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ON THE HAUSDORFF DIMENSION OF MINIMAL INTERVAL EXCHANGE TRANSFORMATIONS WITH FLIPS

ALEXANDRA SKRIPCHENKO AND SERGE TROUBETZKOY

ABSTRACT. We prove linear upper and lower bounds for the Hausdorff dimension set of minimal interval exchange transformations with flips (in particular without periodic points), and a linear lower bound for the Hausdorff dimension of the set of non-uniquely ergodic minimal interval exchange transformations with flips.

1. INTRODUCTION

1.1. The results. We study interval exchange transformations with flips (fiET). An fiET is an piecewise isometry of an interval to itself with a finite number of jump discontinuities, reversing the orientation of at least one of the intervals of continuity. Interval exchanges with flips appear naturally as the first return map to a transversal for vector fields on non-orientable surfaces. Interval exchange transformations without flips (IET) also appear naturally, each IET is the first return map to a transversal for a measured foliation on surface.

The set of fiETs and of IETs on n intervals are naturally parametrized by a subset of \mathbb{R}^{n-1} and some discrete parameters. The ergodic properties of IETs are well known: almost all irreducible IETs are minimal (Keane [Ke]) and almost all IETs are uniquely ergodic (Masur [Ma], Veech [Ve2]). Nogueira has shown that that fiETs have completely different dynamics

Theorem ([No]). Lebesgue almost every interval exchange transformation with flips has a periodic point.

The measure in question is the Lebesgue measure in \mathbb{R}^{n-1} . In this article, we evaluate the Hausdorff dimension of the set MF_n of minimal fiETs on n -interval, this set is subset of fiETs on n -intervals without periodic points. We prove the following

Theorem 1. The Hausdorff dimension of the set MF_n satisfies:

$$n - 3 \leq Hdim(MF_n) < n - 1.$$

We also prove a lower bound for the Hausdorff dimension of non-uniquely ergodic minimal fiETs. Let us denote this set by $NUE_n \subset MF_n$.

Theorem 2. The Hausdorff dimension of the set of non-uniquely ergodic minimal n -fIETs satisfies:

$$\left\lfloor \frac{n}{2} \right\rfloor - 2 \leq Hdim(NUE_n).$$

In the case $n = 6$, using a result of Athreya and Chaika [AtCh] we have much better lower bound:

Proposition 3. For $n = 6$

$$\frac{5}{2} \leq Hdim(NUE_6).$$

The paper is organized as follows. In Section 2 we give main definitions and briefly discuss known results and compare to the case of interval exchange transformations without flips. In Section 3 we describe our main tool - Rauzy induction for interval exchange transformations with flips, and study its combinatorics. We also introduce the notion of the cocycle associated with the Rauzy induction. Section 4 is dedicated to the Markov map associated with the Rauzy induction: we prove key features of this map and of the corresponding Markov partition, in particular we show that this map is uniform expanding in a sense of [AvGoYo]. In Section 5 we prove some distortion estimations for the cocycle based on the so called Kerckhoff lemma (see [Ker]). We mainly follow the approach suggested in [AvGoYo] for interval exchange transformations and applied in [AvRe] for linear involutions and in [AvHuSkr] for systems of isometries. Section 6 is about the roof function: using the estimations from the previous section show that the roof function has exponential tails. The proof is also inspired by the similar result in [AvGoYo]. In Section 7 we prove the upper bound for the Hausdorff dimension announced in Theorem 1. The proof is based on the argument presented in [AvDe]: first, we show that the exponential tail of the roof function implies that the corresponding Markov map is fast decaying in a sense of [AvDe] and then using [AvDe] check that this property implies the estimation we are interested in. Section 8 completes the proof of Theorem 1: we show the lower bound applying the construction described by Nogueira in [No]. Using the same idea, we prove Theorem 2 and Proposition 3.

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2. DEFINITIONS AND KNOWN RESULTS

2.1. Interval exchange transformations. Let $I \subset \mathbb{R}$ be an interval (say, $I = [0, 1)$) and $\{I_\alpha : \alpha \in \mathcal{A}\}$ be a partition of I into subintervals, of the form $[a, b)$, indexed by some alphabet \mathcal{A} on $n \geq 2$ symbols. An *interval exchange*

transformation (IET) is a bijective map from I to I which is a translation on each subinterval I_α .

Such a map f is determined by the following collection of combinatorial and metric data:

- a vector $\lambda = (\lambda_1, \dots, \lambda_n)$ of lengths of the subintervals. Each component λ_α is positive.
- a permutation $\pi = (\pi_0, \pi_1)$ which is a pair of bijections $\pi_i : \mathcal{A} \rightarrow \{1, \dots, n\}$ describing the ordering of the subintervals I_α before and after the map is iterated;

Each IET preserves the measure and the orientation on I .

Definition. Let $f : X \rightarrow X$ be an invertible map. The *orbit* of $x \in X$ is the subset $\{f^k(x) ; k \in \mathbb{Z}\}$.

Definition. A map $f : X \rightarrow X$ is called *minimal* if all orbits are dense in X .

In [Ke] M. Keane proved that almost all IETs for which π is irreducible are minimal. In this article we are only interested in transitive (minimal) IETs, so we assume that π is always irreducible.

Definition. A measure preserving map $f : (X, \mu) \rightarrow (X, \mu)$ is called *uniquely ergodic* if it admits which is necessarily ergodic.

H. Masur in [Ma] and W. Veech in [Ve2] proved that in case of irreducible permutations almost all IET are uniquely ergodic (and so, every invariant measure is a multiple of Lebesgue measure).

2.2. Interval exchange transformations with flips. fIETs are a generalization of IETs. Informally, a fIET f is a piecewise linear map from I to I such that f acts an isometry on each I_α , so that the images of interiors of partition elements do not overlap, which does not preserve the orientation: on some subintervals f inverts it.

We proceed with the precise definition. Consider the interval $I = [0, 1)$, and the partition I_α such that $I_k = [\beta_{k-1}(\lambda), \beta_k(\lambda)]$, where $\beta_i(\lambda) = \lambda_1 + \dots + \lambda_i$ for $i = 1, \dots, n$. We also fix a permutation π and a subset $F \subset \mathcal{A}$. We denote by λ^π the vector $(\lambda_{\pi^{-1}(1)}, \dots, \lambda_{\pi^{-1}(n)})$ and set $\beta^\pi(\lambda) = \beta_i(\lambda^\pi)$.

Definition. The fIET with flip set F is the following map:

$$f(x) = \begin{cases} x - \beta_{i-1} + \beta_{\pi(i)-1}^\pi & \text{if } x \in I_i, i \notin F, \\ \beta_i - (x - \beta_{i-1}) - \beta_{i-1} + \beta_{\pi(i)-1}^\pi, & \text{if } x \in I_i, i \in F. \end{cases}$$

One can see that an n -fIET is determined by the following data:

- a vector $\lambda = (\lambda_1, \dots, \lambda_n)$ of lengths of the subintervals. Each component λ_α is positive;
- a signed permutation $\theta\pi$, where π is as above and $\theta \in \{-1, 1\}^n$. The signed permutation describes the order of the subintervals I_α before and after the map is iterated as well as the subset of flips (that is

marked by $\theta_i = -1$). We sometimes denote the signed permutation corresponding to π by $\hat{\pi}$.

Throughout the article we will reserve the notation fIET to the case when the flip set is non-empty, and use the notation IET when the flip set is empty.

As mentioned above, the dynamics of fIETs is diametrically opposite to the dynamics of classical IETs: typical map has a periodic points ([No]). The first example of a minimal fIET was constructed in [No] (we refer to it as the Nogueira construction). Some examples of minimal and uniquely ergodic fIETs can be also found in [GLMPZh], and a analyse of the number of periodic and minimal components of fIETs can be found in [NoPiTr].

3. RAUZY INDUCTION

A detailed description of Rauzy induction for fIETs can be found in [No] and [GLMPZh]. Here we present a brief scheme of the induction algorithm. In this chapter we will neglect all fIET such that $\lambda_n = \lambda\pi^{-1}(n)$ that form a set of zero measure.

