



**HAL**  
open science

# Gröbner basis. a "pseudo-polynomial" algorithm for computing the Frobenius number

Marcel Morales, Dung Nguyen Thi

► **To cite this version:**

Marcel Morales, Dung Nguyen Thi. Gröbner basis. a "pseudo-polynomial" algorithm for computing the Frobenius number. 2015. hal-01212986v2

**HAL Id: hal-01212986**

**<https://hal.science/hal-01212986v2>**

Preprint submitted on 14 Dec 2015

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Gröbner basis, a "pseudo-polynomial" algorithm for computing the Frobenius number

Marcel Morales

Institut Fourier, Laboratoire de Mathématiques associé au CNRS, UMR 5582,  
Université Joseph Fourier, B.P.74, 38402 Saint-Martin d'Hères cedex, France  
and ESPE Université de Lyon, France  
`marcel.morales@ujf-grenoble.fr`

Nguyen Thi Dung

Thai Nguyen University of Agriculture and Forestry,  
Thai Nguyen, Vietnam

December 14, 2015

ABSTRACT.

Let consider  $n$  natural numbers  $a_1, \dots, a_n$ . Let  $S$  be the numerical semigroup generated by  $a_1, \dots, a_n$ . Set  $A = K[t^{a_1}, \dots, t^{a_n}] = K[x_1, \dots, x_n]/I$ . The aim of this paper is:

1. Give an effective pseudo-polynomial algorithm on  $a_1$ , which computes The Apéry set and the Frobenius number of  $S$ . As a consequence it also solves in pseudo-polynomial time the integer knapsack problem : given a natural integer  $b$ ,  $b$  belongs to  $S$ ?
2. The Gröbner basis of  $I$  for the reverse lexicographic order to  $x_n, \dots, x_1$ , without using Buchberger's algorithm.
3.  $\text{in}(I)$  for the reverse lexicographic order to  $x_n, \dots, x_1$ .
4.  $A$  as a  $K[t^{a_1}]$ -module.

We dont know the complexity of our algorithm. We need to solve the "multiplicative" integer knapsack problem:

Find all positive integer solutions  $(k_1, \dots, k_n)$  of the inequality  $\prod_{i=2}^n (k_i + 1) \leq a_1 + 1$ .

This algorithm is easily implemented. The implementation of this algorithm "frobenius-number-mm", for  $n = 17$ , can be downloaded in  
<https://www-fourier.ujf-grenoble.fr/morales/frobenius-number-mm>

## Introduction

In the sequel we shall use the following notations. Let  $K$  be a field,  $\mathcal{A}$  be a set of  $n$  natural numbers  $\mathcal{A} = \{a_1, \dots, a_n\} \subset \mathbb{N}$ .  $S$  the numerical semigroup generated by  $a_1, \dots, a_n$ , that is  $S = \{k_1 a_1 + \dots + k_n a_n \mid k_i \in \mathbb{N}\}$ . We consider the one-dimensional *toric affine ring*  $A = K[t^{a_1}, \dots, t^{a_n}] \subset K[t]$ , that is  $A = K[t^k \mid k \in S] := K[S]$ .

The ring  $A = K[t^{a_1}, \dots, t^{a_n}] \subset K[t]$  has a presentation as a quotient of the polynomial ring  $K[x_1, \dots, x_n]$ , as follows:

Let  $\varphi : K[x_1, \dots, x_n] \rightarrow K[t^{a_1}, \dots, t^{a_n}]$  defined by

$$\begin{aligned} x_1 &\mapsto t^{a_1} \\ &\vdots \\ x_n &\mapsto t^{a_n} \end{aligned}$$

Let  $I(S)$  be the kernel of  $\varphi$ , that is the ideal ideal formed by all polynomials of  $K[x_1, \dots, x_n]$  such that  $P(t^{a_1}, \dots, t^{a_n}) = 0$ .

The ideal  $I(S)$  has a system of generators formed by binomials which are differences of two monomials with coefficient 1. Note that if we grade the polynomial ring  $K[x_1, \dots, x_n]$  by setting  $\deg(x_i) = a_i$ , the morphism  $\varphi$  is homogeneous, and the ideal  $I(S)$  is homogeneous.

The following theorem is well known, we give here a short proof for the commodity of the reader.

**Theorem 0.1.** *Suppose that  $a_1, \dots, a_n$  are relatively prime numbers then any large integer belongs to  $S$ .*

*Proof.* Suppose that  $n = 2$ , By Bézout's theorem there exist relative integer numbers  $s_1, s_2$  such that  $s_1 a_1 + s_2 a_2 = 1$ . We can assume that  $s_1 > 0, s_2 < 0$ . Let  $k > 0$  big enough we can write  $k = q a_2 + r$  with  $0 \leq r < a_2$ , which implies  $k = q a_2 + r(s_1 a_1 + s_2 a_2) = r s_1 a_1 + (q + r s_2) a_2$ . Since  $k$  is large enough  $(q + r s_2) > 0$ , hence  $k \in S$ .

A similar argument works for  $n > 2$ . □

**Definition 0.2.** Suppose that  $a_1, \dots, a_n$  are relatively prime numbers, the biggest integer number in  $\mathbb{N} \setminus S$  is called the Frobenius number, we denote it by  $g(S)$ . More generally if  $\gcd(a_1, \dots, a_n) = \lambda$  then the biggest integer in  $\lambda \mathbb{N} \setminus S$  is called the Frobenius number, we denote it by  $g(S)$ .

