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Global wellposedness of cubic Camassa-Holm equations

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Abstract

In this paper, we study the Cauchy problem of cubic Camassa-Holm equation or so-called Fokas-Olver-Rosenau-Qiao equation. This equation is derived from a model of shallow water dynamics. We prove the global existence of entropy weak solution to this problem in space $H^1 \cap \dot{W}^{2,1}$. The stability and uniqueness of entropy weak solution are obtained in $W^{1,1}$.

1 Introduction

We consider the cubic Camassa-Holm equation (cCH), or so-called Fokas-Olver-Rosenau-Qiao (FORQ) equation:

$$m_t + [(u^2 - u_x^2)m]_x = 0, \quad m = u - u_{xx}, \quad (1.1)$$

or

$$u_t - u_{xxt} + 3u^2u_x - u_x^3 - 4uu_xu_{xx} + 2u_xu_{xx}^2 + (u_x^2 - u^2)u_{xxx} = 0. \quad (1.2)$$

If written as a first-order equation of u , it is

$$u_t + u^2u_x = \frac{1}{3}u_x^3 - \Phi(u), \quad (1.3)$$

with

$$\Phi(u) = (1 - \partial_x^2)^{-1} \left[\frac{1}{3}u_x^3 \right] + \partial_x(1 - \partial_x^2)^{-1} \left[\frac{2}{3}u^3 + uu_x^2 \right]. \quad (1.4)$$

The initial data

$$u|_{t=0} = u_0(x). \quad (1.5)$$

This equation is a complete integrable water wave equation proposed by several different authors [5, 11, 12]. It is a suitable approximation of incompressible Euler system without swirl. For smooth solutions to the equation, we have

$$(u^2)_t + \left(\frac{1}{2}u^4\right)_x = \frac{2}{3}u_x^3u - 2u\Phi(u), \quad (1.6)$$

$$(u_x^2)_t + (u^2u_x^2)_x = 2u_x^3u_{xx} + \frac{4}{3}u^3u_x - 2u_x\partial_x(1 - \partial_x^2)^{-1}\left(\frac{1}{3}u_x^3\right) - 2u_x(1 - \partial_x^2)^{-1}\left[\frac{2}{3}u^3 + uu_x^2\right]. \quad (1.7)$$

So we can check that the following two quantities are conserved:

$$H_1(u) = \int_{\mathbb{R}} (u^2 + u_x^2)dx, \quad H_2(u) = \int_{\mathbb{R}} (u^4 + 2u^2u_x^2 - \frac{1}{3}u_x^4)dx. \quad (1.8)$$

In fact, these two invariant quantities are the Hamiltonian functionals of (cCH), with the compatible hamiltonian operators

$$\mathcal{J}_2 = -\partial_x m \partial_x^{-1} m \partial_x, \quad \mathcal{J}_1 = -\frac{1}{4}(\partial_x - \partial_x^3). \quad (1.9)$$

With these notations, the equation can be written in the bi-Hamiltonian form

$$m_t = \mathcal{J}_2 \frac{\delta H_1}{\delta m} = \mathcal{J}_1 \frac{\delta H_2}{\delta m}. \quad (1.10)$$

In recent decades, the partial differential equations from the fields of integrable systems have been studied adequately. Among them are KdV equation (KdV), modified KdV equations (mKdV), generalized KdV equations (gKdV), Camassa-Holm equations (CH), generalized Camassa-Holm equations (gCH), and so on. (KdV) and (mKdV) are completely integrable. They have Lax pair formulation of the equations, which allows to apply inverse scattering techniques. However, few equations are completely integrable. From analytical point of view, the Hamiltonian quantities usually provide us the functional spaces for the solutions, which is not sufficient to obtain a solution. It usually requires further regularity to guarantee compactness. For example, (gKdV) is a semilinear dispersive equation. In [9], the authors proved global well-posedness by using contract mapping theorem via dispersive estimates.

Camassa-Holm equations are closely related with KdV equations in the sense that Camassa-Holm equations are the tri-Hamiltonian duality of KdV equations [11]. However, unlike KdV equations, Camassa-Holm equations do not have that strong dispersive effect. In fact, they are more like quasilinear hyperbolic equations. They have the wave breaking phenomena which is not shared by the KdV equations. The problem of global well-posedness of Camassa-Holm equation is equivalent to the problem of how to continue the solution after the singularity appears. There are two methods to deal with it in general case. In [13][14], the author proved global existence of weak solution by compensated compactness method, and also proved a result of weak-strong uniqueness. The other way to obtain global well-posedness is by the method of characteristics. In [2] [3], conservative and dispersive weak solutions are constructed, and in [1], the authors proved the uniqueness of conservative weak solutions.

Different from the Camassa-Holm equation, the cubic Camassa-Holm equation has higher nonlinearity. This difficulty is manifested in the equation of characteristics:

$$\frac{dx}{dt} = (u^2 - u_x^2)(t, x(t)). \quad (1.11)$$

From the two conserved Hamiltonians, it is hard to expect the right-hand-side to be continuous with respect to x . So this characteristic ODE may not be wellposed when singularity occurs. The method of characteristics does not apply here.

Here I would like to mention another interesting equation — Novikov equation

$$u_t - u_{xxt} + 4u^2 u_x - 3u u_x u_{xx} - u^2 u_{xxx} = 0. \quad (1.12)$$

This equation shares the same conserved quantity with (cCH) but has lower order of nonlinearity. An advantage of this is reflected on the equation of characteristics:

$$\frac{dx}{dt} = u^2(t, x(t)). \quad (1.13)$$

The right-hand-side is Hölder continuous with respect to x . So the method of characteristics works well here. For details, see [4].

For the cubic Camassa-Holm equation, local well-posedness of strong solutions has been proved in [6] [7] and [8]. In [7], the authors also revealed the blow-up mechanism of (cCH). They proved that $\inf_{x \in \mathbb{R}} \{(mu_x)(t, x)\}$ will approach $-\infty$ in finite time. It means that the first-order and the second-order derivatives of the solution cannot both be bounded all the time. How to continue the solution after the singularity is an open question, which is our main concern in this paper.

Observe that in (cCH), there is a cubic term of u_x . This implies that the (cCH) is more like a Hamilton-Jacobi equation, rather than a perturbed hyperbolic conservation law. Based on this observation, we take 1st order derivative to (cCH), and then view the obtained equation as a balance law for u_x . This suggests us to use the technique in the theory of scalar conservation laws. Fortunately, the source term is under control and we can prove BV estimates of u_x from the conservative form of equation of m , which leads us to use the Kružhikov's theory [10] to prove global wellposedness.

Denote X to be the functional space

$$X := \{f \in H^1(\mathbb{R}) \cap L^1(\mathbb{R}) | f_x \in BV\}, \quad (1.14)$$

with the norm $\|f\|_X = \|f\|_{H^1} + \|f\|_{L^1} + \|f_x\|_{BV}$.

Our main theorem is:

Theorem 1.1 (Well-posedness of entropy weak solution). *Suppose the initial data $u_0 \in X$. There exists an entropy weak solution to the Cauchy problem (1.3), (1.5), where the definition of the entropy weak solution is in Definition 3.2 in section 3.*

The entropy weak solutions have following stability property. Fix any $T > 0$, and let $u, v : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ be two entropy weak solutions to (1.3), (1.5) with initial data $u_0, v_0 \in X$, respectively. Then for any $t \in (0, T)$

$$\|u(t, \cdot) - v(t, \cdot)\|_{W^{1,1}} \leq e^{M_T t} \|u_0 - v_0\|_{W^{1,1}}, \quad (1.15)$$

where M_T is a constant related with T and $H^1 \cap W^{2,1}$ norms of initial data u_0, v_0 . This implies uniqueness.

