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An ergodic description of ground states

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Abstract

1 Introduction

2 Framework and Main Results

Given a finite spin set F , we introduce the configuration space $\Omega = F^{\mathbb{Z}^d}$. Its elements $\omega \in \Omega$ are written as $\omega = \{\omega_j\}_{j \in \mathbb{Z}^d}$. Endowed with the product topology of discrete topologies, Ω is a compact metrizable space. Indeed, for $j = (j_1, \dots, j_d) \in \mathbb{Z}^d$ one may consider the L^1 norm $\|j\| = |j_1| + \dots + |j_d|$ and define then a metric on Ω by

$$d(\omega, \bar{\omega}) = 2^{-\inf\{\|j\| : \omega_j \neq \bar{\omega}_j\}},$$

which is compatible with the product topology.

Notice that \mathbb{Z}^d acts on Ω by translation. Let $\{e_1, \dots, e_d\}$ be the canonical basis for the lattice \mathbb{Z}^d . For each $i \in \{1, \dots, d\}$, we consider the shift transformation $\theta_i : \Omega \rightarrow \Omega$ given by $\theta_i(\omega) = \{\omega_{j+e_i}\}_{j \in \mathbb{Z}^d}$. Given $j = (j_1, \dots, j_d) \in \mathbb{Z}^d$, we define $\theta^j := \theta_1^{j_1} \circ \theta_2^{j_2} \circ \dots \circ \theta_d^{j_d}$.

Let \mathcal{F} denote the collection of finite subsets of \mathbb{Z}^d . The diameter and the boundary of $A \in \mathcal{F}$ are respectively $\text{diam}(A) = \max\{\|j - k\| : j, k \in A\}$ and $\partial A := \{j \in \mathbb{Z}^d \setminus A : \|j - k\| = 1 \text{ for some } k \in A\}$. For $A \in \mathcal{F}$ and $\omega \in \Omega$, we denote the restriction $\omega|_A$ simply by ω_A .

Definition 2.1. *By an invariant interaction family we mean any collection of continuous maps $\Phi_A : \Omega \rightarrow \mathbb{R}$, indexed by $A \in \mathcal{F}$, such that*

1. $\Phi_{j+A}(\omega) = \Phi_A(\theta^j(\omega))$, for all $j \in \mathbb{Z}^d$ and $\omega \in \Omega$;

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2. $\omega_A = \bar{\omega}_A$ implies $\Phi_A(\omega) = \Phi_A(\bar{\omega})$.

We say that $\{\Phi_A\}_{A \in \mathcal{F}}$ is an invariant short range interaction family if in addition there exists an integer $r > 0$ such that $\Phi_A \equiv 0$ whenever $\text{diam}(A) > r$. In this case, we say also that the invariant interaction family has range $r > 0$. For $r = 1$, one in particular says to have an interaction of nearest neighbors.

Given an invariant interaction family $\{\Phi_A\}_{A \in \mathcal{F}}$, we introduce the associated hamiltonian $H : \mathcal{F} \times \Omega \rightarrow \mathbb{R}$ defined by

$$H(\Lambda, \omega) = H_\Lambda(\omega) := \sum_{A: A \cap \Lambda \neq \emptyset} \Phi_A(\omega), \quad \forall \Lambda \in \mathcal{F}, \omega \in \Omega. \quad (2.1)$$

Notice also that the hamiltonian inherits the invariance of the interaction family, namely, $H_\Lambda \circ \theta^j = H_{j+\Lambda}$ for any $j \in \mathbb{Z}^d$.

Remark 2.2. We say that $\{\Phi_A\}_{A \in \mathcal{F}}$ is an invariant long range interaction family if $\Phi_A \not\equiv 0$ for sets A with arbitrarily large diameter. In this case, one shall assume a summability condition:

$$\sum_{0 \in A} \|\Phi_A\|_\infty < \infty,$$

which is obviously trivial for the short range situation. Notice that the associated hamiltonian is then well defined, since

$$\|H_\Lambda\|_\infty \leq \sum_{j \in \Lambda} \sum_{j \in A} \|\Phi_A\|_\infty = \sum_{j \in \Lambda} \sum_{0 \in A} \|\Phi_A \circ \theta^j\|_\infty = \#\Lambda \sum_{0 \in A} \|\Phi_A\|_\infty < \infty, \quad \forall \Lambda \in \mathcal{F}.$$

