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The Cramer-Rao bound (CRB) gives a lower variance bound that an estimator can achieve. Presented is a mathematical approach to estimate it for specific radar signals, namely Costas signals and their modified versions. The modified versions design consists in widening Costas frequency hops and replacing rectangular pulses with other waveforms (namely linear frequency modulated pulses, phase codes and other Costas codes). The calculation is based on approximations of the ambiguity function (AF) around the mainlobe. The CRBs are then derived based on relationships between the AF and these bounds.

Introduction: The Cramer-Rao bound (CRB) is a well-known tool to evaluate the performances of a given estimator. Estimators that achieve this bound are called efficient estimators. The maximum-likelihood estimator is a well-known estimator that reaches the CRB for high signal-to-noise ratios (SNRs). In a radar context, the CRB bounds the error variance of the estimates of the delay τ and the Doppler ν obtained from the radar measurements. This is useful because it gives an indication of the best achievable performance, independent of the estimation technique. Furthermore, it can be used for different radar waveforms to compare their best performances. The CRB is related to the inverse of the Fisher information matrix (FIM) $\boldsymbol{J}_M$ [1]

$$\mathrm{CRB}(\tau) = [\boldsymbol{J}_M{}^{-1}(\tau,\ \nu)]_{(1,1)} \tag{1}$$

$$\mathrm{CRB}(\nu) = [\boldsymbol{J}_M{}^{-1}(\tau,\ \nu)]_{(2,2)} \tag{2}$$

The FIM is derived from the received data log-likelihood function (LLF) [2]. A relationship between the ambiguity function (AF) $\chi(\tau,\ \nu)$ and the FIM was derived, based on the observation that the AF is the LLF excluding the effect of signal attenuation and noise [1]. It was shown that the CRB depends on the second derivatives of the squared magnitude of the AF $\Theta\ (\tau,\ \nu) = |\chi(\tau,\ \nu)|^2$ [3]

$$\boldsymbol{J}_M(\tau,\ \nu) = -2\mathrm{SNR}\begin{bmatrix} \dfrac{\partial^2\Theta(\tau,\ \nu)}{\partial\tau^2} & \dfrac{\partial^2\Theta(\tau,\ \nu)}{\partial\tau\partial\nu} \\ \dfrac{\partial^2\Theta(\tau,\ \nu)}{\partial\nu\partial\tau} & \dfrac{\partial^2\Theta(\tau,\ \nu)}{\partial\nu^2} \end{bmatrix}_{(\tau,\ \nu)=(0,\ 0)} \tag{3}$$

In this Letter, it is proposed to evaluate the CRB's of Costas signals and their modified versions. A Costas signal of size N consists in N adjacent rectangular pulses of duration t_p, with a random-like frequency hopping from pulse to pulse, with step $\Delta f = 1/t_\mathrm{p}$[4]. The modified versions propose widening frequency hops, that is $\Delta f > 1/t_\mathrm{p}$, by replacing rectangular pulses with other waveforms. According to the nature of these pulse waveforms, several designs were proposed: Costas-linear frequency modulated (LFM) [5], phase-coded Costas [6] and doubly coded Costas [7]. Their main purpose is to achieve the same delay and Doppler resolutions with a smaller Costas size. To evaluate the CRB of these waveforms, closed-form equations describing the AF must be derived.

Ambiguity function: In the following, the AF of basic Costas signals, derived in [4], is extended to any modified version of Costas signals. The complex envelope of basic Costas signal of size N is given by: $u(t) = 1/\sqrt{Nt_\mathrm{p}}\sum_{n=0}^{N-1}\rho_n(t - nt_\mathrm{p})$, where $\rho_n(t) = A(t)\mathrm{e}^{\mathrm{j}2\pi f_n t}$, $f_n = a_n\Delta_f$ (a_n is the frequency order) and $A(t) = \mathrm{Rect}_{t_\mathrm{p}}(t) = \begin{cases} 1, & 0 \le t \le t_\mathrm{p} \\ 0, & \text{elsewhere} \end{cases}$. The corresponding AF is given by [4]