Suppose that  $\gcd(a_1, \dots, a_n) = \lambda$ , let  $\tilde{S}$  be the semigroup generated by  $\frac{a_1}{\lambda}, \dots, \frac{a_n}{\lambda}$ . We have that  $g(S) = \lambda g(\tilde{S})$ . The problem of computing the Frobenius number is open since the end of 19th Century, for  $n = 2$  there is a formula (see section 1), for  $n = 3$  a formula using Euclidean's algorithm for gcd was given in [8]. In 1987, in [6] the first author

translate for the first time the Frobenius problem into an algebraic setting, showing that the Frobenius number is the degree of the Hilbert-Poincaré series written as a rational fraction, moreover By using [8], in the case  $n = 3$  the Hilbert-Poincaré series is completely described by an algorithm using only Euclide's algorithm for gcd, that is of complexity  $\ln(a)$ . An implementation in Pascal was done by the first author to compute a system of generators of the affine monomial curve  $K[t^a, t^b, t^c]$  and its projective closure, which computed the Frobenius number for three natural numbers. In recent works [3] and [9], the computation of Frobenius number, is related to the computation of the Hilbert-Poincaré series. More precisely, in [9] the author deduces the Frobenius number from a Gröbner basis of the ideal  $I(S)$ . We recall that the computation of a Gröbner basis is double exponential complexity by using the Buchberger algorithm.

In this paper we study the Frobenius problem from algebraic point of view, this allows us to give a conceptual frame to our algorithm. As a consequence of our study we get a very fast and simple algorithm to compute the Frobenius number, determine completely the semigroup  $S$  and solve the knapsack integer problem, that is to decide if an integer number belong to  $S$ . We have implemented for  $n \leq 17$ . We need to solve the "multiplicative" integer knapsack problem:

Find all positive integer solutions  $(k_1, \dots, k_n)$  of the inequality  $\prod_{i=2}^n (k_i + 1) \leq a_1 + 1$ .

Moreover let  $G(S)$  be a Gröbner basis for the degree reverse lexicographic order to  $x_n, \dots, x_1$ .  $\prec_{revlex}$  and  $in(I(S))$  be its initial ideal. As an extension we find several algorithms :

1. The set of monomials  $\{x_2^{k_2} \dots x_n^{k_n} \notin in(I(S))\}$ .
2. The system of monomial generators of  $in(I)$ .
3. The Gröbner basis  $G(S)$  of  $I$ , without using Buchberger's algorithm.

They are extensions of the previous work and algorithm by the first author in [6], [7]. The algorithm presented here is implemented and can be downloaded in <https://www-fourier.ujf-grenoble.fr/~morales/>. Note that because of the limitation of the Compiler for the moment the software works only for numbers less than 10000, but an implementation in Mathematica should allow to compute with any number of digits.

In the first section we introduce the Apéry set and we prove some known results.

In the second section we present the connection between Hilbert-Poincaré series and the Frobenius number. This connection was established by the first author for the first time in [6].

In the third section we introduce Noether normalization and we prove the connection between Apéry sets and Noether normalization.

In the last section we develop our algorithm.

In our work in progress, we will extend the above algorithm to compute Gröbner basis of any simplicial monomial ideal.

# 1 Frobenius number, Apéry set

For a survey on the Frobenius number we refer to [10].

**Definition 1.1.** Suppose that  $a_1$  is the smallest among  $a_1, \dots, a_n$ . The Apéry set  $\text{Ap}(S, a_1)$  of the semigroup  $S$  with respect to  $a_1$  is the set  $\text{Ap}(S, a_1) := \{s \in S \mid s - a_1 \notin S\}$ .

*Remark 1.2.* The definition of Apéry set makes sense even if the numbers  $a_1, \dots, a_n$  are not relatively prime numbers. Suppose that  $\gcd(a_1, \dots, a_n) = \lambda$ , let  $\tilde{S}$  be the semigroup generated by  $\frac{a_1}{\lambda}, \dots, \frac{a_n}{\lambda}$ . We have that  $\text{Ap}(S, a_1)$  is obtained from  $\text{Ap}(\tilde{S}, \frac{a_1}{\lambda})$  by multiplication by  $\lambda$ .

**Theorem 1.3.** (*Apéry [1]*) Suppose that  $a_1, \dots, a_n$  are natural numbers such that  $\gcd(a_1, \dots, a_n) = \lambda$ .

1.  $\text{Ap}(S, a_1) := \{\lambda w_0, \dots, \lambda w_{\frac{a_1}{\lambda}-1}\}$ , where  $w_i$  is the smallest element in  $\tilde{S}$  congruent to  $i \pmod{\frac{a_1}{\lambda}}$ .
2.  $g(S) = \max \{s - a_1 \mid s \in \text{Ap}(S, a_1)\}$ .

*Proof.* 1. We can assume that  $a_1, \dots, a_n$  are relatively prime numbers.

