



HAL
open science

Hölderian weak invariance principle under Maxwell and Woodroffe condition

Davide Giraudo

► **To cite this version:**

Davide Giraudo. Hölderian weak invariance principle under Maxwell and Woodroffe condition. 2015.
hal-01211819v1

HAL Id: hal-01211819

<https://hal.science/hal-01211819v1>

Preprint submitted on 5 Oct 2015 (v1), last revised 4 Nov 2018 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

HÖLDERIAN WEAK INVARIANCE PRINCIPLE UNDER MAXWELL AND WOODROOFE CONDITION

DAVIDE GIRAUDO

ABSTRACT. We investigate the weak invariance principle in Hölder spaces under some reinforcement of the Maxwell and Woodroffe condition.

1. INTRODUCTION AND MAIN RESULTS

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and let $T: \Omega \rightarrow \Omega$ be a measure-preserving bijective and bi-measurable function. Let \mathcal{M} be a sub- σ -algebra of \mathcal{F} such that $T\mathcal{M} \subset \mathcal{M}$. If θ is a measure preserving operator and $f: \Omega \rightarrow \mathbb{R}$ a measurable function, we denote $S_n(\theta, f) := \sum_{j=0}^{n-1} f \circ \theta^j$ and

$$W(n, f, \theta, t) := S_{[nt]}(f) + (nt - [nt])f \circ \theta^{[nt]}. \quad (1.1)$$

When $\theta = T$ we shall often write $S_n(f)$ and $W(n, f, t)$. We denote \mathcal{M}_∞ the σ -algebra generated by $\bigcup_{i \in \mathbb{Z}} T^i \mathcal{M}$ and $\mathcal{M}_{-\infty} := \bigcap_{i \in \mathbb{Z}} T^i \mathcal{M}$. We say that the function $f \in \mathbb{L}^1$ is *regular* if f is \mathcal{M}_∞ -measurable and $\mathbb{E}[f | \mathcal{M}_{-\infty}] = 0$.

An important problem in probability theory is the understanding of the asymptotic behavior of the process $(n^{-1/2}W(n, f, t), t \in [0, 1])_{n \geq 1}$. Conditions on the quantities $\mathbb{E}[S_n(f) | T\mathcal{M}]$ and $S_n(f) - \mathbb{E}[S_n(f) | T^{-n}\mathcal{M}]$ have been investigated. The first result in this direction was obtained by Maxwell and Woodroffe [MW00]: if f is regular, \mathcal{M} measurable and

$$\sum_{n=1}^{+\infty} \frac{\|\mathbb{E}[S_n(f) | T\mathcal{M}]\|_2}{n^{3/2}} < \infty, \quad (1.2)$$

then $(n^{-1/2}S_n(f))_{n \geq 1}$ converges in distribution to $\eta^2 N$, where N is normally distributed and independent of η . Then Volný [Vol06] proposed a method to treat the nonadapted case. Peligrad and Utev [PU05] proved the weak invariance principle under condition (1.2). The nonadapted case was addressed in [Vol07]. Peligrad and Utev also showed that condition (1.2) is optimal among conditions on the growth of the sequence $(\|S_n(f) | T\mathcal{M}\|_2)_{n \geq 1}$: if

$$\sum_{n=1}^{+\infty} a_n \frac{\|\mathbb{E}[S_n(f) | T\mathcal{M}]\|_2}{n^{3/2}} < \infty \quad (1.3)$$

for some sequence $(a_n)_{n \geq 1}$ converging to 0, the sequence $(n^{-1/2}S_n(f))_{n \geq 1}$ is not necessarily stochastically bounded (Theorem 1.2. of [PU05]). Volný constructed [Vol10] an example satisfying (1.3) and such that the sequence $(\|S_n(f)\|_2^{-1} S_n(f))_{n \geq 1}$ admits two subsequences which converge weakly to two different distributions.

Date: October 5, 2015.

2010 Mathematics Subject Classification. 60F05; 60F17.

Key words and phrases. Invariance principle, martingales, Hölder spaces, strictly stationary process.

Let us denote by \mathcal{H}_α the space of Hölder continuous functions, that is, the functions $x: [0, 1] \rightarrow \mathbb{R}$ such that $\|x\|_{\mathcal{H}_\alpha} := \sup_{0 \leq s < t \leq 1} |x(t) - x(s)| / (t - s)^\alpha + |x(0)|$ is finite. Since the paths of Brownian motion belong almost surely to \mathcal{H}_α for each $\alpha \in (0, 1/2)$ as well as $W(n, f, \cdot)$, we can investigate the weak convergence of the sequence $(n^{-1/2}W(n, f, \cdot))_{n \geq 1}$ in the space \mathcal{H}_α , for $0 < \alpha < 1/2$. The case of i.i.d. sequences and stationary martingale difference sequences have been addressed respectively by Račkauskas and Suquet (Theorem 1 of [RS03]) and Giraudo (Theorem 2.2 of [Gir15]). In this note, we focus conditions on the sequences $(\mathbb{E}[S_n(f) | T\mathcal{M}])_{n \geq 1}$ and $(S_n(f) - \mathbb{E}[S_n(f) | T^{-n}\mathcal{M}])_{n \geq 1}$.

Theorem 1.1. *Let $p > 2$ and $f \in \mathbb{L}^p$ be a regular function. If*

$$\sum_{k=1}^{\infty} \frac{\|\mathbb{E}[S_k(f) | T\mathcal{M}]\|_p}{k^{3/2}} < \infty, \quad \sum_{k=1}^{\infty} \frac{\|S_k(f) - \mathbb{E}[S_k(f) | T^{-k}\mathcal{M}]\|_p}{k^{3/2}} < \infty, \quad (1.4)$$

then the sequence $(n^{-1/2}W(n, f))_{n \geq 1}$ converges weakly to the process $\eta^2 W$ in $\mathcal{H}_{1/2-1/p}$, where W is the Brownian motion and the random variable η is independent of W .

The expression of η is given in Theorem 1 of [MTK08]. Of course, if f is \mathcal{M} -measurable, all the terms of the second series vanish and we only have to check the convergence of the first series.

Remark 1.2. If the sequence $(f \circ T^j)_{j \geq 0}$ is a martingale difference sequence with respect to the filtration $(T^{-i}\mathcal{M})$, then condition (1.4) is satisfied if and only if the function f belongs to \mathbb{L}^p , hence we recover the result of [Gir15]. However, if the sequence $(f \circ T^j)_{j \geq 0}$ is independent, (1.4) is stronger than the sufficient condition $t^p \mu\{|f| > t\} \rightarrow 0$. This can be explained by the fact that the key maximal inequality (2.7) does not include the quadratic variance term which appears in the martingale inequality. In Remark 1 after Theorem 1 in [PUW07], a version of (2.7) with this term is obtained. In our context it seems that it does not follow from an adaptation of the proof.