The term *Rauzy induction* refers to the operator R on the space of fIETs that associates to each fIET $f = (\lambda, \hat{\pi})$ with irreducible $\hat{\pi}$ another fIET $f' = R(f) = R(\lambda, \hat{\pi})$ which is the first return map induced by f on a subinterval $[0, \nu]$ where ν is as follows:

$$\nu = \begin{cases} \beta_n(\lambda) - \lambda_{\pi^{-1}(n)} & \text{if } \lambda_{\pi^{-1}(n)} < \lambda_n \\ \beta_n(\lambda) - \lambda_n, & \text{if } \lambda_{\pi^{-1}(n)} > \lambda_n. \end{cases}$$

Rauzy induction is not defined in the case that $\lambda_{\pi^{-1}(n)} = \lambda_n$, the set of fIETs satisfying this equality is of Hausdorff dimension $n - 1$ and thus does not interest us.

One can check that

$$(1) \quad f' = (\lambda', \hat{\pi}') = \begin{cases} ((I_a(\hat{\pi}))^{-1}\lambda, a(\hat{\pi})) & \text{if } \lambda_{\pi^{-1}(n)} < \lambda_n \\ ((I_b(\hat{\pi}))^{-1}\lambda, b(\hat{\pi})), & \text{if } \lambda_{\pi^{-1}(n)} > \lambda_n, \end{cases}$$

where the transition matrices $I_a(\hat{\pi}), I_b(\hat{\pi}) \in SL(n, \mathbb{Z})$ and transition maps $a(\hat{\pi}), b(\hat{\pi})$ are defined below.

$$I_a(\hat{\pi}) = E + E_{n, \pi^{-1}(n)},$$

where E is an identity matrix and $E_{i,j}$ is the elementary matrix containing 1 as (i, j) -th element;

$$I_b(\hat{\pi}) = \sum_{i=1}^{\pi^{-1}(n)} E_{i,i} + E_{n, s(\hat{\pi})} + \sum_{i=\pi^{-1}(n)}^{n-1} E_{i, i+1},$$

where $s(\hat{\pi}) = \pi^{-1}(n) + \frac{1+\theta_{\pi^{-1}(n)}}{2}$.

The transition maps are defined as follows: if $\theta_n = 1$, then

$$a(\theta\pi) = \begin{cases} \theta_i\pi_i & \text{if } \pi_i \leq \pi_n \\ \theta_i(\pi_n + 1) & \text{if } \pi_i = n \\ \theta_i(\pi_i + 1) & \text{otherwise,} \end{cases} \quad b(\theta\pi) = \begin{cases} \theta_i\pi_i & \text{if } i \leq \pi_n^{-1} \\ \theta_n\pi_n & \text{if } i = \pi^{-1}(n) + 1 \\ \theta_{i-1}(\pi_{i-1}) & \text{otherwise,} \end{cases}$$

while if $\theta_n = -1$, then

$$a(\theta\pi) = \begin{cases} \theta_i\pi_i & \text{if } \pi_i \leq \pi_n - 1 \\ -\theta_i\pi_n & \text{if } \pi_i = n \\ \theta_i(\pi_i + 1) & \text{otherwise,} \end{cases} \quad b(\theta\pi) = \begin{cases} \theta_i\pi_i & \text{if } i \leq \pi_n^{-1} - 1 \\ -\theta_n\pi_n & \text{if } i = \pi^{-1}(n) \\ \theta_{i-1}(\pi_{i-1}) & \text{otherwise.} \end{cases}$$

In case that the operation a was used, we say that the element n was the *winner* and the element $\pi^{-1}(n)$ was the *loser*; in case that the operation b was applied, the terminology is the opposite.

More generally, if the alphabet \mathcal{A} has n letters and if α and β were the last elements of non-signed permutation π , then, depending on the inequalities written above, we say that α is a winner and β is a loser (or vice versa). Sometimes, since we also work with the signed permutation $\hat{\pi}$, we can specify that α or $-\alpha$ was the winner or the loser.

Remark. We will iterated the Rauzy induction map, however Rauzy induction is not defined everywhere, thus the iteration stops if we arrive to a point outside its domain of definition (see [No] for an example when it stops). In case of an IET (without flips) which satisfies Keane's condition this never happens.

3.1. The Rauzy graph in the case of fIETs. As in case of IETs, one can define Rauzy classes for fIETs. Given pairs $\hat{\pi}$ and $\hat{\pi}'$, we say that $\hat{\pi}'$ is a *successor* of $\hat{\pi}$ if there exist λ, λ' such that $R(\hat{\pi}, \lambda) = (\hat{\pi}', \lambda')$. In the case of IETs, every permutation has exactly two successors; while for fIETs it has four successors.

However, the property that the successors of every irreducible permutations are also irreducible does not hold for fIETs. So, this relation defines a partial order on a set of irreducible permutations, plus a so called *hole*, the set of all $(\hat{\pi}, \hat{\lambda})$ without successors, the hole contains all $(\hat{\pi}, \hat{\lambda})$ with $\hat{\pi}$ reducible; this order can be represented by a directed graph G that is called the *Rauzy graph*. As in case of IETs the connected components of this graph are called *Rauzy diagram* or the *Rauzy class*. Each Rauzy class $\mathcal{R}(\mathcal{A})$ contains a vertex that corresponds to the hole; there are no paths that start in the hole.

A *path* of length $m \geq 0$ in the diagram is a sequence (finite or infinite) v_0, \dots, v_m , of vertices (signed permutations) and a sequence of arrows a_1, \dots, a_m such that a_i starts at v_{i-1} and ends in v_i . The following obvious lemma holds:

Lemma 4. If $\hat{\pi}$ and $\hat{\pi}'$ are in the same Rauzy class then there exists an oriented path in G starting at $\hat{\pi}$ and ending at $\hat{\pi}'$.

As it was mentioned above, the Rauzy induction data contains two parts: metrical (lengths of the intervals) and combinatorial (vertices of the Rauzy diagram). In [No] Nogueira introduced a notion of *expansion* as a way to describe the metrical part of the Rauzy induction. The expansion that can be infinite (if the Rauzy induction can be iterated infinite number of times) and finite (that correspond to the case when the path in the Rauzy graph contains a hole and therefore the induction stops). Using this terminology, we call the path that corresponds to the infinite expansion also *infinite*. The following obvious lemma holds:

Lemma 5. Each infinite path in the Rauzy diagram corresponds to a minimal fIET.

Proof. Infiniteness of the path means the infiniteness of the expansion, so the Rauzy induction can be applied infinite number of times and then the support interval becomes arbitrarily small. This implies that the original fIET was minimal. The proof of the same statement in oriented case can be found in [Vi] (Lemma 4.4 and Corollaries 5.2 and 5.4 imply it); see also Chapter 4.3 in [Yo]. \square

For each Rauzy class \mathcal{R} we denote the set of paths on it by $\Pi(\mathcal{R})$ (and refer to this set of paths as the Rauzy class); for each path γ (finite or infinite) there is the Rauzy operator R_γ that corresponds to it. The matrix of the Rauzy induction, that is the product of I_a and I_b , is also denoted by R_γ . We denote by the matrix $B_\gamma = R_\gamma^T$ sometimes will be referred as *matrix of the cocycle* (the same construction is used in the case of IETs in connection with zippered rectangles as a suspension model, see [Ve2] and [Vi]).

Definition. Let \mathcal{R} be a Rauzy class. A path $\gamma \in \Pi(\mathcal{R})$ is called *complete* if every letter α of the alphabet \mathcal{A} is the winner of some arrow composing γ .

Definition. We say that $\gamma \in \Pi(\mathcal{R})$ is *positive* if all entries of the matrix corresponding to R_γ are positive.

As in case of IETs, the following holds (see [AvGoYo]):

Lemma 6. Every path that is long enough in terms of Rauzy induction is complete and, moreover, positive.

A more precise description of the positive paths in case of fIETs can be found in [No] (Lemma 2.1).