First we prove that for all  $i = 0, \dots, a_1 - 1$ ,  $w_i$  belongs to  $\text{Ap}(S, a_1)$ . Suppose that it is not true, that is  $w_i - a_1 \in S$  for some  $i = 0, \dots, a_1 - 1$ . It follows that  $w_i - a_1 < w_i$  and both  $w_i - a_1, w_i \in S$  are congruent to  $i \pmod{a_1}$ . This is a contradiction with the definition of  $w_i$ . As a consequence  $\text{Ap}(S, a_1)$  has at least  $a_1$  elements in order to prove the claim it will be enough to show that  $\text{Ap}(S, a_1)$  has exactly  $a_1$  elements. Suppose that  $\text{card}(\text{Ap}(S, a_1)) > a_1$ , then there exists two elements  $s_1 < s_2$  in  $\text{Ap}(S, a_1)$  such that both  $s_1 < s_2$  are congruent to  $i \pmod{a_1}$  for some  $i = 0, \dots, a_1 - 1$ , that is  $s_2 = s_1 + ka_1$  with  $k > 0$  a natural integer, hence  $s_2 \notin \text{Ap}(S, a_1)$ , a contradiction.

2. Let  $h \in \mathbb{N}$  such that  $h > \max \{s - a_1 \mid s \in \text{Ap}(S, a_1)\}$ , since  $h$  is congruent to  $i \pmod{a_1}$  for some  $i = 0, \dots, a_1 - 1$ , we can write  $h = w_i + \alpha a_1$ , with  $\alpha \in \mathbb{Z}$ , hence  $h = (w_i - a_1) + (\alpha + 1)a_1$ , since  $h > (w_i - a_1)$  we have  $(\alpha + 1) > 0$ , hence  $\alpha \geq 0$ , which implies that  $h \in S$ . □

**Corollary 1.4.** Let  $n = 2$  suppose that  $a_1, a_2$  are relatively prime numbers then  $g(S) = (a_1 - 1)(a_2 - 1) - 1$ .

*Proof.* We give a combinatorial proof using Apéry sets. Since  $a_1, a_2$  are relatively prime numbers, we have

$$\text{Ap}(S, a_1) := \{0, a_2, \dots, (a_1 - 1)a_2\},$$

hence  $g(S) = (a_1 - 1)a_2 - a_1 = (a_1 - 1)(a_2 - 1) - 1$  □

For  $n = 2$ , we will give an algebraic proof later.

**Corollary 1.5.** *For  $i = 0, \dots, a_1 - 1$  let  $S_i = \{s \in S \mid s \equiv i \pmod{a_1}\}$ . Then  $S$  is the disjoint union of  $S_0, \dots, S_{a_1-1}$ .*

## 2 Frobenius number and Hilbert-Poincaré series

Let  $R := K[x_1, \dots, x_n]$  be the polynomial ring graded by the weights  $\deg x_1 = a_1, \dots, \deg x_n = a_n$ , and  $I \subset K[x_1, \dots, x_n]$  be a graded ideal. Let  $B = R/I$ , the Hilbert-function of  $B$  is defined by  $H_B(l) = \dim_K B_l$ , for all  $l \in \mathbb{Z}$ , and the Hilbert-Poincaré series of  $B$ :

$$P_B(u) = \sum_{l \in \mathbb{Z}} H_B(l) u^l.$$

We recall the following Theorem from [6]

**Theorem 2.1.** *Let  $R := K[x_1, \dots, x_n]$  be the polynomial ring graded by the weights  $\deg x_1 = a_1, \dots, \deg x_n = a_n$ ,  $I \subset K[x_1, \dots, x_n]$  be a graded ideal and  $B := R/I$ . Then*

1. *The Hilbert-Poincaré series of  $B$  is a rational function:*

$$P_B(u) = \frac{Q_B(u)}{(1 - u^{a_1})(1 - u^{a_2}) \dots (1 - u^{a_n})},$$

where  $Q_B(u)$  is a polynomial on  $u$ .

2. *There exists  $h$  polynomials with integer coefficients  $\Phi_{H_B,0}(l), \dots, \Phi_{H_B,h}(l)$  such that  $H_B(lh + i) = \Phi_{H_B,i}(l)$  for  $0 \leq i \leq h - 1$  and  $l$  large enough. We recall that the index of regularity of the Hilbert function is the biggest integer  $l$  such that  $H_B(l) \neq \Phi_{H_B,i}(l)$ , for any  $i$ .*
3. *The index of regularity of the Hilbert function equals the degree of the rational fraction defining the Poincaré series.*

**Corollary 2.2.** [5] *Let  $S$  be the semigroup generated by  $a_1, \dots, a_n$ , and  $A = K[t^{a_1}, \dots, t^{a_n}] \subset K[t]$ . The Frobenius number  $g(S)$  coincides with the degree of the rational fraction defining the Poincaré series  $P_A(u)$  by the theorem 2.1.*

*Proof.* The Hilbert function of  $A$  is given by

$$H_A(l) = \begin{cases} 1 & \text{if } l \in S \\ 0 & \text{if } l \notin S. \end{cases}$$

In particular if  $a_1, \dots, a_n$  are relatively prime,  $H_A(l) = 1$  for  $l$  large enough, and the Frobenius number coincides with the index of regularity of the Hilbert function  $H_A(l)$ , so it is the degree of the rational fraction defining the Poincaré series  $P_A(u)$ .  $\square$

### 3 Gröbner basis

Let  $a_1, \dots, a_n$  be natural numbers,  $\lambda = \gcd(a_1, \dots, a_n)$ . We denote by  $S$  (resp.  $\tilde{S}$ ) the semigroup generated by  $a_1, \dots, a_n$  (resp. by  $a_1/\lambda, \dots, a_n/\lambda$ ). Note that the semigroup rings  $K[S], K[\tilde{S}]$  are isomorphic.