We have the continuity in time

$$\|u(t_2, \cdot) - u(t_1, \cdot)\|_{W^{1,1}} \leq C_T |t_2 - t_1|, \quad \text{for all } t_1, t_2 \in [0, T], \quad (1.16)$$

for some constant C_T . Consequently, the initial data can be interpreted in the following sense:

$$\lim_{t \rightarrow 0^+} \|u(t, \cdot) - u_0(x)\|_{W^{1,1}} = 0. \quad (1.17)$$

Remark 1.1. *This weak solution includes peakon and anti-peakon solutions with the expression $u(t, x) = ae^{-|x-ct|}$, $a, c \in \mathbb{R}$.*

Remark 1.2. *This theorem partially answered the open question proposed in [7]. For an initial data outside $\dot{W}^{2,1}$, whether we can produce a global weak solution is still open.*

The outline of this paper is following. In section 2, we prove the global well-posedness of approximate problem. In section 3, we prove the existence of entropy weak solutions by a compactness argument. The uniqueness and stability results are proved in section 4.

2 Global wellposedness of the approximate equation

We will prove the existence of solutions by using the viscous approximation. Consider the parabolic equation

$$m_{\varepsilon,t} + [(u_{\varepsilon}^2 - u_{\varepsilon,x}^2)m_{\varepsilon}]_x = \varepsilon m_{\varepsilon,xx}. \quad (2.18)$$

It can be written as

$$u_{\varepsilon,t} - u_{\varepsilon,txx} + [(u_{\varepsilon}^2 - u_{\varepsilon,x}^2)(u_{\varepsilon} - u_{\varepsilon,xx})]_x = \varepsilon u_{\varepsilon,xx} - \varepsilon u_{\varepsilon,xxx}, \quad (2.19)$$

or

$$u_{\varepsilon,t} + u_{\varepsilon}^2 u_{\varepsilon,x} = \frac{1}{3} u_{\varepsilon,x}^3 - \Phi(u_{\varepsilon}) + \varepsilon u_{\varepsilon,xx}. \quad (2.20)$$

Rewrite the equation as

$$u_{\varepsilon,t} - \varepsilon u_{\varepsilon,xx} = -u_{\varepsilon}^2 u_{\varepsilon,x} + \frac{1}{3} u_{\varepsilon,x}^3 - \Phi(u_{\varepsilon}), \quad (2.21)$$

and denote the right-hand-side as $F(u_{\varepsilon}, u_{\varepsilon,x})$ for short.

The initial data is the smooth approximation of the original initial data:

$$u_{\varepsilon}(0, x) = u_{0\varepsilon}(x) = J_{\varepsilon} * u_0, \quad (2.22)$$

where

$$J_{\varepsilon}(x) = \frac{C}{\varepsilon} \exp(\varepsilon^2/(x^2 + \varepsilon^2)), \quad C \text{ is a constant such that } \int_{\mathbb{R}} J_{\varepsilon}(x) dx = 1. \quad (2.23)$$

In this section, we assume $\varepsilon > 0$ fixed.

We prove global wellposedness by two steps. First, we prove local wellposedness. Then we prove global wellposedness by a global a priori estimate.

2.1 Local well-posedness

Proposition 2.1 (Local wellposedness). *There exists a time $T_0 > 0$, such that the Cauchy problem (2.21), (2.22) is well-posed, $u \in C([0, T_0]; H^2(\mathbb{R}))$.*

The proof of local wellposedness is standard. We record it here for completeness.

Proof. We first consider the linearized equation

$$u_{\varepsilon,t} - \varepsilon u_{\varepsilon,xx} = -\bar{u}_{\varepsilon}^2 \bar{u}_{\varepsilon,x} + \frac{1}{3} \bar{u}_{\varepsilon,x}^3 - \Phi(\bar{u}_{\varepsilon}), \quad (2.24)$$

or

$$m_{\varepsilon,t} - \varepsilon m_{\varepsilon,xx} = [(-\bar{u}_{\varepsilon}^2 + \bar{u}_{\varepsilon,x}^2) \bar{m}_{\varepsilon}]_x, \quad (2.25)$$

where $\bar{m}_{\varepsilon} = \bar{u}_{\varepsilon} - \bar{u}_{\varepsilon,xx}$.

By Duhamel principle, we can express the solution by the integral form,

$$u_{\varepsilon}(t, x) = S(t)u_{0\varepsilon} + \int_0^t S(t-s) \left(-\bar{u}_{\varepsilon}^2 \bar{u}_{\varepsilon,x} + \frac{1}{3} \bar{u}_{\varepsilon,x}^3 - \Phi(\bar{u}_{\varepsilon}) \right) (s, x) ds \quad (2.26)$$

where $S(t)v = \frac{1}{\sqrt{4\pi\varepsilon t}} e^{-\frac{x^2}{2\pi\varepsilon t}} * v$ is the heat semigroup.

Define a map \mathcal{T} as

$$\mathcal{T}u_\varepsilon = S(t)u_{0\varepsilon} + \int_0^t S(t-s)(-u_\varepsilon^2 u_{\varepsilon,x} + \frac{1}{3}u_{\varepsilon,x}^3 - \Phi(u_\varepsilon))(s,x)ds, \quad (2.27)$$

then (2.26) is equivalent to $u_\varepsilon = \mathcal{T}\bar{u}_\varepsilon$. In the following, we will show that for T sufficiently small, \mathcal{T} is a contraction mapping from $L^\infty(0, T_0; H^2(\mathbb{R}))$ to itself.

(i) \mathcal{T} is a map from $L^\infty(0, T_0; H^2(\mathbb{R}))$ to itself. Suppose $u_\varepsilon \in L^\infty(0, T_0; H^2(\mathbb{R}))$. We prove $\mathcal{T}u_\varepsilon \in L^\infty(0, T_0; H^2(\mathbb{R}))$, where T_0 is a positive number to be determined later. In fact,

$$\begin{aligned} & \|\partial_x^2 \mathcal{T}u_\varepsilon(t, x)\|_{L_{T_0}^\infty L_x^2} \\ & \leq \|\partial_x^2 S(t)u_{0\varepsilon}\|_{L_{T_0}^\infty L_x^2} + \left\| \int_0^t \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi\varepsilon(t-s)}} \partial_x e^{-\frac{(x-y)^2}{2\pi\varepsilon(t-s)}} \partial_y (-u_\varepsilon^2 u_{\varepsilon,x} + \frac{1}{3}u_{\varepsilon,x}^3 - \Phi(u_\varepsilon))(s, y) dy ds \right\|_{L_{T_0}^\infty L_x^2} \\ & \leq \|\partial_x^2 u_{0\varepsilon}\|_{L_x^2} + \left\| \int_0^t \frac{1}{\sqrt{4\pi\varepsilon(t-s)}} \left\| \partial_x e^{-\frac{x^2}{2\pi\varepsilon(t-s)}} \right\|_{L_x^1} \cdot \left\| (-u_\varepsilon^2 u_{\varepsilon,x} + \frac{1}{3}u_{\varepsilon,x}^3 - \Phi(u_\varepsilon))_x(s, \cdot) \right\|_{L_x^2} ds \right\|_{L_{T_0}^\infty} \\ & \leq \|\partial_x^2 u_{0\varepsilon}\|_{L_x^2} + \frac{C}{\sqrt{\varepsilon}} \sqrt{T_0} (\|u_\varepsilon\|_{L_{T_0}^\infty H_x^2}^3 + \|\Phi\|_{L_{T_0}^\infty L_x^2}). \end{aligned} \quad (2.28)$$

Here and in the following, we use C to express an absolute constant which may be different from line by line.