3.2. Acceleration. Rauzy induction has fixed neutral points which means that any absolutely continuous invariant measure is necessarily infinite. In the case of IETs A. Zorich introduced an *accelerated* algorithm of Rauzy induction ([Zo]). It is an analogue of the acceleration of the Euclid algorithm. This idea can be used directly in case of fIETs without any significant changes.

As it was mentioned above, the matrix of Rauzy induction is a product of matrices I_a and I_b for different a and b . Let us for each stage j of Rauzy

induction define the indicator $\epsilon \in \{-1, 1\}$ such that $\epsilon = 1$ if one applied matrix I_a and $\epsilon = -1$ if I_b was the matrix of the corresponding stage of the Rauzy induction. Then, *Zorich induction* is given by the following operator:

$$(\lambda', \pi') = Z(\pi, \lambda) = R^n(\pi, \lambda),$$

where n is the smallest $j \geq 1$ one such that $\epsilon^{(j)} = -\epsilon^{(0)}$.

One can associate Rauzy graphs with the accelerated induction in the same way as for the usual induction. In this paper we work with minimal fIETs. We exclude the hole vertex from the Rauzy graphs (we call this exclusion an “adjustment”).

4. MARKOV MAP: DEFINITIONS AND PROPERTIES

4.1. Definition. The Markov map T is the projectivization of the induction map described above: $T(\lambda, \hat{\pi}) := \left(\frac{\lambda'}{|\lambda'|}, \hat{\pi}'\right)$, where $|\lambda| := \beta_n(\lambda) = \lambda_1 + \dots + \lambda_n$ and $\lambda', \hat{\pi}'$ are defined by the accelerated Rauzy induction: $T(\lambda, \hat{\pi}) = \left(\frac{R_{\gamma}^{-1}\lambda}{\|R_{\gamma}^{-1}\lambda\|}, \hat{\pi}'\right)$ (if the acceleration Rauzy induction is not defined at a point then we define its image by the last possible application of (1)).

4.2. Markov partition. One can check that the map T determines Markov partition $\Delta^{(l)}$ of the parameter space like in case of orientation-preserving IETs (see [Vi]). The only difference is that the induction stops in some Markov cells; we are interested in the set of points where induction can be applied for an infinite time.

4.3. Markov shift. One can also consider the action of the non-accelerated Rauzy induction on the accelerated adjusted Rauzy graph. Then, each vertex of the adjusted Rauzy graph will split into countable number of vertices, and the same happens to the corresponding Markov cell. Then the Rauzy induction corresponds to a Markov shift σ in this coding on a countable alphabet. One can associate in a natural way a graph Γ with such a Markov shift. Γ can be obtained from the Rauzy graph by dividing every vertex into a countable number of vertices and adding a required arrows between these new vertices.

Definition. A countable Markov shift Θ with transition matrix U and set of states \mathcal{S} satisfies the *big images and pre-images property* (BIP) if there exist a finite subset of the states of the Markov shift such that the image under the action of the Markov shift of any state contains some element of this finite set, and furthermore the image of this finite subset contains the whole set of states.

As in case of IETs, the following obvious lemma holds for fIETs:

Lemma 7. The Markov shift σ satisfies the BIP property.

Proof. One can check that it is enough to choose i_j such that each belongs to a different vertex of the accelerated Rauzy graph (for each Rauzy class); note that the number of classes is finite and depends on n (this determines the parameter m in the definition). \square

4.4. The Markov map is uniformly expanding. As it was mentioned above, we use so called Manhattan norm for vectors: $\|v\| = \sum_{i=1}^n v_i$, note that all the vectors we work with are positive. We will also need operator norm $|A| = \sup_{\|v\|=1} Av$ and the norm on continuous functions on compact: $\|f\|_{C^0(\Delta)} = \sup_{x \in \Delta} |f(x)|$.

Definition. Let L be a finite or countable set, let Δ be a parameter space, and let $\{\Delta^{(l)}\}_{(l \in L)}$ be a partition into open sets of a full measure subset of Δ . A map $Q : \cup_l \Delta^{(l)} \rightarrow \Delta$ is a *uniformly expanding* map if:

- there exist a constant $k > 1$ such that for each l , Q is a C^1 diffeomorphism between $\Delta^{(l)}$ and Δ , and there exist constant $C_{(l)}$ such that for all $x \in \Delta^{(l)}$ and all $v \in Q_x \Delta$, $k\|v\| \leq \|DQ(x)v\| \leq C_{(l)}\|v\|$.
- Let $J(x)$ be the inverse of the Jacobian of Q with respect to Lebesgue measure. Denote by \mathcal{H} the set of inverse branches of Q . The function $\log J$ is C^1 on each set $\Delta^{(l)}$ and there exists $C > 0$ such that, for all $h \in \mathcal{H}$,

$$\|D((\log J) \circ h)\|_{C^0(\Delta)} \leq C.$$

We need also one more technical definition (see [AvGoYo]):

Definition. The positive path γ on the Rauzy graph is called a *neat* if starts and ends in the same vertex of the graph and the following condition holds: if $\gamma = \gamma_s \gamma_0 = \gamma_0 \gamma_e$ for some γ_s and γ_e , then either $\gamma = \gamma_0$ or γ_0 is trivial.

Remark. The definition of a neat guarantees that the associated induction matrix is strictly positive.

Starting from this moment we consider a special precompact section: we deal only with paths that contain a neat, i.e. $\gamma = \gamma_s \gamma_{neat} \gamma_e$ for some (arbitrary) γ_s, γ_e . Equivalently, we can say that we additionally accelerate the induction. See section 6.2 for a proof of the correctness of the model.

Lemma 8. The map T is uniformly expanding with respect to the Markov partition $(\Delta^{(l)})$.

Proof. The same lemma was proved for IET in [AvGoYo] (Lemma 4.3) and for linear involutions in [AvRe] (Lemma 6.1). The main idea of the proof remains the same for our case and is briefly described below.

The first part of the definition of uniformly expanding comes from the zippered rectangle suspension model. More precisely, as mentioned above we consider a long enough path γ such that it contains some positive neat. Then the induction matrix R_γ^{-1} is a product of two matrices such that one of them is weakly contracting and another one is strongly contracting with

respect to Hilbert metric. The strong contraction comes from the positivity of the neat.

The second part of the definition of uniformly expanding can be verified as follows. First, we notice that an inverse branch of the map T can be written as $h(\lambda, \hat{\pi}) = (\frac{R_\gamma \lambda}{\|R_\gamma \lambda\|}, \hat{\pi})$. Veech showed that is $J \circ h = \frac{1}{\|R_\gamma \lambda\|^n}$ where n is the number of intervals of the IET [Ve1], Proposition 5.3; the proof holds verbatim for fIETs. So we have:

$$\frac{J \circ h(\lambda, \pi)}{J \circ h(\lambda', \pi)} = \left(\frac{\|R_\gamma \lambda\|}{\|R_\gamma \lambda'\|} \right)^n \leq \sup_{\alpha \in 1, \dots, n} \left(\frac{\lambda_\alpha}{\lambda'_\alpha} \right)^n \leq e^{n \cdot \text{dist}(\lambda, \lambda')}.$$

and thus, $\log J \circ h$ is Lipschitz with respect to the Hilbert metric (which is denoted by dist). \square

5. THE DISTORTION ESTIMATES

5.1. Conditional probabilities. The distortion argument will involve the study of the forward images of the Lebesgue measure under the renormalization map. Following the strategy from [AvGoYo] and [AvRe], we first construct a class of measures which is invariant as a class.

Let us consider the accelerated Rauzy graph and some path γ in it. Let us fix the vertex $\hat{\pi}$ of this graph and the corresponding Rauzy class \mathcal{R} .