Let  $R := K[x_1, \dots, x_n]$  be the polynomial ring graded by the weights  $\deg x_1 = a_1, \dots, \deg x_n = a_n$ . We consider  $\prec_{degrevlex}$  the degree reverse lexicographical order with  $x_n, \dots, x_1$ , but all results are valuable for any monomial order such that the variable  $x_1$  never appears in a minimal system of generators of the initial ideal  $in(I(S))$ . The first statement of the following theorem is an extension to the quasi-homogeneous case of [4].

**Theorem 3.1.** *Let  $A := K[t^{a_1}, \dots, t^{a_n}] \simeq R/I(S)$ .*

1. *The polynomial ring  $K[x_1] \subset A$  is a Noether normalization. Moreover let  $G(S)$  be a Gröbner basis for  $\prec_{revlex}$  and  $in(I(S))$  be the initial ideal then*

$$A \simeq \bigoplus_{x_2^{k_2} \dots x_n^{k_n} \notin in(I(S))} K[t^{a_1}][t^{k_2 a_2 + \dots + k_n a_n}].$$

2. *The Hilbert-Poincaré series is given by:*

$$P_A(t) = \frac{\sum_{x_2^{k_2} \dots x_n^{k_n} \notin in(I(S))} t^{k_2 a_2 + \dots + k_n a_n}}{1 - t^{a_1}}$$

3. *The Frobenius number  $g(\tilde{S}) = \frac{\max\{k_2 a_2 + \dots + k_n a_n \mid x_2^{k_2} \dots x_n^{k_n} \notin in(I(S))\} - a_1}{\lambda}$ .*

*Proof.* 1. We have that for any  $i = 2, \dots, n$ ,  $(t^{a_i})^{a_1} - (t^{a_1})^{a_i} = 0$ , so  $K[t^{a_1}, \dots, t^{a_n}]$  is integral over  $K[t^{a_1}]$ , both rings have dimension one so  $K[t^{a_1}] \subset K[t^{a_1}, \dots, t^{a_n}]$  is a Noether normalization, also both rings are Cohen-Macaulay. By the Auslander-Buchsbaum formula we get that  $K[t^{a_1}, \dots, t^{a_n}]$  is a free  $K[t^{a_1}]$ -module. This is the same to say that  $R/I(S)$  is a free  $K[x_1]$ -module. Since  $R/I(S)$  is a graded  $K[x_1]$ -module, we can use Nakayama's lemma, hence any  $K$ -basis of  $R/(I(S), x_1)$  gives us a basis of  $R/I(S)$  as a free  $K[x_1]$ -module. Let  $G(S)$  be a Gröbner basis for  $\prec_{revlex}$  and  $in(I(S))$  be the initial ideal, by definition of  $\prec_{degrevlex}$ ,  $x_1$  does not divide any of the elements in  $in(I(S))$ . On the other hand Macaulay's theorem [2][Theorem 15.3] says us that the set of monomials not in  $in(I(S))$  is a basis of  $R/I(S)$  as a free  $K[x_1]$ -module.

2. It is clear that the Hilbert-Poincaré series of  $K[t^{a_1}]$  is  $\frac{1}{1-t^{a_1}}$ , the Hilbert-Poincaré series is an additive function, hence we have the formula for the Hilbert-Poincaré series of  $A$ .

3. By 2.1 The Frobenius number of  $S$  is the degree of the Hilbert-Poincaré series of  $A$ . □

We have the following consequence which will be important for our algorithm:

**Corollary 3.2.** *We have that*

1.  $\text{Ap}(S, a_1) = \{k_2 a_2 + \dots + k_n a_n \mid x_2^{k_2} \dots x_n^{k_n} \notin \text{in}(I(S))\}$ . In particular

$$\text{card}\{x_2^{k_2} \dots x_n^{k_n} \notin \text{in}(I(S))\} = \frac{a_1}{\text{gcd}(a_1, \dots, a_n)}.$$

2. Let  $s \in \text{Ap}(S, a_1)$ , such that  $s = k_2 a_2 + \dots + k_n a_n$  and  $x_2^{k_2} \dots x_n^{k_n} \notin \text{in}(I(S))$ . Suppose that  $s = l_2 a_2 + \dots + l_n a_n$  for some natural numbers  $l_2, \dots, l_n$ , then  $x_2^{k_2} \dots x_n^{k_n} \prec_{\text{revlex}} x_2^{l_2} \dots x_n^{l_n}$ .

*Proof.* We can assume that  $\text{gcd}(a_1, \dots, a_n) = 1$ .

