathbf{a}_t are related with those for G through

$$\begin{aligned} \kappa_t(\varphi_t^{-1}g) &= \kappa(g); \\ \mathbf{a}_t(\varphi_t^{-1}g) &= \frac{\mathbf{a}(g)}{t}; \\ \mathbf{m}_t(\varphi_t^{-1}g) &= \mathbf{m}(g). \end{aligned}$$

Because of Lemma 4.2 (or rather the same lemma after a change of notation, and the same proof), we know how the roots of $(\mathfrak{g}_t, \mathbf{a}_\chi)$ evolve with t , and for each $\gamma \in G_t$ we know that

$$\pi_{\lambda,\sigma}^{t,comp}(\gamma) = f \mapsto [k \mapsto \exp(i\lambda + t\rho, \mathbf{a}_t(\gamma^{-1}k))\sigma_t(\mathbf{m}_t(\gamma^{-1}k))f(\kappa_t(\gamma^{-1}k))].$$

Hence

$$\pi_{\lambda,\sigma}^{t,comp}(\varphi_t^{-1}(g)) = f \mapsto [k \mapsto \exp(i\lambda + t\rho, \mathbf{a}_t([\varphi_t^{-1}g]^{-1}k))\sigma_t(\mathbf{m}_t([\varphi_t^{-1}g]^{-1}k))f(\kappa_t([\varphi_t^{-1}g]^{-1}k))].$$

And rearranging, we need only recall that \mathbf{C}_t is a contraction map between σ and σ_t to obtain

$$\begin{aligned} \pi_{\lambda,\sigma}^{t,comp}(\varphi_t^{-1}(g))[\mathbf{C}_t^\sigma f] &= \left[k \mapsto \exp\left\langle i\frac{\lambda}{t} + \rho, t \cdot \mathbf{a}_t(\varphi_t^{-1}[g^{-1}k]) \right\rangle \sigma_t(\mathbf{m}_t([\varphi_t^{-1}g]^{-1}k)) (\mathbf{C}_t f)(\kappa_t(\varphi_t^{-1}[g^{-1}k])) \right] \\ &= \left[k \mapsto \exp\left\langle i\frac{\lambda}{t} + \rho, \mathbf{a}(g^{-1}k) \right\rangle \{ \mathbf{C}_t^\sigma \sigma(g^{-1}k) \} f(\kappa(g^{-1}k)) \right] \\ &= \mathbf{C}_t \left(\pi_{\frac{\lambda}{t},\sigma}^{comp}(g)f \right), \end{aligned}$$

so the proof of lemma 7.4 is complete. \square

Of course the results of section 7.1 and the description of $\mathbf{M}_0(\delta)$ in section 2.3 mean that

Lemma 7.5. *For each $f \in \mathbf{H}$, there is a limit¹³ f_0 to $\mathbf{C}_t f$ as t goes to zero. The vector space $\mathbf{H}_0 := \{f_0 \mid f \in \mathbf{H}\}$ carries an irreducible G_0 -module with class $\mathbf{M}_0(\delta)$.*

¹³The limit is in \mathbf{E} here.

Let me write \mathbf{S}_0^μ for the subspace of \mathbf{E}_μ which gathers the limits (in \mathbf{E}_μ) of the $\mathbf{C}_t^\sigma v$, $v \in \mathbf{S}^\mu$, and note that $\mathbf{H}_0 = C(K, \mathbf{S}_0^\mu)$. The next step is to see how the G_0 -module structure on \mathbf{H}_0 described in section 2.3 emerges from the G -module structure on \mathbf{H} through the contraction process. In section 4.2, I used diffeomorphisms $\alpha_t : G_0 \rightarrow G_t$ to show how operators for $\mathbf{M}(\delta)$ converge to operators for $\mathbf{M}_0(\delta)$. Let me proceed in the same way here and set

$$\tilde{\pi}^t = \pi_{\lambda, \sigma}^{t, \text{comp}} \circ \alpha_t.$$

Now for each g_0 in G_0 , the operator $\tilde{\pi}^t(g_0)$ acts on \mathbf{H}_t , and we want to compare it with $\pi_0(g_0)$ which acts on \mathbf{H}_0 . Point 4. in the statement of Theorem 3.4 above provides a natural way to make the comparison. To see how to prove it, let me come back to discrete series representations for a moment.