Remark 1.3. In [Gir15], the conclusion of Theorem 1.1 was obtained under the condition

$$\sum_{i=1}^{\infty} \|\mathbb{E}[f | T^i\mathcal{M}] - \mathbb{E}[f | T^{i+1}\mathcal{M}]\|_p < \infty. \quad (1.5)$$

Using the construction given in [DV08, Dur09], in any ergodic dynamic system of positive entropy one can construct a function satisfying condition (1.4) but not (1.5) and vice versa.

Remark 1.4. For the ρ -mixing coefficient defined by

$$\rho(n) = \sup \left\{ \text{Cov}(X, Y) / (\|X\|_2 \|Y\|_2), X \in \mathbb{L}^2(\sigma(f \circ T^i, i \leq 0)), Y \in \mathbb{L}^2(\sigma(f \circ T^i, i \geq n)) \right\},$$

Lemma 1 of [PUW07] shows that for an adapted process, condition (1.4) is satisfied if $\sum_{n=1}^{\infty} \rho^{2/p}(2^n)$ converges. However, the conclusion of Theorem 1.1 holds if $t^p \mu\{|f| > t\} \rightarrow 0$ and $\sum_{n=1}^{\infty} \rho(2^n)$ converges (see Theorem 2.3, [Gir14]), which is less restrictive.

It turns out that even in the adapted case, condition (1.4) is sharp among conditions on $\|\mathbb{E}[S_k(f) | T\mathcal{M}]\|_p$ in the following sense.

Theorem 1.5. *For each sequence $(a_n)_{n \geq 1}$ converging to 0 and each real number $p > 2$, there exists a strictly stationary sequence $(f \circ T^j)_{j \geq 0}$ and a sub- σ -algebra \mathcal{M} such that $T\mathcal{M} \subset \mathcal{M}$,*

$$\sum_{n=1}^{\infty} \frac{a_n}{n^{3/2}} \|\mathbb{E}[S_n(f) \mid T\mathcal{M}]\|_p < \infty, \quad (1.6)$$

but the sequence $(n^{-1/2}W(n, f, t))_{n \geq 1}$ is not tight in $\mathcal{H}_{1/2-1/p}$.

Remark 1.6. Using the inequalities in [PUW07] in order to bound $\|\mathbb{E}[S_n(f) \mid T\mathcal{M}]\|_2$, we can see that the constructed f in the proof of Theorem 1.5 satisfies the classical Maxwell and Woodroffe condition (1.2) (the fact that p is strictly greater than 2 is crucial), hence the weak invariance principle in the space of continuous functions takes place.

However, it remains an open question whether condition (1.6) implies the central limit theorem or the weak invariance principle (in the space of continuous functions).

2. PROOFS

The proof of Theorem 1.1 will follow the same strategy as in [PU05]. We start by the adapted case. We want to approximate the partial sum process $(n^{-1/2}W(n, f))_{n \geq 1}$ by a similar process associated to a stationary martingale difference. The approximating martingale is the same as in Section 2.4 of [PU05], and we have to check that it approximates $(n^{-1/2}W(n, f))_{n \geq 1}$ in the sense of the topology of $\mathcal{H}_{1/2-1/p}$. To this aim, we establish a maximal inequality which allows to control the $\mathbb{L}^{p, \infty}$ -norm of the Hölderian norm of the function $t \mapsto W(n, f, T)$. We then exploit ideas of [KV07] to address the non-adapted case.

Notice that condition (1.4) implies by Theorem 1 of [PUW07] that the sequence $(n^{-1/2}S_n(f))_{n \geq 1}$ is bounded in \mathbb{L}^p , nevertheless the counter-example given in Theorem 2.5 of [Gir14] shows that we cannot deduce the weak invariance principle from this.

2.1. A maximal inequality. For $p > 2$, we define

$$\|h\|_{p, \infty} := \sup_{\substack{A \in \mathcal{F} \\ \mu(A) > 0}} \frac{1}{\mu(A)^{1-1/p}} \mathbb{E}[|h| \mathbf{1}_A]. \quad (2.1)$$

This norm is linked to the tail function of h by the following inequalities:

$$\left(\sup_{t > 0} t^p \mu\{|h| > t\} \right)^{1/p} \leq \|h\|_{p, \infty} \leq \frac{p}{p-1} \left(\sup_{t > 0} t^p \mu\{|h| > t\} \right)^{1/p}. \quad (2.2)$$

As a consequence, if N is an integer and h_1, \dots, h_n are functions, then

$$\left\| \max_{1 \leq j \leq N} |h_j| \right\|_{p, \infty} \leq \frac{p}{p-1} N^{1/p} \max_{1 \leq j \leq N} \|h_j\|_{p, \infty}. \quad (2.3)$$

For a positive $n \geq 1$, a function $f: \Omega \rightarrow \mathbb{R}$ and a measure-preserving map θ , we define

$$M(n, f, \theta) := \max_{0 \leq i < j \leq n} \frac{|S_j(\theta, f) - S_i(\theta, f)|}{(j-i)^{1/2-1/p}}. \quad (2.4)$$

By Lemma A.2 of [MSR12], the Hölderian norm of polygonal line is reached at two vertices, hence

$$M(n, f, \theta) = n^{1/2-1/p} \|W(n, f, \theta, \cdot)\|_{\mathcal{H}_{1/2-1/p}} \quad (2.5)$$

Applying Proposition 2.3 of [Gir15], we can find for each $p > 2$ a constant C_p depending only on p such that if $(m \circ T^i)_{i \geq 1}$ is a martingale difference sequence, then for each n ,

$$\|M(n, m, T)\|_{p, \infty} \leq C_p n^{1/p} \|m\|_p. \quad (2.6)$$

In the sequel, fix such a constant C_p and define $K_p := 4 + 2^{1/p}$.

The goal of this subsection is to establish the following maximal inequality.

Proposition 2.1. *Let r be a positive integer. For each measure-preserving map $T: \Omega \rightarrow \Omega$ bijective and bi-measurable, each sub- σ -algebra \mathcal{M} satisfying $T\mathcal{M} \subset \mathcal{M}$, each \mathcal{M} -measurable function $f: \Omega \rightarrow \mathbb{R}$ and each integer n satisfying $2^{r-1} \leq n < 2^r$,*

$$\|M(n, f, T)\|_{p, \infty} \leq C_p n^{1/p} \left(\|f - \mathbb{E}[f | T\mathcal{M}]\|_p + K_p \sum_{j=0}^{r-1} 2^{-j/2} \|\mathbb{E}[S_{2^j}(f) | T\mathcal{M}]\|_p \right). \quad (2.7)$$

The proof is in the same spirit as the proof of Theorem 1 of [PUW07], which is done by dyadic induction. To do so, we start from the following lemma:

Lemma 2.2. *For each positive integer n , each function $h: \Omega \rightarrow \mathbb{R}$ and each measure-preserving map $T: \Omega \rightarrow \Omega$, the following inequality holds:*

$$M(n, h, T) \leq 6 \max_{0 \leq k \leq n} |h \circ T^k| + \frac{1}{2^{1/2-1/p}} M\left(\left\lfloor \frac{n}{2} \right\rfloor, h + h \circ T, T^2\right). \quad (2.8)$$