For $\omega, \bar{\omega} \in \Omega$ and $\Lambda \in \mathcal{F}$, we denote by $\bar{\omega}_\Lambda \omega_{\mathbb{Z}^d \setminus \Lambda}$ the configuration of Ω that coincides with $\bar{\omega}$ on Λ and with ω on $\mathbb{Z}^d \setminus \Lambda$. We also introduce the following notation

$$\Lambda_{n+1} := \{-n, -n+1, \dots, 0, \dots, n-1, n\}^d, \quad \forall n \in \mathbb{N}.$$

Definition 2.3. We say that $\omega \in \Omega$ is a minimizing configuration with respect to the hamiltonian H if

$$H_\Lambda(\omega) \leq H_\Lambda(\bar{\omega}_\Lambda \omega_{\mathbb{Z}^d \setminus \Lambda}), \quad \forall \bar{\omega} \in \Omega, \Lambda \in \mathcal{F}.$$

We define the minimizing ergodic value of the hamiltonian H as the constant

$$\bar{H} := \inf_{\omega \in \Omega} \liminf_{n \rightarrow \infty} \frac{1}{\#\Lambda_n} H_{\Lambda_n}(\omega). \quad (2.2)$$

Our main result concerning these minimizing objects is the following one.

Theorem 2.4. Let H be a hamiltonian defined by an invariant short range interaction family. The set of minimizing configurations with respect to the hamiltonian H is a non-empty invariant closed set. For a minimizing configuration $\omega \in \Omega$, the limit $\lim_{n \rightarrow \infty} \frac{1}{\#\Lambda_n} H_{\Lambda_n}(\omega)$ always exists. Moreover, there are minimizing configurations $\omega \in \Omega$ for which

$$\lim_{n \rightarrow \infty} \frac{1}{\#\Lambda_n} H_{\Lambda_n}(\omega) = \bar{H}.$$

3 Proof of Theorem 2.4

The proof of Theorem 2.4 follows from a series of results regrouped in this section.

Proposition 3.1. *Let H be a hamiltonian defined by an invariant short range interaction family. There exist minimizing configurations for the hamiltonian H . The set of minimizing configurations with respect to H is invariant and closed.*

Proof. For each $\Lambda \in \mathcal{F}$, denote

$$\Omega_\Lambda := \left\{ \omega \in \Omega : H_\Lambda(\omega) \leq H_\Lambda(\bar{\omega}), \forall \bar{\omega} \in \Omega \text{ with } \bar{\omega}_{\mathbb{Z}^d \setminus \Lambda} = \omega_{\mathbb{Z}^d \setminus \Lambda} \right\}.$$

Notice that Ω_Λ is non-empty: for any fixed configuration $\omega \in \Omega$, it clearly contains the minimum points of the map $\bar{\omega}_\Lambda \in \mathbb{F}^\Lambda \mapsto H_\Lambda(\bar{\omega}_\Lambda \omega_{\mathbb{Z}^d \setminus \Lambda}) \in \mathbb{R}$. Besides, each Ω_Λ is a closed set, since the map $\omega \in \Omega \mapsto H_\Lambda(\bar{\omega}_\Lambda \omega_{\mathbb{Z}^d \setminus \Lambda}) - H_\Lambda(\omega) \in \mathbb{R}$ is continuous (actually, locally constant) for all $\bar{\omega}_\Lambda \in \mathbb{F}^\Lambda$.

The family $\{\Omega_\Lambda\}_{\Lambda \in \mathcal{F}}$ is monotone: $\Lambda \subset \Lambda' \Rightarrow \Omega_{\Lambda'} \subset \Omega_\Lambda$. Indeed, suppose that $\omega' \in \Omega_{\Lambda'}$. Let $\bar{\omega} \in \Omega$ be such that $\bar{\omega}_{\mathbb{Z}^d \setminus \Lambda} = \omega'_{\mathbb{Z}^d \setminus \Lambda}$. As $\Lambda \subset \Lambda'$, we get $\bar{\omega}_{\mathbb{Z}^d \setminus \Lambda'} = \omega'_{\mathbb{Z}^d \setminus \Lambda'}$. Hence, $H_{\Lambda'}(\omega') \leq H_{\Lambda'}(\bar{\omega})$. Since

$$H_{\Lambda'} = H_\Lambda + \sum_{\substack{A \cap \Lambda' \neq \emptyset \\ A \cap \Lambda = \emptyset}} \Phi_A$$

and $\Phi_A(\omega') = \Phi_A(\bar{\omega})$ whenever $\emptyset \neq A \cap \Lambda' \subset \Lambda' \setminus \Lambda$, we clearly obtain that $H_\Lambda(\omega') \leq H_\Lambda(\bar{\omega})$, which means that $\omega' \in \Omega_\Lambda$.