1. By the above theorem  $K[t^{a_1}, \dots, t^{a_n}] \simeq \bigoplus_{s_i \in H} K[t^{a_1}][t^{s_i}]$  where  $H = \{k_2 a_2 + \dots + k_n a_n \mid x_2^{k_2} \dots x_n^{k_n} \notin \text{in}(I(S))\}$ , now we prove that  $H = \text{Ap}(S, a_1)$ . Let  $s \in H$  suppose that  $s \notin \text{Ap}(S, a_1)$ , hence  $s - a_1 \in S$ , by the above decomposition there exists unique  $s_i \in H, l \in \mathbb{N}$  such that  $s - a_1 = s_i + l a_1$  that is  $s = s_i + (l + 1)a_1$  a contradiction to the direct sum decomposition. Reciprocally, let  $s \in \text{Ap}(S, a_1)$ , then there exists unique  $s_i \in H, l \in \mathbb{N}$  such that  $s = s_i + l a_1$ , if  $l > 0$  then  $s - a_1 \in S$  a contradiction, hence  $s = s_i \in H$ .
2. If  $s = l_2 a_2 + \dots + l_n a_n$  with  $(k_1, \dots, k_n) \neq (l_1, \dots, l_n)$  and  $x_2^{l_2} \dots x_n^{l_n} \prec_{\text{revlex}} x_2^{k_2} \dots x_n^{k_n}$  then  $x_2^{k_2} \dots x_n^{k_n} \in \text{in}(I(S))$ , a contradiction. □

**Example 3.3.** Let  $n = 2$ , and  $a_1, a_2$  be natural numbers,  $\lambda = \text{gcd}(a_1, a_2)$ , we have that  $K[t^{a_1}, t^{a_2}] \simeq K[x_1, x_2]/(x_2^{a_1/\lambda} - x_1^{a_2/\lambda})$  it is clear that that  $x_2^{a_1/\lambda} - x_1^{a_2/\lambda}$  is a Gröbner basis of the ideal  $(x_2^{a_1/\lambda} - x_1^{a_2/\lambda})$  for  $\prec_{\text{revlex}}$ . We have  $\text{in}(x_2^{a_1/\lambda} - x_1^{a_2/\lambda}) = (x_2^{a_1/\lambda})$ , hence  $K[t^{a_1}, t^{a_2}] \simeq \bigoplus_{k=0}^{a_1/\lambda-1} K[t^{a_1}][t^{ka_2}]$ , the Poincaré series is given by:  $P_A(t) = \frac{\sum_{k=0}^{a_1/\lambda-1} t^{ka_2}}{1 - t^{a_1}}$ . if  $a_1, a_2$  are coprime then the Frobenius number is  $(a_1 - 1)a_2 - a_1 = (a_1 - 1)(a_2 - 1) - 1$ .

## 4 Frobenius number, Hilbert-Poincaré series, the case $n = 3$

This section is a short version of [6] and [7].

Let consider three natural numbers  $a, b, c$  and  $S$  be the semigroup generated by  $a, b, c$ . First, remark that any solution  $\alpha := (u, v, w)$  of the Diophantine equation  $ua + vb + wc = 0$  gives rise to a binomial in the ideal  $I(S)$  in the following way:

we write the vector  $\alpha = \alpha_+ - \alpha_-$ , where the components of both  $\alpha_+, \alpha_-$  are nonnegative



then  $\underline{x}^{\alpha+} - \underline{x}^{\alpha-} \in I(S)$ , where  $\underline{x}^{\alpha+} = x_1^{\alpha+1} x_2^{\alpha+2} x_3^{\alpha+3}$ . Reciprocally if  $\underline{x}^{\alpha} - \underline{x}^{\beta} \in I(S)$  and  $\underline{x}^{\alpha}, \underline{x}^{\beta}$  have not common factors then  $(u, v, w) := \alpha - \beta$  is a solution of the equation  $ua + vb + wc = 0$ .

Second, it is clear that find solutions  $(u, v, w)$  of the Diophantine equation  $ua + vb + wc = 0$  is equivalent to find solutions  $(s, p, r)$  of the Diophantine equation  $sb - pc = ra$ .

Let  $s_0$  be the smallest natural number such that  $(s_0, 0, r_0)$  is solution of the equation  $sb - pc = ra$ . We set  $p_0 = 0$ .

Let  $p_1$  be the smallest natural number such that  $(s_1, p_1, r_1)$  is solution of the equation  $sb - pc = ra$ , with  $0 \leq s_1 < s_0$ . Note that  $s_0 = \frac{a}{\gcd(a,b)}$  and  $p_1 = \frac{\gcd(a,b)}{\gcd(a,b,c)}$ . The numbers  $s_1$  can be got from the extended Euclide's algorithm for the computation of the greatest common divisor of  $a, b$ .

Let consider the Euclides' algorithm with negative rest:

$$\left\{ \begin{array}{l} s_0 = q_2 s_1 - s_2 \\ s_1 = q_3 s_2 - s_3 \\ \dots = \dots \\ s_{m-1} = q_{m+1} s_m \\ s_{m+1} = 0 \end{array} \right.$$

$q_i \geq 2$ ,  $s_i \geq 0 \ \forall i$ .

Let define the sequences:  $p_i, r_i$  ( $0 \leq i \leq m+1$ ), by:

$$p_{i+1} = p_i q_{i+1} - p_{i-1}, r_{i+1} = r_i q_{i+1} - r_{i-1}, (1 \leq i \leq m).$$

Note that from [7] we have for  $i = 0, \dots, m$  that  $s_i p_{i+1} - s_{i+1} p_i = s_0 p_1 = \frac{a}{\gcd(a,b,c)}$ . Let  $\mu$  the unique integer such that  $r_\mu > 0 \geq r_{\mu+1}$ .

