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## A Commentary on Teichmüller's paper "Bestimmung der extremalen quasikonformen Abbildungen bei geschlossenen orientierten Riemannschen Flächen"

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A COMMENTARY ON TEICHMÜLLER'S PAPER  
*BESTIMMUNG DER EXTREMALEN  
QUASIKONFORMEN ABBILDUNGEN BEI  
GESCHLOSSENEN ORIENTIERTEN  
RIEMANNSCHEN FLÄCHEN*

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ABSTRACT. This is a mathematical commentary on Teichmüller's paper *Bestimmung der extremalen quasikonformen Abbildungen bei geschlossenen orientierten Riemannschen Flächen* (Determination of extremal quasiconformal maps of closed oriented Riemann surfaces) [20], (1943). This paper is among the last (and may be the last one) that Teichmüller wrote on the theory of moduli. It contains the proof of the so-called Teichmüller existence theorem for a closed surface of genus  $g \geq 2$ . For this proof, the author defines a mapping between a space of equivalence classes of marked Riemann surfaces (the Teichmüller space) and a space of equivalence classes of certain Fuchsian groups (the so-called Fricke space).<sup>1</sup> After that, he defines a map between the latter and the Euclidean space of dimension  $6g - 6$ . Using Brouwer's theorem of invariance of domain, he shows that this map is a homeomorphism. This involves in particular a careful definition of the topologies of Fricke space, the computation of its dimension, and comparison results between hyperbolic distance and quasiconformal dilatation. The use of the invariance of domain theorem is in the spirit of Poincaré and Klein's use of the so-called "continuity principle" in their attempts to prove the uniformization theorem.

CONTENTS

1. Introduction	2
2. Teichmüller's paper	5
3. Some other approaches	12
References	13

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## 1. INTRODUCTION

We comment on the paper [20] by Oswald Teichmüller. In his article, the author proves a result which Ahlfors called the “existence theorem for extremal quasiconformal mappings,” which is also known as “Teichmüller’s existence theorem” for arbitrary closed orientable surfaces of genus  $g \geq 2$ , and a related result which says that the Teichmüller space of a closed surface of genus  $g$  is homeomorphic to  $\mathbb{R}^{6g-6}$ . (In fact, the two theorems are consequences of each other, but in Teichmüller’s paper, the former is obtained as a consequence of the latter.) The article is a sequel to [24] in which this result was presented as a “conjecture,” valid for a general surface of finite type. In the present paper, the author says that he restricts to the case of closed surfaces “in order to make the essential points clear,” and that “the more general statement shall be proved in a later article.”

The paper [20] was published in 1943, and it is among the last ones that Teichmüller wrote. At the end of this paper, the author refers to the results of the paper [27], which appeared in 1944 (the year after Teichmüller’s death)<sup>2</sup> in the ephemeral journal *Deutsche Mathematik*<sup>3</sup> together with his three other papers [21], [22] and [23].<sup>4</sup>

Concerning the article which is our subject here, let us quote Ahlfors, from the comments he made in his *Collected Works* (vol. II p. 1), concerning his paper [4]. He writes about Teichmüller’s existence theorem: “I found this proof rather hard to read” but “I did not doubt its validity.” He also explains, concerning his own proof of that theorem: “My attempted proof on these lines had a flaw, and even my subsequent correction does not convince me today.” He refers to Bers, and he says that the latter gave in [8] “a very clear version of Teichmüller’s proof.”

In [8] Bers writes : “Our arrangement of the arguments preserves the logical structures of Teichmüller’s proof; the details are carried out differently. More precisely, we work with the most general definition of quasiconformality, we rely on the theory of partial differential equations in some crucial parts of the argument, and we make use of a simple set of moduli for marked Riemann surfaces.” In fact, in the paper [8], written 17 years after the one of Teichmüller, Bers uses results which a priori were not known to Teichmüller, in particular a result of Morrey [18], which is a weak version of the so-called *Ahlfors-Bers*

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<sup>2</sup>Teichmüller was drafted in the German army in July 1939, and never came back to university. He was killed at the Eastern Front in 1943. From 1939 until his death, he continued working on mathematics, but he could write only during his spare time. In his paper [26] (see also the commentary [3]), Teichmüller writes : “Because I only have a limited vacation time at my disposal, I cannot give reasons for many things, but only assert.”

<sup>3</sup>This journal was founded by L. Bieberbach, and it was published between 1936 and 1944.