Therefore, the family of closed sets $\{\Omega_\Lambda\}_{\Lambda \in \mathcal{F}}$ has the finite intersection property: $\bigcap_{i=1}^N \Omega_{\Lambda_i} \supset \Omega_{\bigcup_{i=1}^N \Lambda_i} \neq \emptyset$. By compactness, there exists $\omega \in \Omega_\Lambda$ for all $\Lambda \in \mathcal{F}$.

The set of minimizing configurations with respect to H is exactly $\bigcap_{\Lambda \in \mathcal{F}} \Omega_\Lambda$, which is clearly closed. Besides, since $\theta^{-j}(\Omega_\Lambda) = \Omega_{j+\Lambda}$ for all $j \in \mathbb{Z}^d$, this set is also invariant. \square

Denote the Birkhoff sum by $S_\Lambda \Psi := \sum_{j \in \Lambda} \Psi \circ \theta^j$.

Definition 3.2. *Given an invariant short range interaction family $\{\Phi_A\}_{A \in \mathcal{F}}$, we introduce*

$$\Psi_0 := \sum_{A: 0 \in A} \frac{1}{\#A} \Phi_A.$$

Notice that Ψ_0 is a real valued function since the above sum is actually finite. Moreover, we have the following property.

Lemma 3.3. *The map Ψ_0 satisfies*

$$\sup_{\Lambda \in \mathcal{F}} \sup_{\omega_\Lambda = \bar{\omega}_\Lambda} \frac{1}{\#\partial\Lambda} (S_\Lambda \Psi_0(\omega) - S_\Lambda \Psi_0(\bar{\omega})) < \infty.$$

In order to prove this lemma, given $\Lambda \in \mathcal{F}$ and $r > 0$, we define

$$\text{Int}_r \Lambda := \{j \in \Lambda : d(j, \partial\Lambda) \geq r\}.$$

Proof. Notice that

$$\begin{aligned} S_\Lambda \Psi_0 &= \sum_{j \in \Lambda} \sum_{A: 0 \in A} \frac{1}{\#A} \Phi_A \circ \theta^j = \sum_{j \in \Lambda} \sum_{A: j \in A} \frac{1}{\#A} \Phi_A \\ &= \sum_{A: A \cap \Lambda \neq \emptyset} \sum_{j \in A \cap \Lambda} \frac{1}{\#A} \Phi_A = \sum_{A: A \cap \Lambda \neq \emptyset} \frac{\#(A \cap \Lambda)}{\#A} \Phi_A. \end{aligned} \quad (3.1)$$

Assume the invariant interaction family $\{\Phi_A\}_{A \in \mathcal{F}}$ has range $r > 0$. Whenever $\omega_\Lambda = \bar{\omega}_\Lambda$, it is easy to see that

$$\begin{aligned} S_\Lambda \Psi_0(\omega) - S_\Lambda \Psi_0(\bar{\omega}) &= \sum_{A \cap (\Lambda \setminus \text{Int}_r \Lambda) \neq \emptyset} \frac{\#(A \cap \Lambda)}{\#A} (\Phi_A(\omega) - \Phi_A(\bar{\omega})) \\ &\leq 2 \sum_{A \cap (\Lambda \setminus \text{Int}_r \Lambda) \neq \emptyset} \|\Phi_A\|_\infty \leq 2 \#(\Lambda \setminus \text{Int}_r \Lambda) \sum_{0 \in A} \|\Phi_A\|_\infty, \end{aligned}$$

from which the statement follows immediately. \square

The relation of Ψ_0 with the associated hamiltonian is given below.