As in Section 3.2, B_γ is the matrix of the cocycle corresponding to γ , and B_γ^T denotes its transpose. Let \mathbb{R}_+^n denote the positive cone and $\langle \cdot, \cdot \rangle$ denote the inner product, then define

$$\Lambda_q := \{\lambda \in \mathbb{R}_+^n : \langle \lambda, q \rangle < 1\},$$

and

$$\Delta'_\gamma := B_\gamma^T \mathbb{R}_+^n.$$

For $q = (q_1, \dots, q_n) \in \mathbb{R}_+^n$ we define a measure ν_q on the σ -algebra $\mathcal{A} \subset \mathbb{R}_+^n$ of Borel sets which are positively invariant (i.e. $\mathbb{R}_+ A = A$):

$$\nu_q(A) := \text{Leb}(A \cap \Lambda_q).$$

Equivalently, ν_q can be considered as a measure on the projective space $\mathbb{R}P_+^{n-1}$. Using Proposition 5.4 in [Ve1], one can check that

$$\nu_q(\mathbb{R}_+^n) = \frac{1}{n! q_1 \cdots q_n},$$

$$\nu_q(\Delta'_\gamma) = \frac{1}{n! (B_\gamma q)_1 \cdots (B_\gamma q)_n},$$

and

$$\nu_q(B_\gamma^T A) = \text{Leb}(B_\gamma^T A \cap \Lambda_q) = \text{Leb}(A \cap \Lambda_{B_\gamma q}) = \nu_{B_\gamma q}(A).$$

The measures ν_q are used to calculate the probabilities of realization of different types of combinatorics related to the induction.

Let \mathcal{R} be a Rauzy class and let $\gamma \in \Pi(\mathcal{R})$. Let $\Lambda_{q,\gamma} := \Lambda_{B_\gamma q}$. If $\Gamma \in \Pi(\mathcal{R})$ is a set of paths starting with the same $\hat{\pi} \in \mathcal{R}$, let $\Lambda_{q,\Gamma} = \cup_{\gamma \in \Gamma} \Lambda_{q,\gamma}$. We define

$$P_q(\Gamma|\gamma) := \frac{\text{Leb}(\Lambda_{q,\Gamma_\gamma})}{\text{Leb}(\Lambda_{q,\gamma})} = \frac{\nu_q(\cup_{\gamma' \in \Gamma_\gamma} \Delta'_{\gamma'})}{\nu_q(\Delta'_\gamma)},$$

where $\Gamma_\gamma \subset \Gamma$ is the set of paths starting by γ .

Suppose γ is a path of length one (an arrow), let $\hat{\pi} \in \mathcal{R}$ be the permutation from which γ starts and denote by $\alpha \in \mathcal{A}$ the winner and by $\beta \in \mathcal{A}$ the loser of the first iteration of the Rauzy induction (without acceleration). Then, the conditional probability related to the given combinatorics is defined by

$$P_q(\gamma|\hat{\pi}) := \frac{q_\beta}{(q_\alpha + q_\beta)}.$$

For a long path, for $\mathcal{A}' \subset \mathcal{A}$ and $q \in \mathbb{R}_+^{\mathcal{A}} = \mathbb{R}_+^n$, let $N_{\mathcal{A}'}(q) := \prod_{\alpha \in \mathcal{A}'} q_\alpha$ and, let $N(q) := N_{\mathcal{A}}(q)$, and define

$$P_q(\gamma|\hat{\pi}) := \frac{N(q)}{N(B_\gamma q)}.$$

Let us introduce a partial order on the set of paths: for two given path γ, γ' we say that $\gamma \leq \gamma'$ if $\gamma' = \gamma\gamma_e$ for some γ_e . We say that the subset Γ is *disjoint* if no two elements are comparable with respect to this order. Now, for every family $\Gamma \subset \Pi(\mathcal{R})$ such that any $\gamma \in \Gamma$ start by some element $\gamma_s \in \Gamma_s$, for every $\hat{\pi} \in \mathcal{R}$ we define

$$(2) \quad P_q(\Gamma|\hat{\pi}) := \sum_{\gamma_s \in \Gamma_s} P_q(\Gamma|\gamma_s)P_q(\gamma_s|\hat{\pi}).$$

Note also that

$$(3) \quad P_q(\Gamma|\hat{\pi}) \leq P_q(\Gamma_s|\hat{\pi}) \sup_{\gamma_s \in \Gamma_s} P_q(\Gamma|\gamma_s).$$

5.2. Kerckhoff lemma. In this section we prove the key estimate necessary for the estimation of the distortion properties of the cocycle matrix. The idea that was used for the first time in [Ker] for IETs is the following: in order to control how the induction distorts a vector which was originally was balanced, one has to check that the ratio between the norms of rows (or equivalently, columns) of the matrix of the cocycle (equivalently, of the induction matrix) can rarely be very high. More formally, we have the following:

Lemma 9. Let $\hat{\pi} \in \mathcal{R}$ be irreducible. For any $T > 0, q \in \mathbb{R}_+^{\mathcal{A}}, \alpha \in \mathcal{A}$

$$P_q\left(\{\gamma \in \Gamma_\alpha(\hat{\pi}) : (B_\gamma q)_\alpha > Tq_\alpha\}|\hat{\pi}\right) < T^{-1},$$

where $\Gamma_\alpha(\hat{\pi})$ denotes the set of paths starting at $\hat{\pi}$ with no winner equal to α .

Proof. The proof coincides with the proof of the same statement for IETs, see [Ker][Proposition 1.3] or [No][Proposition 3.5]. \square

Now we apply the Kerckhoff lemma to obtain some more subtle estimations on the distortion. The main idea is as follows: we want to define the first return time to the small subsimplex of the original parameter space; however, some (minimal) orbits that start in the fixed subsimplex will not go back since they stick somewhere close to the boundary; our purpose is to check that the probability of this event is in some sense low.

We mainly follow the strategy suggested in Appendix A of [AvGoYo]. Before we actually state the theorem, let us introduce some useful notation:

$$\begin{aligned} \mathcal{A}' &\subset \mathcal{A}; \\ m_{\mathcal{A}'}(q) &= \min_{\alpha \in \mathcal{A}'} q_\alpha; \\ m(q) &= m_{\mathcal{A}}(q) \\ m_k(q) &= \max_{\{\mathcal{A}' \subset \mathcal{A}: |\mathcal{A}'|=k\}} m_{\mathcal{A}'}(q); \\ M_{\mathcal{A}'}(q) &= \max_{\alpha \in \mathcal{A}'} q_\alpha; \\ M(q) &= \max_{\alpha \in \mathcal{A}} q_\alpha. \end{aligned}$$

The principal result of this section is the following

Theorem 10. Let $\hat{\pi} \in \mathcal{R}$ be irreducible. There exists $C > 1$ such that for all $q \in \mathbb{R}_+^{\mathcal{A}}$

$$P_q\left(\{\gamma : M(B_\gamma q) < C \min\{m(B_\gamma q), M(q)\}\} | \hat{\pi}\right) > C^{-1}.$$

Proof. The main idea of the proof comes from [AvGoYo]: one should consider all the subsets $\mathcal{A}' \subset \mathcal{A}$ of fixed cardinality k and prove that for each $1 \leq k \leq n$ there exists $C > 1$ (depending on k) such that

$$(4) \quad P_q\left(\{\gamma : M(B_\gamma q) < C \min\{m_k(B_\gamma q), M(q)\}\} | \hat{\pi}\right) > C^{-1}.$$

The case $k = n$ implies the desired statement. The proof is by induction on k .

For $k = 1$ we have $M(B_\gamma q) = m_1(B_\gamma q) \geq M(q)$ and so one has to estimate the probability of the following event $E_1 = \{\gamma : M(B_\gamma q) < CM(q)\}$. Let E^c denote the complement of the event E ; then

$$E_1^c = \{\gamma : M(B_\gamma q) \geq CM(q)\} = \bigcap_i \{\gamma : M(B_\gamma q) \geq Cq_i\}, i = 1, \dots, n.$$

But $M(B_\gamma q) = (B_\gamma q)_j$ for some j , so $E_1^c \subset \cup_{j=1}^n E_j$ where $E_j = \{\gamma : (B_\gamma q)_j \geq Cq_j\}$. Then Lemma 9 implies that for any $C > 0$ we have $P_q(E_1^c) < \sum_j P_q(E_j) < nC^{-1}$, and thus $P_q(E_1) \geq 1 - \frac{n}{C} > \frac{1}{C}$ for any $C > n + 1$.