By considering the convolution expression of Φ and Young's inequality, we have

$$\begin{aligned} \|\Phi\|_{L_{T_0}^\infty L_x^2} & \leq \left\| \frac{1}{2} e^{-|\cdot|} \right\|_{L_{T_0}^1 L_x^1} \cdot \left\| \frac{4}{3} u_{\varepsilon,x}^3 + 2u_\varepsilon^2 u_{\varepsilon,x} + 2u_\varepsilon u_{\varepsilon,x} u_{\varepsilon,xx} \right\|_{L_{T_0}^\infty L_x^2} \\ & \leq C \|u_\varepsilon\|_{L_{T_0}^\infty H_x^2}^3 \end{aligned} \quad (2.29)$$

So $\partial_x^3 \mathcal{T}u_\varepsilon(t, x) \in L^\infty(0, T; L^2(\mathbb{R}))$.

$$\begin{aligned} & \|\mathcal{T}u_\varepsilon(t, x)\|_{L_{T_0}^\infty L_x^2} \\ & \leq \|S(t)u_{0\varepsilon}\|_{L_{T_0}^\infty L_x^2} + \left\| \int_0^t \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi\varepsilon(t-s)}} e^{-\frac{(x-y)^2}{2\pi\varepsilon(t-s)}} (-u_\varepsilon^2 u_{\varepsilon,x} + \frac{1}{3}u_{\varepsilon,x}^3 - \Phi(u_\varepsilon))(s, y) dy ds \right\|_{L_{T_0}^\infty L_x^2} \\ & \leq \|u_{0\varepsilon}\|_{L_x^2} + \left\| \int_0^t \frac{1}{\sqrt{4\pi\varepsilon(t-s)}} \left\| e^{-\frac{x^2}{2\pi\varepsilon(t-s)}} \right\|_{L_x^1} \cdot \left\| -u_\varepsilon^2 u_{\varepsilon,x} + \frac{1}{3}u_{\varepsilon,x}^3 - \Phi(u_\varepsilon)(s, \cdot) \right\|_{L_x^2} ds \right\|_{L_{T_0}^\infty} \\ & \leq \|u_{0\varepsilon}\|_{L_x^2} + \frac{C}{\sqrt{\varepsilon}} T_0 (\|u_\varepsilon\|_{L_{T_0}^\infty H_x^2}^3 + \|\Phi\|_{L_{T_0}^\infty L_x^2}). \end{aligned} \quad (2.30)$$

So we can conclude that $\mathcal{T}u_\varepsilon \in L^\infty(0, T_0; H^2(\mathbb{R}))$ by interpolation.

(ii) When T_0 is sufficiently small, \mathcal{T} is a contraction mapping.

$$\begin{aligned}
& \|\mathcal{T}u - \mathcal{T}v\|_{L_{T_0}^\infty H_x^2} \\
& \leq C \left\| \int_0^t \int_{\mathbb{R}} \frac{\partial_x e^{-\frac{(x-y)^2}{2\pi\varepsilon(t-s)}}}{\sqrt{4\pi\varepsilon(t-s)}} \partial_y [-u_\varepsilon^2 u_{\varepsilon,x} + \frac{1}{3}u_{\varepsilon,x}^3 - \Phi(u_\varepsilon) - (-v_\varepsilon^2 v_{\varepsilon,x} + \frac{1}{3}v_{\varepsilon,x}^3 - \Phi(v_\varepsilon))](s, y) dy ds \right\|_{L_{T_0}^\infty L_x^2} \\
& \quad + C \left\| \int_0^t \int_{\mathbb{R}} \frac{e^{-\frac{(x-y)^2}{2\pi\varepsilon(t-s)}}}{\sqrt{4\pi\varepsilon(t-s)}} [-u_\varepsilon^2 u_{\varepsilon,x} + \frac{1}{3}u_{\varepsilon,x}^3 - \Phi(u_\varepsilon) - (-v_\varepsilon^2 v_{\varepsilon,x} + \frac{1}{3}v_{\varepsilon,x}^3 - \Phi(v_\varepsilon))](s, y) dy ds \right\|_{L_{T_0}^\infty L_x^2}
\end{aligned} \tag{2.31}$$

By Young's inequality, the first term

$$\begin{aligned}
& \left\| \int_0^t \int_{\mathbb{R}} \frac{\partial_x e^{-\frac{(x-y)^2}{2\pi\varepsilon(t-s)}}}{\sqrt{4\pi\varepsilon(t-s)}} \partial_y [-u_\varepsilon^2 u_{\varepsilon,x} + \frac{1}{3}u_{\varepsilon,x}^3 - \Phi(u_\varepsilon) - (-v_\varepsilon^2 v_{\varepsilon,x} + \frac{1}{3}v_{\varepsilon,x}^3 - \Phi(v_\varepsilon))](s, y) dy ds \right\|_{L_{T_0}^\infty L_x^2} \\
& \leq \int_0^t \left\| \frac{\partial_x e^{-\frac{x^2}{2\pi\varepsilon(t-s)}}}{\sqrt{4\pi\varepsilon(t-s)}} \right\|_{L_x^1} \left\| \partial_x [-u_\varepsilon^2 u_{\varepsilon,x} + \frac{1}{3}u_{\varepsilon,x}^3 - \Phi(u_\varepsilon) - (-v_\varepsilon^2 v_{\varepsilon,x} + \frac{1}{3}v_{\varepsilon,x}^3 - \Phi(v_\varepsilon))](s, x) \right\|_{L_x^2} ds \\
& \leq \int_0^t \frac{C}{\sqrt{\varepsilon s}} ds (\|u\|_{L_{T_0}^\infty H_x^2} + \|v\|_{L_{T_0}^\infty H_x^2}) \|u - v\|_{L_{T_0}^\infty H_x^2} \\
& \leq \frac{C}{\sqrt{\varepsilon}} \sqrt{T_0} (\|u\|_{L_{T_0}^\infty H_x^2} + \|v\|_{L_{T_0}^\infty H_x^2}) \|u - v\|_{L_{T_0}^\infty H_x^2}.
\end{aligned} \tag{2.32}$$

The second term

$$\begin{aligned}
& \left\| \int_0^t \int_{\mathbb{R}} \frac{e^{-\frac{(x-y)^2}{2\pi\varepsilon(t-s)}}}{\sqrt{4\pi\varepsilon(t-s)}} [-u_\varepsilon^2 u_{\varepsilon,x} + \frac{1}{3}u_{\varepsilon,x}^3 - \Phi(u_\varepsilon) - (-v_\varepsilon^2 v_{\varepsilon,x} + \frac{1}{3}v_{\varepsilon,x}^3 - \Phi(v_\varepsilon))](s, y) dy ds \right\|_{L_{T_0}^\infty L_x^2} \\
& \leq \int_0^t \left\| \frac{e^{-\frac{x^2}{2\pi\varepsilon(t-s)}}}{\sqrt{4\pi\varepsilon(t-s)}} \right\|_{L_x^1} \left\| [-u_\varepsilon^2 u_{\varepsilon,x} + \frac{1}{3}u_{\varepsilon,x}^3 - \Phi(u_\varepsilon) - (-v_\varepsilon^2 v_{\varepsilon,x} + \frac{1}{3}v_{\varepsilon,x}^3 - \Phi(v_\varepsilon))](s, x) \right\|_{L_x^2} ds \\
& \leq CT_0 (\|u\|_{L_{T_0}^\infty H_x^2} + \|v\|_{L_{T_0}^\infty H_x^2}) \|u - v\|_{L_{T_0}^\infty H_x^2}
\end{aligned} \tag{2.33}$$