In section 5, I used the action of G_t on \mathfrak{p} to build an action on $\mathbf{C}^\infty(\mathfrak{p}, W)$ (beware there is an action on the fibers here). This action is defined on the whole space of continuous functions, and it is by restricting it to the G_t -stable vector subspace of square-integrable solutions of the Dirac equation that we get operators for a discrete series representation. In view of the constructions I recalled in section 6 and 7.1, a finite number of trivial steps extends Lemma 5.6 to the following two facts (in italics) :

There is a family of linear maps $\bar{\sigma}_t : M_{t, \chi} \rightarrow \text{End}(\mathbf{E}^\mu)$, weakly continuous w.r.t. the Fréchet topology on \mathbf{E}^μ , such that each $\sigma_t : M_{t, \chi} \rightarrow \text{End}(\mathbf{S}_t^\mu)$ is obtained by restricting $\bar{\sigma}_t$ to \mathbf{E}^μ .

In the sequel I will remove the bar and write σ directly for the maps from \mathbf{E}^μ to itself. Because of Lemma 5.6 and its corollary, which extend to limits of discrete series and nonconnected groups with the obvious modifications, these maps will have the following property :

For each g_0 , in G_0 and each $f \in \mathbf{E}$, there is a limit (in \mathbf{E}) to $\bar{\sigma}_t(\alpha_t g_0) f$ as t goes to zero. When f belongs to \mathbf{H}_0 , this limit is $\mu(k) f$.

Now we can use these observations to extend each of the $\pi_{\lambda, \sigma}^{t, \text{comp}}$ to all of \mathbf{E} , by setting

$$\tilde{\pi}_{\lambda, \sigma}^{t, \text{comp}}(\gamma) f = [k \mapsto \exp \langle i\lambda + t\rho, \mathbf{a}_t(\gamma^{-1}k) \rangle \sigma_t(\mathbf{m}_t(\gamma^{-1}k)) f(\kappa_t(\gamma^{-1}k))],$$

for each γ in G_t and each f in \mathbf{E} .

Then the linear operator defined on all of \mathbf{E} obtained by setting

$$\pi^t = \tilde{\pi}_{\lambda, \mu}^{t, \text{comp}} \circ \alpha_t.$$

extends $\tilde{\pi}^t$.

Theorem 7.1. *For each g_0 in G_0 and each $f \in \mathbf{E}$, there is a limit (in \mathbf{E}) to $\pi_t(g_0) f_t$ as t goes to zero ; the limit is $\pi_0(g_0) f_0$.*

Proof. This again extends Theorem 4.1, and the work done since section 5 is enough to have the same strategy work. If $g_0 = (k, v)$, we want to compare

$$\left[\pi_{\lambda, \sigma}^{t, \text{comp}}(\exp_{G_t}(v)k) f \right] (u) = [u \mapsto \exp \langle -i\lambda - t\rho, \mathbf{a}_t((\exp_{G_t}(v)k)^{-1}u) \rangle \sigma_t(\mathbf{m}_t(\exp_{G_t}(v)k)^{-1}u)) f(\kappa_t(\exp_{G_t}(v)k)^{-1}u)]$$

with

$$\pi_0(g_0)f = \left[u \mapsto e^{i\langle \chi, Ad(k^{-1})v \rangle} f(k^{-1}u) \right].$$

We first rearrange $\left[\pi_{\lambda, \sigma}^{t, comp}(\exp_{G_t}(v)k)f \right]$ as

$$u \mapsto \exp \langle i\lambda + t\rho, \mathbf{a}_t \left[k^{-1}u \exp_{G_t}(-Ad(u^{-1})v) \right] \rangle \sigma_t \left(\mathbf{m}_t \left[k^{-1}u \exp_{G_t}(-Ad(u^{-1})v) \right] \right)^{-1} f \left(\kappa_t(k^{-1}u \exp_{G_t}(-Ad(u^{-1})v)) \right),$$

and imitate the notation in the proof of Theorem 4.1 by setting $\mathfrak{J}_t = \mathbf{a}_t \circ \exp_{G_t}$,
 $\mathfrak{K}_t = \kappa_t \circ \exp_{G_t}$.