<sup>4</sup>Presumably, the last published paper by Teichmüller is [25].

theorem, also known as the *measurable Riemann mapping theorem*. However, in another paper, [9], written 16 years after [8] (and which is among the very last papers that Bers wrote),<sup>5</sup> Bers insists on the fact that all the arguments given by Teichmüller in his paper [20] are correct. The paper [9] is considered by Bers as a postscript to his paper [8]. In the abstract of [9], Bers says that he gives a “simplified version of Teichmüller’s proof (independent of the theory of Beltrami equations with measurable coefficients)<sup>6</sup> of a proposition underlying his continuity argument for the existence part of his theorem on extremal quasiconformal mappings.” He then writes in the introduction:

In proving a basic continuity assertion (Lemma 1 in §14C of [8]) I made use of a property of quasiconformal mappings (stated for the first time in [7] and also in §4F of [8]) which belongs to the theory of quasiconformal mappings with bounded measurable Beltrami coefficients (and seems not to have been known to Teichmüller). Some readers concluded that the use of that theory was indispensable for the proof of Teichmüller’s theorem. This is not so, and Teichmüller’s own argument is correct. This argument can be further simplified and this simplified argument will be presented below. Then we will briefly describe Teichmüller’s argument.

Concerning the general presentation of [20], Teichmüller states precisely his results, with relatively clear and detailed proofs (unlike some of his other papers, e.g. [24] and [27], and the commentaries [6] and [1]). However, Ahlfors declares in [4], concerning Teichmüller’s paper [20], that the proof is an “anticlimax”, in comparison with the same author’s paper [24], which is a “brilliant and unconventional paper.” The presentation here is very close to the one in [28]. This is not surprising because in the two papers, Teichmüller uses a continuity argument and applies Brouwer’s theorem of invariance of domain.

It is probably time now to state Teichmüller’s existence theorem, which is the main result of the paper [20]. The theorem says that in any homotopy (or isotopy) class of homeomorphisms between two closed Riemann surfaces, there exists a quasiconformal homeomorphism which realizes the infimum of the quasiconformal dilatation in the given homotopy class. The uniqueness of such a homeomorphism and a precise description of it in terms of quadratic differentials were already proved in [24].

Let us highlight some of the main ideas contained in the paper [20], besides the proof of this existence theorem:

- (1) The description of the topologies of various spaces of equivalence classes of Fuchsian groups, and in particular of a space

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<sup>5</sup>This paper appears as the antepenultimate in Bers’ *Selected works* edition.

<sup>6</sup>Bers means Morrey’s theory.

which is called today Fricke space, and the computation of their dimension.

- (2) The fact that Teichmüller space is homeomorphic to the Fricke space and that the two spaces are homeomorphic to  $\mathbb{R}^{6g-6}$ .
- (3) A correct use of Brouwer's theorem of invariance of domain in the setting of Riemann surfaces.<sup>7</sup>
- (4) Estimates on the transformation of lengths of curves under a  $K$ -quasiconformal mapping between hyperbolic surfaces. These estimates are used in the proof of the continuity of the map that is involved in Brouwer's theorem of invariance of domain.

The structure of Teichmüller's paper is very clear, with the following sections:

- "Introduction," where the results are stated, the main result being a proof of the Teichmüller existence theorem.
- "Topological determination and uniformization of the surface  $\mathfrak{M}$ ," where the author recalls the notion of *canonical dissection* of a Riemann surface.<sup>8</sup> Using this notion and the uniformization theorem, he establishes a one-to-one correspondence between the space of equivalence classes of marked Riemann surfaces and a space of conjugacy classes of Fuchsian groups uniformizing a closed Riemann surface.
- " $\mathfrak{G}$  as a linear group," where several spaces of equivalence classes of Fuchsian groups are introduced and shown to be topological manifolds.
- "The extremal quasiconformal mappings," where the notion of extremal quasiconformal mapping (in modern terms, *Teichmüller mapping*) and the relation with quadratic differentials are recalled from the paper [24]. Using these notions, Teichmüller reformulates the main result of the paper [24].
- "The fundamental continuity proof," where several fundamental results that involve at the same time quasiconformal geometry and hyperbolic geometry are obtained. The results concern the continuity of a map between the Fricke space and a Euclidean space. These continuity results are essential for the main result of the paper, namely, Teichmüller's existence theorem.
- "Remark," where a fundamental inequality on the effect of hyperbolic length under a quasiconformal mapping is proved. The result complements the continuity result of the previous section.

