Range and critical generations of a random walk on Galton-Watson trees

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In this paper we consider a random walk in random environment on a tree and focus on the frontier case for the underlying branching potential. We study the range R_n of this walk up to time n and obtain its correct asymptotic in probability which is of the order of $n/\log n$. This result is a consequence of the asymptotical behavior of the number of visited sites at generations of order $(\log n)^2$, which turn out to be the most visited generations. Our proof which involves a quenched analysis gives a description of the typical environments responsible for the behavior of R_n.

1 Introduction

Let us consider a random walk with a random environment given by a branching random walk. This branching random walk is governed by a point process $\mathcal{L} := \{A_1, A_2, \cdots, A_N\}$ on the real line, where N is also random in $\mathbb{N} \cup \{\infty\}$. The initial ancestor (i.e. the root), denoted by ϕ, gives birth to N children with displacements $A_1, A_2, \cdots, A_N$ they form the first generation. Then, for any integer $n \geq 1$, each individual in the n-th generation gives birth independently of all others to its own children in the $(n+1)$-th generation. Their displacements are given by independent copies of $\mathcal{L}$.

We thus obtain a genealogical tree, denoted by $\mathbb{T}$, which is a Galton-Watson tree with offspring N. For each vertex (individual or site) $z \in \mathbb{T}$, $A(z)$ denotes its displacement and $V(z)$ its position. If y is the parent of z, write $\overleftarrow{z} = y$, also if y is an ancestor of z, write $y < z$. V can then be written as

$$V(z) = \sum_{\phi < y \leq z} A(y),$$

with $V(\phi) = 0$. In particular $\mathcal{L} = \{V(z), |z| = 1\}$, with $|z|$ the generation of z.

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The branching random walk $(V(z), z \in \mathbb{T})$, serves as a random environment $\mathcal{E}$ (also called random potential). Conditionally on the environment $\mathcal{E} = (V(z), z \in \mathbb{T})$, a random walk $(X_n, n \in \mathbb{N}^*, X_0 = \phi)$ starting from the root and taking values on the vertices of $\mathbb{T}$ can be defined, with probabilities of transition:

$$p^{\mathcal{E}}(z,u) = \begin{cases} \frac{e^{-V(u)}}{e^{-V(z)} + \sum_{v:\overleftarrow{v}=z} e^{-V(v)}}, & \text{if } u \text{ is a child of } z, \\ \frac{e^{-V(z)}}{e^{-V(z)} + \sum_{v:\overleftarrow{v}=z} e^{-V(v)}}, & \text{if } u \text{ is parent of } z. \end{cases} \tag{1.1}$$

For convenience, we add a parent $\overleftarrow{\phi}$ to the root and assume that (1.1) holds also for $z = \phi$ with $p^{\mathcal{E}}(\overleftarrow{\phi}, \phi) = 1$.

Let $\mathbf{P}$ be the probability measure of the environment and $\mathbf{P}^*$, the probability conditioned on the survival set of the tree $\mathbb{T}$ (which is assumed to be supercritical, see (1.2) below). Let $\mathbb{P}^{\mathcal{E}}$, the quenched probability measure of this random walk and $\mathbb{P}(\,) := \int \mathbb{P}^{\mathcal{E}(w)}(\,)\mathbf{P}(dw)$ the annealed probability measure. Similarly we also define $\mathbb{P}^*$ with respect to $\mathbf{P}^*$. In this paper we focus on the boundary case of the environment (in the sense of Biggins-Kyprianou [8]):

$$\mathbf{E}[N] > 1, \quad \mathbf{E}\left[\sum_{|z|=1} e^{-V(z)}\right] = 1, \quad \mathbf{E}\left[\sum_{|z|=1} V(z)e^{-V(z)}\right] = 0. \tag{1.2}$$

Also we need additional hypothesis given below : there exists $\theta > 0$ such that

$$\mathbf{E}\left[\left(\sum_{|z|=1}(1+|V(u)|)e^{-V(u)}\right)^2\right] + \mathbf{E}\left[\sum_{|z|=1} e^{-(1+\theta)V(z)}\right] + \mathbf{E}\left[\sum_{|z|=1} e^{\theta V(z)}\right] < \infty. \tag{1.3}$$

It is proved in [12] that the random walk X is null recurrent under (1.2). Moreover in this case X is very slow, indeed Y. Hu and Z. Shi [14] (see also [13] with G. Faraud) proved that the largest generation visited up to time n, $X_n^* := \max_{k\le n} |X_k|$ behaves in $(\log n)^3$. One of the questions raised by the authors at this time was : is $(\log n)^3$ the typical fluctuation of this walk, that is of $|X_n|$ for example ? If we now look at the largest generation entirely visited $M_n := \max\{k \ge 1 : \{|z| = k\} \subset \{X_i; 0 \le i \le n\}\}$, then it is of order $\log n$ as shown in P. Andreoletti and P. Debs [4], and we could also ask here the same question. It turns out that neither of the two is the good answer. A first result in that direction is obtained in the work of [5]. For any $z \in \mathbb{T}$, define

$$T_z = T_z^1 := \inf\{m \ge 1 : X_m = z\} \text{ and } T_z^k := \inf\{m \ge T_z^{k-1} : X_m = z\}, \forall k \ge 2. \tag{1.4}$$

Then for any generation $\ell \ge 1$, the number of sites visited at this generation up to time n is given by

$$N_n(\ell) := \sum_{|z|=\ell} \mathbb{1}_{T_z<n}.$$

We also introduce the same variable stopped at the n-th return to the root:

$$K_n(\ell) := N_{T_\phi^n}(\ell).$$

It is proved in [5] that the typical generations which maximise the number of distinct visited sites are of the order $(\log n)^2$:

$$\lim_{n\to+\infty} \frac{\mathbb{E}\left(K_n((\log n)^2)\right)}{\mathbb{E}\left(K_n((\log n)^{1+\zeta})\right)} = \infty, \ \forall \zeta \neq 1 \text{ and } \mathbb{E}\left(K_n((\log n)^2)\right) \asymp n/\log n \ [1]. \tag{1.5}$$

They also notice that only the sites such that, along their shortest paths from the root, the branching potentiel $V(\cdot)$ is high enough (typically larger than $\log n$) are of importance. That is to say produce the main contribution for $\mathbb{E}\left(K_n((\log n)^2)\right)$, conversely the sites with low potential are mostly visited but there are very few of them (typically of order $n/(\log n)^2$ compared to $n/(\log n)$). More recently, in [16], it is proved that $(\log n)^2$ is actually the right normalisation for the generation of X at the instant n, and in [15] that the walk can in fact reach height of potential of order $(\log n)^2$.

We now return to our main purpose, the number of distinct visited sites. The main lack in the paper [5] is first that nothing precise is said on the behavior in probability of N_n (neither for K_n), and that their annealed results say nothing on the typical behavior of the potentials leading to this critical $(\log n)^2$-th generation. Our results here bring answers to these points.

We have split our results into three parts, the first subsection below deals with the normalization for the number of distinct visited sites per critical generation as well as for the total number of distinct visited sites up to time n. The second subsection is devoted to a quenched results making a link between the range of X and the behavior of the environment. Finally the third subsection is about the convergence of key random variables, depending only on the random environment $\mathcal{E}$.

1.1 Annealed results

Our first theorem shows that the behavior in probability of the number of distinct sites visited at critical generations is of order $n/(\log n)^3$.

Theorem 1.1. *For any integers $\ell = \ell(n)$ such that $\lim_{n\to+\infty} \frac{\ell}{(\log n)^2} = \gamma > 0$, there exists a positive constant $\lambda(\gamma) > 0$ such that as $n \to \infty$,*

$$\frac{(\log n)^3}{n} N_n(\ell) \xrightarrow{in\ \mathbb{P}^*} \frac{\lambda(\gamma)\sigma^2}{4}, \tag{1.6}$$

where $\sigma^2 := \mathbf{E}\left[\sum_{|x|=1} V^2(x)e^{-V(x)}\right] \in (0,\infty)$ by (1.3).

The function $\lambda(\gamma)$ can be written explicitly but we postpone its definition to Section 1.3 as it is related to the convergence of variables depending only on the environment. This theorem is the consequence of the behavior of K_n and of the local time at the root. To be more precise, let us introduce the derivative martingale (D_m, m) given by

$$D_m := \sum_{|z|=m} V(z)e^{-V(z)}, \tag{1.7}$$

[1] In [5] the lower bound obtained is actually a little smaller than $n/\log n$.

and denote its almost sure limit by D_∞ (see [8] for its existence and [11] for its positivity under $\mathbf{P}^*$). The behavior in probability of K_n is given by

Theorem 1.2. *For any $\ell = \ell(n)$ such that $\lim_{n\to+\infty} \frac{\ell}{(\log n)^2} = \gamma > 0$,*

$$\frac{(\log n)^2}{n} K_n(\ell) \xrightarrow{in\ \mathbb{P}^*} \lambda(\gamma) p^{\mathcal{E}}(\phi, \overleftarrow{\phi}) D_\infty. \tag{1.8}$$

If we compare this results with the behavior in mean (see (1.5) and also the more precise Lemma 4.1), a multiplicative $(\log n)$ appears. It comes from the behavior of the branching potential which typically remain positive in probability (see Section 4).
Also the main difference between $N_n(\ell)$ and $K_n(\ell)$ comes from the normalisation. The additional $\log n$ which appears above for $K_n(\ell)$ comes from the local time of X at the root of the tree studied in [16]:

Proposition 1.3 ([16])**.**

$$\frac{T^n_\phi}{n \log n} \xrightarrow{in\ \mathbb{P}^*} 4 D_\infty p^{\mathcal{E}}(\phi, \overleftarrow{\phi})/\sigma^2. \tag{1.9}$$

Instead of one critical generation, we now turn to consider the total number of visited sites, in other words, the range of the random walk:

$$R_n := \sum_{z\in\mathbb{T}} \mathbb{1}_{T_z \le n}.$$

Following (1.5) and Theorem 1.1 we can ask wether or not critical generations contribute mainly to R_n ? The answer is yes : Proposition 1.4 below states that for non-critical generations, the total number of visited sites contributes to something negligible compared to $n/\log n$, while the range R_n is of order $n/\log n$ in probability, as stated in Theorem 1.5.

Proposition 1.4. *For any $\delta > 0$,*

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \mathbb{P}\left[\left(\sum_{m\le \varepsilon(\log n)^2} \mathbb{1}_{T_z\le n} + \sum_{m\ge (\log n)^2/\varepsilon} \mathbb{1}_{T_z\le n} \right) \ge \delta n/\log n \right] = 0.$$

So as the main contribution comes from generations of order $(\log n)^2$, we have that with high probability, $R_n \approx \sum_{\varepsilon(\log n)^2 \le \ell \le (\log n)^2/\varepsilon} N_n(\ell)$ with $\varepsilon \downarrow 0$. As a consequence we obtain the following result for the range of X :

Theorem 1.5. *We have*

$$\frac{\log n}{n} R_n \xrightarrow{in\ \mathbb{P}^*} \frac{\sigma^2}{4}\Lambda, \tag{1.10}$$

where $\Lambda := \int_0^\infty \lambda(\gamma) d\gamma \in (0,\infty)$.

These first results give a quantitative description of the number of visited sites and of the generations involved, but no description of the underlying environment is given. In the following section we discuss what we have learnt about the typical behavior of the potential that leads to the above behavior of R_n.

1.2 A quenched point of view

Like we said in the first part of the introduction, Andreoletti-Debs [5] observe that the sites where the potential remains small (always lower than $\log n$) have a negligible contribution for the number of visited sites. One of the reasons for this is the fact that the number of such sites is actually negligible on the tree (see their Proposition 1.3). Intuitively these sites are easily accessible as the potential remains low, but the set of these sites still has a low conductance.

Here we give some more details of the sites that the random walk is inclined to visit, i.e. the sites that contribute importantly to the range.

For sites $y,\ z \in \mathbb{T}$, recall that $y \leq z$ means that y belongs to the shortest path from the root ϕ to z. Let $\overline{V}(z) := \max_{\phi<y\leq z} V(y)$. Define for any $a_0 > 1$,

$$A_1 := \left\{ z \in \mathbb{T} :\ \frac{\log n}{a_0} \leq \max_{\phi<y\leq z} \left(\overline{V}(y) - V(y)\right) \leq \log n + g(n) \right\},$$

where $\{g(n), n\}$ is a positive increasing function such that $\lim_{n\to+\infty}(g(n) - \log\log n) = +\infty$. Moreover, for any $a_1 > 0$, let

$$A_2 := \left\{ z \in \mathbb{T} :\ \log n + \log\log n \leq \overline{V}(z) \leq a_1 \log n \sqrt{\log\log n} \right\},$$

and

$$A_3 := \left\{ z \in \mathbb{T} :\ \overline{V}(z) > \max_{\substack{y\leq z;\\ |y|\leq|z|-|z|^{1/3}}} V(y) \right\}.$$

Let us introduce a notation for truncated versions of K_n, R_n and their quenched mean : if A is an event depending only on the environment $\mathcal{E}$, then for any $\ell \geq 1$,

$$K_n^A(\ell) := \sum_{|z|=\ell} \mathbb{1}_{T_z<T_\phi^n} \mathbb{1}_{z\in A},\ R_{T_\phi^n}^A := \sum_{m\geq 0} K_n^A(m), \tag{1.11}$$

$$\mathcal{K}_n^A(\ell) := \mathbb{E}^{\mathcal{E}}\left(K_n^A(\ell)\right),\ \mathscr{R}_{T_\phi^n}^A := \mathbb{E}^{\mathcal{E}}\left(R_{T_\phi^n}^A\right). \tag{1.12}$$

Notice that the above means are easily computable (see section 2), but we are not interested in their expressions for now. The following result proves tightness of the range up to T_ϕ^n minus the truncated quenched mean of $R_{T_\phi^n}$: $\mathscr{R}_n^{A_1\cap A_2\cap A_3}$, this makes appear favorite environments described by potential V.