Proof. First, notice that if $1 \leq j \leq n$, then $j = 2 \lfloor \frac{j}{2} \rfloor$ or $j = 2 \lfloor \frac{j}{2} \rfloor + 1$, hence

$$\left| S_j(h) - S_{2 \lfloor \frac{j}{2} \rfloor}(h) \right| \leq \max_{0 \leq k \leq n} |h \circ T^k|. \quad (2.9)$$

Similarly, we have

$$\left| S_i(h) - S_{2 \lfloor \frac{i+2}{2} \rfloor}(h) \right| \leq 2 \max_{0 \leq k \leq n} |h \circ T^k|. \quad (2.10)$$

It thus follows that

$$M(n, h, T) \leq 4 \max_{0 \leq k \leq n} |h \circ T^k| + \max_{0 \leq i < j \leq n} \frac{\left| S_{2 \lfloor \frac{j}{2} \rfloor}(h) - S_{2 \lfloor \frac{i+2}{2} \rfloor}(h) \right|}{(j-i)^{1/2-1/p}}. \quad (2.11)$$

Notice that if $j \geq i + 4$, then

$$1 \leq \left\lfloor \frac{j}{2} \right\rfloor - \left\lfloor \frac{i+2}{2} \right\rfloor \leq \frac{j-i}{2}, \quad (2.12)$$

and we derive the bound

$$\begin{aligned} \max_{0 \leq i < j \leq n} \frac{\left| S_{2 \lfloor \frac{j}{2} \rfloor}(h) - S_{2 \lfloor \frac{i+2}{2} \rfloor}(h) \right|}{(j-i)^{1/2-1/p}} &\leq \frac{1}{2^{1/2-1/p}} \max_{0 \leq u < v \leq \lfloor \frac{n}{2} \rfloor} \frac{\left| S_v(T^2, h + h \circ T) - S_u(T^2, h + h \circ T) \right|}{(v-u)^{1/2-1/p}} + \\ &+ \max_{\substack{0 \leq i < j \leq n \\ j \leq i+4}} \left| S_{2 \lfloor \frac{j}{2} \rfloor}(h) - S_{2 \lfloor \frac{i+2}{2} \rfloor}(h) \right|. \end{aligned}$$

Since for $j \leq i + 4$, the number of terms of the form $h \circ T^q$ involved in $S_{2[\frac{j}{2}]}(h) - S_{2[\frac{i+2}{2}]}(h)$ is at most 2, we conclude that

$$\begin{aligned} \max_{0 \leq i < j \leq n} \frac{|S_{2[\frac{j}{2}]}(h) - S_{2[\frac{i+2}{2}]}(h)|}{(j-i)^{1/2-1/p}} &\leq \frac{1}{2^{1/2-1/p}} M\left(\left[\frac{n}{2}\right], h + h \circ T, T^2\right) + \\ &+ 2 \max_{0 \leq k \leq n} |h \circ T^k|. \end{aligned}$$

Combining this inequality with (2.11), we obtain (2.8), which concludes the proof of Lemma 2.2. \square

Now, we establish inequality (2.7) by induction on r .

Proof of Proposition 2.1. We check the case $r = 1$. Then necessarily $n = 1$ and the expression $M(n, f, t)$ reduces to f . Since C_p and K_p are greater than 1, the result is a simple consequence of the triangle inequality applied to $f - \mathbb{E}[f | T\mathcal{M}]$ and $\mathbb{E}[f | T\mathcal{M}]$.

Now, assume that Proposition 2.1 holds for some r and let us show that it takes place for $r + 1$. We thus consider an integer n such that $2^r \leq n < 2^{r+1}$, a function $f: \Omega \rightarrow \mathbb{R}$, for each measure-preserving map $T: \Omega \rightarrow \Omega$ bijective and bi-measurable, and a sub- σ -algebra \mathcal{M} satisfying $T\mathcal{M} \subset \mathcal{M}$ and we have to show that (2.7) holds with $r + 1$ instead of r . First, using inequality $M(n, f, t) \leq M(n, f - \mathbb{E}[f | T\mathcal{M}], T) + M(n, \mathbb{E}[f | T\mathcal{M}])$ and Lemma 2.2 with $h := \mathbb{E}[f | T\mathcal{M}]$, we derive

$$\begin{aligned} M(n, f, T) &\leq M(n, f - \mathbb{E}[f | T\mathcal{M}], T) + 6 \max_{0 \leq k \leq n} |\mathbb{E}[f | T\mathcal{M}] \circ T^k| + \\ &+ \frac{1}{2^{1/2-1/p}} M\left(\left[\frac{n}{2}\right], \mathbb{E}[f | T\mathcal{M}] + \mathbb{E}[f | T\mathcal{M}] \circ T, T^2\right), \end{aligned} \quad (2.13)$$

hence taking the norm $\|\cdot\|_{p, \infty}$, we obtain by (2.2) that

$$\begin{aligned} \|M(n, f, T)\|_{p, \infty} &\leq \|M(n, f - \mathbb{E}[f | T\mathcal{M}], T)\|_{p, \infty} + 6(n+1)^{1/p} \frac{p}{p-1} \|\mathbb{E}[f | T\mathcal{M}]\|_p + \\ &+ \frac{1}{2^{1/2-1/p}} \left\| M\left(\left[\frac{n}{2}\right], \mathbb{E}[f | T\mathcal{M}] + \mathbb{E}[f | T\mathcal{M}] \circ T, T^2\right) \right\|_{p, \infty}. \end{aligned} \quad (2.14)$$

By inequality (2.6) and accounting the fact that $6 \cdot (n+1)^{1/p} p / (p-1) \leq C_p n^{1/p}$, we obtain

$$\begin{aligned} \|M(n, f, T)\|_{p, \infty} &\leq C_p n^{1/p} \|f - \mathbb{E}[f | T\mathcal{M}]\|_p + C_p n^{1/p} \|\mathbb{E}[f | T\mathcal{M}]\|_p + \\ &+ \frac{1}{2^{1/2-1/p}} \left\| M\left(\left[\frac{n}{2}\right], \mathbb{E}[f | T\mathcal{M}] + \mathbb{E}[f | T\mathcal{M}] \circ T, T^2\right) \right\|_{p, \infty}. \end{aligned} \quad (2.15)$$

Since $2^{r-1} \leq [n/2] < 2^r$, we may apply the induction hypothesis to the integer $[n/2]$, the function $h := \mathbb{E}[f | T\mathcal{M}] + \mathbb{E}[f | T\mathcal{M}] \circ T$, the operator T^2 and the σ -algebra $T\mathcal{M}$. This gives

$$\begin{aligned} \left[\frac{n}{2}\right]^{-1/p} \left\| M\left(\left[\frac{n}{2}\right], h, T^2\right) \right\|_{p, \infty} &\leq C_p \|h - \mathbb{E}[h | T^3\mathcal{M}]\|_p + \\ &+ C_p K_p \sum_{j=0}^{r-1} 2^{-j/2} \|\mathbb{E}[S_{2^j}(T^2, h) | T^3\mathcal{M}]\|_p. \end{aligned} \quad (2.16)$$