Then of course $\kappa_t(k^{-1}u \exp_{G_t}(-Ad(u^{-1})v)) = k^{-1}u \mathfrak{K}_t[-Ad(u^{-1})v]$, and

$$\left[\pi_{\lambda, \sigma}^{t, comp}(k \exp_{G_t} v) f \right] = u \mapsto \exp \langle -i\lambda - t\rho, \mathfrak{J}_t(-Ad(u^{-1})v) \rangle \sigma_t \left(\mathbf{m}_t \left[k^{-1}u \exp_{G_t}(-Ad(u^{-1})v) \right] \right) f \left(k^{-1}u \mathfrak{K}_t[-Ad(u^{-1})v] \right).$$

But

$$\mathbf{m}_t \left[k^{-1}u \exp_{G_t}(-Ad(u^{-1})v) \right] = \mathbf{m}_t \left[\exp_{G_t}(-Ad(u^{-1})v) \right].$$

So we can rewrite $\sigma_t(\mathbf{m}_t(\exp_{G_t}(v)k)^{-1}u)$ as

$$\sigma_t \circ \alpha_t \left[1, \mathfrak{M}_t(-Ad(u^{-1})v) \right]$$

with

$$\mathfrak{M}_t = \beta \mapsto \log_{G_t} \left(\mathbf{m}_t(\exp_{G_t}(\beta)) \right),$$

a map from \mathfrak{p} to $\mathfrak{m} \cap \mathfrak{p}$. Now, a little playing around with φ_t as in lemma 4.3 shows that

$$\mathfrak{M}_t(\beta) = \frac{1}{t} \log_G \left(\mathbf{m}(\exp_G(-t\beta)) \right).$$

Just as in lemma 4.3, this means that $\mathfrak{M}_t(\beta)$ goes to the Iwasawa projection of β along the decomposition $\mathfrak{g} = \mathfrak{k} \oplus (\mathfrak{m} \cap \mathfrak{p}) \oplus \mathfrak{a} \oplus \mathfrak{n}$.

Set $\beta = -Ad(u^{-1})v$. Then we just saw that

$$\left[\pi_{\lambda, \sigma}^{t, comp}(k \exp_{G_t} v) f \right] = u \mapsto \underbrace{\exp \langle i\lambda + t\rho, \mathfrak{J}_t(\beta) \rangle}_{(9)} \underbrace{(\sigma_t \circ \alpha_t) [\mathfrak{M}_t(\beta)] f \left(k^{-1}u \mathfrak{K}_t[\beta] \right)}_{(9)}.$$

It is time to take the limit, so suppose f is in \mathbf{E} and the above is applied to $f_t = \mathbf{C}_t f$.

- The first underbraced term of course goes to $e^{i\langle \lambda, Ad(k^{-1})v \rangle}$.

- As for the second underbraced term, because there is a limit to $\mathfrak{M}_t(\beta)$, a straightforward extension of lemma 5.6 shows that there is a distance which defines the topology of \mathbf{E}^μ and with respect to which all of the $(\sigma_t \circ \alpha_t) [\mathfrak{M}_t(\beta)]$ (these are operators on \mathbf{E}^μ) are 1-Lipschitz. Rewrite $(\sigma_t \circ \alpha_t) [\mathfrak{M}_t(\beta)]$ as Σ_t . Then we can rewrite

$$(\sigma_t \circ \alpha_t) [\mathfrak{M}_t(\beta)] f_t \left(k^{-1}u \mathfrak{K}_t[\beta] \right) - f_0(k^{-1}u) \text{ as :}$$

$$\underbrace{[f_t(k^{-1}u) - f_0(k^{-1}u)]}_{\text{Term 1}} + \underbrace{[\Sigma_t - Id_E][f_t(k^{-1}u) - f_0(k^{-1}u)]}_{\text{Term 2}} + \underbrace{\Sigma_t[f_t(k^{-1}u \mathfrak{K}_t[\beta]) - f_t(k^{-1}u)]}_{\text{Term 3}} + \underbrace{[\Sigma_t - Id_E][f_0(k^{-1}u)]}_{\text{Term 4}}.$$

The first term goes to zero because it is defined through Lemma 7.5.