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<sup>7</sup>The idea of this method goes back to Poincaré and Klein, who called this the "method of continuity." However, these authors, in using this method, bumped over several difficulties, because they were applying it to Riemann's moduli space, which is not a manifold. Furthermore, a rigorous definition of the notion of dimension was missing. It was Brouwer who proved the needed result with all the needed rigour. For a survey see [2].

<sup>8</sup>This notion was already considered by Poincaré; cf. the historical report [2].

It is a consequence of the proofs in that section. The result shows the continuity of the map from Euclidean space to Fricke space at the origin, and it is also used in the proof that Fricke space is arcwise connected.

- “Construction of a sufficiently regular mapping from  $\mathfrak{M}$  to  $\mathfrak{M}'$ ,” which concerns mappings between marked Riemann surfaces. The author proves that in each homotopy class of homeomorphism between two Riemann surfaces there exists a quasiconformal map.<sup>9</sup>
- “The continuity proof,” which contains the conclusion of the fact that the space denoted by  $\mathfrak{E}$  is homeomorphic to  $\mathbb{R}^{6g-6}$ , based on the so-called “continuity method,” or, Brouwer’s theorem of invariance of domain.
- “Connection to the theory of moduli,” which is a concluding section, where the author makes the connection with the moduli problem as stated by Riemann. The author proves that  $\mathfrak{R}$  is homeomorphic to  $\mathbb{R}^{6g-6}$ , and he mentions relations with his other works on the subject and directions for further investigations.

Throughout this commentary, we shall keep the logical trend of Teichmüller’s paper and state some of the important results with the author’s original wording.

## 2. TEICHMÜLLER’S PAPER

The introduction of the paper is brief and explicit. Teichmüller starts by saying that he is “glad now to be able to actually prove” the existence of an extremal quasiconformal mapping for a given problem (which will be recalled later). This result had the status of a conjecture in the previous paper [24]. In the present paper, the author declares that the theorem will only be proved for closed orientable surfaces of genus  $g > 1$  and, unlike what he did in the paper [24], he will use several times the uniformization theorem. He also recalls that the case  $g = 1$  was completely treated in [24] (§ 25 to § 29). Teichmüller declares that the proof will be based on a “continuity argument.” By this, he means that he will exhibit a mapping between  $\mathbb{R}^{6g-6}$  and another space (in fact, the space which is called today *Teichmüller space*, denoted here by  $\mathfrak{R}$ , as in the paper [27])<sup>10</sup> and he will show that this mapping is injective and continuous. He will then appeal to Brouwer’s *invariance of domain* theorem to conclude that this mapping is a homeomorphism. Teichmüller says that he considers only closed orientable

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<sup>9</sup>Teichmüller assumes, in this papers and others, that the quasiconformal homeomorphisms considered are “sufficiently regular.” In the present construction, they are real-analytic except at a finite set of arcs which are the sides of a triangulation parametrized by real-analytic arcs.

<sup>10</sup>In the paper [24], this paper is denoted by  $R^\sigma$ , with  $\sigma = 6g - 6$ .

surfaces in order to avoid a technical topological theorem that concerns the homotopy equivalence relation between maps between surfaces. He says that to prove that theorem in the more general case would take an unproportional amount of space and distract the reader from the main point of the paper, which is the “continuity argument.” In fact, the technical part would be to show, in the general case, that there exists a “sufficiently regular mapping” in each homotopy class. This is harder for surfaces with distinguished points. He nevertheless promises to treat in the near future the general case, that is, the case of surfaces which are orientable or not, with or without boundary and with or without distinguished points. Let us also note that the proof of the existence theorem in the case of the torus that we mentioned above ( $g = 1$ ) does not use the method of continuity, but that Teichmüller gave a proof of his existence theorem, using the method of continuity, in the case where the surface is a pentagon, that is, a disc with five distinguished points on the boundary, cf. [28]. In the last section of the paper [20], he says: “During my research, I have always kept in mind the aim to give a continuity proof for the existence of quasiconformal mappings similar to the one in the case of the pentagon.”