Notice that $\|h - \mathbb{E}[h \mid T^3\mathcal{M}]\|_p \leq 2\|h\|_p \leq 4\|\mathbb{E}[f \mid T\mathcal{M}]\|_p$, and that $S_{2^j}(T^2, h) = S_{2^{j+1}}(\mathbb{E}[f \mid T\mathcal{M}])$, hence using the fact that $T^3\mathcal{M} \subset T\mathcal{M}$, we derive

$$\begin{aligned} \left[\frac{n}{2}\right]^{-1/p} \left\| M\left(\left[\frac{n}{2}\right], h, T^2\right) \right\|_{p,\infty} &\leq 4C_p \|\mathbb{E}[f \mid T\mathcal{M}]\|_p + \\ &+ C_p K_p \sum_{j=0}^{r-1} 2^{-j/2} \|\mathbb{E}[S_{2^{j+1}}(T, \mathbb{E}[f \mid T\mathcal{M}]) \mid T\mathcal{M}]\|_p \\ &= 4C_p \|\mathbb{E}[f \mid T\mathcal{M}]\|_p \\ &+ 2^{1/2} C_p K_p \sum_{j=1}^r 2^{-j/2} \|\mathbb{E}[S_{2^j}(T, \mathbb{E}[f \mid T\mathcal{M}]) \mid T\mathcal{M}]\|_p \end{aligned}$$

and we infer

$$\begin{aligned} \left\| M\left(\left[\frac{n}{2}\right], h, T^2\right) \right\|_{p,\infty} &\leq \left(\frac{n}{2}\right)^{1/p} (4 - K_p) C_p \|\mathbb{E}[f \mid T\mathcal{M}]\|_p \\ &+ n^{1/p} 2^{1/2-1/p} C_p K \sum_{j=0}^r 2^{-j/2} \|\mathbb{E}[S_{2^j}(T, f) \mid T\mathcal{M}]\|_p. \end{aligned} \quad (2.17)$$

Plugging this into (2.15), we derive

$$\begin{aligned} \|M(n, f, T)\|_{p,\infty} &\leq C_p n^{1/p} \|f - \mathbb{E}[f \mid T\mathcal{M}]\|_p + C_p n^{1/p} (1 + (4 - K_p) 2^{-1/p}) \|\mathbb{E}[f \mid T\mathcal{M}]\|_p + \\ &+ n^{1/p} C_p K_p \sum_{j=0}^r 2^{-j/2} \|\mathbb{E}[S_{2^j}(T, f) \mid T\mathcal{M}]\|_p \end{aligned} \quad (2.18)$$

The definition of K_p implies that $1 + (4 - K_p) 2^{-1/p} = 0$, hence (2.7) is established. This concludes the proof of Proposition 2.1. \square

2.2. Martingale approximation. In this section, we recall the construction of the approximating martingale given in [PU05] and we shall derive tightness of $(n^{-1/2}W(n, f, T))_{n \geq 1}$ in $\mathcal{H}_{1/2-1/p}$.

For a fixed positive integer r we define the functions

$$f_r := \sum_{j=0}^{r-1} f \circ T^j, \quad m_r := \frac{1}{\sqrt{r}} (f_r - \mathbb{E}[f_r \mid T^r\mathcal{M}]). \quad (2.19)$$

Notice that $\mathbb{E}[m_r \mid T^r\mathcal{M}] = 0$, hence the sequence $(m_r \circ T^{ir})_{i \geq 0}$ is a strictly stationary martingale difference sequence for the filtration $(T^{-ir}\mathcal{M})_{i \geq 0}$. Therefore, by Theorem 2.2 of [Gir15], the process $n^{-1/2}W(n, m_r, T^r)$ converges in distribution in $\mathcal{H}_{1/2-1/p}$ to $\eta_r W$, where η_r is independent of the Wiener process W . By the arguments after equation (12) in [PU05], the convergence $\lim_{r \rightarrow \infty} \|\sqrt{\eta_r} - \sqrt{\eta}\|_2 = 0$ takes place. Therefore, we have to check in view of Theorem 4.2 of [Bil68] that

$$\lim_{r \rightarrow \infty} \limsup_{n \rightarrow \infty} c(r, n) = 0, \quad (2.20)$$

where

$$c(r, n) := \left\| \left\| \frac{1}{\sqrt{n}} W(n, f, T) - \frac{1}{\sqrt{\lfloor \frac{n}{r} \rfloor}} W\left(\left\lfloor \frac{n}{r} \right\rfloor, m_r, T^r\right) \right\|_{\mathcal{H}_{1/2-1/p}} \right\|_{p, \infty} \quad (2.21)$$

For each $r, n \geq 1$, we have

$$\begin{aligned} c(r, n) &\leq \left(1 - \sqrt{\frac{n}{\lfloor \frac{n}{r} \rfloor}}\right) \frac{1}{\sqrt{n}} \left\| \left\| W(n, f, T) \right\|_{\mathcal{H}_{1/2-1/p}} \right\|_{p, \infty} + \\ &\quad + \frac{1}{\sqrt{\lfloor \frac{n}{r} \rfloor}} \left\| \left\| W(n, f, T) - W\left(\left\lfloor \frac{n}{r} \right\rfloor, \sqrt{r} \cdot m_r, T^r\right) \right\|_{\mathcal{H}_{1/2-1/p}} \right\|_{p, \infty}. \end{aligned} \quad (2.22)$$

By Proposition 2.1, we have

$$\begin{aligned} &\left(1 - \sqrt{\frac{n}{\lfloor \frac{n}{r} \rfloor}}\right) \frac{1}{\sqrt{n}} \left\| \left\| W(n, f, T) \right\|_{\mathcal{H}_{1/2-1/p}} \right\|_{p, \infty} \leq \\ &\leq \left(1 - \sqrt{\frac{n}{\lfloor \frac{n}{r} \rfloor}}\right) C_p \left(\|f - \mathbb{E}[f | T\mathcal{M}]\|_p + K_p \sum_{j=0}^{+\infty} 2^{-j/2} \|\mathbb{E}[S_{2^j}(f) | T\mathcal{M}]\|_p \right), \end{aligned} \quad (2.23)$$

hence it suffices to show that

$$\lim_{r \rightarrow \infty} \limsup_{n \rightarrow \infty} c'(r, n) = 0, \quad (2.24)$$

where

$$c'(r, n) := \frac{1}{\sqrt{\lfloor \frac{n}{r} \rfloor}} \left\| \left\| W(n, f, T) - W\left(\left\lfloor \frac{n}{r} \right\rfloor, \sqrt{r} \cdot m_r, T^r\right) \right\|_{\mathcal{H}_{1/2-1/p}} \right\|_{p, \infty}. \quad (2.25)$$

By (2.19), we have $\sqrt{r} \cdot m_r = f_r - \mathbb{E}[f_r | T^r \mathcal{M}]$ hence

$$\begin{aligned} c'(r, n) &\leq \frac{1}{\sqrt{\lfloor \frac{n}{r} \rfloor}} \left\| \left\| W(n, f, T) - W\left(\left\lfloor \frac{n}{r} \right\rfloor, f_r, T^r\right) \right\|_{\mathcal{H}_{1/2-1/p}} \right\|_{p, \infty} + \\ &\quad + \frac{1}{\sqrt{\lfloor \frac{n}{r} \rfloor}} \left\| \left\| W\left(\left\lfloor \frac{n}{r} \right\rfloor, \mathbb{E}[f_r | T^r \mathcal{M}], T^r\right) \right\|_{\mathcal{H}_{1/2-1/p}} \right\|_{p, \infty} =: a(r, n) + b(r, n). \end{aligned} \quad (2.26)$$

We shall use the following elementary lemma several times.