The second term goes to zero because of the Lipschitz remark above.

The third goes to zero because $\mathfrak{K}_t(\beta)$ goes to the identity and f_t is defined through pointwise composition of f with a map, namely \mathbf{C}_t^σ , which is Lipschitz with respect to the chosen distance on \mathbf{E}^μ if it is chosen appropriately¹⁴.

The last term goes to zero because $\mathfrak{M}_t(\beta)$ is in \mathfrak{p} $f_0(k^{-1}u)$ is \mathbf{C}_t^σ -invariant and in the corollary to lemma 5.6, it is clear from the proof that when U is in \mathbf{E}^μ , in addition to the convergence (in \mathbf{E}^μ) of $(\sigma_t \circ \alpha_t[k, v])U_t$ to $\mu(k)U_0$ at g_0 fixed, the convergence of $(k, v, x) \mapsto (\sigma_t \circ \alpha_t[k, v])U_t(x)$ to $\mu(k)U_0(x)$ is uniform on compact subsets of $G_0 \times \mathfrak{p}$.

• All in all, from (9) we see that there is a limit, in \mathbf{E} , to $\left[\pi_{\lambda, \sigma}^{t, comp}(\alpha_t[k, v])f_t\right]$ as t goes to zero, and that this limit is $\pi_0(g_0) = u \mapsto e^{i(\chi, Ad(k^{-1})v)}f(k^{-1}u)$, as promised. This is Theorem 7.1. \square

The following statement, a more detailed version of Theorem 3.4, summarizes the contents of sections 4 to 7 :

Theorem 7.2.

Suppose $\delta = (\chi, \mu)$ is a reasonable Mackey datum, W^μ is an irreducible K_χ -module of class μ , and ν is the cardinal of M/M^\sharp (see section 7.1). Denote by $P_\chi = M_\chi A_\chi N_\chi$ the cuspidal parabolic subgroup constructed in section 3.

Set $\mathbf{E} = C(K, C(\mathfrak{m}_\chi \cap \mathfrak{p}, W^\mu)^\nu)$, equipped with the Fréchet topology described above. When f is an element of \mathbf{E} , define its “value at the origin” as the element of $(W^\mu)^\nu$ obtained by evaluating each component of $f(1_K)$ at the origin of $\mathfrak{m}_\chi \cap \mathfrak{p}$.

Then sections 5 to 7 provide a vector subspace $\mathbf{H} \subset \mathbf{E}$, a map $\pi : G \rightarrow \text{End}(\mathbf{E})$, and for each $t > 0$ a vector subspace $\mathbf{H}_t \subset \mathbf{E}$ and a map $\pi_t : G_t \rightarrow \text{End}(\mathbf{E})$, which have the following properties.

1. Embedding of representations inside the fixed space \mathbf{E} . The vector subspace \mathbf{H} is π -stable, and (\mathbf{H}, π) is a tempered irreducible representation of G with class $\mathbf{M}(\delta)$. The vector subspace \mathbf{H}_t is π_t -stable for each $t > 0$, and (\mathbf{H}_t, π_t) is a tempered irreducible representation of G_t with class $\mathbf{M}_t(\delta)$.

2. Existence of a natural family of contraction operators. For each $t > 0$, there is a unique linear map $\mathbf{C}_t \in \text{End}(\mathbf{E})$ which restricts to a contraction map between (\mathbf{H}, π) and (\mathbf{H}_t, π_t) and preserves the value of functions at the origin. The family $(\mathbf{C}_t)_{t>0}$ is weakly continuous.