Proposition 1.6. *For any $\eta > 0$, there exists $a_1 > 0$ such that*

$$\lim_{a_0\to+\infty} \limsup_{n\to+\infty} \mathbb{P}^*\left(\frac{1}{n}\left|R_{T_\phi^n} - \mathscr{R}_{T_\phi^n}^{A_1\cap A_2\cap A_3}\right| \geq \eta\right) = 0.$$

From this result together with the well known fact about the potential : $\mathbf{P}(\inf_{z\in\mathbb{T}} V(z) \geq -\alpha) \geq 1 - e^{-\alpha}$, we are able to draw a typical trajectory of potential that maximises the number of visited

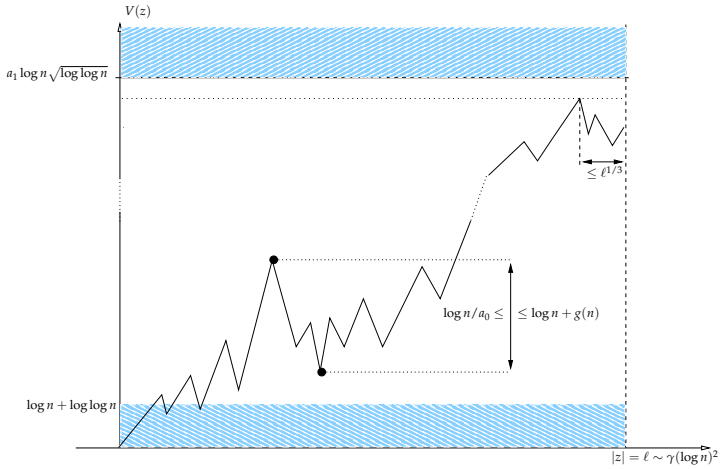

Figure 1: Typical accessible environments within time n

sites (see Figure 1). We finish this subsection by giving the expression of a truncated version of the quenched mean of K_n. For any event $B \subseteq \{z \in \mathbb{T} \ : \ \overline{V}(z) \geq \log n + \log\log n\}$ depending only on the environment (see Section 4)

$$\mathcal{K}_n^B(\ell) = \mathbb{E}^{\mathcal{E}}\left[\sum_{|z|=\ell} \mathbb{1}_{T_z < T_\phi^n} \mathbb{1}_{z\in B}\right] \sim n \sum_{|z|=\ell} a_z \mathbb{1}_{z\in B} =: \tilde{\mathcal{K}}_n^B(\ell), \tag{1.13}$$

where

$$a_z := \mathbb{P}^{\mathcal{E}}(T_z < T_\phi) = \frac{p^{\mathcal{E}}(\phi, \overleftarrow{\phi})}{\sum_{\phi < x \leq z} e^{V(x)}}. \tag{1.14}$$

Notice in particular that $\mathcal{K}_n^{A_1\cap A_2\cap A_3}(\ell) \sim \tilde{\mathcal{K}}_n^{A_1\cap A_2\cap A_3}(\ell)$. The following subsection concerns convergence of $\ell\tilde{\mathcal{K}}_n^B/n$ for well chosen B. This leads to the expressions of constants λ and Λ which appear in the theorems.

1.3 Definitions of λ, Λ and results for the environment

The constants $\lambda(.)$ are obtained from the convergence of functionals of one-dimensional random walk, which is related to the branching potential by the following Biggins-Kyprianou [9] identity usually called many-to-one Lemma :

Lemma 1.7. *In the boundary case* (1.2)*, there exists a sequence of i.i.d. real-valued random variables* $(S_i - S_{i-1}, i \geq 0)$ *with* $S_0 = 0$ *such that for any* $m \geq 1$ *and any Borel function* $g : \mathbb{R}^m \to \mathbb{R}_+$*,*

$$\mathbf{E}\left[\sum_{|x|=m} g\Big(V(x_i), 1 \leq i \leq m\Big)\right] = \mathbf{E}\Big[e^{S_m} g(S_i; 1 \leq i \leq m)\Big]. \tag{1.15}$$

It immediately follows from (1.2) and (1.3) that the sequence $(S_n, n \geq 0)$ is a centered random walk of finite variance $\sigma^2 := \mathbf{E}[\sum_{|z|=1} V(z)^2 e^{-V(z)}]$. For notational simplicity, let

$$\underline{S}_m := \min_{1 \leq i \leq m} S_i, \ \overline{S}_m := \max_{1 \leq i \leq m} S_i.$$

Let us also denote by $\mathcal{R}(\cdot)$ the renewal function associated with the strict descending ladder heights of (S_i, i), it can be expressed as

$$\mathcal{R}(u) = \sum_{k=0}^{\infty} \mathbf{P}(S_k < \underline{S}_{k-1}, S_k \geq -u), \ \forall u \geq 0. \tag{1.16}$$

It is known (see [1]) that the following joint convergence in law holds

$$\left\{ \left(\frac{S_{\lfloor nt \rfloor}}{\sqrt{\sigma^2 n}}, t \in [0,1] \right) ; \sum_{i=0}^{n} e^{-S_i} | \underline{S}_n > 0 \right\} \Longrightarrow \{(\mathfrak{m}_t, t \in [0,1]), \mathcal{H}_\infty\}, \tag{1.17}$$

where $(\mathfrak{m}_t, t \in [0,1])$ is a Brownian meander independent of $\mathcal{H}_\infty \in [1, \infty)$. In fact, in the sense of [6], the associated random walk conditioned to stay positive, denoted $(\zeta_n, n \geq 0)$, is a Markov chain with probabilities of transition $p(x, dy) := \frac{\mathcal{R}(y)}{\mathcal{R}(x)} \mathbb{1}_{y>0} \mathbf{P}_x(S_1 \in dy)$, with $\mathbf{P}(\zeta_0 = 0) = 1$. Consequently $\mathcal{H}_\infty$ can be defined as

$$\mathcal{H}_\infty := \sum_{j=0}^{\infty} e^{-\zeta_j}.$$

Also we denote

$$\mathbf{c}_1^+ := \lim_{n \to \infty} \sqrt{n} \mathbf{P}(\underline{S}_n \geq 0), \ \mathbf{c}_2^+ := \lim_{n \to \infty} \sqrt{n} \mathbf{P}(\underline{S}_n > 0), \tag{1.18}$$

and the renewal theorem implies the existence of $\mathbf{c_0} \in (\mathbf{0}, +\infty)$ such that

$$\mathbf{c}_0 := \lim_{u \to +\infty} \frac{\mathcal{R}(u)}{u}. \tag{1.19}$$

We also introduce two functions which appears in the definition of $\lambda(\cdot)$. The first one involves the discrete random walk (S_j, j). For any $j \geq 1$ and $x \geq 1$, define

$$\mathcal{G}_j(x) := \mathbf{E}\left[\frac{e^{S_j}}{x + \sum_{1 \leq i \leq j} e^{S_i}} ; \overline{S}_j \leq 0 \right], \text{ with } \mathcal{G}_0(x) := \frac{1}{x}. \tag{1.20}$$

The second function depends on Brownian meander $(\mathfrak{m}_s, 0 \leq s \leq 1)$. Let $\overline{\mathfrak{m}}_s := \sup_{0 \leq t \leq s} \mathfrak{m}_t$ and $\underline{\mathfrak{m}}_{[s,1]} := \inf_{0 \leq t \leq s} \mathfrak{m}_t$ for any $s \in [0,1]$. Take $a \geq 0$ and $b \geq 0$, for any $(x, h) \in \mathbb{R}_+^2$, let

$$\Psi^{a,b}(x,h) := \mathbf{c}_2^+ \mathbf{P}\left(\sigma \mathfrak{m}_1 > (\sqrt{2}b - x) \vee h, \sigma(\overline{\mathfrak{m}}_1 - \mathfrak{m}_1) \leq (\sqrt{2}a - h)_+ \wedge x, \max_{0 \leq s \leq 1} \sigma(\mathfrak{m}_s - \underline{\mathfrak{m}}_{[s,1]}) \leq \sqrt{2}a \right).$$

Finally, let

$$\mathcal{C}_{a,b} := 2\mathbf{c}_1^+\mathbf{c}_2^+\mathbf{E}\left(\Psi^{a,b}(\sigma\mathfrak{m}_1, \sigma(\overline{\mathfrak{m}}_1 - \mathfrak{m}_1)); \max_{0\le s\le 1}\sigma(\overline{\mathfrak{m}}_s - \mathfrak{m}_s) \le \sqrt{2}a\right), \text{ and} \tag{1.21}$$

$$\mathscr{C}_{a,b} := \mathcal{C}_{a,b}\sum_{j=0}^{+\infty}\mathbf{E}\left[\mathcal{G}_j(\mathcal{H}_\infty)\right]. \tag{1.22}$$

$\mathcal{C}_{a,b}$ is well defined positive and finite. Note also that $\mathcal{G}_j(x) \le \mathcal{G}_j := \mathbf{E}[e^{S_j}\mathbb{1}_{\overline{S}_j\le 0}]$ for any $j \ge 1$ and $x \ge 1$ so $\mathscr{C}_{a,b}$ is finite [see (A.14)]. We are now ready to define λ, for any $\gamma > 0$,

$$\lambda(\gamma) := \mathbf{c}_0\frac{\mathscr{C}_{\gamma^{-1/2},\gamma^{-1/2}}}{\gamma}.$$

The integrability of λ is stated in Lemma A.1 of the Appendix, so Λ in Theorem (1.5) is well defined, i.e.

$$\Lambda = \int_0^{+\infty}\lambda(x)dx \in (0,\infty). \tag{1.23}$$

In what follows we state two results which involve only the environment, and which are used later in the proofs of the theorems. Let us define the following variable, for any $a, b \ge 0$, define for any $m \ge 0$

$$W_m(F_{a\sqrt{m}+a_m,b\sqrt{m}+b_m}) := \sum_{|z|=m} e^{-V(z)}F_{a\sqrt{m}+a_m,b\sqrt{m}+b_m}(z)$$

where

$$F_{a\sqrt{m}+a_m,b\sqrt{m}+b_m}(z) := \sqrt{m}\frac{e^{V(z)}}{\sum_{\phi<y\le z}e^{V(y)}}\mathbb{1}_{\overline{V}(z)\ge b\sqrt{m}+b_m}\mathbb{1}_{\max_{\phi<y\le z}(\overline{V}(y)-V(y))\le a\sqrt{m}+a_m}, \tag{1.24}$$

with $a_m = o(\sqrt{m})$ and $b_m = o(\sqrt{m})$.
$W_m(F_{a\sqrt{m}+a_m,b\sqrt{m}+b_m})$ is related to our problem in the following way. Recalling the truncated variable defined in (1.13), one sees that for $\ell \sim \gamma(\log n)^2$,

$$\frac{\ell}{n}\tilde{\mathcal{K}}_n^{\left\{z\in\mathbb{T}:\overline{V}(z)\ge\log n+\log\log n,\ \max_{y\le z}(\overline{V}(y)-V(y))\le\log n+g(n)\right\}}(\ell) = p^{\mathcal{E}}(\phi, \overleftarrow{\phi})\sqrt{\ell}W_\ell(F_{a\sqrt{\ell}+a_\ell,b\sqrt{\ell}+b_\ell}),$$

with $a = b = \gamma^{-1/2}$. This is why the understanding of the asymptotical behavior of $\sqrt{m}W_m(F_{a\sqrt{m}+a_m,b\sqrt{m}+b_m})$ is an important point in the behavior of the range. We now state a result about W

Proposition 1.8.

$$\sqrt{m}W_m(F_{a\sqrt{m}+a_m,b\sqrt{m}+b_m}) \xrightarrow{in\ \mathbf{P}^*} \mathbf{c}_0\mathscr{C}_{a,b}D_\infty. \tag{1.25}$$

Note that the variable $W_m(F_{a\sqrt{m}+a_m,b\sqrt{m}+b_m})$ looks like, in some ways, to the additive martingale $W_m := \sum_{|z|=m}e^{-V(z)}$ (so we call it martingale-like variable). However, as $F_{a\sqrt{m}+a_m,b\sqrt{m}+b_m}$ is not simply a bounded functional of the rescaled trajectory $(\frac{V(y)}{\sqrt{n}}, \phi < y \le z)$, we can not use the result of Madaule

[17] in order to get the correct asymptotic of this martingale-like variable. Nevertheless, we can still borrow some of the ideas of [3] to obtain the convergence.