$$\chi(\tau,\ \nu) = \sum_{n=0}^{N-1}\frac{\mathrm{e}^{\mathrm{j}2\pi n\nu t_\mathrm{p}}}{N}\left[\Phi_{nn}(\tau,\ \nu) + \sum_{\substack{m=0 \\ m\neq n}}^{N-1}\Phi_{nm}(\tau - (n-m)t_\mathrm{p},\ \nu)\right] \tag{4}$$

where $\Phi_{nm}(\tau,\ \nu) = 1/t_\mathrm{p}\int_{-\infty}^{+\infty}\rho_n^*(t)\rho_m(t-\tau)\mathrm{e}^{\mathrm{j}2\pi\nu t}\mathrm{d}t,\ |\tau| \le t_\mathrm{p}$. We propose to extend (4) for other pulse waveforms. For modified Costas signals, $A(t)$ will be the novel pulse waveform, namely LFM pulse, phase code or Costas code and $\Phi_{nm}(\tau,\ \nu)$ changes to

$$\begin{aligned} \Phi_{nm}(\tau,\ \nu) &= \frac{1}{t_\mathrm{p}}\int_{-\infty}^{+\infty}A^*(t)\mathrm{e}^{-\mathrm{j}2\pi f_n t}A(t-\tau)\mathrm{e}^{\mathrm{j}2\pi f_m(t-\tau)}\mathrm{e}^{\mathrm{j}2\pi\nu t}\,\mathrm{d}t \\ \Phi_{nm}(\tau,\ \nu) &= \frac{1}{t_\mathrm{p}}\mathrm{e}^{-\mathrm{j}2\pi f_m\tau}\int_{-\infty}^{+\infty}A^*(t)A(t-\tau)\mathrm{e}^{\mathrm{j}2\pi(\nu+(f_m-f_n))t}\,\mathrm{d}t \\ \Phi_{nm}(\tau,\ \nu) &= \frac{1}{t_\mathrm{p}}\mathrm{e}^{-\mathrm{j}2\pi f_m\tau}\chi_A(\tau,\ \nu + (f_m - f_n)),\ |\tau| \le t_\mathrm{p} \end{aligned} \tag{5}$$

where $\chi_A(\tau,\ \nu)$ is the AF of the pulse waveform. The overall AF is then given by

$$\begin{aligned} \chi(\tau,\ \nu) = \frac{1}{Nt_\mathrm{p}}\sum_{n=0}^{N-1}\mathrm{e}^{+\mathrm{j}2\pi n\nu t_\mathrm{p}}\big[\chi_A(\tau,\ \nu)\mathrm{e}^{-\mathrm{j}2\pi f_n\tau} + \\ \sum_{\substack{m=0 \\ m\neq n}}^{N-1}\chi_A(\tau-(n-m)t_\mathrm{p},\ \nu+(f_m-f_n))\mathrm{e}^{-\mathrm{j}2\pi f_m(\tau-(n-m)t_\mathrm{p})}\big] \end{aligned} \tag{6}$$

In [4], it was stated that the mainlobe mainly depends on the first term of the sum and sidelobes on the second one. Since the CRB reflects the sharpness of the mainlobe, an approximation of the AF around it, that is $|\tau| \le t_\mathrm{p}$, is proposed

$$\chi(\tau,\ \nu) = \frac{1}{Nt_\mathrm{p}}\chi_A(\tau,\ \nu)\sum_{n=0}^{N-1}\mathrm{e}^{+\mathrm{j}2\pi(n\nu t_\mathrm{p} - f_n\tau)},\ |\tau| \le t_\mathrm{p} \tag{7}$$

The AF depends on the pulse AF $\chi_A(\tau,\ \nu)$, the adjacent pulses order n and the frequency at each pulse f_n. In the following, the closed-forms of the pulse AF $\chi_A(\tau,\ \nu)$ of investigated pulse waveform $A(t)$ is given.