When T_0 is small enough, we have

$$\|\mathcal{T}u - \mathcal{T}v\|_{L_{T_0}^\infty H_x^2} \leq \frac{C}{\sqrt{\varepsilon}} \sqrt{T_0} (\|u\|_{L_{T_0}^\infty H_x^2} + \|v\|_{L_{T_0}^\infty H_x^2}) \|u - v\|_{L_{T_0}^\infty H_x^2}. \tag{2.34}$$

When T_0 is small enough, we can obtain that the map \mathcal{T} is a contraction mapping. By contraction mapping theorem, there is a unique fixed point of the map \mathcal{T} . So we proved the local wellposedness. \square

2.2 A priori estimates

1. L^1 estimate of m_ε . Assume $\eta(v) = |v|$, $\eta_\delta(v) = J_\delta(v) * \eta(v)$, so $\eta_\delta(v) \rightarrow \eta(v)$ as $\delta \rightarrow 0$. Multiply equation (2.18) with $\eta'_\delta(m_\varepsilon)$, take integral and let $\delta \rightarrow 0$,

$$\int_{\mathbb{R}} \eta(m_\varepsilon(t, x)) dx - \int_{\mathbb{R}} \eta(m_\varepsilon(0, x)) dx \leq \varepsilon \int_0^t \int_{\mathbb{R}} -\eta''(m_\varepsilon) m_{\varepsilon, x}^2 dx \leq 0. \quad (2.35)$$

So we have

$$\|m_\varepsilon(t, \cdot)\|_{L^1} \leq \|m_{\varepsilon, 0}\|_{L^1}. \quad (2.36)$$

2. H^1 estimate. We have

$$(u_\varepsilon^2)_t + \left(\frac{1}{2}u_\varepsilon^4\right)_x = \frac{2}{3}u_{\varepsilon, x}^3 u_\varepsilon - 2u_\varepsilon \Phi(u_\varepsilon) + 2\varepsilon u_{\varepsilon, xx} u_\varepsilon, \quad (2.37)$$

$$\begin{aligned} (u_{\varepsilon, x}^2)_t + (u_\varepsilon^2 u_{\varepsilon, x}^2)_x &= 2u_{\varepsilon, x}^3 u_{\varepsilon, xx} + \frac{4}{3}u_\varepsilon^3 u_{\varepsilon, x} - 2u_{\varepsilon, x} \partial_x (1 - \partial_x^2)^{-1} \left(\frac{1}{3}u_{\varepsilon, x}^3\right) \\ &\quad - 2u_{\varepsilon, x} (1 - \partial_x^2)^{-1} \left[\frac{2}{3}u_\varepsilon^3 + uu_{\varepsilon, x}^2\right] + 2\varepsilon u_{\varepsilon, x} u_{\varepsilon, xxx}. \end{aligned} \quad (2.38)$$

Take the summation and then integrate,

$$\frac{d}{dt} \int_{\mathbb{R}} u_\varepsilon^2 + u_{\varepsilon, x}^2 dx = -2\varepsilon \int_{\mathbb{R}} u_{\varepsilon, x}^2 + u_{\varepsilon, xx}^2 dx \quad (2.39)$$

So we have the energy estimate,

$$\int_{\mathbb{R}} u_\varepsilon^2(t, x) + u_{\varepsilon, x}^2(t, x) dx \leq \int_{\mathbb{R}} u_\varepsilon^2(0, x) + u_{\varepsilon, x}^2(0, x) dx. \quad (2.40)$$

3. L^1 estimate of u_ε and $u_{\varepsilon, xx}$. Multiply $\eta'_\delta(u_\varepsilon)$ to the equation of u_ε , take integration, and let $\delta \rightarrow 0$,

$$\begin{aligned} &\int_{\mathbb{R}} \eta(u_\varepsilon(t, x)) dx - \int_{\mathbb{R}} \eta(u_\varepsilon(0, x)) dx \\ &\leq \int_0^t \int_{\mathbb{R}} \frac{1}{3} |u_{\varepsilon, x}|^3 + |\Phi(u_\varepsilon)| dx + \varepsilon \int_{\mathbb{R}} \eta'(u_\varepsilon) u_{\varepsilon, xx} dx \\ &\leq \int_0^t \int_{\mathbb{R}} \frac{1}{3} |u_{\varepsilon, x}|^3 dx + \int_0^t \int_{\mathbb{R}} \left| (e^{-|x|} * \frac{1}{3} u_{\varepsilon, x}^3) \right| dx + \int_0^t \int_{\mathbb{R}} \left| e^{-|x|} * \partial_x \left[\frac{2}{3} u_\varepsilon^3 + u_\varepsilon u_{\varepsilon, x}^2 \right] \right| dx \\ &\quad - \varepsilon \int_0^t \int_{\mathbb{R}} \eta''(u_\varepsilon) u_{\varepsilon, x}^2 dx \\ &\leq \int_0^t (2\|u_\varepsilon\|_{H^1}^3 + \|u_\varepsilon\|_{H^1}^2 \|u_{\varepsilon, xx}\|_{L^1} + \|u_{\varepsilon, x}\|_{L^\infty} \|u_{\varepsilon, x}\|_{L^2}^2 + 4\|u_\varepsilon\|_{H^1}^3) dt \end{aligned} \quad (2.41)$$

So

$$\|u_\varepsilon(t, \cdot)\|_{L^1} - \|u_\varepsilon(0, \cdot)\|_{L^1} \leq \int_0^t 6\|u_\varepsilon\|_{H^1}^3 + \|u_\varepsilon\|_{H^1}^2 (\|u_\varepsilon\|_{L^1} + \|m_\varepsilon\|_{L^1}) dt. \quad (2.42)$$

$$\|u_\varepsilon(t)\|_{L^1} \leq \|u_{\varepsilon,0}\|_{L^1} e^{E_0 t} + (6E_0^{1/2} + \|m_{\varepsilon,0}\|_{L^1})(e^{E_0 t} - 1). \quad (2.43)$$

Combining the estimate of m , we obtain the L^1 estimate of u_{xx}

$$\|u_{\varepsilon,xx}(t)\|_{L^1} \leq \|m_{\varepsilon,0}\|_{L^1} + \|u_{\varepsilon,0}\|_{L^1} e^{E_0 t} + (6E_0^{1/2} + \|m_{\varepsilon,0}\|_{L^1})(e^{E_0 t} - 1). \quad (2.44)$$

Above three estimates are all uniform with respect to ε . Next, we derive the high order energy estimates, which are depend on ε .