Now, we make the induction step. Let us assume that Equation (4) holds for some $1 \leq k < n$ with some constant C_0 . We denote by Γ the set of shortest possible paths γ starting at $\hat{\pi}$ with $M(B_\gamma q) < C_0 \min\{m_k(B_\gamma q), M(q)\}$ (by the length of the path we mean the number of steps of the accelerated induction). We will construct a family of paths such that the statement holds

for all paths from this family for some subset \mathcal{A}'' of cardinality $k + 1$ (it is enough because m_k is a maximum taken over all subsets of fixed cardinality).

Recall that

$$m_k(B_\gamma q) = \max_{\{\mathcal{A}' \subset \mathcal{A}: |\mathcal{A}'|=k\}} m_{\mathcal{A}'}(B_\gamma q) = m_{\mathcal{A}'}(B_\gamma q)$$

for some \mathcal{A}' with cardinality k ; so there exists a non-empty set of paths $\Gamma_1 \subset \Gamma$ such that if $\gamma \in \Gamma_1$ then $m_k(B_\gamma q) = m_{\mathcal{A}'}(B_\gamma q)$. Since non-empty sets of finite paths have positive probability we can find $C_1 > 1$ such that $P_q(\Gamma_1|\hat{\pi}) > C_1^{-1}$.

For each $\gamma_s \in \Gamma_1$ consider all paths of $\gamma := \gamma_s \gamma_e$ with minimal length such that γ ends at a permutation $\hat{\pi}_e$, such that the top or the bottom row of $\hat{\pi}_e$ (possibly both) ends with some element that does not belong to $\mathcal{A}' \cup -\mathcal{A}'$. Let Γ_2 be the collection of paths γ obtained in this way. Since Γ_2 is non-empty and non-empty sets of finite paths have positive probability there exists $A_2 > 1$ such that $P_q(\Gamma_2|\hat{\pi}) > A_2^{-1}$.

On the other hand, Γ_2 is the set of shortest paths with the described combinatorics which implies that all paths containing loops can be excluded from the set of possible γ_e , and so the length of γ_e is naturally bounded (because we always choose such a path that two consequent steps of the Rauzy induction have different winners, and the set \mathcal{A}' has bounded cardinality). The last argument implies that there exists a constant B_2 such that $M(B_\gamma q) < B_2 M(B_{\gamma_s} q)$ for $\gamma = \gamma_s \gamma_e$. Let $C_2 := \max(A_2, B_2)$. Then, both inequalities are satisfied: $P_q(\Gamma_2|\pi) > C_2^{-1}$ and $M(B_\gamma q) < C_2 M(B_{\gamma_s} q)$.

Let Γ_3 be a set of paths $\gamma := \gamma_s \gamma_e$, where $\gamma_s \in \Gamma_2$, $(B_\gamma q)_\alpha \leq 2n \cdot (B_{\gamma_s} q)_\alpha$ for all $\alpha \in \mathcal{A}'$ and also several combinatorial conditions hold:

- the winner of the last arrow of γ_e belongs either to \mathcal{A}' or to $-\mathcal{A}'$;
- the winners of other arrows does not belong to \mathcal{A}' and to $-\mathcal{A}'$.

This set of paths is essentially the set described in Lemma 9 (the winners of arrows should not belong to \mathcal{A}') with exactly one difference: one has to consider a conditional probability $P_q(\Gamma_3|\gamma_s)$ with respect to γ_s , not some fixed permutation $\hat{\pi}$. More precisely, for fixed γ_s let $\Gamma_3^{\gamma_s} := \{\gamma_e : \gamma_s \gamma_e \in \Gamma_3\}$. Suppose that γ_s ends at the permutation $\hat{\pi}$, then $P_q(\Gamma_3|\gamma_s) = P_{B_{\gamma_s} q}(\Gamma_3^{\gamma_s}|\hat{\pi})$. So, we apply Lemma 9 with $T = \frac{1}{2n}$ and see that $P_q((\Gamma_3|\gamma_s)^c) \leq \frac{1}{2n}$ and so $P_q(\Gamma_3|\gamma_s) \geq 1 - \frac{1}{2n} = \frac{2n-1}{2n} > \frac{1}{2}$.

This implies that $P_q(\Gamma_3|\hat{\pi}) = P_q(\Gamma_3|\gamma_s)P_q(\gamma_s|\hat{\pi}) > (2C_2)^{-1}$.

Let $\gamma := \gamma_s \gamma_e \in \Gamma_3$. If $M(B_\gamma q) > 2n \cdot M(B_{\gamma_s} q)$ we consider $\gamma' = \gamma_s \gamma_e$ the minimal length prefix of γ for which the same inequality holds: $M(B_{\gamma'} q) > 2n \cdot M(B_{\gamma_s} q)$. Then, there exists $\alpha \notin \pm \mathcal{A}'$ such that $M(B_{\gamma'} q) = (B_{\gamma'} q)_\alpha \leq 4n \cdot M(B_{\gamma_s} q)$ (it follows from the fact that γ_1 is of minimal length and that the Mq can at most double after one step of the non-accelerated Rauzy induction). Together with the assumption that the statement holds for k it implies that

$$m_{\mathcal{A}'}(B_{\gamma_1} q) > (C_0 C_2 4n)^{-1} M(B_{\gamma_1} q).$$

If $M(B_\gamma q) \leq 2nM(B_{\gamma_s} q)$ the loser α of the last arrow of γ satisfies the following inequality:

$$(B_{\gamma_1} q)_\alpha \geq (C_0 C_2 4n)^{-1} M(B_{\gamma_1} q)$$

(by the same calculation as above). Our α did not belong to \mathcal{A}' .

In any case, we construct the family Γ_4 (it contains γ' presented above) and \mathcal{A}' of cardinality $k + 1$ for which the Inequality (4) holds. \square

Now we prove some more subtle distortion estimate.

Theorem 11. For every $\hat{\gamma} \in \Pi(\mathcal{R})$ there exist $\delta > 0, C > 0$ such that for every $\hat{\pi} \in \mathcal{R}, q \in \mathbb{R}_+^A$ and for every $T > 1$

$$P_q(\gamma \text{ cannot be written as } \gamma_s \hat{\gamma} \gamma_\epsilon \text{ and } M(B_\gamma q) > TM(q)|\hat{\pi}) \leq CT^{-\delta}.$$

Remark. The restriction on the paths means that we only consider paths that do not contain $\hat{\gamma}$ as a proper part.

The most important point of the argument we use is that the estimates that we prove in Theorem 11 are uniform with respect to q .

The proof of the theorem is based on the following two lemmas that can be considered as the corollaries of Theorem 10.

Lemma 12. There exists $C' > 1$ such that for any permutation $\hat{\pi}$

$$P_q(\{\gamma : M(B_\gamma q) > C' M(q), m(B_\gamma q) < M(q)\}|\hat{\pi}) < 1 - \frac{1}{C'}.$$

Proof. From Theorem 10 we know the lower bound of the probability of the following event:

$$P_q(X \cup Y|\hat{\pi}) > \frac{1}{C},$$

where $X := X_1 \cap X_2$ is defined by

$$X_1 := \{\gamma : M(B_\gamma q) < Cm(B_\gamma q)\}$$

$$X_2 := \{\gamma : m(B_\gamma q) < M(q)\}$$

and $Y := Y_1 \cap Y_2$ is given by

$$Y_1 := \{\gamma : M(B_\gamma q) < CM(q)\}$$

$$Y_2 := \{\gamma : M(q) \leq m(B_\gamma q)\}.$$

Suppose $x \in X_1^c \cap X_2$, then $x \notin X_1$ and thus $x \notin X$. Furthermore $x \notin Y_2$ since $X_2 \cap Y_2 = \emptyset$, and thus $x \notin Y$. Thus

$$P_q(X_1^c \cap X_2|\hat{\pi}) < 1 - \frac{1}{C}.$$

Then the lemma follows for any $C' > C$ since

$$\{\gamma : M(B_\gamma q) > C' M(q), m(B_\gamma q) < M(q)\} \subset X_1^c \cap X_2.$$

\square

We write $\hat{\gamma} \sqsubset \gamma$ if there exist non-empty γ_s and γ_e such that $\gamma = \gamma_s \hat{\gamma} \gamma_e$, otherwise we write $\hat{\gamma} \not\sqsubset \gamma$. The next lemma follows from the previous one and is also important for the proof of Theorem 11.