Lemma 2.3. *Let $p > 2$ and let f be a function in \mathbb{L}^p . Then*

$$\lim_{n \rightarrow \infty} n^{-1/p} \left\| \max_{0 \leq k \leq n} |f \circ T^k| \right\|_{p, \infty} = 0. \quad (2.27)$$

Proof. Let $R > 0$ be fixed; then

$$\max_{0 \leq k \leq n} |f \circ T^k| \leq R + \max_{0 \leq k \leq n} |(f \mathbf{1}_{\{|f| > R\}}) \circ T^k|,$$

hence using inequality (2.3), we get

$$n^{-1/p} \left\| \max_{0 \leq k \leq n} |f \circ T^k| \right\|_{p, \infty} \leq \frac{R}{n^{1/p}} + \frac{p}{p-1} \left(\frac{n+1}{n}\right)^{1/p} \|f \mathbf{1}_{\{|f| > R\}}\|_{p, \infty}.$$

Taking the lim sup as n goes to infinity and using the fact that $\|g\|_{p,\infty} \leq \|g\|_p$ for any function g , we infer that

$$\limsup_{n \rightarrow \infty} n^{-1/p} \left\| \max_{0 \leq k \leq n} |f \circ T^k| \right\|_{p,\infty} \leq \frac{p}{p-1} \|f \mathbf{1}_{\{|f| > R\}}\|_p.$$

We conclude by monotone convergence as R is arbitrary. \square

- *Control of $a(r, n)$* : let us define for $n \geq r$ the sets

$$I := \left\{ \frac{i}{n} \mid i \in \{0, \dots, n\} \right\} \quad \text{and} \quad J := \left\{ \frac{j}{\lfloor \frac{n}{r} \rfloor} \mid j \in \{0, \dots, \lfloor \frac{n}{r} \rfloor\} \right\}. \quad (2.28)$$

Notice that the random function $W(n, f, T) - W(\lfloor n/r \rfloor, f_r, T^r)$ is piecewise linear, and the vertices of its graph are at points of abscissa in $I \cup J$, hence

$$a(r, n) = \frac{1}{\sqrt{\lfloor \frac{n}{r} \rfloor} r} \left\| \sup_{\substack{s, t \in I \cup J \\ s \neq t}} \left\{ \frac{|W(n, f, T, t) - W(\lfloor \frac{n}{r} \rfloor, f_r, T^r, t)|}{|s - t|^{1/2-1/p}} - \frac{|W(n, f, T, s) - W(\lfloor \frac{n}{r} \rfloor, f_r, T^r, s)|}{|s - t|^{1/2-1/p}} \right\} \right\|_{p,\infty} =: \max \{a'(r, n), a''(r, n)\}, \quad (2.29)$$

where in $a'(r, n)$ (respectively $a''(r, n)$), the supremum is restricted to the $s, t \in I \cup J$ such that $|t - s| \geq 1/n$ (respectively $< 1/n$), which entails

$$a'(r, n) \leq 2 \frac{n^{1/2-1/p}}{\sqrt{\lfloor \frac{n}{r} \rfloor} r} \sup_{t \in I \cup J} \left\| W(n, f, T, t) - W\left(\left\lfloor \frac{n}{r} \right\rfloor, f_r, T^r, t\right) \right\|_{p,\infty}. \quad (2.30)$$

For each $i \in \{0, \dots, n\}$, we have

$$\left| W\left(n, f, T, \frac{i}{n}\right) - W\left(\left\lfloor \frac{n}{r} \right\rfloor, f_r, T^r, \frac{i}{n}\right) \right| = \left| \sum_{l=0}^{i-1} f \circ T^l - \sum_{l=0}^{\lfloor \frac{n}{r} \rfloor \frac{i}{n} - 1} f_r \circ T^{lr} - \right. \quad (2.31)$$

$$\left. - \left(\left\lfloor \frac{n}{r} \right\rfloor \frac{i}{n} - \left\lfloor \frac{n}{r} \right\rfloor \frac{i}{n} \right) f_r \circ T^{r \lfloor \frac{n}{r} \rfloor \frac{i}{n}} \right| \leq \left| \sum_{l=r \lfloor \frac{n}{r} \rfloor \frac{i}{n}}^{i-1} f \circ T^l \right| + r \max_{0 \leq k \leq n} |f \circ T^k|, \quad (2.32)$$

and since the number of indices in the sum is at most $r(1 + 1/n) \leq 2r$, we derive that

$$\sup_{t \in I} \left| W(n, f, T, t) - W\left(\left\lfloor \frac{n}{r} \right\rfloor, f_r, T^r, t\right) \right| \leq 3r \max_{0 \leq k \leq n} |f \circ T^k|. \quad (2.33)$$

Treating the supremum over J in a similar way, we obtain, in view of (2.30),

$$a'(r, n) \leq 6\sqrt{r} \cdot \frac{n^{1/2-1/p}}{\sqrt{\lfloor \frac{n}{r} \rfloor}} \left\| \max_{0 \leq k \leq n} |f \circ T^k| \right\|_{p,\infty}, \quad (2.34)$$

hence

$$\limsup_{n \rightarrow \infty} a'(r, n) \leq 6r \limsup_{n \rightarrow \infty} \frac{1}{n^{1/p}} \left\| \max_{0 \leq k \leq n} |f \circ T^k| \right\|_{p, \infty}. \quad (2.35)$$

By Lemma 2.3, it follows that

$$\limsup_{n \rightarrow \infty} a'(r, n) = 0. \quad (2.36)$$

Next, we bound $a''(r, n)$ by

$$\begin{aligned} & \frac{1}{\sqrt{\lfloor \frac{n}{r} \rfloor} r} \left\| \sup_{\substack{s, t \in I \cup J \\ |t-s| \leq 1/n}} \frac{|W(n, f, T, t) - W(n, f, T, s)|}{|s-t|^{1/2-1/p}} \right\|_{p, \infty} \\ & + \frac{1}{\sqrt{\lfloor \frac{n}{r} \rfloor} r} \left\| \sup_{\substack{s, t \in I \cup J \\ |t-s| \leq 1/n}} \frac{|W(\lfloor \frac{n}{r} \rfloor, f_r, T^r, t) - W(\lfloor \frac{n}{r} \rfloor, f_r, T^r, s)|}{|s-t|^{1/2-1/p}} \right\|_{p, \infty}. \end{aligned} \quad (2.37)$$