**Theorem 4.1.** ([5], [6] and [7]) *The set*

$$x_2^{s_\mu} - x_1^{r_\mu} x_3^{p_\mu}, x_3^{p_{\mu+1}} - x_1^{-r_{\mu+1}} x_2^{s_{\mu+1}}, x_2^{s_\mu - s_{\mu+1}} x_3^{p_{\mu+1} - p_\mu} - x_1^{r_\mu - r_{\mu+1}}$$

is a Gröbner basis of  $I(S)$  for  $\prec_{revlex}$  with  $x_3, x_2, x_1$ . In particular  $\text{in}(I(S)) = (x_2^{s_\mu}, x_3^{p_{\mu+1}}, x_2^{s_\mu - s_{\mu+1}} x_3^{p_{\mu+1}})$ , and

$$\begin{aligned} \mathbb{N}^2 \setminus \text{exp}(\text{in}(I(S))) &= \{(k, l) \in \mathbb{N}^2 \mid 0 \leq k < s_\mu - s_{\mu+1}, 0 \leq l < p_{\mu+1}\} \cup \\ &\{(k, l) \in \mathbb{N}^2 \mid s_\mu - s_{\mu+1} \leq k < s_\mu, 0 \leq l < p_{\mu+1} - p_\mu\}. \end{aligned}$$

$$K[t^a, t^b, t^c] \simeq \bigoplus_{(k,l) \in \mathbb{N}^2 \setminus \text{exp}(\text{in}(I(S)))} K[t^a][t^{kb+lc}].$$

In particular the Poincaré series is given by:

$$P_A(t) = \frac{\sum_{(k,l) \in \mathbb{N}^2 \setminus \text{exp}(\text{in}(I(S)))} t^{kb+lc}}{1 - t^a}$$

and if the numbers  $a, b, c$  are relatively prime the Frobenius number is

$$g(S) = \max \{kb + lc \mid (k, l) \in \mathbb{N}^2 \setminus \exp(\text{in}(I(S)))\} - a_1.$$

*Proof.* we can give a new and shorter proof than the one given in the general case in [7]. We can assume that the numbers  $a, b, c$  are relatively prime. Let consider the three elements of  $I(S)$ :  $x_2^{s_\mu} - x_1^{r_\mu} x_3^{p_\mu}$ ,  $x_3^{p_{\mu+1}} - x_1^{-r_{\mu+1}} x_2^{s_{\mu+1}}$ ,  $x_2^{s_{\mu} - s_{\mu+1}} x_3^{p_{\mu+1} - p_\mu} - x_1^{r_{\mu} - r_{\mu+1}}$ . It then follows that  $J := (x_2^{s_\mu}, x_3^{p_{\mu+1}}, x_2^{s_{\mu} - s_{\mu+1}} x_3^{p_{\mu+1}}) \subset \text{in}(I(S))$ . Now we count the numbers of monomial not in  $J$ ,

$$\text{card}(\mathbb{N}^2 \setminus \exp(J)) = (s_\mu - s_{\mu+1})p_{\mu+1} + s_{\mu+1}(p_{\mu+1} - p_\mu) = s_\mu p_{\mu+1} - s_{\mu+1} p_\mu = a.$$

On the other hand by Corollary 3.2,  $\text{card}(\mathbb{N}^2 \setminus \exp(\text{in}(I(S)))) = a$  this implies  $\text{in}(I(S)) = J$ , hence the set  $x_2^{s_\mu} - x_1^{r_\mu} x_3^{p_\mu}$ ,  $x_3^{p_{\mu+1}} - x_1^{-r_{\mu+1}} x_2^{s_{\mu+1}}$ ,  $x_2^{s_{\mu} - s_{\mu+1}} x_3^{p_{\mu+1} - p_\mu} - x_1^{r_{\mu} - r_{\mu+1}}$  is a Gröbner basis of  $I(S)$  for  $\prec_{\text{revlex}}$  with  $x_3, x_2, x_1$ . The other claims follows from the Theorem 3.1  $\square$

## 5 Algorithm for the case $n \geq 4$

For  $n = 3$ , we have seen that the algorithm use only Euclide's algorithm. Let  $n \geq 4$ . Let  $a_1, \dots, a_n$  be relatively prime natural numbers,  $S$  the semigroup generated by  $a_1, \dots, a_n$ . Let  $R := K[x_1, \dots, x_n]$  be the polynomial ring graded by the weights  $\deg x_1 = a_1, \dots, \deg x_n = a_n$ . We consider  $\prec_{\text{degrevlex}}$  the degree reverse lexicographical order with  $x_n, \dots, x_1$ ,  $A = K[t^{a_1}, \dots, t^{a_n}] \simeq R/I(S)$ .

*Remark 5.1.* If  $x_2^{k_2} \dots x_n^{k_n} \in \text{in}(I(S))$  is part of a minimal generating system of  $\text{in}(I(S))$ , then all the monomials with exponents in the cube  $[0; k_2] \times \dots \times [0; k_n]$  are not in  $\text{in}(I(S))$  except  $x_2^{k_2} \dots x_n^{k_n}$ . that is we have  $(k_2 + 1) \times \dots \times (k_n + 1) - 1$  monomials not in  $\text{in}(I(S))$ , hence  $(k_2 + 1) \times \dots \times (k_n + 1) - 1 \leq a_1$ , which implies  $S_{n-1} + S_{n-2} + \dots + S_2 + S_1 \leq a_1$ , where  $S_i$  is the symmetric polynomial of degree  $i$  in the variables  $k_2, \dots, k_n$ . In particular  $k_2 + \dots + k_n \leq a_1$ .