In addition, note that $\mathscr{C}_{a,b}$ is continuous and monotone on $(a,b) \in \mathbb{R}_+^2$ and that $W_m(F_{a\sqrt{m},b\sqrt{m}})$ is also monotone on $(a,b) \in \mathbb{R}_+^2$. It follows that (1.25) holds uniformly for $W_m(F_{a\sqrt{m},b\sqrt{m}})$ in $(a,b) \in \mathbb{R}_+^2$ in the following sense: for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbf{P}^* \left(\sup_{a\geq 0, b\geq 0} \left| \sqrt{m} W_m(F_{a\sqrt{m},b\sqrt{m}}) - \mathbf{c}_0 \mathscr{C}_{a,b} D_\infty \right| \geq \varepsilon \right) = 0. \tag{1.26}$$

This induces the following corollary.

Corollary 1.9. *For any $\beta \geq 0$, define*

$$W_m^*(\beta) := \sum_{|z|=m} \frac{1}{\sum_{\phi<y\leq z} e^{V(y)}} \mathbb{1}_{\max_{\phi<y\leq z}(\overline{V}(y)-V(y))\leq\beta} \mathbb{1}_{\overline{V}(z)\geq\beta},$$

then

$$\lim_{\beta\to\infty} \sum_{m=1}^{\infty} W_m^*(\beta) = \Lambda D_\infty, \text{ in } \mathbf{P}^*\text{-probability}. \tag{1.27}$$

Notice that $W_m^*(\beta) = W_m(F_{\beta,\beta})/\sqrt{m}$. This corollary still holds if we replace $\mathbb{1}_{\overline{V}(z)\geq\beta}$ by $\mathbb{1}_{\overline{V}(z)\geq\beta\pm O(\log\beta)}$ in the definition of $W_m^*(\beta)$, it is used in the proof of of Theorem 1.5.

1.4 Final remarks

(1.26) suggests that uniformity may also occur in probability for $K_n(\ell)$, meaning that the "for any ℓ" in Theorem 1.1 could actually be placed inside the probability. Unfortunately, this uniformity can not be obtained by the way of our proofs and we believe in fact that this is not true and that the right normalisation for $\max_\ell N_n(\ell)$ could be different from $n/(\log n)^3$.

The rest of the paper is organized as follows :

In Section 2 we use results of Sections 3 and 4 to give the main steps of the proofs of theorems and propositions stated in Section 1.1.

In Section 3 we focus on the environment and show Proposition 1.8 and Corollary 1.9. This section is independent of the other sections using only the Appendix.

In Section 4 we compute the annealed mean of K_n and give an upper bound for the mean of the quenched variance. Also we prove lemmata used in section 2 and finish with the proof of Proposition 1.6.

In the *Appendix* we collect and prove many estimations for centered random walk $(S_n, n \geq 0)$.

In this paper, we use c or c' for constants which may change from line to line. We write $c(x)$ when that constant depends on some parameter x.

2 Proof of the theorems

This section is devoted to proving Theorems 1.2 and 1.5, i.e. the convergence in probability of $K_n(\ell)$. Theorem 1.1 follows immediately from Theorems 1.2 and Proposition 1.3, so we feel free to omit its proof.

Our arguments are based on the study of truncated versions of K_n. This decomposition of K_n appears naturally when computing the mean of K_n as well as the mean of its quenched variance. We therefore start with this decomposition.

2.1 Quenched expectation and truncated versions of K_n

The quenched mean of K_n, i.e. its expectation given the environment $\mathcal{E}$, can be easily given by the following nice expression:

$$\mathcal{K}_n(\ell) := \mathbb{E}^{\mathcal{E}}\left[\sum_{|z|=\ell} \mathbb{1}_{T_z<T_\phi^n}\right] = \sum_{|z|=\ell} \mathbb{P}^{\mathcal{E}}(T_z < T_\phi^n) = \sum_{|z|=\ell} (1-(1-a_z)^n),$$

with, recall, $a_z = p^{\mathcal{E}}(\phi, \overleftarrow{\phi})(\sum_{\phi<y\leq z} e^{V(y)})^{-1}$.

To exclude the sites that make few contribution to K_n, we add restrictions for the potentials on the above sum. First it is known (see [2]) that $\mathbf{P}(\inf_{u\in T} V(u) < -\alpha) \leq e^{-\alpha}$. So for any $\varepsilon > 0$, we can choose $\alpha > 0$ such that

$$\mathbf{P}\left(\inf_{u\in T} V(u) < -\alpha\right) \leq e^{-\alpha} \leq \varepsilon. \tag{2.1}$$

Let $\underline{V}(z) := \min_{\phi<y\leq z} V(y)$, it is then natural to consider the set

$$B_1 := \{z \in \mathbb{T} : \underline{V}(z) \geq -\alpha\}.$$

Secondly, in [16], a reflecting barrier is introduced, for any $r > 0$, let

$$\mathcal{L}_r := \left\{z \in \mathbb{T} : \sum_{\phi<u\leq z} e^{V(u)-V(z)} > r, \max_{\phi<y<z} \sum_{\phi<u\leq y} e^{V(u)-V(y)} \leq r\right\}.$$

This reflecting barrier allows to reduce the number of interesting sites for the walk in the following sense : let f be a positive increasing function such that $\lim_{n\to+\infty} f(n) = +\infty$, then

$$\lim_{n\to+\infty} \mathbb{P}\left(\exists k \leq T_\phi^n, X_k \in \mathscr{L}_{\frac{nf(n)}{\log n}}\right) = 0. \tag{2.2}$$

The above result is a direct consequence of Theorem 2.8 (in [16]) together with Proposition 1.3. Following this idea, we introduce the following sets

$$B_2 := \left\{ z \in \mathbb{T} : \max_{\phi<y\le z} \sum_{\phi<u\le y} e^{V(u)-V(y)} \le n \right\} =: \{z \in \mathbb{T} : z < \mathcal{L}_n\},$$

then according to (2.2)

$$\lim_{n\to+\infty} \mathbf{P}\left(\forall k \le T_\phi^n, X_k \in B_2\right) = 1.$$

Also, for any $\delta > 0$ let $s_n := n(\log n)^{-1-\delta}$ and

$$B_2^\delta := \left\{ z \in \mathbb{T} : \max_{\phi<y\le z} \sum_{\phi<u\le y} e^{V(u)-V(y)} \le s_n \right\} = \{z \in \mathbb{T} : z < \mathcal{L}_{s_n}\}.$$

We will see that for our specific problem, we can restrict the set B_2 to B_2^δ for chosen δ. We also denote $B := B_1 \cap B_2$ and $B^\delta := B_1 \cap B_2^\delta$, and finally introduce the last restriction over the values of $\overline{V}$, let

$$B_3 := \{z \in \mathbb{T} : \ \overline{V}(z) \ge \log n + \log\log n\}.$$

Recall the definitions given in (1.11), because of (2.1) and (2.2), one sees that with high probability, $K_n(\ell) \sim K_n^B(\ell)$. The following lemma shows that we can also add B_3 to the restrictions,

Lemma 2.1. *For $\ell = \ell(n)$ such that $\lim_{n\to+\infty} \frac{\ell}{(\log n)^2} = \gamma > 0$, we have*

$$\mathbb{E}\left[K_n^{B\setminus B_3}(\ell)\right] = o\left(\frac{n}{(\log n)^2}\right). \tag{2.3}$$

Also, arguments presented later show that $\mathbb{E}[K_n^{B\cap B_3}(\ell)] = \Theta(\frac{n}{(\log n)^2})$, which implies that mainly sites in $B \cap B_3$ count. We postpone the proof of this lemma to Section 4. Now we only need to consider $K_n^{B\cap B_3}(\ell)$. For convenience, let $C := B \cap B_3$ and $C^\delta := B^\delta \cap B_3$.

Here is our strategy to obtain Theorem 1.1. We first show that for suitable $\delta > 0$, with high probability, $K_n(\ell) \approx K_n^C(\ell) \approx K_n^{C^\delta}(\ell)$, while the last quantity can be approached by its quenched mean by bounding its quenched variance. We stress on the fact that replacement of C by C^δ helps to correctly bound the quenched variance, it appears that the price of this replacement is negligible (as shown in the Lemma 2.2). Then the study of the quenched variable $\mathcal{K}_n^{C^\delta}(\ell) = \mathbb{E}^{\mathcal{E}}\left[K_n^{C^\delta}(\ell)\right]$ can be found in Proposition 1.8.

Lemma 2.2. *For $\ell = \ell(n)$ such that $\lim_{n\to+\infty} \frac{\ell}{(\log n)^2} = \gamma > 0$ and for any $\delta > 0$, we have*

$$\mathbb{E}[K_n^C(\ell) - K_n^{C^\delta}(\ell)] = o\left(\frac{n}{(\log n)^2}\right). \tag{2.4}$$

The next step is to approach $K_n^{C^\delta}(\ell)$ by its quenched mean $\mathcal{K}_n^{C^\delta}(\ell)$. Observe that for any $z \in B_3$, $a_z \le e^{-\overline{V}(z)} \le \frac{1}{n\log n}$, so $1-(1-a_z)^n \sim na_z$. Similarly as (1.12), for any event A defined from the environment only, let

$$\widetilde{\mathcal{K}}_n^A(\ell) := \sum_{|z|=\ell} na_z \mathbb{1}_{z\in A}. \tag{2.5}$$

So in particular $\mathcal{K}_n^{C^\delta}(\ell)$ and $\widetilde{\mathcal{K}}_n^{C^\delta}(\ell)$ are close. Then $K_n^{C^\delta}$ can be approached by $\widetilde{\mathcal{K}}_n^{C^\delta}$ in the following way

Proposition 2.3. *For any $\eta > 0$ and $\delta > 3$,*

$$\lim_{n\to+\infty} \mathbb{P}\left(|K_n^{C^\delta}(\ell) - \widetilde{\mathcal{K}}_n^{C^\delta}(\ell)| \ge \eta \frac{n}{(\log n)^2} \right) = 0. \tag{2.6}$$

We prove this proposition in Section 4.

2.2 Convergence of $K_n(\ell)$ and R_n

We now turn to the proof of Theorems 1.2 and 1.5. Recall that $s_n = n(\log n)^{-1-\delta}$. Observe that on $B_1 = \{\inf_{u\in\mathbb{T}} V(u) \ge -\alpha\}$,

$$C^\delta \subset \{z\in\mathbb{T} : \max_{\phi<y\le z}(\overline{V}(y)-V(y)) \le \log s_n, \overline{V}(z) \ge \log n + \log\log n\}$$

and that

$$\{z\in\mathbb{T} : \max_{\phi<y\le z}(\overline{V}(y)-V(y)) \le \log s_n - \log|z|, \overline{V}(z) \ge \log n + \log\log n\} \subset C^\delta.$$

Recalling the definition of F in (1.24), one sees that on B_1,

$$\frac{n}{\sqrt{\ell}} W_\ell(F_{\log s_n - \log\ell, \log n+\log\log n}) \le \frac{\widetilde{\mathcal{K}}_n^{C^\delta}(\ell)}{p^{\mathcal{E}}(\phi, \overleftarrow{\phi})} \le \frac{n}{\sqrt{\ell}} W_\ell(F_{\log s_n, \log n+\log\log n}). \tag{2.7}$$

This implies that $\frac{\ell}{n}\widetilde{\mathcal{K}}_n^{C^\delta}(\ell) \sim p^{\mathcal{E}}(\phi,\overleftarrow{\phi})\sqrt{\ell} W_\ell(F_{\gamma^{-1/2}\sqrt{\ell}+a_\ell, \gamma^{-1/2}\sqrt{\ell}+b_\ell})$. So as a consequence of Proposition 1.8, for $\ell = \ell(n)$ such that $\lim_{n\to+\infty}\frac{\ell}{(\log n)^2} = \gamma > 0$ and for any $\eta, \alpha > 0$,

$$\limsup_{n\to\infty} \mathbf{P}\left(\left| \frac{(\log n)^2}{n} \widetilde{\mathcal{K}}_n^{C^\delta}(\ell) - p^{\mathcal{E}}(\phi,\overleftarrow{\phi})\lambda(\gamma) D_\infty \right| \ge \eta \right) \le \mathbf{P}\left(\inf_{u\in\mathbb{T}} V(u) < -\alpha \right) \tag{2.8}$$

with $\lambda(\gamma) = \mathbf{c}_0 \frac{\mathscr{C}_{\gamma^{-1/2},\gamma^{-1/2}}}{\gamma} > 0$.