The aim of the section called “Topological determination and uniformization of the surface  $\mathfrak{M}$ ” is to give several equivalent definitions of the object that will be denoted by  $\underline{\mathfrak{M}}$  in the rest of the paper. The author fixes a base surface  $\mathfrak{M}_0$  and defines an equivalence relation on the set of pairs  $(\mathfrak{M}, H)$  where  $\mathfrak{M}$  is a Riemann surface which has the same topological type as  $\mathfrak{M}_0$  and  $H : \mathfrak{M}_0 \rightarrow \mathfrak{M}$  an orientation-preserving homeomorphism. The equivalence relation is such that  $(\mathfrak{M}, H)$  is equivalent to  $(\mathfrak{M}', H')$  if  $\mathfrak{M} = \mathfrak{M}'$  and  $H' \circ H^{-1}$  is homotopic to the identity.<sup>11</sup> He calls such an equivalence class a *topologically determined Riemann surface*.<sup>12</sup> Teichmüller denotes by  $\underline{\mathfrak{M}}$  the equivalence class of pairs  $(\mathfrak{M}, H)$ . He then introduces the natural notions of topological and of conformal maps between two topologically determined Riemann surfaces. Using this language, the Teichmüller space of  $\mathfrak{M}_0$  is then the set  $\underline{\mathfrak{M}}$  of “topologically determined Riemann surfaces up to conformal maps.”

Teichmüller then recalls the notion of *canonical dissection* of the surface  $\mathfrak{M}_0$ . This is a collection of  $2g$  simple loops on this surface,  $A_1, \dots, A_g, B_1, \dots, B_g$ , with the same base point and disjoint except at this base point, and satisfying, as elements of the fundamental group

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<sup>11</sup>Teichmüller uses *isotopy*, and he notes that by a result of Mangler (cf. [17]), isotopy is equivalent to homotopy in the setting he considers.

<sup>12</sup>Interpreting the equality  $\mathfrak{M} = \mathfrak{M}'$  as meaning that  $\mathfrak{M}$  is conformally equivalent to  $\mathfrak{M}'$  by a map  $f : \mathfrak{M} \rightarrow \mathfrak{M}'$  and the relation “ $H' \circ H^{-1}$  is homotopic to the identity” as saying that the map  $f \circ H$  is homotopic to  $H'$ , we get the definition of Teichmüller space to which we are used today.

of the surface, the relation

$$(1) \quad \prod_{i=1}^g A_i B_i A_i^{-1} B_i^{-1} = 1,$$

where 1 denotes the identity element of the group. The term *dissection* is used because when the surface is cut along the union of these curves, we get a polygon with  $4g$  sides. In particular, the set of homotopy classes of loops  $A_1, \dots, A_g, B_1, \dots, B_g$  is a set of generators for the fundamental group of the surface  $\mathfrak{M}_0$ . A dissection of  $\mathfrak{M}_0$  induces a dissection of any topologically determined Riemann surface  $(\mathfrak{M}, H)$ , via the topological map  $H$ , which is well defined up to an inner automorphism of the fundamental group of  $\mathfrak{M}$ . Conversely, a dissection on  $\mathfrak{M}$  defines a topological determination  $H$  of that surface, that is, a topologically marked surface  $(\mathfrak{M}, H)$ . Teichmüller appeals to a theorem of Mangler, contained in the same paper [17], for the fact that a topologically determined surface defines a surface with a marking of its fundamental group, up to an inner automorphism.<sup>13</sup>

Using the uniformization theorem, Teichmüller identifies the universal cover of the Riemann surface  $\mathfrak{M}$  with upper half-plane  $\mathbb{H}$  and the fundamental group of  $\mathfrak{M}$  to a discrete subgroup of  $\mathrm{PSL}(2, \mathbb{R})$  of orientation-preserving isometries of  $\mathbb{H}$ . With this, he obtains the following (we use his own words, in the translation of [20]):

*We have a one-to-one correspondence between, on the one hand, the topologically determined Riemann surfaces  $\mathfrak{M}$  that are given only up to conformal mappings and, on the other hand, certain collections of linear mappings  $A_1, \dots, A_g, B_1, \dots, B_g$  of  $\mathbb{H}$  onto itself that are determined only up to a common transformation with a linear mapping of  $\mathbb{H}$  onto itself.*

The term “certain collections” refers to a property satisfied by the elements  $\{A_i, B_i\}_{1 \leq i \leq g}$ , namely, that they are different from the identity and are of hyperbolic type, that they satisfy the relation (1) when the elements are considered as sitting in  $\mathrm{PSL}(2, \mathbb{R})$ , that they generate a discrete subgroup (denoted by  $\mathfrak{G}$ ) which acts properly discontinuously on  $\mathbb{H}$  and such that the quotient is a closed surface of genus  $g$ . Such a collection of elements is called an “admissible collection” and the set of all admissible collections forms a space which will be denoted by  $\mathfrak{E}$ . Teichmüller quotes another result of Mangler [17] which implies that the natural map between the space of topologically determined surfaces up to conformal maps (that is, Teichmüller space) and Fricke space is surjective.