3. Convergence of vectors under the contraction. For each $f \in \mathbf{E}$, there is a limit (in \mathbf{E}) to $\mathbf{C}_t f$ as t goes to zero. Define \mathbf{H}_0 as $\left\{ \lim_{t \rightarrow 0} \mathbf{C}_t f \mid f \in \mathbf{H} \right\}$.

¹⁴More precisely, in the proof of Lemma 5.6, choose the compact subsets (A_n) of \mathfrak{p} so that they contain zero and are star domains.

4. Weak convergence of operators. Suppose f_0 is in \mathbf{H}_0 , f is an element of \mathbf{H} such that $\lim_{t \rightarrow 0} \mathbf{C}_t f = f_0$, and set $f_t = \mathbf{C}_t f$. Then for each g_0 in G_0 , there is a limit to $\pi_t(\alpha_t(g_0))f_t$ as t goes to zero, this limit depends only on f_0 , and it belongs to \mathbf{H}_0 . Call it $\pi_0(g_0)f_0$.

5. The limit produces the appropriate representation of G_0 . The representation (\mathbf{H}_0, π_0) of G_0 thus obtained is irreducible unitary, and its equivalence class is $\mathbf{M}_0(\delta)$.

8 Concluding remarks

8.1. The contents of sections 4-7 (especially Theorem 7.2) show how, starting from a Hilbert space for $\mathbf{M}(\delta)$, the contraction process wears away everything but a carrier space for $\mathbf{M}_0(\delta)$. However, when Mackey hoped for a result relating the representation theories of G and G_0 , his aim was more ambitious and he hoped that new results on G could follow :

We feel sure that some such result exists and that a routine if somewhat lengthy investigation will tell us what it is. We also feel that a further study of the apparently rather close relationship between the representation theory of a semisimple Lie group and that of its associated semi-direct product will throw valuable light on the much more difficult semisimple case.

Higson's constructions certainly throw valuable light on the semisimple case, at least in the case of complex groups, since he shows that the structure of the reduced group C^* -algebras is constant along the deformation from G to G_0 , and how an apparently deep fact on the reduced C^* -algebra of G (the Connes-Kasparov isomorphism) follows from this. I leave it to the reader to decide whether the correspondence and contraction process described in these notes throws any light on the semisimple case. But here are some features of the semisimple case which we met on the way :

8.1.1. When realizing a discrete series representation as the space of square-integrable solutions of the Dirac equation for sections of a homogeneous bundle on G/K , lemma 5.5 says that the finite-dimensional subspace carrying the minimal K -type of the representation consists of sections which are entirely determined by their value at the identity coset¹⁵. I think it is an interesting fact that there are solutions of the Dirac equation which are entirely determined by their value at one given point, that this determines the subspace carrying the minimal K -type, and that these sections are enough to determine the whole representation-theoretic structure of the space of solutions through Vogan's theorem. This is nice, especially because while Theorem 5.1 says these approach constant functions as the contraction is performed, to my knowledge no explicit construction is known for the square-integrable harmonic section with a given value at the identity coset.

8.1.2. Suppose G/K is hermitian symmetric. Lemma 2.3 shows how the linear map ϕ_t sends G/K , viewed as the G -adjoint orbit of a distinguished elliptic element λ_0 , to the G_t -adjoint orbit G_t/K of the same element. Since they are coadjoint orbits, both G/K

¹⁵Of course it is not necessary to bring G_0 into the picture to prove lemma 5.5 !

and G_t/K are symplectic manifolds, and because the Kirillov-Kostant-Souriau form on a coadjoint orbit, or any constant multiple of it, is invariant under the coadjoint action, we see that φ_t provides a symplectic diffeomorphism between G/K and G_t/K if we take care to define the symplectic structures so that they coincide at the tangent space at the identity cosets. Now, we saw on Figure 1 how the G_t -adjoint orbit of λ_0 draws closer and closer to the affine space $\lambda_0 + \mathfrak{p}$. This might throw some light on a theorem of McDuff [34] which says that G/K and \mathfrak{p} are diffeomorphic as symplectic manifolds, so that there exist global Darboux coordinates on G/K . The two proofs of this result that I know of [34, 35] use a variation on Moser's homotopy method [36] obtain a deformation between both symplectic forms. I think the geometrical setting in these notes is a rather nice way to understand Deltour's proof.