We have the following algorithm for the Frobenius number:

**Algorithm 5.2.** Frobenius MM-DD:

**Input:**  $a_1, \dots, a_n$

**Ouput:** the Apéry set of  $S$  with respect to  $a_1$ . The Frobenius number of  $S$ .

**Begin**

1.  $sum = 1$ ; test=false, testsum=false,  $Ap = \{0\}$ ,  $Apmod = \{0\}$ .
2. while  $(sum \leq a_1)$  and  $(testsum=false)$  do
  - (a) for each monomial  $x_2^{k_2} \dots x_n^{k_n}$  with  $k_2 + \dots + k_n = sum$

- (b) if  $\prod_{i=2}^n (k_i + 1) \leq a_1 + 1$  then
    - i. compute  $qsum := k_2 a_2 + \dots + k_n a_n$  and the congruence class mod  $a_1$ , that is  $k_2 a_2 + \dots + k_n a_n \bmod a_1$
    - ii. If  $qsum \bmod a_1$  doesn't belongs to  $Apmod$  then  $Ap = Ap \cup \{qsum\}$ ,  $Apmod = Apmod \cup \{qsum \bmod a_1\}$ .
    - iii. If  $qsum \bmod a_1$  belongs to  $Apmod$ , let  $h \in Ap$  such that  $h = qsum \bmod a_1$ 
      - A. test=true
      - B. If  $qsum < h$  then  $Ap = (Ap \setminus \{h\}) \cup \{qsum\}$ .
  - (c) end if ( $\prod_{i=2}^n (k_i + 1) \leq a_1 + 1$ )
  - (d) If either  $\text{card}(Apmod) = a_1$  or for all monomials  $x_2^{k_2} \dots x_n^{k_n}$  with  $k_2 + \dots + k_n = sum$ , we have that test=true then testsum=true.
  - (e) Otherwise testsum=false.  $sum = sum + 1$ .
3.  $frob + a_1 = \max \{p \in Ap\}$ .

**End** (of the algorithm).

The knapsack integer problem: Let  $b \in \mathbb{N}$ , in order to know if  $b \in S$  we run the above algorithm, let  $h \in Ap$  such that  $b = h \bmod a_1$ , then  $b \in S$  if and only if  $b \geq h$ .

**Algorithm 5.3.** Frobenius NoINI MM-DD:

**Input:**  $a_1, \dots, a_n$

**Output:** the Apéry set of  $S$  with respect to  $a_1$ . The Frobenius number of  $S$ . The set of monomials in the variables  $x_2, \dots, x_n$  not in the ideal  $in(I(S))$ .  $Sumnoini$  the least upper bound for the usual degree of a minimal system of monomial generators of the ideal  $in(I(S))$

**Begin**

1.  $sum = 1$ ; test=false, testsum=false,  $Ap = \{0\}$ ,  $Apmod = \{0\}$ ,  $Lnoini = \{1\}$ .
2. while ( $sum \leq a_1$ ) and (testsum=false) do
  - (a) for each monomial  $x_2^{k_2} \dots x_n^{k_n}$  with  $k_2 + \dots + k_n = sum$
  - (b) if  $\prod_{i=2}^n (k_i + 1) \leq a_1 + 1$  then
    - i. compute  $qsum := k_2 a_2 + \dots + k_n a_n$  and the congruence class mod  $a_1$ , that is  $k_2 a_2 + \dots + k_n a_n \bmod a_1$
    - ii. If  $qsum \bmod a_1$  doesn't belongs to  $Apmod$  then  $Ap = Ap \cup \{qsum\}$ ,  $Apmod = Apmod \cup \{qsum \bmod a_1\}$ ,  $Lnoini = Lnoini \cup \{x_2^{k_2} \dots x_n^{k_n}\}$ .
    - iii. If  $qsum \bmod a_1$  belongs to  $Apmod$ , let  $h \in Ap$  such that  $h = qsum \bmod a_1$ , and  $x_2^{j_2} \dots x_n^{j_n} \in Lnoini$  which  $qsum$  is  $h$ .
      - A. test=true

B. If  $x_2^{k_2} \dots x_n^{k_n} \prec_{revlex} x_2^{j_2} \dots x_n^{j_n}$  then  $Ap = (Ap \setminus \{h\} \cup \{qsum\})$ , and  
 $Lnoini = (Lnoini \setminus \{x_2^{j_2} \dots x_n^{j_n}\} \cup \{x_2^{k_2} \dots x_n^{k_n}\})$ .

- (c) end if ( $\prod_{i=2}^n (k_i + 1) \leq a_1 + 1$ )
  - (d) If either  $\text{card}(Apmo) = a_1$  or for all monomials  $x_2^{k_2} \dots x_n^{k_n}$  with  $k_2 + \dots + k_n = \text{sum}$ , we have that  $\text{test}=\text{true}$  then  $\text{testsum}=\text{true}$ .
  - (e) Otherwise  $\text{testsum}=\text{false}$ .  $\text{sum} = \text{sum} + 1$ .
3.  $\text{frob} + a_1 = \max \{p \in Ap\}$ .