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<sup>13</sup>This is related to the fact that the group of homotopy classes of homeomorphisms (that is, the mapping class group) of the closed surface is in natural one-to-one correspondence with the outer automorphism group of the fundamental group, a result attributed to Dehn and Nielsen.



The third section, titled “ $\mathfrak{G}$  as a linear group”, is about topology. Teichmüller introduces several spaces of equivalence classes of matrices and he equips them with topologies, showing that these spaces are topological manifolds and computing their dimension. This will be used in the “method of continuity”, or “invariance of domain,” which is a crucial ingredient in the proof of the main result of the paper.

Teichmüller introduces a space  $\mathfrak{E}$  equipped with a topology induced by its inclusion in a  $6g - 6$ -dimensional topological manifold  $\mathfrak{D}$ . This space  $\mathfrak{E}$  is called today the *Fricke space*. An important fact which is shown in this section is that for every  $\mathfrak{t}$  in  $\mathfrak{E}$ , there exists an element  $\mathfrak{G}$  in  $\mathfrak{E}$  generated by a collection of elements  $A_1, A_2, B_1, B_2, \dots, A_g, A_g, B_g, B_g$  such that a finite number of invariants  $\lambda$  of these elements and of certain of their products determine  $\mathfrak{t}$  uniquely and in a continuous manner. Let us see this in some detail.

Teichmüller recalls that to every hyperbolic element  $A$  of  $\mathrm{PSL}(2, \mathbb{R})$  one can associate an attractive fixed point, a repulsive fixed point and an invariant, which is the real number  $\lambda > 1$  such that  $A$  is conjugate to the transformation  $z \mapsto \lambda z$  of the upper half-plane.<sup>14</sup> He briefly recalls the notion of an  $n$ -dimensional topological manifold and he introduces three sets,  $\mathfrak{A}$ ,  $\mathfrak{B}$  and  $\mathfrak{C}$ .

The set  $\mathfrak{A}$  is the set of all  $6g$ -tuples of elements of  $\mathrm{PSL}(2, \mathbb{R})$ . This is a  $6g$ -dimensional manifold. The set  $\mathfrak{B}$  is the subset of  $6g$ -tuples of  $\mathfrak{A}$  that satisfy the relation (1) and such that the first two elements,  $A_1$  and  $B_1$  are hyperbolic with four distinct fixed points. The set  $\mathfrak{C}$  is the set of admissible collections  $A_1, \dots, A_g, B_1, \dots, B_g$  of elements  $\mathrm{PSL}(2, \mathbb{R})$ .

Applying the *implicit function theorem* to

$$D = \prod_{i=1}^g A_i B_i A_i^{-1} B_i^{-1},$$

seen as a map  $\mathfrak{A} \rightarrow \mathrm{PSL}(2, \mathbb{R})$ , Teichmüller shows that  $\mathfrak{B}$  is a  $6g - 3$ -dimensional manifold. He then defines the spaces  $\mathfrak{D}$  and  $\mathfrak{E}$  which are the quotients of  $\mathfrak{B}$  and  $\mathfrak{C}$  respectively by the natural actions on them of the group  $\mathrm{PSL}(2, \mathbb{R})$  by conjugation.

He concludes with the following:

*The set  $\mathfrak{X}$  of all classes of conformally equivalent topologically determined Riemann surfaces of genus  $g$  is in one-to-one correspondence with  $\mathfrak{E}$ .*

The first set that is considered here is what became known later on as Teichmüller space and the second space is Fricke space. Let us note that it is at this stage that Teichmüller introduces the notation  $\mathfrak{X}$ .

Teichmüller equips  $\mathfrak{D}$  with the topology induced by  $\mathfrak{B}$  and he shows that this topology is Hausdorff, after introducing several technical lemmas. He then shows that this space is a  $(6g - 6)$ -topological manifold

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<sup>14</sup>The value of  $\lambda$  is the exponential of the hyperbolic length of the simple closed curve associated to  $A$ .

and he announces the result saying that  $\mathfrak{E}$  is an open connected subset of  $\mathfrak{D}$  which is homeomorphic to  $\mathbb{R}^{6g-6}$ . He says that this statement “was not known previously.” He ends this section by showing that the elements of  $\mathfrak{E}$  depend continuously on a finite number of invariants. Let us state his result in his own terms:

*There is a finite set of a priori listable expressions in the  $A_i$ ,  $B_i$ , as e.g.  $A_1$ ,  $A_1B_1$ ,  $(A_1B_1)^2B_2$ , etc. with the following property: The class  $\mathfrak{k}$  of the collection  $\mathfrak{a}$  is uniquely determined by the invariants  $\lambda_k$  of those finitely many hyperbolic mappings, and for every neighborhood  $\mathfrak{U}$  of  $\mathfrak{k}$  there is a  $\delta > 0$  such that the classes  $\mathfrak{k}'$  of all collections  $\mathfrak{a}'$  whose invariants  $\lambda'_k$  satisfy the inequalities  $(\lambda'_k - \lambda_k) < \delta$ , lie in the neighborhood  $\mathfrak{U}$ .*

Here,  $\mathfrak{k} \in \mathfrak{E}$  and  $\mathfrak{a} \in \mathfrak{C}$ .<sup>15</sup>

The next section is called “The extremal quasiconformal mappings.” The author recalls a construction he presented in detail in [24] of what is called today the *Teichmüller maps*, i.e. the maps referred to in the title of this section. Such a map is determined by a holomorphic quadratic differential (which the author calls an “*everywhere finite quadratic differential*”)<sup>16</sup>  $d\zeta^2$  on  $\mathfrak{M}_0$  and a real number  $K \geq 1$ . We denote such a map  $E(K, d\zeta^2)$ .<sup>17</sup> Away from the zeros of  $d\zeta^2$ , a Teichmüller map is affine with respect to *natural coordinates* of  $d\zeta^2$ , with constant “dilatation quotient”<sup>18</sup> equal to  $K$ . Teichmüller also recalls the fact that such a map is extremal and unique in its homotopy class. We note by the way that unlike the existence theorem, the proofs of the uniqueness theorem that were given in the first three decades after Teichmüller’s work are modeled on those that he gave in ([24] Chapter 27).<sup>19</sup> Such a map naturally defines a new element of  $\mathfrak{R}$  denoted by  $\mathfrak{M}(K, d\zeta^2)$ . This element depends on  $d\zeta^2$  “up to a common *positive*

<sup>15</sup>In his proof, Teichmüller uses a representative of  $\mathfrak{k}$  in  $\mathfrak{C}$  which is sometimes called *normalized Fuchsian model* (cf. [14] p. 47). Let us also note that, in his proof, Teichmüller shows that “ $\mathfrak{k}$  is uniquely and continuously determined by” precisely  $24g - 20$  invariants, whereas in [14] (Proposition 6.17), there is a similar result with only  $14g - 11$  invariants. It seems obvious to the authors of the present report that Teichmüller was aiming directly to the proof of the main theorem, and did not try to obtain the best estimates for the intermediate results.

<sup>16</sup>The word “finite” originates in the fact that the area of the surface, for the singular flat metric associated to the quadratic differential, is finite. This is a condition on the poles. It is true that in the present paper of Teichmüller, all quadratic differentials considered are holomorphic and therefore the area is always finite, but the terminology originates in his paper [24], and in that paper, the quadratic differentials are allowed to have poles at distinguished points.

<sup>17</sup>Teichmüller uses a slightly different notation, by taking first a basis for the vector space of quadratic differentials.

<sup>18</sup>In the literature, this quantity is usually called *maximal dilatation*.

<sup>19</sup>We can quote here Ahlfors from [4], talking about his own paper: “[...] a complete proof of the uniqueness part of Teichmüller’s theorem was included. Like

factor.” In other words, a positive constant multiple of  $d\zeta^2$ , together with the same  $K$ , define the same extremal quasiconformal map. Using the bijective correspondence that he obtained between  $\mathfrak{R}$  and  $\mathfrak{E}$ , Teichmüller defines a mapping between  $\mathbb{R}^{6g-6}$  and  $\mathfrak{E}$ . The element of  $\mathfrak{E}$  which corresponds to  $\underline{\mathfrak{M}}(K, d\zeta^2)$  is denoted by  $\mathfrak{e}(K, d\zeta^2)$ . Teichmüller concludes this section by announcing the following key result:

$$\mathfrak{e}(K, d\zeta^2) \text{ depends continuously on } (K, d\zeta^2).$$

He stresses the fact that continuity makes sense because a topology has been defined on  $\mathfrak{D}$  and therefore on  $\mathfrak{E}$ .