Let $s, t \in I \cup J$ be such that $0 < t - s < 1/n$. Then we either have $k \leq ns < nt \leq k+1$ or $k-1 \leq ns \leq k < nt < k+1$ for some $k \in \{0, \dots, n-1\}$. In the first case,

$$\begin{aligned} \frac{|W(n, f, T, t) - W(n, f, T, s)|}{|s-t|^{1/2-1/p}} &= (nt - ns) \frac{|f \circ T^k|}{|s-t|^{1/2-1/p}} \\ &\leq n(t-s)^{1-(1/2-1/p)} |f \circ T^k| \\ &\leq n^{1/2-1/p} \max_{0 \leq j \leq n} |f \circ T^j|, \end{aligned}$$

and in the second one, we have

$$\begin{aligned} \frac{|W(n, f, t) - W(n, f, s)|}{|s-t|^{1/2-1/p}} &\leq \frac{|W(n, f, t) - W(n, f, k/n)|}{|t-k/n|^{1/2-1/p}} + \frac{|W(n, f, k/n) - W(n, f, s)|}{|k/n-s|^{1/2-1/p}} \\ &= \frac{(nt-k)|f \circ T^k|}{|t-k/n|^{1/2-1/p}} + \frac{(k/ns)|f \circ T^{k-1}|}{(k/n-s)^{1/2-1/p}} \\ &\leq 2n^{1/2-1/p} \max_{0 \leq j \leq n} |f \circ T^j|. \end{aligned}$$

As a consequence, the following inequality holds:

$$\frac{1}{\sqrt{\lfloor \frac{n}{r} \rfloor} r} \left\| \sup_{\substack{s, t \in I \cup J \\ |t-s| \leq 1/n}} \frac{|W(n, f, t) - W(n, f, s)|}{|s-t|^{1/2-1/p}} \right\|_{p, \infty} \leq \frac{2\sqrt{n}}{\sqrt{\lfloor \frac{n}{r} \rfloor} r} \left\| \max_{0 \leq j \leq n} |f \circ T^j| \right\|_{p, \infty} n^{-1/p}. \quad (2.38)$$

Using a similar bound for the second term in (2.37), we obtain by Lemma 2.3, that for each $r \geq 1$,

$$\lim_{n \rightarrow \infty} a''(r, n) = 0. \quad (2.39)$$

By (2.29), (2.36) and (2.39), we finally obtain

$$\lim_{r \rightarrow \infty} \limsup_{n \rightarrow \infty} a(r, n) = 0. \quad (2.40)$$

- *Control of $b(r, n)$:* by Proposition 2.1, we have the following upper bound:

$$b(n, r) \leq C_p \left(\frac{2}{\sqrt{r}} \|\mathbb{E}[S_r(f) | T\mathcal{M}]\|_p + \frac{K_p}{\sqrt{r}} \sum_{j=0}^{+\infty} 2^{-j/2} \|\mathbb{E}[S_{r2^j}(f) | T\mathcal{M}]\|_p \right). \quad (2.41)$$

To conclude, we recall Lemma 2.8 of [PU05]:

Lemma 2.4. *Let $(V_n)_{n \geq 1}$ be a subadditive sequence such that $\sum_{n=1}^{\infty} V_n n^{-3/2} < \infty$. Then*

$$\lim_{r \rightarrow \infty} \frac{1}{\sqrt{r}} \sum_{k=0}^{+\infty} \frac{V_{r2^k}}{2^{k/2}} = 0. \quad (2.42)$$

In particular, $V_r/\sqrt{r} \rightarrow 0$ as $r \rightarrow \infty$.

Since the sequence $(\|\mathbb{E}[S_n(f) | T\mathcal{M}]\|_p)_{n \geq 1}$ is subadditive, from inequality (2.41) and Lemma 2.4 we derive

$$\lim_{r \rightarrow \infty} \limsup_{n \rightarrow \infty} b(n, r) = 0. \quad (2.43)$$

Combining (2.40) with (2.43), we obtain (2.24).

This concludes the proof of Theorem 1.1 in the adapted case.

2.3. The non-adapted case. In [Vol06], a method to prove the central limit theorem under the condition

$$\sum_{k=1}^{\infty} \frac{\|\mathbb{E}[S_k(f) | T\mathcal{M}]\|_2}{k^{3/2}} < \infty, \quad \sum_{k=1}^{\infty} \frac{\|S_k(f) - \mathbb{E}[S_k(f) | T^{-k}\mathcal{M}]\|_2}{k^{3/2}} < \infty \quad (2.44)$$

is proposed. The idea is the following: one writes $f = f' + f''$, where $f' = \mathbb{E}[f | T\mathcal{M}]$ and applies a transformation V to the process $(U^i f'')$ in such a way that $(U^i V f'')$ is a adapted sequence. The mapping V is defined as

$$Vh := \sum_{i \in \mathbb{Z}} U^{-i} P_0 U^{-i} h, \quad (2.45)$$

where $P_0(h) := \mathbb{E}[h | \mathcal{M}] - \mathbb{E}[h | T\mathcal{M}]$.

Notice that the operator V is not necessarily a point mapping (see section 3 of [KV07]). Therefore, deducing the non-adapted case from the adapted one is not immediate.

Volný proved the functional central limit theorem under (2.44) in [Vol07]. The idea is to write the maximal inequality (5) in [PU05] with the notion of contraction. We follow this approach and begin by recalling the definition of contraction operators. Let H be a subspace of \mathbb{L}^p for which $UH \subset H$. We associate to the operator U a semigroup of contraction operators $(P_{T^k})_{k \geq 1}$ on H which satisfy:

- (1) $P_{T^k} = P_T^k$ for each $k \geq 1$;
- (2) $P_T U = I$ (where I is the identity operator);
- (3) if $P_T f = 0$, then $(U^i f)_{i \geq 0}$ is a martingale difference sequence.

Writing $P_T =: P$, we are able to write Proposition 2.1 in a more general form.

Proposition 2.5. *There exist constants C_p and K_p depending only on p such that for each $f \in H$ and each $n \geq 1$,*

$$\|M(n, f, T)\|_{p, \infty} \leq C_p n^{1/p} \left(\|f - U^{-1}P(f)\|_p + K_p \sum_{j=1}^{\lfloor \log_2 n \rfloor} 2^{-j/2} \left\| \sum_{i=0}^{2^j-1} P^i f \right\|_p \right). \quad (2.46)$$

The proof can be done in a similar way as that of Proposition 2.1. The later corresponds to the particular case $H = \mathbb{L}^p(\mathcal{M})$, $P_T(f) = \mathbb{E}[Uf \mid \mathcal{M}]$ and the operator U is then replaced by U^{-1} . From this, we may deduce the following:

Corollary 2.6. *Let $f \in H$ be such that*

$$\sum_{n=1}^{\infty} \frac{\|\sum_{i=1}^n P_T^i f\|_p}{n^{3/2}} < \infty. \quad (2.47)$$

Then the sequence $(n^{-1/2}W(n, f, T))_{n \geq 1}$ is tight in $\mathcal{H}_{1/2-1/p}$. In particular, if $f \in \mathbb{L}^p$ is \mathcal{M}_∞ -measurable, $\mathbb{E}[f \mid \mathcal{M}] = 0$ and

$$\sum_{k=1}^{\infty} \frac{\|S_k(f) - \mathbb{E}[S_k(f) \mid T^{-k}\mathcal{M}]\|_p}{k^{3/2}} < \infty, \quad (2.48)$$

then the sequence $(n^{-1/2}W(n, f, T))_{n \geq 1}$ is tight in $\mathcal{H}_{1/2-1/p}$.