**End** (of the algorithm).

**Algorithm 5.4.** Frobenius INI,GB MM-DD:

This algorithm needs to run previously the algorithm Frobenius NOINI MM-DD.

**Input:**  $a_1, \dots, a_n, Ap, Apmo, Lnoini$ .

**Output:** the Apéry set of  $S$  with respect to  $a_1$ . The Frobenius number of  $S$ . The set of monomials in the variables  $x_2, \dots, x_n$  not in the ideal  $\text{in}(I(S))$ .

**Begin**

- 1.  $\text{sum} = 1$ ;  $\text{test}=\text{false}$ ,  $\text{testsum}=\text{false}$ ,  $Lini = \emptyset$ .
- 2. while ( $\text{sum} \leq \text{Sumnoini}$ ) and ( $\text{testsum}=\text{false}$ ) do
  - (a) for each monomial  $x_2^{k_2} \dots x_n^{k_n}$  with  $k_2 + \dots + k_n = \text{sum}$
  - (b) if  $x_2^{k_2} \dots x_n^{k_n}$  belongs to the ideal generated by  $Lini$  then  $\text{testini} = \text{true}$ .
  - (c) If  $\text{testini} = \text{false}$  then
  - (d) if  $\prod_{i=2}^n (k_i + 1) \leq a_1 + 1$  then
  - (e) compute  $qsum(\mathbf{k}) := k_2 a_2 + \dots + k_n a_n$  and the congruence class mod  $a_1$ , that is  $k_2 a_2 + \dots + k_n a_n \bmod a_1$ 
    - i. Test if  $x_2^{k_2} \dots x_n^{k_n}$  is a generator of  $\text{in}(I)$ :  
let  $x_2^{j_2} \dots x_n^{j_n} \in Lnoini$  such that its  $qsum(\mathbf{k}) = qsum(\mathbf{j}) \bmod a_1$ . If  $\mathbf{j} \neq \mathbf{k}$   
and  $\mathbf{j}, \mathbf{k}$  have disjoint supports then  $Lini = Lini \cup \{x_2^{k_2} \dots x_n^{k_n}\}$
    - ii.  $\text{test}=\text{true}$
  - (f) end if ( $\prod_{i=2}^n (k_i + 1) \leq a_1 + 1$ )
  - (g) end if ( $\text{testini} = \text{false}$ )
  - (h) If either  $\text{card}(Apmo) = a_1$  or for all monomials  $x_2^{k_2} \dots x_n^{k_n}$  with  $k_2 + \dots + k_n = \text{sum}$ , we have that  $\text{test}=\text{true}$  then  $\text{testsum}=\text{true}$ .
  - (i) Otherwise  $\text{testsum}=\text{false}$ .  $\text{sum} = \text{sum} + 1$ .
- 3.  $\text{frob} + a_1 = \max \{p \in Ap\}$ .

**End** (of the algorithm).

In the following example we compute the Gröbner basis by my software and by Cocoa, it runs in above 7 seconds in both softwares. The Gröbner basis has 571 generators.

**Example 5.5.** ( $a_1 = 1030, a_2 = 1031, a_3 = 1034, a_4 = 1039, a_5 = 1046, a_6 = 1055, a_7 = 1066, a_8 = 1079, a_9 = 1094, a_{10} = 1111, a_{11} = 1130, a_{12} = 1151, a_{13} = 1373, a_{14} = 1393, a_{15} = 1423, a_{16} = 1433, a_{17} = 1493$ ).

If we compute only the Apéry number our algorithm is much faster than Cocoa. In this example the Apéry number is 5145.

## References

- [1] R. Apéry, Sur les branches superlinéaires des courbes algébriques, C.R. Acad. Sci. Paris 222 (1946), 1198-1200.
- [2] D. Eisenbud, Commutative algebra with a view toward algebraic geometry. Graduate Texts in Math., vol. 150 (1995), Springer-Verlag, Berlin and New York.
- [3] Einstein, David, et al. "Frobenius numbers by lattice point enumeration.." Integers 7.1 (2007)
- [4] Monique Lejeune-Jalabert, Effectivité des calculs polynomiaux, Cours de DEA 1984-1985, Institut Fourier, Université de Grenoble I.
- [5] Morales Marcel, *Fonctions de Hilbert, genre géométrique d'une singularité quasi-homogène Cohen-Macaulay*. CRAS Paris, t.301, série A n° 14 (1985).
- [6] Morales M.- Syzygies of monomial curves and a linear diophantine problem of Frobenius, Preprint Max Planck Institut für Mathematik (Bonn-RFA) (1987)
- [7] M. Morales, Équations des variétés monomiales en codimension deux, J. Algebra **175** (1995) 1082-1095.
- [8] J. Rodseth, On a linear Diophantine problem of Frobenius, J. reine angew. Math., 301(1978) 171-178
- [9] B.H. Roune, Solving Thousand Digit Frobenius Problems Using Grobner Bases, Journal of Symbolic Computation volume 43(1) 2008,1-7.
- [10] J.L. Ramírez Alfonsín, The Diophantine Frobenius Problem, Oxford Lectures Series in Mathematics and its Applications 30, Oxford University Press, (2005), 256 pages.