4. H^2 estimate.

$$u_{\varepsilon,xx} - \varepsilon u_{\varepsilon,xxx} = \partial_x^2 F(u_\varepsilon, u_{\varepsilon,x}), \quad (2.45)$$

where the right-hand-side is

$$\begin{aligned} \partial_x^2 F(u_\varepsilon, u_{\varepsilon,x}) &= \partial_x(-2u_\varepsilon u_{\varepsilon,x}^2 - u_\varepsilon^2 u_{\varepsilon,xx} + u_{\varepsilon,x}^2 u_{\varepsilon,xx} - \partial_x \Phi(u_\varepsilon)) \\ &= -6u_\varepsilon u_{\varepsilon,x} u_{\varepsilon,xxx} - 2u_{\varepsilon,x}^3 - u_\varepsilon^2 u_{\varepsilon,xxx} + 2u_{\varepsilon,x} u_{\varepsilon,xx}^2 + u_{\varepsilon,x}^2 u_{\varepsilon,xxx} - \partial_x^2 \Phi(u_\varepsilon). \end{aligned} \quad (2.46)$$

Multiply $u_{\varepsilon,xx}$ and then take the integral,

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} \frac{1}{2} u_{\varepsilon,xx}^2 dx + \varepsilon \int_{\mathbb{R}} u_{\varepsilon,xxx}^2 dx &\leq \int_{\mathbb{R}} |u_{\varepsilon,xxx}(-2u_\varepsilon u_{\varepsilon,x}^2 - u_\varepsilon^2 u_{\varepsilon,xx} + u_{\varepsilon,x}^2 u_{\varepsilon,xx} - \partial_x \Phi(u_\varepsilon))| dx \\ &\leq \int_{\mathbb{R}} \frac{\varepsilon}{2} u_{\varepsilon,xxx}^2 + \frac{2}{\varepsilon} (-2u_\varepsilon u_{\varepsilon,x}^2 - u_\varepsilon^2 u_{\varepsilon,xx} + u_{\varepsilon,x}^2 u_{\varepsilon,xx} - \partial_x \Phi(u_\varepsilon))^2 dx \end{aligned} \quad (2.47)$$

So we obtain

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{R}} \frac{1}{2} u_{\varepsilon,xx}^2 dx + \frac{\varepsilon}{2} \int_{\mathbb{R}} u_{\varepsilon,xxx}^2 dx \\ &\leq \int_{\mathbb{R}} \frac{8}{\varepsilon} [4u_\varepsilon^2 u_{\varepsilon,x}^4 + u_\varepsilon^4 u_{\varepsilon,xx}^2 + u_{\varepsilon,x}^4 u_{\varepsilon,xx}^2 + (\partial_x \Phi)^2(u_\varepsilon)] dx \\ &\leq \frac{32}{\varepsilon} (\|u_\varepsilon\|_{H^1}^2 \|u_{\varepsilon,xx}\|_{L^1}^2 \|u_\varepsilon\|_{H^1}^2 + \|u_\varepsilon\|_{H^1}^4 \|u_\varepsilon\|_{H^2}^2 + \|u_{\varepsilon,xx}\|_{L^1}^4 \|u_\varepsilon\|_{H^2}^2 + \|\partial_x \Phi(u_\varepsilon)\|_{L^2}^2). \end{aligned} \quad (2.48)$$

Considering (1.4), we have

$$\begin{aligned} \|\partial_x \Phi(u_\varepsilon)\|_{L^2}^2 &\leq \left\| \frac{4}{3} u_{\varepsilon,x}^3 + 2u_\varepsilon^2 u_{\varepsilon,x} + 2u_\varepsilon u_{\varepsilon,x} u_{\varepsilon,xx} \right\|_{L_x^2} \left\| \frac{1}{2} e^{-|x|} \right\|_{L_x^1} \\ &\leq \frac{4}{3} \|u_{\varepsilon,xx}\|_{L_x^1}^2 \|u_\varepsilon\|_{H_x^1} + 2\|u_\varepsilon\|_{H_x^1}^3 + 2\|u_\varepsilon\|_{H_x^1} \|u_{\varepsilon,xx}\|_{L_x^1} \|u_\varepsilon\|_{H_x^2} \\ &\leq \left(\frac{4}{3} \|u_{\varepsilon,xx}\|_{L_x^1}^2 + 2\|u_\varepsilon\|_{H_x^1}^2 + 2\|u_\varepsilon\|_{H_x^1} \|u_{\varepsilon,xx}\|_{L_x^1} \right) \|u_\varepsilon\|_{H_x^2}. \end{aligned} \quad (2.49)$$

Therefore,

$$\begin{aligned} &\frac{d}{dt} \frac{1}{2} \|u_\varepsilon\|_{H_x^2}^2 + \frac{\varepsilon}{2} \|u_{\varepsilon,xxx}(t, \cdot)\|_{L_x^2}^2 \\ &\leq \frac{C}{\varepsilon} \left[\|u_\varepsilon\|_{H_x^1}^4 \|u_{\varepsilon,xx}\|_{L_x^1}^2 + \|u_{\varepsilon,xx}\|_{L_x^1}^4 + \|u_\varepsilon\|_{H_x^1}^4 + \|u_\varepsilon\|_{H_x^1}^2 \|u_{\varepsilon,xx}\|_{L_x^1}^2 \right. \\ &\quad \left. + \|u_\varepsilon\|_{H_x^2}^2 (\|u_\varepsilon\|_{H_x^1}^4 + \|u_{\varepsilon,xx}\|_{L_x^1}^4 + 1) \right]. \end{aligned} \quad (2.50)$$

By Gronwall inequality, we have for any fixed T , $\|u_\varepsilon\|_{L_T^\infty H_x^2}$ and $\|u_{\varepsilon,x}\|_{L_T^2 H_x^2}$ are bounded by some constant $C(\varepsilon, T)$, which is related with the initial data, ε and T .

5. H^ℓ estimate. We prove H^ℓ ($\ell > 2$) estimate by a induction argument. Assume that for any $T, \varepsilon > 0$, there is a constant $C(\varepsilon, T)$, such that $\|u_\varepsilon\|_{L_T^\infty H_x^{\ell-1}} + \|u_\varepsilon\|_{L_T^2 H_x^\ell} < C(\varepsilon, T)$, for every $t \in [0, T]$. Then we derive H^ℓ estimate. Take ℓ th order derivative

$$(\partial_x^\ell u_\varepsilon)_t - \varepsilon \partial_x^\ell u_{\varepsilon,xx} = \partial_x^\ell F(u_\varepsilon, u_{\varepsilon,x}). \quad (2.51)$$

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}} \frac{1}{2} |\partial_x^\ell u_\varepsilon|^2 dx + \varepsilon \int_{\mathbb{R}} |\partial_x^{\ell+1} u_\varepsilon|^2 dx = \int_{\mathbb{R}} |\partial_x^\ell F(u_\varepsilon, u_{\varepsilon,x}) \partial_x^\ell u_\varepsilon| dx \\ & \leq \frac{2}{\varepsilon} \int_{\mathbb{R}} |\partial_x^{\ell-1} F(u_\varepsilon, u_{\varepsilon,x})|^2 dx + \frac{\varepsilon}{2} \int_{\mathbb{R}} |\partial_x^{\ell+1} u_\varepsilon|^2 dx \end{aligned} \quad (2.52)$$

So

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} \frac{1}{2} |\partial_x^\ell u_\varepsilon|^2 dx + \frac{\varepsilon}{2} \int_{\mathbb{R}} |\partial_x^{\ell+1} u_\varepsilon|^2 dx & \leq \frac{2}{\varepsilon} \|\partial_x^{\ell-1} F(u_\varepsilon, u_{\varepsilon,x})\|_{L_x^2}^2 \\ & \leq \frac{2}{\varepsilon} \left[\|u_\varepsilon^2 + u_{\varepsilon,x}^2\|_{L_x^\infty} \|\partial_x^\ell u_\varepsilon\|_{L_x^2}^2 + \|Q(u_\varepsilon, \dots, \partial_x^{\ell-1} u_\varepsilon)\|_{L_x^2}^2 \right] \end{aligned} \quad (2.53)$$

We only care about the term with $\partial_x^{\ell-1} u_\varepsilon$ in the term Q , since the lower order terms like $\partial_x^i u_\varepsilon \partial_x^j u_\varepsilon \partial_x^k u_\varepsilon$ with $i, j, k < \ell - 1$ can be estimated by

$$\|\partial_x^i u_\varepsilon \partial_x^j u_\varepsilon \partial_x^k u_\varepsilon\|_{L_x^2} \leq \|\partial_x^i u_\varepsilon\|_{L_x^\infty} \|\partial_x^j u_\varepsilon\|_{L_x^\infty} \|\partial_x^k u_\varepsilon\|_{L_x^2} \leq \|u_\varepsilon\|_{H_x^{\ell-1}}^3 < [C(\varepsilon, T)]^3. \quad (2.54)$$