For basic Costas signals, $A(t)$ is a rectangular pulse $A(t) = \mathrm{Rect}_{t_\mathrm{p}}(t)$ and $\chi_A(\tau,\ \nu)$ is given by [8]

$$\chi_A(\tau,\ \nu) = (t_\mathrm{p} - |\tau|)\frac{\sin(\pi\nu(t_\mathrm{p} - |\tau|))}{\pi\nu(t_\mathrm{p} - |\tau|)}\mathrm{e}^{-\mathrm{j}\pi\nu(t_\mathrm{p}+\tau)} \tag{8}$$

For an LFM pulse of bandwidth B_p, $A(t)$ is equal to $A(t) = \mathrm{Rect}_{t_\mathrm{p}}(t)\mathrm{e}^{\mathrm{j}\pi(B_\mathrm{p}/t_\mathrm{p})t^2}$ and $\chi_A(\tau,\ \nu)$ is given by [8]

$$\chi_A(\tau,\ \nu) = (t_\mathrm{p} - |\tau|)\frac{\sin(\pi(\nu t_\mathrm{p} - \tau B_\mathrm{p})(1 - |\tau|/t_\mathrm{p}))}{\pi(\nu t_\mathrm{p} - \tau B_\mathrm{p})(1 - |\tau|/t_\mathrm{p})}\mathrm{e}^{\mathrm{j}\pi\nu\tau} \tag{9}$$

For phase-coded Costas signals, a phase code of length L and time chip $t_\mathrm{c} = t_\mathrm{p}/L$ is used to encode the Costas pulses. The complex envelope is given by $A(t) = \sum_{l=0}^{L-1}q_l\mathrm{Rect}_{t_\mathrm{c}}(t - lt_\mathrm{c})$ and $\chi_A(\tau,\ \nu)$ is given by [9]

$$\begin{aligned} \chi_A(\tau,\ \nu) = \sum_{k=0}^{L-1}\sum_{n=0}^{L-1}q_k^*q_n\Big[\frac{1}{\pi\nu}\sin(\pi\nu(t_\mathrm{c} - |\tau + (n-k)t_\mathrm{c}|)) \\ \mathrm{e}^{\mathrm{j}\pi\nu(\tau+(n+k)t_\mathrm{c})}1_{(|\tau+(n-k)t_\mathrm{c}|<t_\mathrm{c})}\Big] \end{aligned} \tag{10}$$

For $|\tau| \le t_\mathrm{c}$ $(n = k)$, we have

$$\chi_A(\tau,\ \nu) = \frac{\sin(\pi\nu(t_\mathrm{c} - |\tau|))}{\pi\nu}\mathrm{e}^{\mathrm{j}\pi\nu\tau}\sum_{k=0}^{L-1}|q_k|^2\mathrm{e}^{\mathrm{j}2\pi\nu kt_\mathrm{c}} \tag{11}$$

Furthermore, for phase codes $|q_l|^2 = 1$, the pulse AF simplifies to

$$\chi_A(\tau,\ \nu) = \frac{\sin(\pi\nu(t_\mathrm{c} - |\tau|))}{\pi\nu}\frac{\sin(\pi\nu Lt_\mathrm{c})}{\sin(\pi\nu t_\mathrm{c})}\mathrm{e}^{\mathrm{j}\pi\nu(\tau+(L-1)t_\mathrm{c})} \tag{12}$$

For doubly coded Costas signals, the pulse waveform is another Costas signal of size L, pulse duration t_s and frequency spacing Δf_s. The complex envelope is $A(t) = \sum_{l=0}^{L-1}\rho_\mathrm{s}(t - lt_\mathrm{s})$, where $\rho_\mathrm{s}(t) = \mathrm{Rect}_{t_\mathrm{s}}(t)\mathrm{e}^{\mathrm{j}2\pi f_l t}$, $f_l = l\Delta f_\mathrm{s}$ and $t_\mathrm{s} = t_\mathrm{p}/L$. Hence, according to (7), the pulse AF is given by

$$\chi_A(\tau,\ \nu) = \chi_\mathrm{s}(\tau,\ \nu)\sum_{l=0}^{L-1}\mathrm{e}^{+\mathrm{j}2\pi(l\nu t_\mathrm{s} - f_\mathrm{s}\tau)},\ |\tau| \le t_\mathrm{s} \tag{13}$$