Proposition 3.4. *Suppose $\{\Phi_A\}_{A \in \mathcal{F}}$ is an invariant interaction family with range $r > 0$. If H is the associated hamiltonian, then*

$$\|H_\Lambda - S_\Lambda \Psi_0\|_\infty \leq \#(\Lambda \setminus \text{Int}_r \Lambda) \sum_{0 \in A} \|\Phi_A\|_\infty.$$

Proof. Using (3.1), we obtain

$$H_\Lambda - S_\Lambda \Psi_0 = \sum_{A \cap \text{Int}_r \Lambda \neq \emptyset} \left[1 - \frac{\#(A \cap \Lambda)}{\#A} \right] \Phi_A + \sum_{A \cap (\Lambda \setminus \text{Int}_r \Lambda) \neq \emptyset} \left[1 - \frac{\#(A \cap \Lambda)}{\#A} \right] \Phi_A.$$

Notice that if $A \in \mathcal{F}$ has diameter less than r and intersects $\text{Int}_r \Lambda$, then $A \subset \Lambda$. Thus, the first term in the right side of the above equation is equal to zero. Hence, we get

$$\|H_\Lambda - S_\Lambda \Psi_0\|_\infty \leq \sum_{A \cap (\Lambda \setminus \text{Int}_r \Lambda) \neq \emptyset} \|\Phi_A\|_\infty \leq \#(\Lambda \setminus \text{Int}_r \Lambda) \sum_{0 \in A} \|\Phi_A\|_\infty. \quad \square$$

We remark that, for $\Lambda_n = (-n, n)^d \cap \mathbb{Z}^d$, one has $\#(\Lambda_n \setminus \text{Int}_r \Lambda_n) \leq C_d r^d n^{d-1}$, for some constant $C_d > 0$ which depends just on the dimension d . In particular, we immediately obtain the following corollary.

Corollary 3.5. *Let H be a hamiltonian defined by an invariant short range interaction family. Then*

$$\bar{H} = \inf_{\omega \in \Omega} \liminf_{n \rightarrow \infty} \frac{1}{\#\Lambda_n} S_{\Lambda_n} \Psi_0(\omega).$$

We may now prove the second statement of Theorem 2.4.

Proposition 3.6. *Let H be a hamiltonian defined by an invariant short range interaction family. Then, for any minimizing configuration ω with respect to H , the limit*

$$\lim_{n \rightarrow \infty} \frac{1}{\#\Lambda_n} H_{\Lambda_n}(\omega)$$

does exist.

Proof. Let ω be a minimizing configuration for H . By Proposition 3.4, it is equivalent to show that the limit $\lim_{n \rightarrow \infty} \frac{1}{\#\Lambda_n} S_{\Lambda_n} \Psi_0(\omega)$ exists. Notice also that we may assume without loss of generality that $\Psi_0 \geq 0$. So denote

$$L = \liminf_{n \rightarrow \infty} \frac{1}{\#\Lambda_n} S_{\Lambda_n} \Psi_0(\omega)$$

and, given $\epsilon > 0$, consider a positive integer N such that $\frac{1}{\#\Lambda_N} S_{\Lambda_N} \Psi_0(\omega) < L + \epsilon$.

Consider integers m and ℓ with $m \geq 1$ and $\ell = 0, 1, \dots, N-1$. Notice that, for suitable integers $j_1, \dots, j_{m^d} \in \Lambda_{mN}$, we may write

$$S_{\Lambda_{mN+\ell}} \Psi_0 = \sum_{k=1}^{m^d} S_{\Lambda_N} \Psi_0 \circ \theta^{j_k} + \sum_{j \in \Lambda_{mN+\ell} \setminus \Lambda_{mN}} \Psi_0 \circ \theta^j.$$

Define then $\omega_{N,k} := (\theta^{j_k}(\omega))_{j_k + \Lambda_N} \omega_{\mathbb{Z}^d \setminus (j_k + \Lambda_N)}$ and $\bar{\omega}_{N,k} := \theta^{-j_k}(\omega_{N,k})$. As ω is a minimizing configuration, one thus gets

$$H_{j_k + \Lambda_N}(\omega) \leq H_{j_k + \Lambda_N}(\omega_{N,k}) = H_{\Lambda_N}(\bar{\omega}_{N,k}).$$