When $\ell = 3$, there is term $u_{\varepsilon,x} u_{\varepsilon,xx}^2$. To estimate this term, we need to use the boundedness of $\|u_{\varepsilon,xxx}\|_{L_T^2 L_x^2}$.

$$\begin{aligned} \|u_{\varepsilon,x} u_{\varepsilon,xx}^2\|_{L_T^2 L_x^2} & \leq \|u_{\varepsilon,x}\|_{L_T^\infty L_x^\infty} \|u_{\varepsilon,xx}\|_{L_T^2 L_x^\infty} \|u_{\varepsilon,xx}\|_{L_T^\infty L_x^2} \\ & \leq \|u_{\varepsilon,x}\|_{L_T^\infty L_x^\infty} \|u_{\varepsilon,xx}\|_{L_T^2 H_x^1} \|u_{\varepsilon,xx}\|_{L_T^\infty L_x^2} < \infty. \end{aligned} \quad (2.55)$$

When $\ell \geq 4$, Q is linear with respect to $\partial_x^{\ell-1} u_\varepsilon$.

$$\|\partial_x^i u_\varepsilon \partial_x^j u_\varepsilon \partial_x^{\ell-1} u_\varepsilon\|_{L_t^\infty L_x^2} \leq \|\partial_x^i u_\varepsilon\|_{L_t^\infty L_x^\infty} \|\partial_x^j u_\varepsilon\|_{L_t^\infty L_x^\infty} \|\partial_x^{\ell-1} u_\varepsilon\|_{L_t^\infty L_x^2} \leq \|u_\varepsilon\|_{L_t^\infty H_x^{\ell-1}}^3 < [C(\varepsilon, T)]^3. \quad (2.56)$$

So in every case, we can obtain $\|u_\varepsilon\|_{L_T^\infty H_x^\ell} + \|u_\varepsilon\|_{L_T^2 H_x^{\ell+1}}$ is bounded.

2.3 Global wellposedness and regularity

Then from the above a priori estimates, we have

Proposition 2.2 (Global well-posedness and regularity). *If ε is a fixed positive constant. For any $T > 0$, $\ell \geq 2$. If the initial data $u_{0\varepsilon} \in H^\ell(\mathbb{R})$, then there is a unique solution $u_\varepsilon(t, x) \in L^\infty(0, T; H^\ell(\mathbb{R})) \cap L^2(0, T; H^{\ell+1}(\mathbb{R}))$, $u_{\varepsilon,t} \in L^\infty(0, T; H^{\ell-2}(\mathbb{R})) \cap L^2(0, T; H^{\ell-1}(\mathbb{R}))$.*

3 Existence of entropy weak solutions to (cCH)

Definition 3.1 (Weak solution). *We call a function $u : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ a weak solution of the Cauchy problem (1.3), (1.5) if*

- (i) $u \in L^\infty(\mathbb{R}_+; H^1(\mathbb{R}) \cap \dot{W}^{1,4}(\mathbb{R}))$, and
- (ii) u satisfies (1.3), (1.5) in distributional sense, that is

$$\iint_{\mathbb{R}_+ \times \mathbb{R}} u \phi_t + \frac{1}{3} u^3 \phi_x + \frac{1}{3} u_x^3 \phi - \Phi(u) \phi dx dt + \int_{\mathbb{R}} u_0(x) \phi(0, x) dx = 0, \quad (3.57)$$

where

$$\Phi(u) = (1 - \partial_x^2)^{-1} \left[\frac{4}{3} u_x^3 + 2u^2 u_x + 2uu_x u_{xx} \right] = \frac{1}{2} e^{-|x|} * \left[\frac{4}{3} u_x^3 + 2u^2 u_x + 2uu_x u_{xx} \right]. \quad (3.58)$$

Definition 3.2 (Entropy weak solution). *We call a function $u : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ an entropy weak solution of the Cauchy problem (1.3), (1.5) provided*

- (i) u is a weak solution in the sense of Definition 3.1,
- (ii) $u \in L^\infty(0, T; X)$ for any $T > 0$, and
- (iii) for any convex C^2 entropy $\eta : \mathbb{R} \rightarrow \mathbb{R}$ with corresponding entropy flux $q : \mathbb{R} \rightarrow \mathbb{R}$ defined by $q'(u) = \eta'(u)u^2$ there holds

$$\partial_t \eta(u_x) + \partial_x (u^2 \eta(u_x) - q(u_x)) + \eta'(u_x) \partial_x \Phi(u) + 2uu_x (u_x \eta'(u_x) - \eta(u_x)) \leq 0,$$

that is, for any $\varphi \in C_c^\infty([0, \infty) \times \mathbb{R})$, $\varphi \geq 0$,

$$\begin{aligned} & \iint_{\mathbb{R}_+ \times \mathbb{R}} \eta(u_x) \varphi_t + (u^2 \eta(u_x) - q(u_x)) \varphi_x - \eta'(u_x) \partial_x \Phi(u) \varphi - 2uu_x (u_x \eta'(u_x) - \eta(u_x)) \varphi dx dt \\ & + \int_{\mathbb{R}} \eta(u_0(x)) \varphi(0, x) dx \geq 0. \end{aligned} \quad (3.59)$$

Our main theorem in this section is

Theorem 3.1 (Existence). *Suppose the initial data $u_0 \in X$. There exists at least one entropy weak solution to the Cauchy problem (1.3), (1.5).*

Proof. We take a sequence of positive numbers $\{\varepsilon_k\}_{k=1}^\infty$ tending to zero. Then we obtain a sequence of solutions $\{u_{\varepsilon_k}\}_{k=1}^\infty$. From section 2.2, we know this sequence satisfies uniform $L^1, H^1, \dot{W}^{1,\infty}$, and $\dot{W}^{2,1}$ estimates.

So by Helly's theorem, as $k \rightarrow \infty$,

$$u_{\varepsilon_k, x} \rightarrow u_x \quad \text{a.e. in } \mathbb{R}_+ \times \mathbb{R}, \quad (3.60)$$

and hence

$$u_{\varepsilon_k, x} \rightarrow u_x \quad \text{in } L_{loc}^p(\mathbb{R}_+ \times \mathbb{R}) \text{ for all } p \in [1, \infty). \quad (3.61)$$

By the $\dot{W}^{1,\infty}$ estimate and the energy estimate, there holds

$$u_{\varepsilon_k, x} \rightarrow u_x \quad \text{in } L^p((0, T) \times \mathbb{R}) \text{ for all } T > 0, \text{ for all } p \in [1, \infty). \quad (3.62)$$

Further more, we can obtain that $u_{\varepsilon_k} \rightarrow u$ pointwise. So in order to prove (3.57), it suffices to prove the convergence of term $\Phi(u_{\varepsilon_k})$. In fact,

$$\begin{aligned} & \|\Phi(u_{\varepsilon_k}) - \Phi(u)\|_{L^p} \\ &= \left\| \frac{1}{2} e^{-|x|} * \frac{1}{3} (u_{\varepsilon_k, x}^3 - u_x^3) + \frac{1}{2} (\text{sgn } x) e^{-|x|} * \left[\left(\frac{2}{3} u_{\varepsilon_k}^3 + u_{\varepsilon_k} u_{\varepsilon_k, x}^2 \right) - \left(\frac{2}{3} u^3 + u u_x^2 \right) \right] \right\|_{L^p} \\ &\leq 2(\|u_{\varepsilon_k, x}^3 - u_x^3\|_{L^p} + \|u_{\varepsilon_k}^3 - u^3\|_{L^p} + \|u_{\varepsilon_k} u_{\varepsilon_k, x}^2 - u u_x^2\|_{L^p}) \rightarrow 0. \end{aligned} \quad (3.63)$$