We are now ready to prove Theorem 1.2: it suffices to show that for any $\eta > 0$,

$$\lim_{n\to+\infty} \mathbb{P}\Big(\Big| \frac{(\log n)^2}{n} K_n(\ell) - \lambda(\gamma) p^{\mathcal{E}}(\phi,\overleftarrow{\phi}) D_\infty \Big| \ge \eta \Big) = 0. \tag{2.9}$$

Proof of (2.9). Let $\mathbf{p}_n := \mathbb{P}\Big(\Big|\frac{(\log n)^2}{n}K_n(\ell) - \lambda(\gamma)p^{\mathcal{E}}(\phi, \overleftarrow{\phi})D_\infty\Big| \geq \eta\Big)$. Observe that

$$\mathbf{p}_n \leq \mathbf{P}\left(\inf_{u\in T} V(u) < -\alpha\right) + \mathbb{P}\left(\bigcup_{i=1}^{n}\{X_i \in \mathcal{L}_n\}\right) + \mathbb{P}\left(\Big|\frac{(\log n)^2}{n}K_n^B(\ell) - \lambda(\gamma)p^{\mathcal{E}}(\phi, \overleftarrow{\phi})D_\infty\Big| \geq \eta\right).$$

By (2.1) and (2.2), for any $\varepsilon > 0$, we can choose $\alpha > 0$ such that

$$\limsup_{n\to\infty} \mathbf{p}_n \leq \varepsilon + \limsup_n \mathbb{P}\Big(\Big|\frac{(\log n)^2}{n}K_n^B(\ell) - \lambda(\gamma)p^{\mathcal{E}}(\phi, \overleftarrow{\phi})D_\infty\Big| \geq \eta\Big). \tag{2.10}$$

It then follows from (2.3) and (2.4) that

$$\limsup_{n\to\infty} \mathbf{p}_n \leq \varepsilon + \limsup_n \mathbb{P}\Big(\Big|\frac{(\log n)^2}{n}K_n^{C^\delta}(\ell) - \lambda(\gamma)p^{\mathcal{E}}(\phi, \overleftarrow{\phi})D_\infty\Big| \geq \eta/2\Big),$$

so by Proposition 2.3,

$$\limsup_{n\to\infty} \mathbf{p}_n \leq \varepsilon + \limsup_n \mathbb{P}\Big(\Big|\frac{(\log n)^2}{n}\widetilde{\mathcal{K}}_n^{C^\delta}(\ell) - \lambda(\gamma)p^{\mathcal{E}}(\phi, \overleftarrow{\phi})D_\infty\Big| \geq \eta/4\Big).$$

Applying (2.8) with the same $\alpha > 0$ chosen in (2.10), one sees that

$$\limsup_{n\to\infty} \mathbf{p}_n \leq 2\varepsilon.$$

Letting $\varepsilon \downarrow 0$ concludes (2.9). □

It remains to show the convergence of the range R_n. Once again by Proposition 2.2, we only need to demonstrate that

$$\frac{R_{T_\phi^n}}{n} \xrightarrow{\text{in } \mathbb{P}^*} \Lambda p(\phi, \overleftarrow{\phi})D_\infty, \tag{2.11}$$

with $R_{T_\phi^n} = \sum_{m=0}^{\infty} K_n(m)$. First, we say that only the critical generations really count in this sum, and that the truncated versions of $(K_n(m), m)$ gives the main contribution.

Lemma 2.4. *We have*

$$\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty}\frac{1}{n}\left\{\mathbb{E}\left[\sum_{m\leq\varepsilon(\log n)^2} K_n^B(m)\right] + \mathbb{E}\left[\sum_{m\geq(\log n)^2/\varepsilon} K_n^B(m)\right]\right\} = 0, \tag{2.12}$$

And for any $\varepsilon > 0$,

$$\lim_{n\to\infty}\frac{1}{n}\left\{\mathbb{E}\left[\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} K_n^{B\setminus B_3}(m)\right] + \mathbb{E}\left[\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} K_n^{C\setminus C^\delta}(m)\right]\right\} = 0. \tag{2.13}$$

Notice here that Proposition 1.4 follows from (2.12) and Proposition 1.3. As non-critical generations are negligible, we can borrow the previous arguments for $K_n(\ell)$ to show the convergence for $R_{T_\phi^n}$.

Proof of Theorem 1.5 (i.e. (2.11)). For any $\eta > 0$, let us consider $\mathbb{P}\Big(|R_{T_\phi^n} - \Lambda p^{\mathcal{E}}(\phi, \overleftarrow{\phi})D_\infty n| \geq \eta n\Big)$. For any $\alpha > 0$,

$$\mathbb{P}\Big(|R_{T_\phi^n} - \Lambda p^{\mathcal{E}}(\phi, \overleftarrow{\phi})D_\infty n| \geq \eta n\Big) \leq \mathbf{P}\left(\inf_{u\in\mathbb{T}} V(u) < -\alpha\right) + \mathbb{P}\left(\bigcup_{k\leq n}\{X_k \in \mathcal{L}_n\}\right) + \mathbb{P}\left(\Big|\sum_{m=0}^{\infty} K_n^B(m) - \Lambda p^{\mathcal{E}}(\phi, \overleftarrow{\phi})D_\infty n\Big| \geq \eta n\right).$$

Again by (2.1) and (2.2),

$$\limsup_{n\to\infty} \mathbb{P}\Big(|R_{T_\phi^n} - \Lambda p^{\mathcal{E}}(\phi, \overleftarrow{\phi})D_\infty n| \geq \eta n\Big) \leq e^{-\alpha} + \limsup_{n\to\infty} \mathbb{P}\left(\Big|\sum_{m=0}^{\infty} K_n^B(m) - \Lambda p^{\mathcal{E}}(\phi, \overleftarrow{\phi})D_\infty n\Big| \geq \eta n\right). \tag{2.14}$$

Here we only need to consider the generations of order $(\log n)^2$. In fact, for any $\varepsilon > 0$,

$$\begin{aligned}&\mathbb{P}\left(\Big|\sum_{m=1}^{\infty} K_n^B(m) - \Lambda p^{\mathcal{E}}(\phi, \overleftarrow{\phi})D_\infty n\Big| \geq \eta n\right)\\ \leq&\mathbb{P}\left(\sum_{\substack{m\geq(\log n)^2/\varepsilon,\\ \text{or } m\leq\varepsilon(\log n)^2}} K_n^B(m) \geq \eta n/2\right) + \mathbb{P}\left(\Big|\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} K_n^B(m) - \Lambda p^{\mathcal{E}}(\phi, \overleftarrow{\phi})D_\infty n\Big| \geq \eta n/2\right),\end{aligned}$$

where the first probability on the right hand side is negligible because of (2.12). For the second probability, we consider only the sites $z \in C^\delta$ and obtain that

$$\begin{aligned}&\mathbb{P}\left(\Big|\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} K_n^B(m) - \Lambda p^{\mathcal{E}}(\phi, \overleftarrow{\phi})D_\infty n\Big| \geq \eta n/2\right) \leq \mathbb{P}\left(\Big|\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} K_n^{B\setminus B_3}(m)\Big| \geq \eta n/6\right) +\\ &\mathbb{P}\left(\Big|\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} K_n^{C^\delta}(m) - \Lambda p^{\mathcal{E}}(\phi, \overleftarrow{\phi})D_\infty n\Big| \geq \eta n/6\right) + \mathbb{P}\left(\Big|\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} K_n^{C\setminus C^\delta}(m)\Big| \geq \eta n/6\right).\end{aligned}$$

In view of (2.13), we obtain that

$$\limsup_{n\to\infty} \mathbb{P}\left(\Big|\sum_{m=1}^{\infty} K_n^B(m) - \Lambda p^{\mathcal{E}}(\phi, \overleftarrow{\phi})D_\infty n\Big| \geq \eta n\right) \leq o_\varepsilon(1) + \limsup_{n\to\infty} \mathbb{P}\left(\Big|\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} K_n^{C^\delta}(m) - \Lambda p^{\mathcal{E}}(\phi, \overleftarrow{\phi})D_\infty n\Big| \geq \eta n/6\right). \tag{2.15}$$

Recall that the quenched mean of $K_n^{C^\delta}(m)$ is denoted $\mathcal{K}_n^{C^\delta}(m)$. Then,

$$\mathbb{P}\left(\Big|\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} K_n^{C^\delta}(m) - \Lambda p^{\mathcal{E}}(\phi, \overleftarrow{\phi})D_\infty n\Big| \geq \frac{\eta n}{6}\right) \leq \mathbb{P}\left(\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \Big|K_n^{C^\delta}(m) - \mathcal{K}_n^{C^\delta}(m)\Big| \geq \eta n/12\right)$$
$$+ \mathbf{P}\left(\Big|\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \mathcal{K}_n^{C^\delta}(m) - \Lambda p^{\mathcal{E}}(\phi, \overleftarrow{\phi})D_\infty n\Big| \geq \eta n/12\right).$$

On the one hand, by Markov inequality,

$$\mathbb{P}\left(\Big|\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} K_n^{C^\delta}(m) - \mathcal{K}_n^{C^\delta}(m)\Big| \geq \frac{\eta n}{12}\right) \leq \frac{144}{\eta^2 n^2}\mathbb{E}\left[\Big(\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \Big|K_n^{C^\delta}(m) - \mathcal{K}_n^{C^\delta}(m)\Big|\Big)^2\right],$$

which by Cauchy-Schwartz inequality is bounded by

$$\frac{144}{\eta^2 n^2}\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} 1 \sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \mathbf{E}\left(Var^{\mathcal{E}}(K_n^{C^\delta}(m))\right)$$

and applying Lemma 4.5 with $\delta > 5$ to this term implies that

$$\limsup_{n\to\infty} \mathbb{P}\left(\Big|\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} K_n^{C^\delta}(m) - \mathcal{K}_n^{C^\delta}(m)\Big| \geq \eta n/12\right) = 0.$$

On the other hand, replacing $\mathcal{K}_n^{C^\delta}$ by $\widetilde{\mathcal{K}}_n^{C^\delta}$ (see (2.5)), one sees that

$$\mathbf{P}\left(\Big|\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \mathcal{K}_n^{C^\delta}(m) - \Lambda p^{\mathcal{E}}(\phi, \overleftarrow{\phi})D_\infty n\Big| \geq \eta n/12\right)$$
$$\leq \mathbf{P}\left(\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \Big|\mathcal{K}_n^{C^\delta}(m) - \widetilde{\mathcal{K}}_n^{C^\delta}(m)\Big| \geq \eta n/24\right) + \mathbf{P}\left(\Big|\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \widetilde{\mathcal{K}}_n^{C^\delta}(m) - \Lambda p^{\mathcal{E}}(\phi, \overleftarrow{\phi})D_\infty n\Big| \geq \eta n/24\right)$$
$$\leq \mathbf{P}\left(\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \widetilde{\mathcal{K}}_n^{C^\delta}(m) \geq \eta n \log n/24\right) + \mathbf{P}\left(\Big|\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \widetilde{\mathcal{K}}_n^{C^\delta}(m) - \Lambda p^{\mathcal{E}}(\phi, \overleftarrow{\phi})D_\infty n\Big| \geq \eta n/24\right),$$

where the last inequality follows from the fact that $0 \leq \widetilde{\mathcal{K}}_n^{C^\delta}(m) - \mathcal{K}_n^{C^\delta}(m) \leq \widetilde{\mathcal{K}}_n^{C^\delta}(m)/\log n$. Consequently,

$$\mathbb{P}\left(\Big|\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} K_n^{C^\delta}(m) - \Lambda p^{\mathcal{E}}(\phi, \overleftarrow{\phi})D_\infty n\Big| \geq \eta n/6\right) \leq o_n(1) + \tag{2.16}$$
$$\mathbf{P}\left(\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \widetilde{\mathcal{K}}_n^{C^\delta}(m) \geq \eta n \log n/24\right) + \mathbf{P}\left(\Big|\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \widetilde{\mathcal{K}}_n^{C^\delta}(m) - \Lambda p^{\mathcal{E}}(\phi, \overleftarrow{\phi})D_\infty n\Big| \geq \eta n/24\right).$$

Moving back to (2.15), we deduce that

$$\limsup_{n\to\infty} \mathbb{P}\left(\Big|\sum_{m=1}^{\infty} K_n^B(m) - \Lambda p^{\mathcal{E}}(\phi, \overleftarrow{\phi})D_\infty n\Big| \geq \eta n\right) \leq o_\varepsilon(1) + \tag{2.17}$$

$$\limsup_{n\to\infty}\left\{\mathbf{P}\left(\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \widetilde{\mathcal{K}}_n^{C^\delta}(m) \geq \frac{\eta n \log n}{24}\right) + \mathbf{P}\left(\Big|\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \widetilde{\mathcal{K}}_n^{C^\delta}(m) - \Lambda p^{\mathcal{E}}(\phi, \overleftarrow{\phi})D_\infty n\Big| \geq \eta n/24\right)\right\}.$$

Now, observe that

$$\begin{aligned}\mathbf{P}\left(\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \widetilde{\mathcal{K}}_n^{C^\delta}(m) \geq \frac{\eta n \log n}{24}\right) \leq& \mathbf{P}\left(\Big|\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \widetilde{\mathcal{K}}_n^{C^\delta}(m) - \Lambda p^{\mathcal{E}}(\phi, \overleftarrow{\phi})D_\infty n\Big| \geq \eta n/24\right)\\ &+ \mathbf{P}\left(\Lambda p^{\mathcal{E}}(\phi, \overleftarrow{\phi})D_\infty \geq \eta(\log n - 1)/24\right)\\ =& \mathbf{P}\left(\Big|\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \widetilde{\mathcal{K}}_n^{C^\delta}(m) - \Lambda p^{\mathcal{E}}(\phi, \overleftarrow{\phi})D_\infty n\Big| \geq \eta n/24\right) + o_n(1).\end{aligned}$$

So in view of (2.14) and (2.17), we have

$$\begin{aligned}\limsup_{n\to\infty} \mathbb{P}\Big(|R_{T_\phi^n} - \Lambda p^{\mathcal{E}}(\phi, \overleftarrow{\phi})D_\infty n| \geq \eta n\Big) \leq& e^{-\alpha} + o_\varepsilon(1)\\ &+ 2\limsup_{n\to\infty} \mathbf{P}\left(\Big|\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \widetilde{\mathcal{K}}_n^{C^\delta}(m) - \Lambda p^{\mathcal{E}}(\phi, \overleftarrow{\phi})D_\infty n\Big| \geq \eta n/24\right).\end{aligned} \tag{2.18}$$

We are left to prove that

$$\limsup_{\varepsilon\downarrow 0} \limsup_{n\to\infty} \mathbf{P}\left(\Big|\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \widetilde{\mathcal{K}}_n^{C^\delta}(m) - \Lambda p^{\mathcal{E}}(\phi, \overleftarrow{\phi})D_\infty n\Big| \geq \eta n/24\right) \leq e^{-\alpha}, \tag{2.19}$$

which together with (2.18), letting $\alpha \to \infty$ yields $\limsup_{n\to\infty} \mathbb{P}\Big(|R_{T_\phi^n} - \Lambda p^{\mathcal{E}}(\phi, \overleftarrow{\phi})D_\infty n| \geq \eta n\Big) = 0$.