Recall that $\|S_{\Lambda_N} \Psi_0 \circ \theta^j - H_{j + \Lambda_N}\|_\infty \leq \Gamma N^{d-1}$ for every $j \in \mathbb{Z}^d$, where $\Gamma = \Gamma(d, r, \{\Phi_A\}) = C_d r^d \sum_{0 \in A} \|\Phi_A\|_\infty$. Therefore, we have that

$$\begin{aligned} S_{\Lambda_{mN+\ell}} \Psi_0(\omega) &\leq \sum_{k=1}^{m^d} H_{\Lambda_N}(\bar{\omega}_{N,k}) + m^d \Gamma N^{d-1} + \sum_{j \in \Lambda_{mN+\ell} \setminus \Lambda_{mN}} \Psi_0 \circ \theta^j(\omega) \\ &\leq \sum_{k=1}^{m^d} S_{\Lambda_N} \Psi_0(\bar{\omega}_{N,k}) + 2m^d \Gamma N^{d-1} + 2^d \left[(m+1)^d - m^d \right] N^d \|\Psi_0\|_\infty. \end{aligned}$$

Notice that $(\bar{\omega}_{N,k})_{\Lambda_N} = \omega_{\Lambda_N}$. We use now Lemma 3.3 to guarantee that there exists a constant $\kappa > 0$ such that $S_{\Lambda_N} \Psi_0(\bar{\omega}_{N,k}) - S_{\Lambda_N} \Psi_0(\omega) \leq \kappa \#\partial\Lambda_N$.

We obtain that

$$\begin{aligned} S_{\Lambda_{mN+\ell}} \Psi_0(\omega) &\leq \\ &\leq m^d \left[S_{\Lambda_N} \Psi_0(\omega) + \kappa \#\partial\Lambda_N + 2\Gamma N^{d-1} + 2^d \left(\left(1 + \frac{1}{m}\right)^d - 1 \right) N^d \|\Psi_0\|_\infty \right], \end{aligned}$$

which yields

$$\begin{aligned} \frac{1}{\#\Lambda_{mN+\ell}} S_{\Lambda_{mN+\ell}} \Psi_0(\omega) &\leq \\ &\leq \frac{1}{\#\Lambda_N} S_{\Lambda_N} \Psi_0(\omega) + \kappa \frac{\#\partial\Lambda_N}{\#\Lambda_N} + 2\frac{\Gamma}{N} + \left(\left(1 + \frac{1}{m}\right)^d - 1 \right) \|\Psi_0\|_\infty. \end{aligned}$$

Hence, it follows that

$$\limsup_{n \rightarrow \infty} \frac{1}{\#\Lambda_n} S_{\Lambda_n} \Psi_0(\omega) \leq L + \epsilon + \kappa \frac{\#\partial\Lambda_N}{\#\Lambda_N} + 2\frac{\Gamma}{N}.$$

Since the integer N can be taken arbitrarily large and $\epsilon > 0$ can be chosen as close as one wants to zero, the proof is complete. \square

We will adopt an ergodic point of view. To that end, denote by \mathcal{M} the set of Borel probability measures, equipped with the weak* topology. We consider the compact convex subset of invariant probabilities

$$\mathcal{M}_\theta := \left\{ \mu \in \mathcal{M} : \mu \circ \theta^j = \mu, \quad \forall j \in \mathbb{Z}^d \right\}. \quad (3.2)$$

We have then the following characterization of the minimizing ergodic value \bar{H} .

Proposition 3.7. *Let H be a hamiltonian defined by an invariant short range interaction family. Then*

$$\bar{H} = \min_{\mu \in \mathcal{M}_\theta} \int_{\Omega} \Psi_0(\omega) d\mu(\omega).$$

Proof. By the ergodic decomposition theorem (see, for example, Theorem 2.3.3 in [3]), one may suppose that $\mu \in \mathcal{M}_\theta$ is ergodic. Therefore, by Birkhoff's ergodic theorem (see, for instance, Theorem 2.1.5 in [3]), any configuration ω belonging to the support of μ satisfies

$$\lim_{n \rightarrow \infty} \frac{1}{\#\Lambda_n} S_{\Lambda_n} \Psi_0(\omega) = \int_{\Omega} \Psi_0(\omega) d\mu(\omega).$$

So thanks to Corollary 3.5, we have that $\bar{H} \leq \inf_{\mu \in \mathcal{M}_\theta} \int \Psi_0 d\mu$.