Lemma 13. For any $\hat{\gamma} \in \Pi(\mathcal{R})$ there exist $M \geq 0, \rho < 1$ such that for any $\hat{\pi} \in \mathcal{R}, q \in \mathbb{R}_+^A$

$$P_q(\gamma : \hat{\gamma} \not\sqsubset \gamma \text{ and } M(B_\gamma q) > 2^M M(q) | \hat{\pi}) \leq \rho.$$

Proof. Fix M_0 large enough (we will choose it precisely later) and let $M := 2M_0$. We consider the set

$$\Gamma := \{\gamma : \gamma \text{ of minimal length satisfying } \hat{\gamma} \not\sqsubset \gamma \text{ and } M(B_\gamma q) > 2^M M(q)\}.$$

As mentioned above, $M(B_\gamma q)$ can not increase more than twice for a path of the length one. So any path in Γ can be written as $\gamma = \gamma_1 \gamma_2$ where γ_1 is the shortest path such that

$$M(B_{\gamma_1} q) > 2^{M_0} M(q),$$

and neither γ_1 nor γ_2 coincide with γ .

Let us denote the set of such γ_1 by Γ_1 . It follows directly from minimality that Γ_1 is disjoint in terms of [AvGoYo] which means that any path is not a part of some other path from the same set.

Now consider the subset $\tilde{\Gamma}_1$ of Γ_1 consisting of all γ_1 such that $m(B_{\gamma_1} q) \geq M(q)$ (or, equivalently, $M_{\mathcal{A}'}(B_{\gamma_1} q) \geq M(q)$, for all non-empty \mathcal{A}').

By Lemma 12 choosing M_0 large enough we have that

$$P_q(\Gamma_1 \setminus \tilde{\Gamma}_1 | \hat{\pi}) < 1 - \frac{1}{C}$$

with some constant $C > 1$.

Now we use the strategy from [AvRe]. We fix some permutation $\hat{\pi}_e$ and consider the shortest path $\gamma_{\hat{\pi}_e}$ starting at $\hat{\pi}_e$ and containing $\hat{\gamma}$ (if there are several such paths choose one). We define γ_s by $\gamma_{\hat{\pi}_e} = \gamma_s \hat{\gamma}$. Then, if M_0 is large enough, we can assume that

$$(5) \quad |B_{\gamma_{\hat{\pi}_e}}| < 2^{M_0-1}.$$

If $\hat{\pi}_e$ is the end of some $\gamma_1 \in \Gamma_1$, then $P_q(\Gamma | \gamma_1) = P_{B_{\gamma_1} q}(\Gamma^{\gamma_1} | \hat{\pi}_e)$, where $\Gamma^{\gamma_1} := \{\gamma_e : \gamma = \gamma_1 \gamma_e \in \Gamma\}$. So, since γ does not contain $\hat{\gamma}$ as a proper part it follows that

$$(6) \quad P_q(\Gamma | \gamma_1) \leq 1 - P_{B_{\gamma_1} q}(\gamma_{\hat{\pi}_e} | \hat{\pi}_e)$$

because Γ^{γ_1} and the set of $\gamma_{\hat{\pi}_e}$ do not intersect and together they fill not more than the set of possible continuations of γ_1 .

If $\gamma_1 \in \tilde{\Gamma}_1$, $P_q(\Gamma | \gamma_1)$ can be estimated directly in terms of the measures of subsimplices of the original simplex: if $N(q) = q_1 \cdots q_n$, then

$$P_{B_{\gamma_1} q}(\gamma_{\hat{\pi}_e} | \hat{\pi}_e) = \frac{N(B_{\gamma_1} q)}{N(B_{\gamma_{\hat{\pi}_e}} B_{\gamma_1} q)}.$$

We need to make several estimates, first all by the definition of $\tilde{\Gamma}_1$ we have $m(B_{\gamma_1}q) \geq M(q)$ and thus $N(B_{\gamma_1}q) \geq (M(q))^n$. Next Inequality (5) implies that

$$N(B_{\gamma_{\hat{\pi}_e}}B_{\gamma_1}q) < (2^{M_0-1}2^M M(q))^n,$$

because it follows from the definition of Γ_1 that $M(B_{\gamma_1}q) < 2^M M_q$. So, it is easy to see now that

$$(7) \quad P_{B_{\gamma_1}q}(\gamma_{\hat{\pi}_e}|\hat{\pi}_e) \geq 2^{-3nM_0}.$$

We start with the case $P_q(\tilde{\Gamma}_1|\hat{\pi}) \geq \frac{1}{2C}$. Starting with the definition of P_q (Equation (2)) we have

$$\begin{aligned} P_q(\Gamma|\hat{\pi}) &= \sum_{\gamma_1 \in \Gamma_1} P_q(\Gamma|\gamma_1)P_q(\gamma_1|\hat{\pi}) \\ &= \sum_{\gamma_1 \in \tilde{\Gamma}_1} P_q(\Gamma|\gamma_1)P_q(\gamma_1|\hat{\pi}) + \sum_{\gamma_1 \in \Gamma_1 \setminus \tilde{\Gamma}_1} P_q(\Gamma|\gamma_1)P_q(\gamma_1|\hat{\pi}) \\ &\leq \left(\sup_{\gamma_1 \in \tilde{\Gamma}_1} P_q(\Gamma|\gamma_1) \right) \cdot \sum_{\gamma_1 \in \tilde{\Gamma}_1} P_q(\gamma_1|\hat{\pi}) + \left(\sup_{\gamma_1 \in \Gamma_1 \setminus \tilde{\Gamma}_1} P_q(\Gamma|\gamma_1) \right) \cdot \sum_{\gamma_1 \in \Gamma_1 \setminus \tilde{\Gamma}_1} P_q(\gamma_1|\hat{\pi}) \\ &= \left(\sup_{\gamma_1 \in \tilde{\Gamma}_1} P_q(\Gamma|\gamma_1) \right) \cdot P_q(\tilde{\Gamma}_1|\hat{\pi}) + \left(\sup_{\gamma_1 \in \Gamma_1 \setminus \tilde{\Gamma}_1} P_q(\Gamma|\gamma_1) \right) \cdot P_q(\Gamma_1 \setminus \tilde{\Gamma}_1|\hat{\pi}). \end{aligned}$$

In the last line we used the fact that Γ_1 is disjoint.

Inequalities (6), (7) imply that $\sup_{\gamma_1 \in \tilde{\Gamma}_1} P_q(\Gamma|\gamma_1) < 1 - 2^{-3nM_0}$, using this and the facts that $\sup_{\gamma_1 \in \Gamma_1 \setminus \tilde{\Gamma}_1} P_q(\Gamma|\gamma_1) \leq 1$ and $P_q(\Gamma_1 \setminus \tilde{\Gamma}_1|\hat{\pi}) \leq 1 - P_q(\tilde{\Gamma}_1|\hat{\pi})$ along with the assumption $P_q(\tilde{\Gamma}_1|\hat{\pi}) \geq \frac{1}{2C}$ yields

$$\begin{aligned} P_q(\Gamma|\hat{\pi}) &\leq P_q(\tilde{\Gamma}_1|\hat{\pi}) \cdot (1 - 2^{-3nM_0}) + 1 - P_q(\tilde{\Gamma}_1|\hat{\pi}) \\ &= 1 - P_q(\tilde{\Gamma}_1|\hat{\pi}) \cdot 2^{-3nM_0} \\ &\leq 1 - \frac{2^{-3nM_0}}{2C}. \end{aligned}$$

Now consider the case $P_q(\tilde{\Gamma}_1|\hat{\pi}) < \frac{1}{2C}$, then by Inequality (3) $P_q(\Gamma_1|\hat{\pi}) < 1 - \frac{1}{C} + \frac{1}{2C} = 1 - \frac{1}{2C}$. So, Lemma 13 holds with $\rho = 1 - \frac{2^{-3nM_0}}{2C}$. \square

Now we turn to the proof of the theorem.