In the next section, called “The fundamental continuity proof,” Teichmüller proves the result we just mentioned. Even though the details are technical, the idea is rather clear. Indeed, according to the section “ $\mathfrak{G}$  as a linear group”, we only need to show that the invariants corresponding to the group elements that determine the point  $\mathfrak{e}(K, d\zeta^2)$  depend continuously on  $K$  and  $d\zeta^2$ . For this, the author considers two points of  $\mathfrak{R}$ ;  $\underline{\mathfrak{M}}(K, d\zeta^2)$  and  $\underline{\mathfrak{M}}(\tilde{K}, d\tilde{\zeta}^2)$ , determined respectively by the Teichmüller maps  $E(K, d\zeta^2)$  and  $E(\tilde{K}, d\tilde{\zeta}^2)$  such that  $\tilde{K}$  is close to  $K$  and  $d\tilde{\zeta}^2$  is close to  $d\zeta^2$ . He then lifts the composition of the two Teichmüller maps  $E(\tilde{K}, d\tilde{\zeta}^2)E(K, d\zeta^2)^{-1}$  to a quasiconformal mapping of  $\mathbb{H}$  which conjugates the normalized Fuchsian group  $\mathfrak{G}$  to a normalized Fuchsian group  $\tilde{\mathfrak{G}}$ . Let us recall that these two groups determine respectively  $\underline{\mathfrak{M}}(K, d\zeta^2)$  and  $\underline{\mathfrak{M}}(\tilde{K}, d\tilde{\zeta}^2)$ . He concludes using a notion of *average of the local dilatation*<sup>20</sup> that the invariants that characterize  $\underline{\mathfrak{M}}(\tilde{K}, d\tilde{\zeta}^2)$  are close to the invariants characterizing  $\underline{\mathfrak{M}}(K, d\zeta^2)$ . It is interesting to observe that Teichmüller uses, for the needs of the average of the local dilatation, an idea already present in the paper [24], § 35, which permits, among other things, to compare hyperbolic length with quasiconformal dilatation and which leads to the so-called Wolpert inequality (cf. [29]).

In the section titled “Remark,” Teichmüller recalls the idea of the proof of the result of the preceding section and he states a consequence of this proof which will be useful in particular to show that  $\mathfrak{E}$  is arcwise connected. This result will also be useful to show the continuity at the origin of the map between  $\mathbb{R}^{6g-6}$  and the Fricke space, which we already alluded to. Let us state this idea explicitly.

*Let  $\mathfrak{M}'$  be a topologically determined Riemann surface of genus  $g > 1$ , corresponding to the point  $\mathfrak{e}' \in \mathfrak{E}$ . Then for every neighborhood  $\mathfrak{U}$  of  $\mathfrak{e}'$  in  $\mathfrak{D}$  there exists a  $\delta > 0$  with*

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all other known proofs of the uniqueness it was modeled on Teichmüller’s own proof [...].”

<sup>20</sup>This notion is contained in Teichmüller’s paper, and the name is given to it by Bers in [9].

*the following property: If a quasiconformal mapping  $A$  from  $\mathfrak{M}'$  onto a second topologically determined surface  $\mathfrak{M}$  has a dilatation quotient  $D$  satisfying  $D \leq 1 + \delta$  everywhere, then the point  $\tilde{\mathfrak{e}}$  of  $\mathfrak{E}$  belonging to  $\mathfrak{M}$  lies in  $\mathfrak{U}$ .*

Let us only note that to prove this fact it is not necessary to consider what we called the average of the local dilatation.

The next section is called “Construction of a sufficiently regular mapping from  $\mathfrak{M}$  to  $\mathfrak{M}'$ .” Let us explain the term “sufficiently regular.” Teichmüller starts by defining the notion of a “*sufficiently regular triangulation*” of a Riemann surface  $\mathfrak{M}$ , as a triangulation whose vertices are parametrized by maps which are real analytic except at the vertices. Then, a map between  $\mathfrak{M}$  and  $\mathfrak{M}'$  is said to be sufficiently regular if it is a homeomorphism which transforms a sufficiently regular triangulation of  $\mathfrak{M}$  into a sufficiently regular triangulation of  $\mathfrak{M}'$  and which is real analytic in the interior of the triangles. Furthermore, he imposes that the dilatation quotient is bounded at the vertices.<sup>21</sup>