For $\epsilon > 0$, consider a configuration $\omega_\epsilon \in \Omega$ and an arbitrarily large integer $n_\epsilon > 0$ such that $\frac{1}{\#\Lambda_{n_\epsilon}} S_{\Lambda_{n_\epsilon}} \Psi_0(\omega_\epsilon) < \bar{H} + \epsilon$ and define a Borel probability measure

$$\mu_\epsilon := \frac{1}{\#\Lambda_{n_\epsilon}} \sum_{j \in \Lambda_{n_\epsilon}} \delta_{\theta^j(\omega_\epsilon)} \in \mathcal{M}.$$

Let μ be any weak* accumulation probability for the family $\{\mu_\epsilon\}_{\epsilon > 0}$ when ϵ goes to zero. Clearly by construction, $\int_{\Omega} \Psi_0(\omega) d\mu(\omega) \leq \bar{H}$. So in order to obtain the opposite inequality, it is enough to argue that μ is invariant. However, notice that, for every $i = 1, 2, \dots, n$ and for all $f \in C^0(\Omega)$, one has

$$\left| \int (f \circ \theta_i - f) d\mu_\epsilon \right| \leq \frac{1}{\#\Lambda_{n_\epsilon}} 2 \#\partial\Lambda_{n_\epsilon} \|f\|_\infty \rightarrow 0 \quad \text{as } n_\epsilon \rightarrow \infty,$$

which indeed shows the invariance of μ . \square

By a *minimizing probability* we mean an invariant probability μ that minimizes the average value $\int \Psi_0 d\mu$, namely, such that $\bar{H} = \int \Psi_0 d\mu$. Their existence is guaranteed by the previous proposition. Moreover, by the ergodic decomposition theorem, there always exist ergodic minimizing probabilities.

Recall now that a point at the support of an invariant probability is said to be *generic* if it belongs to a subset of full measure. We may complete the proof of Theorem 2.4 with the following result.

Theorem 3.8. *Suppose $\{\Phi_A\}_{A \in \mathcal{F}}$ is an invariant interaction family with range $r > 0$. Then there are generic points ω at the support of any ergodic minimizing probability which are minimizing configurations for the associated hamiltonian H and satisfy*

$$\lim_{n \rightarrow \infty} \frac{1}{\#\Lambda_n} H_{\Lambda_n}(\omega) = \bar{H}.$$

Proof. For $W \in \mathbb{F}^{\Lambda_M}$, $M \in \mathbb{N}$, consider the characteristic function $\chi_W : \Omega \rightarrow \{0, 1\}$ which has value 1 at a point $\bar{\omega} \in \Omega$ if, and only if, $\bar{\omega}_{\Lambda_M} = W$.

Let $\mu \in \mathcal{M}_\theta$ be an ergodic minimizing probability. Let us denote by $b(\phi)$ the subset of Ω of full measure for which Birkhoff's ergodic theorem holds with respect to the integrable map $\phi : \Omega \rightarrow \mathbb{R}$. Obviously, any point

$$\omega \in \text{supp}(\mu) \cap b(\Psi_0) \cap \bigcap_{W \in \mathbb{F}^{\Lambda_M}, M \in \mathbb{N}} b(\chi_W)$$

is generic and verify

$$\lim_{n \rightarrow \infty} \frac{1}{\#\Lambda_n} H_{\Lambda_n}(\omega) = \lim_{n \rightarrow \infty} \frac{1}{\#\Lambda_n} S_{\Lambda_n} \Psi_0(\omega) = \int \Psi_0 d\mu = \bar{H}.$$

Suppose on the contrary that ω is not a minimizing configuration. Hence, there shall exist $\tilde{\omega} \in \Omega$, $\tilde{N} \in \mathbb{N}$ and $\tilde{\eta} > 0$ such that

$$\tilde{\omega}_{\mathbb{Z}^d \setminus \Lambda_{\tilde{N}}} = \omega_{\mathbb{Z}^d \setminus \Lambda_{\tilde{N}}} \quad \text{and} \quad H_{\Lambda_{\tilde{N}}}(\tilde{\omega}) < H_{\Lambda_{\tilde{N}}}(\omega) - \tilde{\eta}.$$

Since in particular $\omega \in \bigcap_{M \in \mathbb{N}} b(\chi_{\omega_{\Lambda_M}})$, we have that

$$\lim_{n \rightarrow \infty} \frac{1}{\#\Lambda_n} S_{\Lambda_n} \chi_{\omega_{\Lambda_{\tilde{N}+r}}}(\omega) = \mu \left(\bigcup_{\tilde{\omega} \in \Omega} \omega_{\Lambda_{\tilde{N}+r}} \tilde{\omega}_{\mathbb{Z}^d \setminus \Lambda_{\tilde{N}+r}} \right) =: \lambda_{\tilde{N}+r} > 0.$$

Therefore, for n large enough, one guarantees that $S_{\Lambda_n} \chi_{\omega_{\Lambda_{\tilde{N}+r}}}(\omega) > \#\Lambda_n \frac{\lambda_{\tilde{N}+r}}{2}$.