Proof of Theorem 11. Let M and ρ be as in the Lemma 13. For a given $T > 1$ let k be the maximal integer such that $T \geq 2^{k(M+1)}$. Let γ be the shortest path that does not include $\hat{\gamma}$ as a proper part and such that $M(B_\gamma q) > 2^{k(M+1)}M(q)$. Then γ can be written in the following way: $\gamma = \gamma_1 \cdots \gamma_i \cdots \gamma_k$, where for each i $\gamma_{(i)} = \gamma_1 \cdots \gamma_i$ is the shortest path such that

$$(8) \quad M(B_{\gamma_i}q) > 2^{i(M+1)}M(q).$$

Then all such $\gamma_{(i)}$ comprise a set $\Gamma_{(i)}$ for each i , and these sets are disjoint because each path has to be the shortest one satisfying (8). Now, Lemma

13 and (8) imply that for all $\gamma_{(i)} \in \Gamma_{(i)}$

$$P_q(\Gamma_{(i+1)} | \gamma_{(i)}) \leq \rho.$$

So $P_q(\Gamma | \hat{\pi}) < \rho^k$. The result follows from the definition of k . \square

6. THE ROOF FUNCTION

6.1. Definition. The construction of the roof function that we present in this section is based on the idea of renormalization provided by Veech in [Ve2]. Fix some positive complete path γ_* starting and ending at the same permutation $\hat{\pi}$, and the subsimplex of the parameter space that corresponds to this path Δ_{γ_*} . We are interested in the first return map to the subsimplex Δ_{γ_*} . The connected components of the domain of this first return map are given by the $\Delta_{\gamma\gamma_*}$ where γ is a path that contains γ_* as a part, but does not start with $\gamma_*\gamma_*$. Thus the first return map T restricted to such a component satisfies

$$(9) \quad T(\lambda, \hat{\pi}) = \left(\frac{R_\gamma^{-1}\lambda}{\|R_\gamma^{-1}\lambda\|}, \hat{\pi} \right),$$

where R is the matrix of the Rauzy induction.

Definition. The roof function is the return time to the connected component described above:

$$r(\lambda, \hat{\pi}) := -\log \|R_\gamma^{-1}\lambda\|.$$

Remark. As it was mentioned in [AvGoYo] and [AvRe], with such a definition one works with the precompact sections because the path γ_* is positive.

6.2. Correctness of the model. In this subsection we follow the strategy from [AvHuSkr].

In the suspension model we work with, the orbits that do not come back to the fixed precompact section are not considered. We need to show that they do not contribute to the Hausdorff dimension of the fractal we are studying. To see this note that the following properties of the Markov map hold:

- (1) the BIP property implies that each small simplex of the Markov partition (let's say that it Δ_γ where γ is a corresponding complete path in the Rauzy graph) is mapped on the whole parameter space X , and the map is surjective;
- (2) the Markov map T is uniformly expanding and so the Jacobian of the map from Δ_γ to X is bounded.

Let us recall that for each Rauzy class the subset of the parameter space that gave rise to minimal interval exchange transformations with flips MF_n has a fractal structure for the following reason: the point belongs to MF_n iff the Rauzy induction can be applied infinitely many times to the corresponding nIET and never arrives to the hole. So, now we denote by $MF_n(\Delta_\gamma) = \Delta_\gamma \cap MF_n$. It is a standard calculation in Hausdorff dimension to show that

the properties mentioned above imply that $Hdim(MF_n(\Delta_{\gamma_*})) = Hdim(MF_n)$, where $Hdim$ is the Hausdorff dimension (see also [AvHuSkr] where the same statement was proved for minimal systems of isometries).

The same argument can be used for Δ_γ and $\Delta_{\gamma'}$, where $\gamma' = \gamma\hat{\gamma}\gamma$ for some suitable $\hat{\gamma}$. Therefore, the orbits that escape the control do not contribute to the Hausdorff dimension of the fractal we study, and our suspension model is correct.

6.3. Exponential tails. In this section we prove that the roof function constructed above has *exponential tails*. We follow the strategy from [AvGoYo].

Definition. A function f has *exponential tails* if there exists $\sigma > 0$ such that $\int_\Delta e^{\sigma f} dLeb < \infty$.

Theorem 14. The roof function r defined above has exponential tails.

Proof. This theorem is a direct corollary of Theorem 11. The main idea is the same as was used in the case of IETs (see [AvGoYo]): $-\log \|(B_{\gamma_*}^T)^{-1}\lambda\|$ is the ‘‘Teichmüller’’ time needed to renormalize the support interval to unit length. Then time is divided into pieces of exponential size. For each piece, we apply Theorem 11.

Indeed, in the previous section we constructed the set of Lebesgue measures ν_q on Λ_q that depended on vector q . Let us consider $q_0 = (1, \dots, 1)$ and the corresponding measure ν_{q_0} . Let us recall that our parameter space for a given Rauzy class \mathcal{R} can be viewed as $\mathbb{R}_+^A \times \mathcal{R}$ with the renormalization condition $\sum_{i=1}^n \lambda_i = 1$. In particular for a given permutation we define $\Delta_{\hat{\pi}} = \mathbb{R}_+^A \times \hat{\pi}$ with the same renormalization condition. Fix $\hat{\pi}$ and consider the natural projection of Λ_q to the set $\Delta_{\hat{\pi}}$. The pushforward ν of the measure ν_{q_0} under this projection is a smooth function on the parameter space of the Markov map $T(\lambda, \hat{\pi})$ (see [AvRe] or [AvGoYo]). Thus, in order to prove the theorem, it is enough to show that

$$(10) \quad \nu\{x \in \Delta_{\gamma_*} : r(x) \geq \log T\} \leq CT^{-\delta}$$

for some C and some δ .

The connected component of the domain of the Markov map $T(\lambda, \hat{\pi})$ that intersects the set $W = \{x : \{x \in \Delta_{\gamma_*} : r(x) \geq \log T\} \leq CT^{-\delta}\}$ is of the form Δ_γ for some γ . such that γ can not be a concatenation of more than three copies of γ_* and

$$M(B_\gamma q_0) > C^{-1}T,$$

for some constant C that depends on γ_* . This first requirement on γ follows from the fact that we work with the first return maps while if γ is a concatenation of four copies of γ_* , one can take $\gamma_*\gamma_*$ as a path of the first return (and all other properties will be the same); the second statement follows from the definition of the roof function and the definition of the set W .

Now we estimate the measure of the interesting set in terms of probabilities of corresponding events: $\nu\{x \in \delta_{\gamma_*} : r(x) \geq \log T\} \leq P_{q_0}(\gamma \text{ does not contain$

some $\hat{\gamma}$ as a proper set and $M(B_\gamma q_0) > C^{-1}T|\hat{\pi}| < CT^{-\delta}$. The statement of the theorem follows now from Theorem 11. \square

7. THE UPPER BOUND PROOF

7.1. Fast decaying Markov maps. Let Δ be a measurable space and $T : \Delta \rightarrow \Delta$ be a Markov map. We will denote the corresponding Markov partition by $\Delta^{(l)}, l \in \mathbb{Z}$.

Definition. We say that T is *fast decaying* if there exists $C_1 > 0, \alpha_1 > 0$ such that

$$(11) \quad \sum_{\mu(\Delta^{(l)}) \leq \varepsilon} \mu(\Delta^{(l)}) \leq C_1 \varepsilon^{\alpha_1}$$

for all $0 < \varepsilon < 1$.

Lemma 15. Exponential tails of the roof function implies fast decaying property of the Markov map.