Here, $\chi_\mathrm{s}(\tau,\ \nu)$ is the AF of the pulse waveform of the secondary Costas signal, which is a rectangular pulse. From (8), with secondary Costas parameters t_s and Δf_s, we obtain

$$\chi_\mathrm{s}(\tau,\ \nu) = (t_\mathrm{s} - |\tau|)\frac{\sin(\pi\nu(t_\mathrm{s} - |\tau|))}{\pi\nu(t_\mathrm{s} - |\tau|)}\mathrm{e}^{-\mathrm{j}\pi\nu(t_\mathrm{s}+\tau)},\ |\tau| < t_\mathrm{s} \tag{14}$$

CRBs calculation: We propose to calculate the CRBs according to (2) and (3), from theoretical expressions of the signals' AF. Some assumptions are used: the AF is continuous and derivable around its maximum $(\tau,\ \nu) = (0,\ 0)$ with $|\chi(\tau,\ \nu)|_{(\tau,\ \nu)=(0,\ 0)} = 1$ and $\partial|\chi(\tau,\ \nu)|/$

$\partial\tau|_{(\tau,\nu)=(0,0)} = \partial|\chi(\tau,\ \nu)|/\partial\nu|_{(\tau,\nu)=(0,0)} = 0$. Hence, the second derivatives simplify to: $\partial^2|\chi(\tau,\ \nu)|^2/\partial a_1\partial a_2|_{(\tau,\nu)=(0,0)} = 2\mathrm{Re}(\partial^2\chi(\tau,\ \nu)/\partial a_1\partial a_2|_{(\tau,\nu)=(0,0)})$, where $(a_1,\ a_2) \in \{\tau,\ \nu\}$. The total AF $\chi(\tau,\ \nu)$ depends on the pulse AF $\chi_A(\tau,\ \nu)$ and $\chi_N(\tau,\ \nu) = \sum_{n=0}^{N-1} \mathrm{e}^{+\mathrm{j}2\pi(n\nu t_\mathrm{p} - f_n\tau)}$ which are maximum at (0, 0) and equal to t_p and N, respectively. Hence, second derivatives are given by $d_{a_1a_2} = \dfrac{\partial^2|\chi(\tau,\ \nu)|^2}{\partial a_1\partial a_2}|_{(\tau,\nu)=(0,0)} = \dfrac{2}{Nt_\mathrm{p}}\left[N\mathrm{Re}\left(\dfrac{\partial^2\chi_A(\tau,\ \nu)}{\partial a_1\partial a_2}|_{(\tau,\nu)=(0,0)}\right) + t_\mathrm{p}\mathrm{Re}\left(\dfrac{\partial^2\chi_N(\tau,\ \nu)}{\partial a_1\partial a_2}|_{(\tau,\nu)=(0,0)}\right)\right]$, $(a_1,\ a_2) \in \{\tau,\ \nu\}$. Then, algebraic manipulations lead to the derivation results summarised in Table 1.