For (2.19), we have

$$\begin{aligned}\mathbf{P}\left(\Big|\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \widetilde{\mathcal{K}}_n^{C^\delta}(m) - \Lambda p^{\mathcal{E}}(\phi, \overleftarrow{\phi})D_\infty n\Big| \geq \frac{\eta n}{24}\right) \leq& \mathbf{P}\left(\inf_{u\in\mathbb{T}} V(u) < -\alpha\right) +\\ \mathbf{P}\left(\Big|\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \sum_{|z|=m} n a_z \mathbb{1}_{\overline{V}(z)\geq \log n + \log\log n, z<\mathcal{L}_{s_n}} - \Lambda p^{\mathcal{E}}(\phi, \overleftarrow{\phi})D_\infty n\Big| \geq \eta n/48\right),&\end{aligned}$$

where the first term on the right hand side is bounded by $e^{-\alpha}$. Again by Lemma 2.4, we have

$$\begin{aligned}\mathbf{P}\left(\Big|\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \widetilde{\mathcal{K}}_n^{C^\delta}(m) - \Lambda p^{\mathcal{E}}(\phi, \overleftarrow{\phi})D_\infty n\Big| \geq \frac{\eta n}{24}\right) \leq& e^{-\alpha} + o_{n,\varepsilon}(1) +\\ &\mathbf{P}\left(\Big|\sum_{m=1}^{\infty} \sum_{|z|=m} n a_z \mathbb{1}_{\overline{V}(z)\geq \log n + \log\log n, z<\mathcal{L}_{s_n}} - \Lambda p^{\mathcal{E}}(\phi, \overleftarrow{\phi})D_\infty n\Big| \geq \eta n/96\right).\end{aligned}$$

As $\sum_{|z|=m} na_z \mathbb{1}_{\overline{V}(z)\geq \log n+\log\log n, z<\mathcal{L}_{s_n}} = np^{\mathcal{E}}(\phi, \overleftarrow{\phi})W^*_m(\log n + \log\log n)$, (2.19) follows immediately from Corollary 1.9 . □

3 Convergence of martingale-like variables $W_m(F_{a\sqrt{m}+a_m, b\sqrt{m}+b_m})$

This section is devoted to proving Proposition 1.8 and Corollary 1.9 which only concern the environment. The main idea is borrowed from [3], on the Seneta-Heyde norm of the additive martingale W_m in the boundary case (1.2). To do so, we need to introduce a change of measure and the corresponding spinal decomposition.

3.1 Lyons' change of measures and spinal decomposition

For $\alpha > 0$, define the truncated variables adapted to $\{\mathcal{F}_n := \sigma((z, V(z)); |z| \leq n); n \geq 0\}$, the natural filtration of the branching random walk:

$$W_n^{(\alpha)}(F) := \sum_{|z|=n} e^{-V(z)} F_{a\sqrt{m}+a_m, b\sqrt{m}+b_m}(z) \mathbb{1}_{\underline{V}(z)\geq -\alpha},$$
$$D_n^{(\alpha)} := \sum_{|z|=n} \mathcal{R}(\alpha + V(z)) e^{-V(z)} \mathbb{1}_{\underline{V}(z)\geq -\alpha}, \ \forall n \geq 0.$$

See (1.24) for the definition of F. For any $a \in \mathbb{R}$, let $\mathbf{P}_a$ be the probability measure such that $\mathbf{P}_a(\{V(z), z \in \mathbb{T}\} \in \cdot) = \mathbf{P}(\{a + V(z), z \in \mathbb{T}\} \in \cdot)$. For $a \geq -\alpha$, we introduce the change of measure as follows:

$$\mathbf{Q}_a^{(\alpha)}|_{\mathcal{F}_n} := \frac{D_n^{(\alpha)}}{\mathcal{R}(\alpha + a)e^{-a}} \mathbf{P}_a|_{\mathcal{F}_n}. \tag{3.1}$$

The fact that $D_n^{(\alpha)}$ is a non-negative martingale which converges a.s. to some limit $D_\infty^{(\alpha)}$ has been proved by Biggins and Kyprianou [8]. So $\mathbf{Q}_a^{(\alpha)}$ is well define. Following their spirits, we present a spinal decomposition of the branching random walk under $\mathbf{Q}_a^{(\alpha)}$: we start with one individual w_0 (i.e., the root ϕ), located at $V(w_0) = a$. Then for any $n \geq 0$,

1. in the n-th generation, each individual u except w_n, gives birth independently of all others to its children of the $n+1$-th generation whose positions constitute a point process distributed as $(V(z), |z| = 1)$ under $\mathbf{P}_{V(u)}$;
2. w_n produces independently its children in the $n+1$-th generation, whose positions are given by a point process distributed as $(V(z), |z| = 1)$ under $\mathbf{Q}^{(\alpha)}_{V(w_n)}$;
3. Among the children of w_n, w_{n+1} is chosen to be z with probability proportional to $\mathcal{R}(\alpha + V(z))e^{-V(z)}\mathbb{1}_{\underline{V}(z)\geq -\alpha}$.

In this description, the infinite ray $(w_n, n \geq 0)$ is call the spine under $\mathbf{Q}_a^{(\alpha)}$. For simplicity, we write $\mathbf{Q}^{(\alpha)}$ for $\mathbf{Q}_0^{(\alpha)}$.

Fact 3.1 ([8]). *Assume* (1.2). *Let* $\alpha \geq 0$. *For any* $n \geq 1$ *and* $|z| = n$,

$$\mathbf{Q}^{(\alpha)}(w_n = z|\mathcal{F}_n) = \frac{\mathcal{R}(\alpha + V(z))e^{-V(z)}\mathbb{1}_{\underline{V}(z)\geq -\alpha}}{D_n^{(\alpha)}}. \tag{3.2}$$

The spine process $(V(w_n), n \geq 0)$ *under* $\mathbf{Q}^{(\alpha)}$ *is distributed as the random walk* $(S_n, n \geq 0)$ *under* $\mathbf{P}$ *conditioned to stay above* $-\alpha$. *In other words, for any* $n \geq 1$ *and any measurable function* $g : \mathbb{R}^n \to \mathbb{R}_+$,

$$\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\Big[g(V(w_k), 1 \leq k \leq n)\Big] = \frac{1}{\mathcal{R}(\alpha)}\mathbf{E}\Big[g(S_k, 1 \leq k \leq n)\mathcal{R}(\alpha + S_n); \underline{S}_n \geq -\alpha\Big]. \tag{3.3}$$

In our case, $\min_{|z|=n} V(z) \to \infty$ a.s. As a consequence of (1.19), $D_\infty^{(\alpha)} = \mathfrak{c}_0 D_\infty$ on $\{\inf_{z\in T} V(z) \geq -\alpha\}$. The L^1 convergence of $D_n^{(\alpha)}$ is assured because of (1.3) (see [8] and [11]).

3.2 Convergence in probability of $\frac{W_n^{(\alpha)}(F)}{D_n^{(\alpha)}}$ under $\mathbf{Q}^{(\alpha)}$

Following the arguments of Section 5 in [3], as $D_\infty > 0$ under $\mathbf{P}^*$, $\mathbf{Q}^{(\alpha)}$ is absolutely continuous with respect to $\mathbf{P}$, so we only need to prove the following convergence under $\mathbf{Q}^{(\alpha)}$,

$$\sqrt{n}\frac{W_n^{(\alpha)}(F_{a\sqrt{n}+a_n, b\sqrt{n}+b_n})}{D_n^{(\alpha)}} \longrightarrow \mathscr{C}_{a,b}, \text{ in probability,} \tag{3.4}$$

and focus on the cases when $a > 0$ and $b > 0$ (when $a = 0$ or $b = 0$, it is trivial by first moment estimation). The proof is based on the computations of the first and second moments of this random variable. However, as F is not a bounded functional, we have to add some restrictions to the sites. Let us first introduce some notations. For any $|z| = n$ and $0 \leq m \leq n$, let z_m be the ancestor of z in the m-th generation and define

$$\begin{aligned} v_z &:= \inf\{k : V(z_k) = \underline{V}(z) = \min_{y\leq z} V(y)\}, \\ \mathrm{Y}_z &:= \inf\{k : V(z_k) = \overline{V}(z) = \max_{y\leq z} V(y)\},\ \underline{V}(z_{[m,n]}) := \min_{m\leq k\leq n} V(z_k). \end{aligned}$$

Similarly, we also define v_S, Y_S and $\underline{S}_{[m,n]}$ for one-dimensional random walk (S_k, k). Take $n_0 := \lfloor n - n^{1/3} \rfloor$, define

$$G^+(z) := \sqrt{n}\frac{e^{V(z)}}{\sum_{\phi<y\leq z} e^{V(y)}}\mathbb{1}_{\{\mathrm{Y}_z > n_0, \overline{V}(z)\geq b\sqrt{n}+b_n, \max_{y\leq z}(\overline{V}(y)-V(y))\leq a\sqrt{n}+a_n\}},\ G(z) := G^+(z)\mathbb{1}_{\underline{V}(z_{[\mathrm{Y}_z,n]})\geq b\sqrt{n}/2}.$$

Moreover, following [3], let us introduce the events E_n^z for $|z| = n$ as follows. Let $\Omega(y) := \{u \in \mathbb{T} : u \neq y, \overleftarrow{u} = \overleftarrow{y}\}$ be the collection of brothers of y. If (k_n, n) is a positive sequence such that $k_n = o(n^{1/2})$ and $(\log n)^6 = o(k_n)$, let

$$E_n^z := E_{n,1}^z \cap E_{n,2}^z \cap E_{n,3}^z,$$

where

$$E^z_{n,1} = \{k_n^{1/3} \le V(z_{k_n}) \le k_n\} \cap \bigcap_{i=k_n}^{n} \{V(z_i) \ge k_n^{1/6}\};$$

$$E^z_{n,2} = \bigcap_{i=k_n}^{n} \{ \sum_{y \in \Omega(z_{i+1})} [1 + (V(y) - V(z_i))_+] e^{-[V(y)-V(z_i)]} \le e^{V(z_i)/2} \};$$

$$E^z_{n,3} = \{ \sum_{i=k_n}^{n} \sum_{y \in \Omega(z_{i+1})} \sum_{|u|=n, u \ge y} \mathcal{R}(\alpha + V(u)) e^{-V(u)} \mathbb{1}_{\underline{V}(u) \ge -\alpha} \le \frac{1}{n^2} \}.$$

In particular, for w_n, write E_n (resp. $E_{n,i}$) instead of $E_n^{w_n}$ (resp. $E_{n,i}^{w_n}$). Let $H(z) := G(z)\mathbb{1}_{E_n^z}$, and notice that $0 \le H(z) \le G(z) \le G^+(z) \le F(z)$. We use F instead of $F_{a\sqrt{n}+a_n, b\sqrt{n}+b_n}$ for convenience. Similarly as $W_n^{(\alpha)}(F)$, we write also $W_n^{(\alpha)}(G)$, $W_n^{(\alpha)}(G^+)$ and $W_n^{(\alpha)}(H)$.

The convergence (3.4) follows from the following lemma.