Suppose we start with the symplectic form ω_t on G_t/K defined from the Kirillov-Kostant-Souriau form on $Ad^*(G_t) \cdot \lambda_0$, and define a symplectic structure Ω_t on \mathfrak{p} as $u_t^* \omega_t$, times what it needs to have $\Omega_t(0)$ coincide with $\Omega_1(0)$. Then just as it was the case for the riemannian metrics in section 5, lemma 2.2 implies that

$$\Omega_t = \frac{1}{t^2} z_t^* \Omega_1$$

and of course Ω_t converges to the constant form Ω_0 on \mathfrak{p} as t goes to zero (as before the convergence holds, say, for the topology of uniform convergence of the coefficients in affine coordinates on \mathfrak{p}).

The family (Ω_t) is the main ingredient in Deltour's proof of McDuff's theorem : although he does not work with G_t , he uses this very family of symplectic forms and an adaptation of Moser's homotopy method to the noncompact setting to prove that there is an isotopy $(\Psi_t)_{t \in [0,1]}$ of \mathfrak{p} , with $\Psi_0 = id_{\mathfrak{p}}$, such that $\Psi_t^* \Omega_t = \Omega_0$. Very natural indeed if one brings G_t into the picture !

8.2 Here is a list of questions which should get reasonable answers after some work, and to which I hope to come back in the near future :

8.2.1. When an irreducible tempered representation of G with real infinitesimal character is neither in the discrete series nor a limit of discrete series, does it have a realization as a space of functions, or sections of a homogeneous bundle, on G/K ? This would be a natural follow-up on these notes and would permit to extend Theorem 3.4 to the whole tempered dual.

8.2.2. Suppose $\delta = (\chi, \mu)$ is a Mackey datum and $\mathbf{V}_{M_\chi}(\mu)$ belongs to the discrete series of M_χ . Then J. A. Wolf wrote down in [40] the details for a realization of $\mathbf{M}(\delta)$ as a space of sections of a bundle on $U_\chi := G/(K_\chi A_\chi N_\chi)$ which are square-integrable on each fiber of the natural projection $U_\chi \rightarrow K/K_\chi$, and satisfy a partial differential equation gathering the Dirac equations on each of these fibers. It is natural to expect that an easy adaptation of the methods in Section 5 to this realization will lead to another setting for Theorem 3.4 concerning $\mathbf{M}(\delta)$. Does everything go through without any pain ?

8.2.3. The next one was asked by Mackey : how are the (global, distribution) characters of $\mathbf{M}_0(\delta)$ and \mathbf{M}_δ related ? Since the character of an irreducible representation depend

only on its isomorphism class and not on its possible geometric realizations, sections 4 to 7 are not going to be of much help here. A possible direction for understanding this is Kirillov's character formula, which roughly relates the global character to the Euclidean Fourier transform of the Dirac distribution on a coadjoint orbit of G . Rossman proved that the distribution characters of (generic) tempered irreducible representations of G can indeed be exhibited in this way. To my knowledge (which might be faulty on that point), this has not been done for G_0 (for a study of the coadjoint orbits of G_0 , see [37], however).

8.2.4. What is the relationship between the Plancherel measure of \widehat{G} and that of \widehat{G}_0 ? This question would call for Harish-Chandra's full work, so it might prove tricky. A simpler question would be : how are the Plancherel decompositions of $\mathbf{L}^2(G/K)$ and $\mathbf{L}^2(G_0/K)$ related? Since this calls only for the spherical principal series, section 4 might be of some help ; this would amount to answering question 4.3.3 above. In view of chapter 3 in [18], of section 4 above, and of section 3.6 in [21], this calls for a look at Harish-Chandra's c -function and its behaviour as one "goes to infinity in the Weyl chamber".

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