Table 1: CRB $d_{a_1a_2}$ coefficients $(a_1,\ a_2) \in \{\tau,\ \nu\}$

	Standard Costas	Costas-LFM
$d_{\tau\tau}$	$-\dfrac{8\pi^2}{N}\sum_{n=0}^{N-1} f_n^2$	$-\dfrac{2\pi^2}{3}B_\mathrm{p}^2 - \dfrac{8\pi^2}{N}\sum_{n=0}^{N-1} f_n^2$
$d_{\nu\nu}$	$-\dfrac{8\pi^2}{3}t_p^2 - \dfrac{8\pi^2}{N}t_p^2\sum_{n=0}^{N-1} n^2$	$-\dfrac{2\pi^2}{3}t_\mathrm{p}^2 - \dfrac{8\pi^2}{N}t_\mathrm{p}^2\sum_{n=0}^{N-1} n^2$
$d_{\tau\nu}$	$\dfrac{8\pi^2}{N}t_\mathrm{p}\sum_{n=0}^{N-1} nf_n$	$\dfrac{2\pi^2}{3}t_\mathrm{p}B_\mathrm{p} + \dfrac{8\pi^2}{N}t_\mathrm{p}\sum_{n=0}^{N-1} nf_n$
$d_{\nu\tau}$	$\dfrac{8\pi^2}{N}t_\mathrm{p}\sum_{n=0}^{N-1} nf_n$	$\dfrac{2\pi^2}{3}t_\mathrm{p}B_\mathrm{p} + \dfrac{8\pi^2}{N}t_\mathrm{p}\sum_{n=0}^{N-1} nf_n$
	Phase-coded Costas	**Doubly coded Costas**
$d_{\tau\tau}$	$-\dfrac{8\pi^2}{N}\sum_{n=0}^{N-1} f_n^2$	$-\dfrac{8\pi^2}{L}\sum_{l=0}^{L-1} f_l^2 - \dfrac{8\pi^2}{N}\sum_{n=0}^{N-1} f_n^2$
$d_{\nu\nu}$	$-\dfrac{8}{3}\pi^2t_\mathrm{p}^2 - 2\pi^2t_\mathrm{c}^2 + 4\pi^2t_\mathrm{p}t_\mathrm{c} - \dfrac{8\pi^2}{N}t_\mathrm{p}^2\sum_{n=0}^{N-1} n^2$	$-\dfrac{8}{3}\pi^2t_\mathrm{s}^2 - \dfrac{8\pi^2}{L}t_\mathrm{s}^2\sum_{l=0}^{L-1} l^2 - \dfrac{8\pi^2}{N}t_\mathrm{p}^2\sum_{n=0}^{N-1} n^2$
$d_{\tau\nu}$	$\dfrac{8\pi^2}{N}t_\mathrm{p}\sum_{n=0}^{N-1} nf_n$	$\dfrac{8\pi^2}{L}t_\mathrm{s}\sum_{l=0}^{L-1} lf_l + \dfrac{8\pi^2}{N}t_\mathrm{p}\sum_{n=0}^{N-1} nf_n$
$d_{\nu\tau}$	$\dfrac{8\pi^2}{N}t_\mathrm{p}\sum_{n=0}^{N-1} nf_n$	$\dfrac{8\pi^2}{L}t_\mathrm{s}\sum_{l=0}^{L-1} lf_l + \dfrac{8\pi^2}{N}t_\mathrm{p}\sum_{n=0}^{N-1} nf_n$

Finally, the CRBs are given by

$$\mathrm{CRB}(\tau) = \frac{-1}{2\mathrm{SNR}}\frac{d_{\nu\nu}}{d_{\tau\tau}d_{\nu\nu} - d_{\tau\nu}d_{\nu\tau}} \tag{15}$$

$$\mathrm{CRB}(\nu) = \frac{-1}{2\mathrm{SNR}}\frac{d_{\tau\tau}}{d_{\tau\tau}d_{\nu\nu} - d_{\tau\nu}d_{\nu\tau}} \tag{16}$$