Lemma 3.2. *Let $\alpha \ge 0$, we have*

$$\lim_n \sqrt{n} \mathbf{E}_{\mathbf{Q}^{(\alpha)}} \Big[\frac{W_n^{(\alpha)}(H)}{D_n^{(\alpha)}} \Big] = \lim_n \sqrt{n} \mathbf{E}_{\mathbf{Q}^{(\alpha)}} \Big[\frac{W_n^{(\alpha)}(F)}{D_n^{(\alpha)}} \Big] = \mathscr{C}_{a,b}, \tag{3.5}$$

$$\lim_n n \mathbf{E}_{\mathbf{Q}^{(\alpha)}} \Big[\Big(\frac{W_n^{(\alpha)}(H)}{D_n^{(\alpha)}} \Big)^2 \Big] = \mathscr{C}_{a,b}^2. \tag{3.6}$$

3.2.1 First moment estimate: proof of (3.5)

(3.5) follows from the following lemma.

Lemma 3.3.

$$\lim_n \sqrt{n} \mathbf{E}_{\mathbf{Q}^{(\alpha)}} \Big[\frac{W_n^{(\alpha)}(G^+)}{D_n^{(\alpha)}} \Big] = \mathscr{C}_{a,b}, \tag{3.7}$$

$$\lim_n \sqrt{n} \mathbf{E}_{\mathbf{Q}^{(\alpha)}} \Big[\frac{W_n^{(\alpha)}(F-G)}{D_n^{(\alpha)}} \Big] = 0, \tag{3.8}$$

$$\lim_n \sqrt{n} \mathbf{E}_{\mathbf{Q}^{(\alpha)}} \Big[\frac{W_n^{(\alpha)}(G-H)}{D_n^{(\alpha)}} \Big] = 0. \tag{3.9}$$

Proof. By (3.2), one sees that

$$\mathbf{E}_{\mathbf{Q}^{(\alpha)}} \Big[\frac{F(w_n)}{\mathcal{R}(\alpha + V(w_n))} \Big| \mathcal{F}_n \Big] = \frac{W_n^{(\alpha)}(F)}{D_n^{(\alpha)}}. \tag{3.10}$$

This identity holds for H, G, G^+ and will be used several times in the sequel.

Proof of (3.8): For any $|x| = n$, define

$$h_1(x) := \sqrt{n}\frac{e^{V(x)}}{\sum_{\phi<y\le x} e^{V(y)}}\mathbb{1}_{\{Y_x\le n_0, \overline{V}(x)\ge b\sqrt{n}+b_n,\ \max_{y\le x}(\overline{V}(y)-V(y))\le a\sqrt{n}+a_n\}},$$

$$h_2(x) := \sqrt{n}\frac{e^{V(x)}}{\sum_{\phi<y\le x} e^{V(y)}}\mathbb{1}_{\{Y_x> n_0, \overline{V}(x)\ge b\sqrt{n}+b_n, \underline{V}(x_{[Y_x,n]})\le b\sqrt{n}/2,\ \max_{y\le x}(\overline{V}(y)-V(y))\le a\sqrt{n}+a_n\}}.$$

It is clear that $0 \le F - G \le h_1 + h_2$. So to obtain (3.8), it suffices to show that for $i = 1, 2$,

$$\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\Big[\frac{W_n^{(\alpha)}(h_i)}{D_n^{(\alpha)}}\Big] = \frac{o_n(1)}{\sqrt{n}}. \tag{3.11}$$

For h_1, by (3.3), $\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\Big[W_n^{(\alpha)}(h_1)(D_n^{(\alpha)})^{-1}\Big]$ equals

$$\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\Big[\frac{\sqrt{n}e^{V(w_n)}}{\mathcal{R}(\alpha+V(w_n))\sum_{1\le j\le n} e^{V(w_j)}}\mathbb{1}_{Y_{w_n}\le n_0, \overline{V}(w_n)\ge b\sqrt{n}+b_n, \max_{j\le n}(\overline{V}(w_j)-V(w_j))\le a\sqrt{n}+a_n}\Big]$$
$$=\frac{1}{\mathcal{R}(\alpha)}\mathbf{E}\Big[\frac{\sqrt{n}e^{S_n}}{\sum_{1\le j\le n} e^{S_j}}; Y_S \le n_0, \overline{S}_n \ge b\sqrt{n}+b_n, \max_{j\le n}(\overline{S}_j - S_j) \le a\sqrt{n}+a_n, \underline{S}_n \ge -\alpha\Big].$$

Partitioning on the values of Y_S implies that

$$\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\Big[\frac{W_n^{(\alpha)}(h_1)}{D_n^{(\alpha)}}\Big] = \sum_{k=0}^{n_0}\frac{1}{\mathcal{R}(\alpha)}\mathbf{E}\Big[\frac{\sqrt{n}e^{S_n}}{\sum_{1\le j\le n} e^{S_j}}\mathbb{1}_{Y_S=k, \overline{S}_n\ge b\sqrt{n}+b_n, \max_{j\le n}(\overline{S}_j-S_j)\le a\sqrt{n}+a_n, \underline{S}_n\ge -\alpha}\Big]$$
$$\le \sum_{k=0}^{n_0}\frac{\sqrt{n}}{\mathcal{R}(\alpha)}\mathbf{E}\Big[e^{S_n-S_k}\mathbb{1}_{Y_S=k, \underline{S}_k\ge -\alpha}\Big].$$

By Markov property at time k,

$$\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\Big[\frac{W_n^{(\alpha)}(h_1)}{D_n^{(\alpha)}}\Big] \le \sum_{k=0}^{n_0}\frac{\sqrt{n}}{\mathcal{R}(\alpha)}\mathbf{P}\Big(\underline{S}_k \ge -\alpha, S_k = \overline{S}_k\Big)\mathbf{E}\Big[e^{S_{n-k}}\mathbb{1}_{\overline{S}_{n-k}\le 0}\Big],$$

which by (A.4) and (A.14) gives that

$$\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\Big[\frac{W_n^{(\alpha)}(h_1)}{D_n^{(\alpha)}}\Big] \le \sum_{k=0}^{n_0}\frac{\sqrt{n}}{\mathcal{R}(\alpha)}\frac{c(1+\alpha)}{(k+1)(n-k)^{3/2}} \le \frac{1}{n^{2/3}}.$$

Thus (3.11) holds for $i = 1$.

For h_2, similarly, by (3.10) and (3.3), one has

$$\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\Big[\frac{W_n^{(\alpha)}(h_2)}{D_n^{(\alpha)}}\Big] \le \frac{1}{\mathcal{R}(\alpha)}\mathbf{E}\Big[\frac{\sqrt{n}e^{S_n}}{\sum_{1\le j\le n} e^{S_j}}; Y_S > n_0, \underline{S}_{[Y_S,n]} \le b\sqrt{n}/2, \overline{S}_n \ge b\sqrt{n}+b_n, \underline{S}_n \ge -\alpha\Big]$$
$$\le \frac{\sqrt{n}}{\mathcal{R}(\alpha)}\sum_{k=n_0+1}^{n}\mathbf{P}\Big[Y_S = k, \underline{S}_{[k,n]} \le b\sqrt{n}/2, S_k \ge b\sqrt{n}+b_n, \underline{S}_k \ge -\alpha\Big].$$

Applying Markov property at time k implies that

$$\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\Big[\frac{W_n^{(\alpha)}(h_2)}{D_n^{(\alpha)}}\Big] \le \frac{\sqrt{n}}{\mathcal{R}(\alpha)} \sum_{k=n_0+1}^{n} \mathbf{P}\Big[\underline{S}_k \ge -\alpha, S_k = \overline{S}_k\Big]\mathbf{P}\Big(\underline{S}_{n-k} \le -b\sqrt{n}/2 - b_n\Big)$$

which by (A.4) and then (A.18) is bounded by $\sum_{k=n_0+1}^{n-1} \frac{c(1+\alpha)}{k} e^{-c'\sqrt{n}} = cn^{-1/6}e^{-c'n^{1/2}}$. So (3.11) holds also for $i = 2$. This completes the proof of (3.8).

Proof of (3.7): It follows from (3.10) and (3.3) that

$$\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\Big[\frac{W_n^{(\alpha)}(G^+)}{D_n^{(\alpha)}}\Big] = \frac{\sqrt{n}}{\mathcal{R}(\alpha)}\mathbf{E}\Big[\frac{e^{S_n}}{\sum_{1\le j\le n} e^{S_j}}; \Upsilon_S > n_0, \overline{S}_n \ge b\sqrt{n} + b_n, \max_{j\le n}(\overline{S}_j - S_j) \le a\sqrt{n} + a_n, \underline{S}_n \ge -\alpha\Big].$$

Arguing over the value of Υ_S yields that $\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\Big[\frac{W_n^{(\alpha)}(G^+)}{D_n^{(\alpha)}}\Big] = \frac{\sqrt{n}}{\mathcal{R}(\alpha)} \sum_{k=n_0+1}^{n} \sigma_k$ where

$$\sigma_k := \mathbf{E}\Big[\frac{e^{S_n}}{\sum_{1\le j\le n} e^{S_j}}; \Upsilon_S = k, \overline{S}_n \ge b\sqrt{n} + b_n, \max_{j\le n}(\overline{S}_j - S_j) \le a\sqrt{n} + a_n, \underline{S}_n \ge -\alpha\Big].$$

Let $T_i = S_{i+k} - S_k$, then

$$\sigma_k = \mathbf{E}\Big[\frac{e^{T_{n-k}}}{\sum_{1\le j\le k} e^{S_j - S_k} + \sum_{1\le j\le n-k} e^{T_j}}; \overline{S}_{k-1} < S_k, S_k \ge b\sqrt{n} + b_n, \underline{S}_k \ge -\alpha,$$
$$\underline{T}_{n-k} \ge (-\alpha - S_k) \vee (-a\sqrt{n} - a_n), \overline{T}_{n-k} \le 0, \max_{i\le k}(\overline{S}_i - S_i) \le a\sqrt{n} + a_n\Big].$$

In the next steps we prove that σ_k can be approximated by σ'_k which is defined as follows

$$\sigma'_k := \mathbf{E}\Big[\frac{e^{T_{n-k}}}{\sum_{1\le j\le k} e^{S_j - S_k} + \sum_{1\le j\le n-k} e^{T_j}}; \overline{S}_{k-1} < S_k, S_k \ge b\sqrt{n} + b_n, \underline{S}_k \ge -\alpha,$$
$$\overline{T}_{n-k} \le 0, \max_{i\le k}(\overline{S}_i - S_i) \le a\sqrt{n} + a_n\Big].$$

Observe that $0 \le \sigma'_k - \sigma_k \le \mathbf{P}(\overline{S}_{k-1} < S_k, \underline{S}_k \ge -\alpha, \underline{T}_{n-k} \le (-a\sqrt{n} - a_n) \vee (-\alpha - b\sqrt{n} - b_n))$. By independence of S and T,

$$\begin{aligned}\sigma'_k - \sigma_k \le& \mathbf{P}\left(\underline{S}_k \ge -\alpha, S_k = \overline{S}_k\right)\mathbf{P}\left(\underline{T}_{n-k} \le (-a\sqrt{n} - a_n) \vee (-b\sqrt{n} - b_n)\right)\\ \le& \frac{c(1+\alpha)}{k} e^{-c'\sqrt{n}},\end{aligned}$$

because of (A.4) and (A.18). Hence, $\sum_{k=n_0+1}^{n-1}(\sigma'_k - \sigma_k) = \frac{o_n(1)}{n}$. This implies that

$$\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\Big[\frac{W_n^{(\alpha)}(G^+)}{D_n^{(\alpha)}}\Big] = \frac{\sqrt{n}}{\mathcal{R}(\alpha)} \sum_{k=n_0+1}^{n} \sigma'_k + \frac{o_n(1)}{n^{1/2}}. \tag{3.12}$$

By independence of S and T again,

$$\sigma_k' = \mathbf{E}\Big[\mathcal{G}_{n-k}(\sum_{1\le j\le k} e^{S_j - S_k}); \overline{S}_{k-1} < S_k, S_k \ge b\sqrt{n} + b_n, \underline{S}_k \ge -\alpha, \max_{i\le k}(\overline{S}_i - S_i) \le a\sqrt{n} + a_n\Big]$$

where $\mathcal{G}_\cdot(x)$ is defined in (1.20). By (A.13), for $n - n^{1/3} < k \le n$, $\sigma_k' = \frac{\mathcal{C}_{a,b}\mathcal{R}(\alpha)\mathbf{E}[\mathcal{G}_{n-k}(\mathcal{H}_\infty)]}{k} + \frac{o_n(1)}{n}$. Further, as $|\mathcal{G}_{n-k}(x) - \mathcal{G}_{n-k}(y)| \le \mathcal{G}_{n-k}|\frac{1}{x} - \frac{1}{y}|$, uniformly on $k \in (n - n^{1/3}, n] \cap \mathbb{Z}$,

$$\sigma_k' = \frac{\mathcal{C}_{a,b}\mathcal{R}(\alpha)\mathbf{E}[\mathcal{G}_{n-k}(\mathcal{H}_\infty)]}{n} + \frac{o_n(1)}{n}.$$

Plugging it into (3.12) implies that for sufficiently large n,

$$\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\Big[\frac{W_n^{(\alpha)}(G^+)}{D_n^{(\alpha)}}\Big] = \frac{\mathcal{C}_{a,b}}{\sqrt{n}}\sum_{j=0}^{\infty}\mathbf{E}[\mathcal{G}_j(\mathcal{H}_\infty)] + \frac{o_n(1)}{\sqrt{n}} = \frac{\mathscr{C}_{a,b}}{\sqrt{n}} + \frac{o_n(1)}{\sqrt{n}},$$

which ends the proof of (3.7).