So the limit function u is a weak solution to (cCH). Then we shall prove the entropy inequality. Since for $\eta \in C^2$ with $\eta'' > 0$, the approximate solution u_ε satisfies

$$\begin{aligned} \partial_t \eta(u_{\varepsilon, x}) + (u_\varepsilon^2 \eta(u_{\varepsilon, x}) - q(u_{\varepsilon, x}))_x + 2u_\varepsilon u_{\varepsilon, x} (u_{\varepsilon, x} \eta'(u_{\varepsilon, x}) - \eta(u_{\varepsilon, x})) \\ + \eta'(u_{\varepsilon, x}) \partial_x \Phi(u_\varepsilon) = \varepsilon \eta'(u_{\varepsilon, x}) (u_{\varepsilon, x})_{xx}. \end{aligned} \quad (3.64)$$

Multiply a positive test function,

$$\begin{aligned} & - \iint_{\mathbb{R}_+ \times \mathbb{R}} \eta(u_{\varepsilon, x}) \varphi_t + [u_\varepsilon^2 \eta(u_\varepsilon) - q(u_{\varepsilon, x})] \varphi_x dx dt - \int_{\mathbb{R}} \eta(u_0(x)) \varphi(0, x) dx \\ & \quad + \iint_{\mathbb{R}_+ \times \mathbb{R}} 2u_\varepsilon u_{\varepsilon, x} (u_{\varepsilon, x} \eta'(u_{\varepsilon, x}) - \eta(u_{\varepsilon, x})) \varphi + \eta'(u_{\varepsilon, x}) \partial_x \Phi(u_\varepsilon) dx dt \\ &= - \iint_{\mathbb{R}_+ \times \mathbb{R}} \varepsilon \eta(u_{\varepsilon, x})_x \varphi_x dx dt - \varepsilon \iint_{\mathbb{R}_+ \times \mathbb{R}} \eta'' u_{\varepsilon, xx}^2 \varphi dx dt \\ &\leq - \iint_{\mathbb{R}_+ \times \mathbb{R}} \varepsilon \eta'(u_{\varepsilon, x}) u_{\varepsilon, xx} \varphi_x dx dt. \end{aligned} \quad (3.65)$$

Since $u_{\varepsilon, xx}(t, \cdot) \in L^1(\mathbb{R})$ and φ is compact support, the right-hand-side approaches 0 as $\varepsilon \rightarrow 0$. So u is an entropy weak solution. □

4 $W^{1,1}$ stability and uniqueness

Suppose u and v are two entropy weak solutions with initial data u_0 and v_0 respectively. We estimate $W^{1,1}$ norm of their difference.

1 ($|u - v|_{L^1}$ estimate). Since $u, v \in \dot{W}^{2,1}$, u, v are Lipschitz functions. The derivatives u_x, v_x exist almost everywhere. From the equation, we know u_t, v_t are also well-defined. The equations for u and v are

$$\begin{aligned} u_t + u^2 u_x &= \frac{1}{3} u_x^3 - \Phi(u), \\ v_t + v^2 v_x &= \frac{1}{3} v_x^3 - \Phi(v). \end{aligned} \quad (4.66)$$

Take their difference

$$(u-v)_t + u^2(u-v)_x + (u+v)(u-v)v_x = \frac{1}{3}(u_x - v_x)(u_x^2 + u_x v_x + v_x^2) - (\Phi(u) - \Phi(v)). \quad (4.67)$$

Multiply $\text{sgn}(u-v)$ and then take integral,

$$\frac{d}{dt} \int_{\mathbb{R}} |u-v| dx \leq \int_{\mathbb{R}} (|2uu_x| + |v_x||u+v|)|u-v| dx + \frac{1}{3} |u_x - v_x| |u_x^2 + u_x v_x + v_x^2| + |\Phi(u) - \Phi(v)| dx. \quad (4.68)$$

By the expression of Φ ,

$$\begin{aligned} & \|\Phi(u) - \Phi(v)\|_{L_x^1} \\ & \leq \frac{1}{3} \left\| \frac{1}{2} e^{-|x|} \right\|_{L_x^1} \|u_x - v_x\|_{L_x^1} \|u_x^2 + u_x v_x + v_x^2\|_{L_x^\infty} \\ & \quad + \left\| \frac{1}{2} e^{-|x|} \right\|_{L_x^1} \left(\frac{2}{3} \|u-v\|_{L_x^1} \|u^2 + uv + v^2\|_{L_x^\infty} + \|u-v\|_{L_x^1} \|u_x^2\|_{L_x^\infty} + \|v(u_x + v_x)\|_{L_x^\infty} \|u_x - v_x\|_{L_x^1} \right) \end{aligned} \quad (4.69)$$

Considering the boundedness of the $\dot{W}^{2,1}$ -norm of u, v , we have

$$\frac{d}{dt} \|(u-v)(t, \cdot)\|_{L_x^1} \leq K(T, \|u_0\|_{H_x^1}, \|v_0\|_{H_x^1}, \|u_0\|_{\dot{W}_x^{2,1}}, \|v_0\|_{\dot{W}_x^{2,1}}) \|(u-v)(t, \cdot)\|_{W_x^{1,1}} \quad (4.70)$$

where K is a constant related with $T, \|u_0\|_{H_x^1}, \|v_0\|_{H_x^1}, \|u_0\|_{\dot{W}_x^{2,1}}, \|v_0\|_{\dot{W}_x^{2,1}}$.

2 ($|u_x - v_x|_{L^1}$ estimate). Denote $\eta_k(u) = |u - k|$. Consider the equation

$$(u_x)_t + (u^2 u_x - \frac{1}{3} u_x^3)_x + \Phi_x = 0. \quad (4.71)$$

Multiply $\eta'_k(u_x)$,

$$|u_x - k|_t + \text{sgn}(u_x - k)(u^2 u_x - \frac{1}{3} u_x^3)_x + \text{sgn}(u_x - k) \partial_x \Phi = 0. \quad (4.72)$$

$$\int_{\mathbb{R}_+ \times \mathbb{R}} |u_x - k| \varphi_t + \text{sgn}(u_x - k)(u^2 u_x - \frac{1}{3} u_x^3 - u^2 k + \frac{1}{3} k^3) \varphi_x - \text{sgn}(u_x - k)(\partial_x \Phi(u) + 2k u u_x) \varphi dx dt \geq 0. \quad (4.73)$$

$$\int_{\mathbb{R}_+ \times \mathbb{R}} |u_x - v_y| \varphi_t + \text{sgn}(u_x - v_y)(u^2 u_x - \frac{1}{3} u_x^3 - u^2 v_y + \frac{1}{3} v_y^3) \varphi_x - \text{sgn}(u_x - v_y)(\partial_x \Phi(u) + 2v_y u u_x) \varphi dx dt \geq 0. \quad (4.74)$$