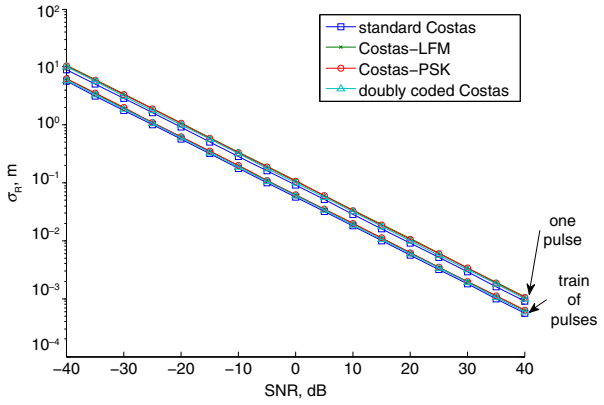

Fig. 1 *Range CRB for standard and modified Costas signals*

Simulation results: In the following, we numerically evaluate CRBs for the waveforms above with the signals bandwidth $B = 500$ MHz and duration $\sim T = Nt_\mathrm{p} = 900$ ns. Results are presented in Figs. 1 and 2 (one pulse case) where R and V refer to range and velocity with standard deviations $\sigma_\mathrm{R} = \dfrac{c}{2}\sqrt{\mathrm{CRB}(\tau)}$ and $\sigma_\mathrm{V} = \dfrac{c}{2f_\mathrm{c}}\sqrt{\mathrm{CRB}(\nu)}$. The carrier frequency is $f_\mathrm{c} = 79$ GHz. It can be seen that the CRB allows the evaluation of performances for each signal as well as comparision with other signals. In this case, signals achieve almost the same performances. For example, for SNR = 15 dB, σ_R is about 1.8 cm and σ_V is about 133 m/s.

To make velocity errors smaller, the signal duration must be increased as well as the Doppler resolution and hence the mainlobe frequency width. A well-known way to do this is to use a pulse train of K pulses with time repetition T_r instead of a single pulse. The novel AF $\chi_K(\tau,\ \nu)$, when $|\tau| \le T$, is given by [8]

$$\chi_K(\tau,\ \nu) = \chi(\tau,\ \nu)\frac{\sin(K\pi\nu T_\mathrm{r})}{K\sin(\pi\nu T_\mathrm{r})}\mathrm{e}^{\mathrm{j}\pi\nu(K-1)T_\mathrm{r}},\ |\tau| \le T \tag{17}$$

$\chi(\tau,\ \nu)$ is the single waveform AF described in (7). The improvement can be seen in Figs. 1 and 2 (train of pulses case) when previous waveforms are used in a pulse train of $K = 100$ pulses and $T_\mathrm{r} = 5T$. The effective duration is $T_K = KT_\mathrm{r}$ instead of $T = Mt_\mathrm{p}$, hence the improvement. For SNR = 15 dB, σ_R is now about 1 cm and σ_V is about 0.7 m/s.

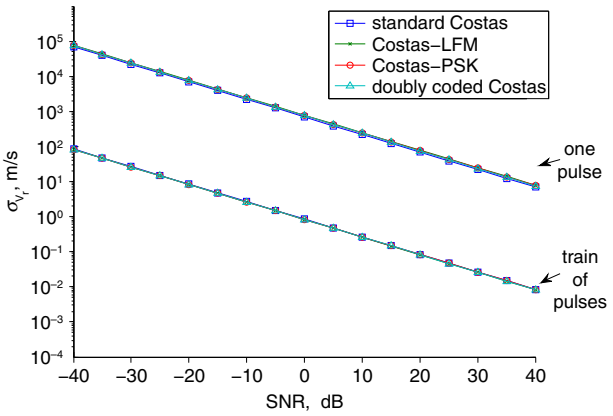

Fig. 2 *Velocity CRB for standard and modified Costas signals*

Conclusion: In this Letter, CRBs for delay and Doppler estimations are derived for basic and modified Costas signals. The calculation is based on relationships between the CRB and Fischer inverse matrix (FIM) which is also related to the AF of the signals. This last is first derived for modified Costas signals based on the AF of basic Costas signals. The CRBs are then derived using second derivatives calculations. CRBs allow the evaluation and comparison of the performances of different waveforms. In future work, other bounds will be studied in order to evaluate the threshold SNR for these waveforms.

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One or more of the Figures in this Letter are available in colour online.

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