Proof of (3.9): by (3.10), we only need to prove that

$$LHS := \mathbf{E}_{\mathbf{Q}^{(\alpha)}}\Big[\frac{\sqrt{n}G(w_n)}{\mathcal{R}(\alpha + V(w_n))}\mathbb{1}_{E_n^c}\Big] = o_n(1). \tag{3.13}$$

First, we have

$$\begin{aligned} LHS \le & n\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\Big[\frac{e^{V(w_n)-\overline{V}(w_n)}\mathbb{1}_{E_n^c}}{\mathcal{R}(\alpha + V(w_n))}; \Upsilon_{w_n} > n_0, \overline{V}(w_n) \ge b\sqrt{n} + b_n, \underline{V}(w_{[\Upsilon_{w_n},n]}) \ge b\sqrt{n}/2\Big] \\ \le & LHS_1 + LHS_2 + LHS_3 \end{aligned}$$

where

$$\begin{aligned} LHS_1 :=& n\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\Big[\frac{e^{V(w_n)-\overline{V}(w_n)}\mathbb{1}_{E_{n,1}^c}}{\mathcal{R}(\alpha + V(w_n))}; \Upsilon_{w_n} > n_0, \overline{V}(w_n) \ge b\sqrt{n} + b_n, \underline{V}(w_{[\Upsilon_{w_n},n]}) \ge b\sqrt{n}/2\Big], \\ LHS_2 :=& n\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\Big[\frac{e^{V(w_n)-\overline{V}(w_n)}\mathbb{1}_{E_{n,1}\cap E_{n,2}^c}}{\mathcal{R}(\alpha + V(w_n))}; \Upsilon_{w_n} > n_0, \overline{V}(w_n) \ge b\sqrt{n} + b_n, \underline{V}(w_{[\Upsilon_{w_n},n]}) \ge b\sqrt{n}/2\Big], \\ LHS_3 :=& n\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\Big[\frac{e^{V(w_n)-\overline{V}(w_n)}\mathbb{1}_{E_{n,1}\cap E_{n,2}\cap E_{n,3}^c}}{\mathcal{R}(\alpha + V(w_n))}; \Upsilon_{w_n} > n_0, \overline{V}(w_n) \ge b\sqrt{n} + b_n, \underline{V}(w_{[\Upsilon_{w_n},n]}) \ge b\sqrt{n}/2\Big]. \end{aligned}$$

Each term LHS_i, $i = 1, 2, 3$, are treated separately.

For LHS_1, we have

$$\begin{aligned} LHS_1 \le & \frac{n}{\mathcal{R}(\alpha)}\mathbf{E}[e^{S_n - \overline{S}_n}; S_{k_n} \in [k_n^{1/3}, k_n], \underline{S}_{[k_n,\Upsilon_S]} \le k_n^{1/6}, \Upsilon_S > n_0, \underline{S}_n \ge -\alpha] \\ & + \frac{n}{\mathcal{R}(\alpha)}\mathbf{E}[e^{S_n - \overline{S}_n}; S_{k_n} \notin [k_n^{1/3}, k_n], \Upsilon_S > n_0, \underline{S}_n \ge -\alpha] =: \xi_1 + \xi_1'. \end{aligned}$$

By Markov property at $\Upsilon_S = k$,

$$\begin{aligned}\xi_1 \leq & \frac{n}{\mathcal{R}(\alpha)} \sum_{k=n_0+1}^{n} \mathbf{E}\left[e^{S_{n-k}} \mathbb{1}_{\overline{S}_{n-k}\leq 0}\right] \mathbf{P}\Big(S_{k_n} \in [k_n^{1/3}, k_n], \underline{S}_{[k_n,k]} \leq k_n^{1/6}, S_k > \overline{S}_{k-1}, \underline{S}_k \geq -\alpha\Big) \\ \leq & \frac{n}{\mathcal{R}(\alpha)} \sum_{k=n_0+1}^{n} \frac{c}{(n-k+1)^{3/2}} \mathbf{P}\Big(S_{k_n} \in [k_n^{1/3}, k_n], \underline{S}_{[k_n,k]} \leq k_n^{1/6}, S_k > \overline{S}_{k-1}, \underline{S}_k \geq -\alpha\Big), \end{aligned} \tag{3.14}$$

where the second inequality holds because of (A.14). It remains to obtain an upper bound for $\mathbf{P}\Big(S_{k_n} \in [k_n^{1/3}, k_n], \underline{S}_{[k_n,k]} \leq k_n^{1/6}, S_k > \overline{S}_{k-1}, \underline{S}_k \geq -\alpha\Big)$. One sees that by (A.22), for any $n_0 < k \leq n$,

$$\begin{aligned} & \mathbf{P}\Big(S_{k_n} \in [k_n^{1/3}, k_n], \underline{S}_{[k_n,k]} \leq k_n^{1/6}, S_k > \overline{S}_{k-1}, \underline{S}_k \geq -\alpha\Big) \\ \leq & \mathbf{P}\Big(\min_{k/2<j\leq k} S_j \leq k_n^{1/6}, S_k > \overline{S}_{k-1}, \underline{S}_k \geq -\alpha\Big) + \mathbf{P}\Big(\underline{S}_{[k_n,k/2]} \leq k_n^{1/6}, \underline{S}_{(k/2,k]} > k_n^{1/6}, S_k > \overline{S}_{k-1}, \underline{S}_k \geq -\alpha\Big) \\ = & \frac{o_n(1)}{n} + \mathbf{P}\Big(\underline{S}_{[k_n,k/2]} \leq k_n^{1/6} < \underline{S}_{(k/2,k]}, S_k > \overline{S}_{k-1}, \underline{S}_k \geq -\alpha\Big). \end{aligned}$$

By (3.14), to conclude that $\xi_1 = o_n(1)$, it suffices to show that uniformly on $k \in [n_0, n] \cap \mathbb{Z}$,

$$\mathbf{P}\Big(\underline{S}_{[k_n,k/2]} \leq k_n^{1/6} < \underline{S}_{(k/2,k]}, S_k > \overline{S}_{k-1}, \underline{S}_k \geq -\alpha\Big) = \frac{o_n(1)}{n}.$$

Considering the first hitting time of $\underline{S}_k$ which should be before $k/2$, one has

$$\begin{aligned} & \mathbf{P}\Big(\underline{S}_{[k_n,k/2]} \leq k_n^{1/6} < \underline{S}_{(k/2,k]}, S_k > \overline{S}_{k-1}, \underline{S}_k \geq -\alpha\Big) \\ \leq & \sum_{0\leq j\leq k/2} \mathbf{P}\Big(\underline{S}_{[k_n,k/2]} \leq k_n^{1/6}, S_k > \overline{S}_{k-1}, \underline{S}_{j-1} > S_j = \underline{S}_k \geq -\alpha\Big). \end{aligned} \tag{3.15}$$

For any $k_n/2 \leq j \leq k/2$ and $n_0 \leq k \leq n$, by Markov property at time j,

$$\begin{aligned} & \sum_{k_n/2\leq j\leq k/2} \mathbf{P}\Big(\underline{S}_{[k_n,k/2]} \leq k_n^{1/6}, S_k > \overline{S}_{k-1}, \underline{S}_{j-1} > S_j = \underline{S}_k \geq -\alpha\Big) \\ \leq & \sum_{k_n/2\leq j\leq k/2} \mathbf{P}\Big(\underline{S}_{j-1} > S_j \geq -\alpha\Big) \mathbf{P}\Big(\underline{S}_{k-j} \geq 0, S_{k-j} = \overline{S}_{k-j}\Big) \end{aligned}$$

which by (A.8) and (A.4) is bounded by $\sum_{k_n/2\leq j\leq k/2} \frac{c(1+\alpha)^2}{j^{3/2}n} = \frac{o_n(1)}{n}$. Also when $j \leq k_n/2$, applying Markov property at time $2k/3$ then at time j implies that

$$\begin{aligned} & \mathbf{P}\Big(\underline{S}_{[k_n,k/2]} \leq k_n^{1/6}, S_k > \overline{S}_{k-1}, \underline{S}_{j-1} > S_j = \underline{S}_k \geq -\alpha\Big) \\ \leq & \mathbf{P}\Big(\underline{S}_{[k_n,k/2]} \leq k_n^{1/6}, \underline{S}_{j-1} > S_j = \underline{S}_{2k/3} \geq -\alpha\Big) \mathbf{P}\Big(S_{k/3} > \overline{S}_{k/3-1}\Big) \\ \leq & \mathbf{P}\Big(\underline{S}_{j-1} > S_j \geq -\alpha\Big) \mathbf{P}\Big(\underline{S}_{7k/12} \geq 0, \underline{S}_{[k_n/2,k/2]} \leq k_n^{1/6}\Big) \mathbf{P}\Big(S_{k/3} > \overline{S}_{k/3-1}\Big). \end{aligned}$$

By time reversal together with (A.2), $\mathbf{P}\Big(\overline{S}_{k/3-1} < S_{k/3}\Big) \le c/\sqrt{k/3}$. Also, in view of (A.8) and (A.17), for any $n_0 \le k \le n$,

$$\sum_{j\le k_n/2} \mathbf{P}\Big(\underline{S}_{[k_n,k/2]} \le k_n^{1/6}, S_k > \overline{S}_{k-1}, \underline{S}_{j-1} > S_j = \underline{S}_k \ge -\alpha\Big) \le \sum_{j\le k_n/2} \frac{c(1+\alpha)^2 k_n^{1/6}}{(j+1)^{3/2} n k_n^{1/2}} = \frac{o_n(1)}{n}.$$

Going back to (3.15),

$$\mathbf{P}\Big(\underline{S}_{[k_n,k/2]} \le k_n^{1/6} < \underline{S}_{(k/2,k]}, S_k > \overline{S}_{k-1}, \underline{S}_k \ge -\alpha\Big) = \frac{o_n(1)}{n},$$

which leads to $\xi_1 = o_n(1)$.

For ξ_1', applying Markov property at time $\mathrm{Y}_S = k$,

$$\xi_1' \le \frac{n}{\mathcal{R}(\alpha)} \sum_{k=n_0+1}^{n} \mathbf{E}[e^{S_{n-k}}; \overline{S}_{n-k} \le 0]\mathbf{P}\Big(S_{k_n} \notin [k_n^{1/3}, k_n], \underline{S}_k \ge -\alpha, S_k = \overline{S}_k\Big),$$

which again by Markov property at time k_n yields that

$$\xi_1' \le \frac{n}{\mathcal{R}(\alpha)} \sum_{k=n_0+1}^{n} \mathbf{E}[e^{S_{n-k}}; \overline{S}_{n-k} \le 0]\mathbf{E}\Big(\mathbf{P}_{S_{k_n}}\Big(\underline{S}_{k-k_n} \ge -\alpha, S_{k-k_n} = \overline{S}_{k-k_n}\Big), \underline{S}_{k_n} \ge -\alpha, S_{k_n} \notin [k_n^{1/3}, k_n]\Big),$$

and recall that $\mathbf{P}_u$ is for the distribution of the random walk starting from u. By (A.4), $\mathbf{P}_{S_{k_n}}\Big(\underline{S}_{k-k_n} \ge -\alpha, S_{k-k_n} = \overline{S}_{k-k_n}\Big) \le c(1+\alpha+S_{k_n})/(k-k_n)$. This together with (A.14) yields

$$\xi_1' \le \frac{cn}{\mathcal{R}(\alpha)} \sum_{k=n_0+1}^{n} \frac{1}{(n-k+1)^{3/2}} \mathbf{E}\Big[\frac{(1+\alpha+S_{k_n})}{k-k_n}; \underline{S}_{k_n} \ge -\alpha, S_{k_n} \notin [k_n^{1/3}, k_n]\Big].$$

We now split the above expectation into two terms, first by Markov's inequality,

$$\mathbf{E}\Big[\frac{(1+\alpha+S_{k_n})}{k-k_n}; \underline{S}_{k_n} \ge -\alpha, S_{k_n} \ge k_n\Big] \le \frac{c}{n(1+\alpha+k_n)^3}\mathbf{E}\Big[(1+\alpha+S_{k_n})^4\Big] \le \frac{c}{nk_n},$$

and also by (A.3)

$$\mathbf{E}\Big[\frac{(1+\alpha+S_{k_n})}{k-k_n}; \underline{S}_{k_n} \ge -\alpha, S_{k_n} \le k_n^{1/3}\Big] \le \sum_{l=-\alpha}^{k_n^{1/3}} \mathbf{E}\Big[\frac{(1+\alpha+S_{k_n})}{k-k_n}; \underline{S}_{k_n} \ge -\alpha, S_{k_n} \in [l, l+1]\Big]$$
$$\le \sum_{l=-\alpha}^{k_n^{1/3}} \frac{c(1+\alpha+l)}{n}\mathbf{P}\Big(\underline{S}_{k_n} \ge -\alpha, S_{k_n} \in [l, l+1]\Big) \le \frac{c}{nk_n^{1/2}}.$$

These two inequalities gives $\xi_1' = o_n(1)$.