Similarly, for v ,

$$\int_{\mathbb{R}_+ \times \mathbb{R}} |v_y - k| \varphi_s + \text{sgn}(v_y - k)(v^2 v_y - \frac{1}{3} v_y^3 - v^2 k + \frac{1}{3} k^3) \varphi_y - \text{sgn}(v_y - k)(\partial_y \Phi(v) + 2k v v_y) \varphi dy ds \geq 0. \quad (4.75)$$

$$\int_{\mathbb{R}_+ \times \mathbb{R}} |v_y - u_x| \varphi_s + \text{sgn}(v_y - u_x)(v^2 v_y - \frac{1}{3} v_y^3 - v^2 u_x + \frac{1}{3} u_x^3) \varphi_y - \text{sgn}(v_y - u_x)(\partial_y \Phi(v) + 2u_x v v_y) \varphi dy ds \geq 0. \quad (4.76)$$

Combine (4.74) and (4.76),

$$\begin{aligned}
& \iint\limits_{\mathbb{R}_+^2 \times \mathbb{R}^2} |u_x - v_y|(\varphi_t + \varphi_s) + \operatorname{sgn}(u_x - v_y)[u^2(u_x - v_y) - \frac{1}{3}(u_x^3 - v_y^3)]\varphi_x \\
& - \operatorname{sgn}(u_x - v_y)[-v^2(u_x - v_y) + \frac{1}{3}(u_x^3 - v_y^3)]\varphi_y \\
& - \operatorname{sgn}(u_x - v_y)(\partial_x \Phi(u) - \partial_y \Phi(v) + 2v_y u u_x - 2u_x v v_y)\varphi dx dy ds dt \geq 0.
\end{aligned} \tag{4.77}$$

Take $\varphi(t, x, s, y) = \psi(\frac{t+s}{2}, \frac{x-y}{2})\delta_h(\frac{t-s}{2})\delta_h(\frac{x-y}{2})$,

$$\begin{aligned}
& \iint\limits_{\mathbb{R}_+ \times \mathbb{R}} |u_x - v_x|\psi_t(t, x) + \operatorname{sgn}(u_x - v_x)[(u^2 + v^2)(u_x - v_x) - \frac{2}{3}(u_x^3 - v_x^3)]\psi_x(t, x) \\
& - \operatorname{sgn}(u_x - v_x)(\partial_x \Phi(u) - \partial_x \Phi(v) + 2u_x v_x(u - v))\psi(t, x) dx dt \geq 0.
\end{aligned} \tag{4.78}$$

$$\begin{aligned}
& \iint\limits_{\mathbb{R}_+ \times \mathbb{R}} |u_x - v_x|\psi_t(t, x) + |u_x - v_x|[(u^2 + v^2) - \frac{2}{3}(u_x^2 + u_x v_x + v_x^2)]\psi_x(t, x) \\
& - \operatorname{sgn}(u_x - v_x)(\partial_x \Phi(u) - \partial_x \Phi(v) + 2u_x v_x(u - v))\psi(t, x) dx dt \geq 0.
\end{aligned} \tag{4.79}$$

By the $\dot{W}^{2,1}$ estimate and the energy estimate, we can find a constant L such that

$$\left| (u^2 + v^2) - \frac{2}{3}(u_x^2 + u_x v_x + v_x^2) \right| < L. \tag{4.80}$$

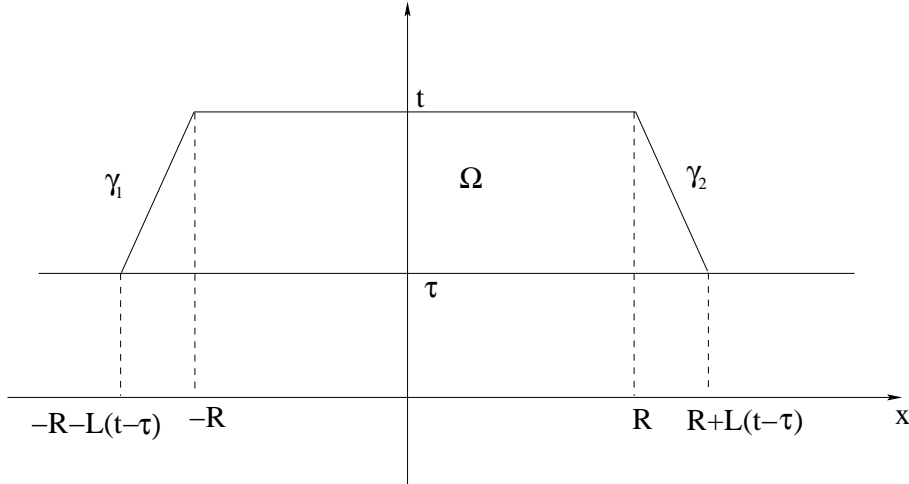


Figure 1: The domain in consideration.

Consider a trapezoid Ω as in Figure 1. The straight line γ_1 and γ_2 have slopes as $-L$ and L separately.

Then we take ψ to be a smooth approximation to the characteristic function of the trapezoid

Ω . We obtain

$$\begin{aligned} \int_{-R}^R |u_x(t, x) - v_x(t, x)| dx &\leq \int_{-R-L(t-\tau)}^{R+L(t-\tau)} |u_x(\tau, x) - v_x(\tau, x)| dx \\ &+ \iint_{\Omega} |\partial_x \Phi(u) - \partial_x \Phi(v) + 2u_x v_x (u - v)| dx dt \end{aligned} \quad (4.81)$$

For the source term, we have the following estimate

$$\begin{aligned} &|\partial_x \Phi(u) - \partial_x \Phi(v) + 2u_x v_x (u - v)| \\ &= \left| \partial_x (1 - \partial_x^2)^{-1} \left[\frac{1}{3} (u_x - v_x) (u_x^2 + v_x^2 + u_x v_x) \right] + (1 - \partial_x^2)^{-1} \left[\frac{2}{3} (u - v) (u^2 + uv + v^2) \right. \right. \\ &\quad \left. \left. + u(u_x - v_x)(u_x + v_x) + (u - v)v_x^2 \right] - \frac{2}{3} (u - v) (u^2 + uv + v^2) - (u - v)u_x^2 \right. \\ &\quad \left. - v(u_x - v_x)(u_x + v_x) + 2u_x v_x (u - v) \right|. \end{aligned} \quad (4.82)$$

Take integral,

$$\begin{aligned} &\int_{\mathbb{R}} |\partial_x \Phi(u) - \partial_x \Phi(v) + 2u_x v_x (u - v)| dx \\ &\leq \sup_{t \in [0, T]} \left(\left\| \frac{1}{3} (u_x^2 + v_x^2 + u_x v_x) \right\|_{L^\infty} + \left\| \frac{2}{3} (u^2 + uv + v^2) \right\|_{L^\infty} + \|u_x^2\|_{L^\infty} + \|u(u_x + v_x)\|_{L^\infty} \right. \\ &\quad \left. + 2\|u_x v_x\|_{L^\infty} \|u - v\|_{W^{1,1}} \right) \end{aligned} \quad (4.83)$$

We let $R \rightarrow +\infty$ in (4.81), and add (4.70)

$$\|u(t, \cdot) - v(t, \cdot)\|_{W^{1,1}} \leq \|u(0, \cdot) - v(0, \cdot)\|_{W^{1,1}} + \int_0^t M_T \|u(s, \cdot) - v(s, \cdot)\|_{W^{1,1}} ds, \quad (4.84)$$

where M_T is a constant of order $O(e^{CT})$, and related with $\|u_0\|_{H^1 \cap \dot{W}^{2,1}}$ and $\|v_0\|_{H^1 \cap \dot{W}^{2,1}}$. By Gronwall's inequality, we obtain

$$\|u(t, \cdot) - v(t, \cdot)\|_{W^{1,1}} \leq e^{M_T t} \|u(0, \cdot) - v(0, \cdot)\|_{W^{1,1}}. \quad (4.85)$$

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