In the section called “The continuity proof,” Teichmüller recalls and describes in detail the continuous and injective map between  $\mathbb{R}^{6g-6}$  and  $\mathfrak{E}$ . He denotes the image by  $\mathfrak{E}^*$ . Given that this set is included in a  $6g - 6$ -dimensional manifold, he uses Brouwer’s theorem of invariance of domain in order to conclude that the map is a homeomorphism. This proves in particular that the set  $\mathfrak{E}^*$  is open in  $\mathfrak{D}$ . Then, using the existence of a sufficiently regular mapping between two points of  $\mathfrak{E}$ , using the notion of Riemannian metrics on surfaces representing conformal structures which he developed in his paper [24], § 16, he shows that  $\mathfrak{E}$  is arcwise connected. Finally, using the arcwise connectedness of  $\mathfrak{E}$ , he concludes this section by showing that  $\mathfrak{E}^* = \mathfrak{E}$  and therefore that this space is homeomorphic to  $\mathbb{R}^{6g-6}$ .

In the last section, called “Connection to the theory of modules,” Teichmüller establishes relations between the present work and his previous works. He also recalls his motivation, namely, the solution of the so-called *problem of moduli*. This problem originates in Riemann’s work, and it consists in giving a precise meaning to Riemann’s statement that the set of equivalence classes of Riemann surfaces, where the equivalence relation is that of conformal equivalence, has  $3g - 3$

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<sup>21</sup>Regarding the existence of a quasiconformal map between two points of  $\mathfrak{A}$ , we can quote Bers, from his paper [8]: “Had we demanded that the homeomorphism  $f$  be continuously differentiable everywhere, the proof would be somewhat laborious. Since we use a very general definition of quasiconformality, the proof [that there exists a quasiconformal homeomorphism between two arbitrary marked Riemann surfaces] presents no difficulties and may be omitted.” We note that Teichmüller considered maps which are differentiable except may be on a finite set of arcs, and that Bers’ remark holds also in this more general case.

complex moduli. More generally, the problem became that of understanding the space of moduli. We refer the reader to the paper [2] for the history of this problem. Teichmüller considers that the *problem of moduli* consists in turning the space  $\mathfrak{M}$  into an analytic manifold. He recalls that he outlined a solution to that problem in his paper [27] (see also the commentary in [1]).

Teichmüller, in this last section, states several results of which he does not give the details, and which make connections between the work done in this paper and his other papers. It is interesting to note that in the paper [27], in which Teichmüller presents a version of Teichmüller space equipped with a complex-analytic structure, he declares that he is not sure that this space (which he also denoted by  $\mathfrak{A}$ )<sup>22</sup> coincides with the Teichmüller space that he studied in his paper [24] and whose study is continued in the paper [20] that we are commenting on here. He writes, in particular, in [27]: “The space  $\mathfrak{M}$  consists of at most countably many connected parts. I believe that  $\mathfrak{M}$  is in fact simply connected.” In the paper [20], there is no more questioning about this fact, and Teichmüller knows that the spaces considered are the same. In particular, the space  $\mathfrak{M}$  is connected and is homeomorphic to a  $(6g - 6)$ -dimensional Euclidean space.

Furthermore, Teichmüller says that the bijective correspondence between  $\mathfrak{M}$  and  $\mathfrak{E}$  that was given at the beginning of the paper is continuous, when we consider on  $\mathfrak{M}$  the topology induced by the *Teichmüller metric*. This completes the statement known today as the *Teichmüller theorem*, namely, that  $\mathfrak{M}$  is homeomorphic to  $\mathbb{R}^{6g-6}$ .

### 3. SOME OTHER APPROACHES

We mention briefly some later (but still old) approaches to Teichmüller’s existence theorem. We already mentioned the works of Ahlfors [4] and Bers [8], [9]. Gerstenhaber and Rauch inaugurated an approach using the theory of minimal surfaces, more precisely, the method of minimizing the so-called energy integral, or Douglas-Dirichlet functional [11] [12]. Hamilton in [13] gave a proof of the existence theorem based on the geometry of Banach spaces (the spaces of integrable Beltrami differentials and the space of all Beltrami differentials) and maps between them. The proof works for general Riemann surfaces with or without distinguished points. The Gerstenhaber-Rauch approach is surveyed in [10], together with other approaches. Krushkal’ also developed a variational proof, cf. [15] and the book [16]. There are several modern proofs and generalizations of Teichmüller’s existence theorem, in particular to infinite-dimensional Teichmüller spaces, and it is not possible to mention them here.

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<sup>22</sup>In [27], Teichmüller space is denoted by  $\mathfrak{A}$ , and, as a space equipped with its complex analytic structure, it is denoted by  $\mathfrak{C}$ .

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