Denote $A_n := \{j \in \Lambda_n : \omega_{\Lambda_{\tilde{N}+r}+j} = \omega_{\Lambda_{\tilde{N}+r}}\}$. Let then $B_n \subset A_n$ be a maximal subcollection of indices such that $(\Lambda_{\tilde{N}+r}+j) \cap (\Lambda_{\tilde{N}+r}+k) = \emptyset$ whenever $j, k \in B_n$. Since for all $j \in A_n$ there must exist $k \in B_n$ such that $(\Lambda_{\tilde{N}+r}+j) \cap (\Lambda_{\tilde{N}+r}) \neq \emptyset$, it follows that $S_{\Lambda_n} \chi_{\omega_{\Lambda_{\tilde{N}+r}}}(\omega) = \#A_n \leq \#B_n \cdot \#\Lambda_{\tilde{N}+r}$, which yields for n large enough

$$\frac{1}{\#\Lambda_n} \#B_n > \frac{1}{\#\Lambda_{\tilde{N}+r}} \frac{\lambda_{\tilde{N}+r}}{2}.$$

Thus, for n sufficiently large, let us introduce the configuration $\omega^n \in \Omega$ as $\omega^n_{\Lambda_{\tilde{N}+j}} = \tilde{\omega}_{\Lambda_{\tilde{N}}}$ for all $j \in B_n$, and $\omega^n_{\mathbb{Z}^d \setminus \sqcup_{j \in B_n} (\Lambda_{\tilde{N}+j})} = \omega_{\mathbb{Z}^d \setminus \sqcup_{j \in B_n} (\Lambda_{\tilde{N}+j})}$. From the construction, one gets that $H_{\Lambda_{\tilde{N}+j}}(\omega^n) = H_{\Lambda_{\tilde{N}}}(\tilde{\omega})$ for each $j \in B_n$. Notice then

that

$$\begin{aligned}
H_{\Lambda_n}(\omega^n) &= \sum_{j \in B_n} H_{\Lambda_{\tilde{N}+j}}(\omega^n) + \sum_{\substack{A \cap \Lambda_n \neq \emptyset, \\ A \cap \cup_{j \in B_n} (\Lambda_{\tilde{N}+j}) = \emptyset}} \Phi_A(\omega^n) \\
&< \#B_n (H_{\Lambda_{\tilde{N}}}(\omega) - \tilde{\eta}) + \sum_{\substack{A \cap \Lambda_n \neq \emptyset, \\ A \cap \cup_{j \in B_n} (\Lambda_{\tilde{N}+j}) = \emptyset}} \Phi_A(\omega) \\
&= H_{\Lambda_n}(\omega) - \tilde{\eta} \#B_n + \sum_{j \in B_n} (H_{\Lambda_{\tilde{N}}}(\omega) - H_{\Lambda_{\tilde{N}}}(\theta^j(\omega))).
\end{aligned}$$

By the very definition of A_n , one obtains that $H_{\Lambda_{\tilde{N}}}(\omega) = H_{\Lambda_{\tilde{N}}}(\theta^j(\omega))$ for any $j \in B_n$. In this way, let us also assume that n is large enough in order that

$$\frac{1}{\#\Lambda_n} H_{\Lambda_n}(\omega^n) < \bar{H} + \frac{1}{\#\Lambda_{\tilde{N}+r}} \frac{\lambda_{\tilde{N}+r}}{4} \tilde{\eta}.$$

Hence, for n sufficiently large, it is not difficult to see that

$$\frac{1}{\#\Lambda_n} H_{\Lambda_n}(\omega^n) < \bar{H} - \frac{1}{\#\Lambda_{\tilde{N}+r}} \frac{\lambda_{\tilde{N}+r}}{4} \tilde{\eta},$$

which contradicts the definition (2.2) of the constant \bar{H} . □

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