Proof. First, one can check directly that $JT(\lambda, \hat{\pi}) = e^{n \cdot r(\lambda, \hat{\pi})}$. We claim that the lemma follows from this formula and the fact that the measure of subsimplices (Markov cylinders) are proportional to $|DT|$. The scheme of the proof is as follows: the measure of a subsimplex is proportional to the inverse of the Jacobian, thus one begins by replacing the measures of subsimplices in the sum of Equation (11) by the corresponding jacobians; using the above formula the jacobians are then replaced by the exponential of the roof function; so we only need to evaluate the following sum:

$$\sum_{a : r(a) \geq N} e^{-nr(a)},$$

where by a we denoted a point of $\Delta^{(l)}$ since the roof function is locally constant. The last sum can be evaluated using the exponential tails of the roof function (namely, the convergence of the corresponding integral): first, the exponential tail implies that $\text{Card}(Y(N)) \leq Ce^{(n-\sigma)N}$, where $Y(N)$ is the set of partition subsets for which $r(a)$ is between N and $N+1$ (see [AvHuSkr2], Lemma 17); then the sum we are interested in it can be estimated from above by a geometric series with ratio $e^{-\sigma}$. \square

7.2. The proof of the upper bound in Theorem 1. We need first to fix some notations. Fix $m \geq 1$ and consider $\underline{l} = (l_1, l_2, \dots, l_m)$ where all l_i are positive integers; by $\Delta^{\underline{l}}$ we denote $x \in \Delta$ such that $T^{j-1}(x) \in \Delta_{l_j}$ for all $1 \leq j \leq m$. The collection of such $\Delta^{\underline{l}}$ is a partition, we call m the *depth* of this partition. Avila and Delecroix in [AvDe] proved the following

Theorem 16 (AD,2013). Assume that T is fast decaying. For $m \geq 1$, let $X_m \subset \Delta$ be a union of some subsimplices ($\Delta^{\underline{l}}$) of depth m . Let

$$X := \liminf_{m \rightarrow \infty} X_m \quad \text{and} \quad \delta := - \lim_{m \rightarrow \infty} \frac{1}{m} \ln \mu(X_m).$$

Then $HD(X) \leq p - 1 - \min(\delta, \alpha_1)$, where α_1 is the fast decay constant.

Let us consider the Markov map T and the corresponding Markov partition. Fix the total number of the steps of Rauzy induction to be m . Let X_m be the union of all the Markov cells of the partition of depth m that do not correspond to the hole (by hole here we mean the set of parameters for which periodic orbits already appeared before step n and the induction stopped). Then, one can see that the set $X = \liminf_{m \rightarrow \infty} X_m$ is exactly the set $MF_n(\Delta_{\gamma_*})$ of parameters that give rise to minimal fIETs.

Now, in order to deduce the upper bound of Theorem 1 from Theorem 16 we only have to check that $\delta > 0$. It follows directly from the fact that the size of simplices of the partition decreases exponentially fast because the map T is uniformly expanding (see [MeNo] for details). This completes the proof of the upper bound.

Remark. Our proof implies that the obtained estimation holds for MF_n restricted to any Rauzy class.

8. LOWER BOUNDS

In this section we show the lower bound in Theorem 1 and prove Theorem 2 and Proposition 3.

8.1. Nogueira construction. Let us recall the construction of minimal fIETs suggested in [No]. We take arbitrary m -IET (without flips) $S : [0, \frac{1}{3}] \rightarrow [0, \frac{1}{3}]$ and construct a fIET T in the following way:

$$Tx = \begin{cases} Sx + \frac{1}{3}, & \text{if } x \in [0, \frac{1}{3}], \\ \frac{4}{3} - x, & \text{if } x \in [\frac{1}{3}, \frac{2}{3}], \\ 1 - x, & \text{if } x \in [\frac{2}{3}, 1]. \end{cases}$$

One can easily check that $T^3 = S$ and therefore T^3 is minimal if S is. Since all but countably many IETs (without flips) are minimal the lower bound of Theorem 1 follows.

8.2. Non-uniquely ergodic case. The idea is the same as above but we fix S with the maximal possible number of invariant ergodic measures (we denote this number by k). Let us denote these ergodic measures by μ_1, \dots, μ_k . Consider a probability vector $\vec{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_k)$ and the corresponding measure $\mu_{\vec{\varepsilon}} = \sum \varepsilon_i \mu_i$. Using Lemma 1 from [Ka], we have the following:

Lemma 17. There exists an m -IET $S_{\vec{\varepsilon}}$ such that $(S, \mu_{\vec{\varepsilon}})$ is metrically isomorphic to $(S_{\vec{\varepsilon}}, Leb)$.

We need to check is that the map $\vec{\varepsilon} \rightarrow S_{\vec{\varepsilon}}$ is almost surely invertible.

Lemma 18. For a set of full measure of $\vec{\varepsilon}$ the length vectors $(\lambda_1^{\vec{\varepsilon}}, \dots, \lambda_n^{\vec{\varepsilon}})$ are distinct.

Proof. Assume that for two vectors $\vec{\varepsilon} \neq \vec{\varepsilon}'$ the lengths coincide. Let μ^i denote the μ_i measure of the interval $[0, \lambda_1 + \dots + \lambda_i]$. Then the assumption implies that for each $i = 1, \dots, n$

$$\varepsilon_1 \mu_1^i + \dots + \varepsilon_k \mu_k^i = \varepsilon'_1 \mu_1^i + \dots + \varepsilon'_k \mu_k^i$$

and thus

$$(12) \quad \sum_{j=1}^k (\varepsilon_j - \varepsilon'_j) \mu_j^i = 0.$$

But

$$\sum_{j=1}^k \varepsilon_j = \sum_{j=1}^k \varepsilon'_j$$

and so Equation (12) is equivalent to

$$(13) \quad \sum_{j=1}^{k-1} (\varepsilon_j - \varepsilon'_j) C_j = 0,$$

where $C_j := \mu_j^i - \mu_n^i$. Since the C_j are constants (S is fixed) the set of $\vec{\varepsilon}$ for which (12) holds is a subspace of codimension 1 in the parameter space, and therefore the statement of the lemma holds. \square

Now, one can apply Nogueira's construction to get a family of non-uniquely ergodic fIETs. The Hausdorff dimension of this subset is not smaller than $k - 1$ where k is the number of invariant measures of IET S' constructed above. S' is $(n - 2)$ -IET. It was proven by Sataev in [Sa] that $k = g$ where g is the genus of a translation surface associated with IET (see, for example, [Vi]). Therefore, $2g = n - 2 - r + 1$ where r is the number of singularities of the translation surface; on the other hand, r can be estimated using the Euler characteristics of the surface and, in particular, the minimal value of r is 1 (and this value is always obtained). So it implies that $2g \leq n - 2$ and so

$$Hdim(NUE_n) \geq \left\lfloor \frac{n-2}{2} \right\rfloor - 1.$$

Remark. If one is interested in lower bound of $Hdim(NUE_n)$ for a particular combinatorics determined by the Rauzy class of IET S' , it is easy to see that for any integer i between $\left\lfloor \frac{n+1}{4} \right\rfloor$ and $\left\lfloor \frac{n-2}{2} \right\rfloor$ one can find a Rauzy class of IET S' constructed above such that the lower bound of the Hausdorff dimension of non-uniquely ergodic minimal fIETs is $i - 1$.

8.3. Non-uniquely ergodic fIETs on 6 intervals. One can also combine Nogueira construction presented above with the following result by J. Athreya and J. Chaika:

Theorem ([AtCh]). The Hausdorff dimension of the set of non-uniquely ergodic 4-IET on $[0, 1)$ with the permutation $\pi_0 = (4, 3, 2, 1)$ is $\frac{5}{2}$.

Proposition 3 follows since the Nogueira construction increases the number of intervals by 2.

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FACULTY OF MATHEMATICS, NATIONAL RESEARCH UNIVERSITY HIGHER SCHOOL OF
ECONOMICS, VAVILOVA ST. 7, 112312 MOSCOW, RUSSIA

E-mail address: `sashaskrip@gmail.com`

AIX MARSEILLE UNIVERSITÉ, CNRS, CENTRALE MARSEILLE, I2M, UMR 7373,
13453 MARSEILLE, FRANCE

Mailing address: I2M, Luminy, Case 907, F-13288 Marseille CEDEX 9, France

E-mail address: `serge.tribetzkoy@univ-amu.fr`