Proof. We define $H := \{h \in \mathbb{L}^p(\mathcal{M}_\infty), \mathbb{E}[h \mid \mathcal{M}] = 0\}$ and $P_{T^k}h := U^{-k}h - \mathbb{E}[U^{-k}h \mid \mathcal{M}]$. It is checked in the proof of Proposition 2 of [Vol07] that such a P_T satisfies the conditions (1)-(2) and (3) of the definition of a semigroup of contractions. We then conclude in a similar way as in the adapted case. \square

End of the proof of Theorem 1.1. The proof of the convergence of the finite-dimensional distributions under condition (2.44) is addressed in Theorem 1 of [Vol07]. It remains to check tightness. We define $f' := \mathbb{E}[f \mid \mathcal{M}]$ and $f'' := f - \mathbb{E}[f \mid \mathcal{M}]$ and we have to check that the sequences $(n^{-1/2}W(n, f', T))_{n \geq 1}$ and $(n^{-1/2}W(n, f'', T))_{n \geq 1}$ are tight in $\mathcal{H}_{1/2-1/p}$. Tightness of the first sequence follows from the results of Subsection 2.2. That of the second sequence is a consequence of Corollary 2.6. This concludes the proof of Theorem 1.1. \square

2.4. Counter-example. We take a similar construction as in the proof of Proposition 1 of [PUW07]. We consider a non-negative sequence $(a_n)_{n \geq 1}$, and a sequence $(u_k)_{k \geq 1}$ of real numbers such that

$$u_1 = 1, u_2 = 2, u_k^{p/2+1} + 1 < u_{k+1} \text{ for } k \geq 3 \text{ and } a_t \leq k^{-2} \text{ for } t \geq u_k. \quad (2.49)$$

Notice that since $p > 2$, the conditions (2.49) are more restrictive than that of the proof of Proposition 1 of [PUW07]. If $i = u_j$ for some $j \geq 1$, then we define $p_i := cj/u_j^{1+p/2}$ and $p_i = 0$ otherwise. Let $(Y_k)_{k \geq 0}$ be a discrete time Markov chain with the state space \mathbb{Z}^+ and transition matrix given by $p_{k, k-1} = 1$ for $k \geq 1$ and $p_{0, j-1} := p_j$, $j \geq 1$. We shall also consider a random variable τ which takes its values among non-negative integers, and whose distribution is given by $\mu(\tau = j) = p_j$. Then the stationary distribution exists and is given by

$$\pi_j = \pi_0 \sum_{i=j+1}^{\infty} p_i, j \geq 1, \text{ where } \pi_0 = 1/\mathbb{E}[\tau]. \quad (2.50)$$

We start from the stationary distribution $(\pi_j)_{j \geq 0}$ and we take $g(x) := \mathbf{1}_{x=0} - \pi_0$, where $\pi_0 = \mu \{Y_0 = 0\}$. We then define $f \circ T^j = X_j := g(Y_j)$.

It is already checked in [PUW07] that the sequence $(X_j)_{j \geq 0}$ satisfies (1.6), where $\mathcal{M} = \sigma(X_k, k \leq j)$ and $S_n = \sum_{j=1}^n X_j$. To conclude the proof, it remains to check that the sequence $(n^{-1/2}W(n, f, T))_{n \geq 1}$ is not tight in $\mathcal{H}_{1/2-1/p}$. To this aim, we define

$$T_0 = 0, T_k = \min \{t > T_{k-1} \mid Y_t = 0\}, \quad \tau_k = T_k - T_{k-1}, k \geq 1. \quad (2.51)$$

Then $(\tau_k)_{k \geq 1}$ is an independent sequence and each τ_k is distributed as τ and

$$S_{T_k} = \sum_{j=1}^k (1 - \pi_0 \tau_j) = k - \pi_0 T_k. \quad (2.52)$$

Let us fix some integer K greater than $\mathbb{E}[\tau]$. Let $\delta > 0$ be fixed and n an integer such that $1/n < \delta$. Then the inequality

$$\begin{aligned} \frac{1}{(nK)^{1/p}} \max_{\substack{0 \leq i < j \leq nK \\ j-i \leq n\delta}} \frac{|S_j - S_i|}{(j-i)^{1/2-1/p}} &\geq \frac{1}{(nK)^{1/p}} \mathbf{1} \{T_n \leq Kn\} \times \\ &\times \max_{1 \leq k \leq n} \frac{|S_{T_k} - S_{T_{k-1}}|}{(T_k - T_{k-1})^{1/2-1/p}} \mathbf{1} \{|T_k - T_{k-1}| \leq n\delta\} \end{aligned} \quad (2.53)$$

takes place. By (2.51) and (2.52), this can be rewritten as

$$\begin{aligned} \frac{1}{(nK)^{1/p}} \max_{\substack{0 \leq i < j \leq nK \\ j-i \leq n\delta}} \frac{|S_j - S_i|}{(j-i)^{1/2-1/p}} &\geq \frac{1}{(nK)^{1/p}} \mathbf{1} \{T_n \leq Kn\} \times \\ &\times \max_{1 \leq k \leq n} \frac{|1 - \pi_0 \tau_k|}{\tau_k^{1/2-1/p}} \mathbf{1} \{\tau_k \leq n\delta\}. \end{aligned} \quad (2.54)$$

Defining for a fixed C the event

$$A_n(C) := \left\{ \frac{|1 - \pi_0 \tau|}{\tau^{1/2-1/p}} \geq C(Kn)^{1/p} \right\} \cap \{\tau \leq n\delta\}, \quad (2.55)$$

we obtain by the remark before equation (2.52)

$$\mu \left\{ \frac{1}{(nK)^{1/p}} \max_{\substack{0 \leq i < j \leq nK \\ j-i \leq n\delta}} \frac{|S_j - S_i|}{(j-i)^{1/2-1/p}} \geq C \right\} \geq 1 - (1 - \mu(A_n(C)))^n - \mu \{T_n > Kn\}. \quad (2.56)$$

By the law of large numbers, we obtain, accounting $K > \mathbb{E}[\tau]$, that

$$\limsup_{n \rightarrow \infty} \mu \left\{ \frac{1}{(nK)^{1/p}} \max_{\substack{0 \leq i < j \leq nK \\ j-i \leq n\delta}} \frac{|S_j - S_i|}{(j-i)^{1/2-1/p}} \geq C \right\} \geq \limsup_{n \rightarrow \infty} 1 - (1 - \mu(A_n(C)))^n. \quad (2.57)$$

We choose $C := \pi_0/(2K^{1/p})$. Considering the integers n of the form $[u_j^{(p+2)/2}]$, we obtain in view of (2.57) :

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mu \left\{ \frac{1}{(nK)^{1/p}} \max_{\substack{0 \leq i < j \leq nK \\ j-i \leq n\delta}} \frac{|S_j - S_i|}{(j-i)^{1/2-1/p}} \geq \frac{\pi_0}{2K^{1/p}} \right\} &\geq \\ &\geq \limsup_{j \rightarrow \infty} 1 - \left(1 - \mu(A_{[u_j^{(p+2)/2}]}(\pi_0/(2K^{1/p}))) \right)^{u_j^{(p+2)/2}}. \end{aligned} \quad (2.58)$$

Since $\tau \geq 1$ almost surely, the following inclusions take place for $n > (2/\pi_0)^p$:

$$\begin{aligned} A_n(\pi_0/(2K^{1/p})) &\supset \left\{ \pi_0 \tau^{1/2+1/p} - \tau^{-1/2+1/p} \geq \pi_0/(2K^{1/p})(Kn)^{1/p} \right\} \cap \{\tau \leq n\delta\} \\ &\supset \left\{ \tau^{1/2+1/p} \geq \frac{1 + \pi_0 n^{1/p}/2}{\pi_0} \right\} \cap \{\tau \leq n\delta\} \\ &\supset \left\{ \tau^{1/2+1/p} \geq n^{1/p} \right\} \cap \{\tau \leq n\delta\} \\ &= \left\{ n^{2/(p+2)} \leq \tau \leq n\delta \right\}. \end{aligned}$$

Consequently, for j large enough,

$$\mu(A_{[u_j^{(p+2)/2}]}(\pi_0/(2K^{1/p}))) \geq \mu \left\{ \left[u_j^{(p+2)/2} \right]^{2/(p+2)} \leq \tau \leq \left[u_j^{(p+2)/2} \right] \delta \right\} \quad (2.59)$$

Since τ take only integer values among u_i 's and $\left[u_j^{(p+2)/2} \right] \delta < u_{j+1}$ (by (2.49)), we obtain in view of (2.58), that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mu \left\{ \frac{1}{(nK)^{1/p}} \max_{\substack{0 \leq i < j \leq nK \\ j-i \leq n\delta}} \frac{|S_j - S_i|}{(j-i)^{1/2-1/p}} \geq \frac{\pi_0}{2K^{1/p}} \right\} &\geq \\ &\geq \limsup_{j \rightarrow \infty} 1 - (1 - \mu\{\tau = u_j\})^{[u_j^{(p+2)/2}]} \\ &= 1 - \liminf_{j \rightarrow \infty} \left(1 - cju_j^{-1-p/2} \right)^{[u_j^{(p+2)/2}]}. \end{aligned} \quad (2.60)$$

Noticing that for a fixed J ,

$$\liminf_{j \rightarrow \infty} \left(1 - cju_j^{-1-p/2} \right)^{[u_j^{(p+2)/2}]} \leq \limsup_{j \rightarrow \infty} \left(1 - cJu_j^{-1-p/2} \right)^{[u_j^{(p+2)/2}]} = e^{-cJ}, \quad (2.61)$$

we deduce that the last term of (2.60) is equal to 1, which finishes the proof of Theorem 1.5.

REFERENCES

- [Bil68] P. Billingsley, *Convergence of probability measures*, John Wiley & Sons Inc., New York, 1968. MR 0233396 (38 #1718) 6
- [Dur09] Olivier Durieu, *Independence of four projective criteria for the weak invariance principle*, ALEA Lat. Am. J. Probab. Math. Stat. **5** (2009), 21–26. MR 2475604 (2010c:60109) 2
- [DV08] Olivier Durieu and Dalibor Volný, *Comparison between criteria leading to the weak invariance principle*, Ann. Inst. Henri Poincaré Probab. Stat. **44** (2008), no. 2, 324–340. MR 2446326 (2010a:60075) 2

- [Gir14] Davide Giraud, *Holderian weak invariance principle for stationary mixing sequences*, 2014, <https://hal.archives-ouvertes.fr/hal-01075583>. 2, 3
- [Gir15] ———, *Holderian weak invariance principle under a Hannan type condition*, 2015. 2, 4, 6
- [KV07] Jana Klicnarová and Dalibor Volný, *An invariance principle for non-adapted processes*, C. R. Math. Acad. Sci. Paris **345** (2007), no. 5, 283–287. MR 2353682 (2008i:60057) 3, 10
- [MSR12] Jurgita Markevičiūtė, Charles Suquet, and Alfredas Račkauskas, *Functional central limit theorems for sums of nearly nonstationary processes*, Lith. Math. J. **52** (2012), no. 3, 282–296. MR 3020943 3
- [MTK08] Michael C. Mackey and Marta Tyran-Kamińska, *Central limit theorems for non-invertible measure preserving maps*, Colloq. Math. **110** (2008), no. 1, 167–191. MR 2353903 (2009j:60074) 2
- [MW00] Michael Maxwell and Michael Woodroffe, *Central limit theorems for additive functionals of Markov chains*, Ann. Probab. **28** (2000), no. 2, 713–724. MR 1782272 (2001g:60164) 1
- [PU05] Magda Peligrad and Sergey Utev, *A new maximal inequality and invariance principle for stationary sequences*, Ann. Probab. **33** (2005), no. 2, 798–815. MR 2123210 (2005m:60047) 1, 3, 6, 10
- [PUW07] Magda Peligrad, Sergey Utev, and Wei Biao Wu, *A maximal L_p -inequality for stationary sequences and its applications*, Proc. Amer. Math. Soc. **135** (2007), no. 2, 541–550 (electronic). MR 2255301 (2007m:60047) 2, 3, 4, 11, 12
- [RS03] Alfredas Račkauskas and Charles Suquet, *Necessary and sufficient condition for the Lamperti invariance principle*, Teor. Īmovīr. Mat. Stat. (2003), no. 68, 115–124. MR 2000642 (2004g:60050) 2
- [Vol06] Dalibor Volný, *Martingale approximation of non adapted stochastic processes with nonlinear growth of variance*, Dependence in probability and statistics, Lecture Notes in Statist., vol. 187, Springer, New York, 2006, pp. 141–156. MR 2283254 (2008b:60070) 1, 10
- [Vol07] ———, *A nonadapted version of the invariance principle of Peligrad and Utev*, C. R. Math. Acad. Sci. Paris **345** (2007), no. 3, 167–169. MR 2344817 (2008k:60078) 1, 10, 11
- [Vol10] ———, *Martingale approximation and optimality of some conditions for the central limit theorem*, J. Theoret. Probab. **23** (2010), no. 3, 888–903. MR 2679961 (2011k:60122) 1

NORMANDIE UNIVERSITÉ, UNIVERSITÉ DE ROUEN, LABORATOIRE DE MATHÉMATIQUES RAPHAËL SALEM, CNRS, UMR 6085, AVENUE DE L'UNIVERSITÉ, BP 12, 76801 SAINT-ETIENNE DU ROUVRAY CEDEX, FRANCE.
E-mail address: davide.giraud01@univ-rouen.fr