For LHS_3, let $\mathcal{G}_\infty$ be the sigma-field generated by the spine and all siblings of the spine. We know from ([3] eq. (4.9)) that

$$\mathbf{Q}^{(\alpha)}\Big(E_{n,1} \cap E_{n,2} \cap E_{n,3}^c \Big| \mathcal{G}_\infty\Big) \le O(n^3 e^{-k_n^{1/6}/3}),$$

wich implies that

$$\begin{aligned} LHS_3 \leq & n\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\Big[\frac{e^{V(w_n)-\overline{V}(w_n)}}{\mathcal{R}(\alpha+V(w_n))}\times \mathbf{Q}^{(\alpha)}\Big(E_{n,1}\cap E_{n,2}\cap E_{n,3}^c\Big|\mathcal{G}_\infty\Big)\Big] \\ \leq & O(n^4 e^{-k_n^{1/6}/3}) = o_n(1). \end{aligned}$$

For LHS_2, we follow the same lines as in ([3] page 18, below (4.8)) using the same notations.

$$LHS_2 \leq c\sum_{i=k_n}^{n} n\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\Big[\frac{e^{V(w_n)-\overline{V}(w_n)}}{\mathcal{R}(\alpha+V(w_n))}h(V(w_i))\mathbb{1}_{E_{n,1}}; \Upsilon_{w_n} > n_0, \underline{V}(w_{[\Upsilon_{w_n},n]}) \geq b\sqrt{n}/2\Big],$$

where $h(x) := \frac{1}{x^3(x+\alpha+1)}$ (this inequality holds because of (1.3)). Applying (3.3) then partitioning on the values of Υ_S yields that

$$\begin{aligned} LHS_2 \leq & c\left(\sum_{k=n_0+1}^{n}\sum_{i=k_n}^{k} + \sum_{k=n_0+1}^{n}\sum_{i=k+1}^{n}\right)\frac{n}{\mathcal{R}(\alpha)}\mathbf{E}[e^{S_n-S_k}h(S_i)\mathbb{1}_{S_i\geq k_n^{1/6}}; \Upsilon_S = k, \underline{S}_{[k,n]} \geq b\sqrt{n}/2, \underline{S}_n \geq -\alpha] \\ =: & \xi_2 + \xi_2'. \end{aligned}$$

For the first sum, applying Markov property at time i then at k and using (A.4) yields that

$$\begin{aligned} \xi_2 \leq & \frac{cn}{\mathcal{R}(\alpha)}\sum_{k=n_0+1}^{n}\sum_{i=k_n}^{k}\mathbf{E}\left[e^{S_{n-k}}; \overline{S}_{n-k}\leq 0\right]\mathbf{E}\left[h(S_i)\mathbb{1}_{\{\underline{S}_i\geq-\alpha\}\cap\{S_i\geq k_n^{1/6}\}}\mathbf{P}_{S_i}\left(\underline{S}_{k-i}\geq -\alpha, S_{k-i} = \overline{S}_{k-i}\right)\right] \\ \leq & \frac{cn}{\mathcal{R}(\alpha)}\sum_{k=n_0+1}^{n}\sum_{i=k_n}^{k}\frac{1}{(n-k+1)^{3/2}}\frac{\mathbf{E}[h(S_i)(1+\alpha+S_i)\mathbb{1}_{\{\underline{S}_i\geq-\alpha\}\cap\{S_i\geq k_n^{1/6}\}}]}{k-i+1} \end{aligned}$$

By (A.3), $\mathbf{E}[h(S_i)(1+\alpha+S_i)\mathbb{1}_{\{\underline{S}_i\geq-\alpha\}\cap\{S_i\geq k_n^{1/6}\}}] = o_n(1)/i^{3/2}$. We hence deduce that $\xi_2 = o_n(1)/\sqrt{k_n}$

For ξ_2', observe that $h(S_i) \leq h(b\sqrt{n}/2) \leq c/n$. Applying Markov property at time k then using (A.14) and (A.4) implies that

$$\begin{aligned} \xi_2' \leq & c\sum_{k=n_0+1}^{n}\sum_{i=k+1}^{n}\frac{n}{\mathcal{R}(\alpha)}\mathbf{E}[e^{S_{n-k}}/n; \overline{S}_{n-k}\leq 0]\mathbf{P}[S_k = \overline{S}_k \geq \sqrt{n}, \underline{S}_k \geq -\alpha] \\ \leq & c\sum_{k=n_0+1}^{n}\frac{1}{(n-k+1)^{3/2}k} = o_n(1). \end{aligned}$$

Collecting all the estimations for the LHS_i, this ends the proof of (3.9). □

3.2.2 Second moment estimate: proof of (3.6)

Recall the definitions of H and G Section 3.2. In view of (3.5), it suffices to show that

$$\limsup_{n\to\infty}\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\left(\frac{\sqrt{n}W_n^{(\alpha)}(H)}{D_n^{(\alpha)}}\right)^2\right] \leq \mathscr{C}_{a,b}^2. \tag{3.16}$$

By (3.10),

$$LHS_{(3.16)} := \mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\left(\frac{\sqrt{n}W_n^{(\alpha)}(H)}{D_n^{(\alpha)}}\right)^2\right] = \mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\frac{\sqrt{n}W_n^{(\alpha)}(H)}{D_n^{(\alpha)}} \times \frac{\sqrt{n}H(w_n)}{\mathcal{R}(\alpha + V(w_n))}\right]$$
$$\leq \mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\frac{\sqrt{n}W_n^{(\alpha)}(G)}{D_n^{(\alpha)}} \times \frac{\sqrt{n}H(w_n)}{\mathcal{R}(\alpha + V(w_n))}\right].$$

For convenience, let

$$W_n^{(\alpha),[k_n,n]}(G) := e^{-V(w_n)}G(w_n)\mathbb{1}_{\underline{V}(w_n)\geq -\alpha} + \sum_{i=k_n}^{n-1}\sum_{y\in\Omega(w_{i+1})}\sum_{|z|=n, z\geq y} e^{-V(z)}G(z)\mathbb{1}_{\underline{V}(z)\geq -\alpha},$$

$$W_n^{(\alpha),[0,k_n)}(G) := \sum_{i=0}^{k_n-1}\sum_{y\in\Omega(w_{i+1})}\sum_{|z|=n, z\geq y} e^{-V(z)}G(z)\mathbb{1}_{\underline{V}(z)\geq -\alpha},$$

with $\Omega(\omega_{i+1}) = \{|x| = i+1 : \overleftarrow{x} = \omega_i,\ x \neq \omega_{i+1}\}$. In the similar way, we define $D_n^{(\alpha),[0,k_n)}$ and $D_n^{(\alpha),[k_n,n]}$. With such a decomposition, one sees that

$$LHS_{(3.16)} \leq \mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\frac{\sqrt{n}W_n^{(\alpha),[k_n,n]}(G)}{D_n^{(\alpha)}} \times \frac{\sqrt{n}H(w_n)}{\mathcal{R}(\alpha + V(w_n))}\right] + \mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\frac{\sqrt{n}W_n^{(\alpha),[0,k_n)}(G)}{D_n^{(\alpha)}} \times \frac{\sqrt{n}H(w_n)}{\mathcal{R}(\alpha + V(w_n))}\right].$$

For the first expectation above, as $H \leq \sqrt{n}\mathbb{1}_{V(w_n)\geq b\sqrt{n}/2}\mathbb{1}_{E_n}$ and $W_n^{(\alpha),[0,k_n)}(G) \leq \sqrt{n}D_n^{(\alpha),[0,k_n)}$,

$$\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\frac{\sqrt{n}W_n^{(\alpha),[k_n,n]}(G)}{D_n^{(\alpha)}} \times \frac{\sqrt{n}H(w_n)}{\mathcal{R}(\alpha + V(w_n))}\right] \leq \mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\frac{nD_n^{(\alpha),[k_n,n]}}{D_n^{(\alpha)}} \times \frac{n}{\mathcal{R}(\alpha + b\sqrt{n}/2)}\mathbf{1}_{E_n}\right]$$
$$\leq \frac{c}{\sqrt{n}},$$

since $E_{n,3} = \{D_n^{(\alpha),[k_n,n]} \leq 1/n^2\}$ and $\mathbf{E}_{\mathbf{Q}^{(\alpha)}}[(D_n^{(\alpha)})^{-1}] = \mathcal{R}(\alpha)^{-1} \leq 1$. Recall also that $\mathcal{R}(u) = \Theta(u+1)$. For the second expectation,

$$\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\frac{\sqrt{n}W_n^{(\alpha),[0,k_n)}(G)}{D_n^{(\alpha)}} \times \frac{\sqrt{n}H(w_n)}{\mathcal{R}(\alpha + V(w_n))}\right] \leq \mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\frac{\sqrt{n}W_n^{(\alpha),[0,k_n)}(G)}{D_n^{(\alpha),[0,k_n)}} \times \frac{\sqrt{n}G(w_n)\mathbb{1}_{E_n}}{\mathcal{R}(\alpha + V(w_n))}\right]$$
$$\leq \mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\frac{\sqrt{n}W_n^{(\alpha),[0,k_n)}(G)}{D_n^{(\alpha),[0,k_n)}} \times \mathbb{1}_{\{V(w_{k_n})\in[k_n^{1/3},k_n]\}}\right] \times \sup_{u\in[k_n^{1/3},k_n]}\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\frac{\sqrt{n}G(w_n)\mathbb{1}_{E_n}}{\mathcal{R}(\alpha + V(w_n))}\middle| V(w_{k_n}) = u\right]$$
$$=: RHS_1 \times \sup_{u\in[k_n^{1/3},k_n]} RHS_2(u)$$

where the second inequality follows from Markov property at k_n. Next we are going to show that

$$\limsup_{n\to\infty} RHS_1 \leq \mathscr{C}_{a,b}, \text{ and} \tag{3.17}$$

$$\limsup_{n\to\infty} \sup_{u\in[k_n^{1/3},k_n]} RHS_2(u) \leq \mathscr{C}_{a,b}. \tag{3.18}$$

For RHS_1, note that by Markov property

$$RHS_1 \times \inf_{u\in[k_n^{1/3},k_n]} \mathbf{Q}^{(\alpha)}(E_n|V(w_{k_n})=u) \le \mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\frac{\sqrt{n}W_n^{(\alpha),[0,k_n)}(G)}{D_n^{(\alpha),[0,k_n)}}\times \mathbb{1}_{E_n}\right].$$

It is proved in Lemma 4.7 of [3] that for sufficiently large n,

$$\inf_{u\in[k_n^{1/3},k_n]} \mathbf{Q}^{(\alpha)}(E_n|V(w_{k_n})=u) = 1+o_n(1), \tag{3.19}$$

therefore,

$$\begin{aligned} RHS_1 \le& (1+o_n(1))\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\frac{\sqrt{n}W_n^{(\alpha),[0,k_n)}(G)}{D_n^{(\alpha),[0,k_n)}}\times \mathbb{1}_{E_n}\right] \\ \le& (1+o_n(1))\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\frac{\sqrt{n}W_n^{(\alpha),[0,k_n)}(G)}{D_n^{(\alpha),[0,k_n)}}\times \mathbb{1}_{E_n}\mathbb{1}_{D_n^{(\alpha)}\ge n^{-3/2}}\right] + 2n\mathbf{Q}^{(\alpha)}\left((D_n^{(\alpha)})^{-1}>n^{3/2}\right), \end{aligned}$$

since $W_n^{(\alpha),[0,k_n)}(G) \le \sqrt{n}D_n^{(\alpha),[0,k_n)}$. Again by Markov inequality with $\mathbf{E}_{\mathbf{Q}^{(\alpha)}}[(D_n^{(\alpha)})^{-1}] = \mathcal{R}(\alpha)^{-1} \le 1$,

$$\begin{aligned} RHS_1 \le& (1+o_n(1))\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\frac{\sqrt{n}W_n^{(\alpha),[0,k_n)}(G)}{D_n^{(\alpha)}}\times \mathbb{1}_{E_n}\mathbb{1}_{D_n^{(\alpha)}\ge n^{-3/2}}\right] + \frac{2}{\sqrt{n}} \\ \le& (1+o_n(1))\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\frac{\sqrt{n}W_n^{(\alpha)}(F)}{D_n^{(\alpha)}}\right] + \frac{2}{\sqrt{n}}. \end{aligned}$$

So (3.17) follows from (3.5).

It remains to prove (3.18). Let $m := n-k_n$ and $m_0 := n_0 - k_n$, for any $u \in [k_n^{1/3}, k_n]$, $RHS_2(u)$ is bounded by

$$\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\frac{n}{\mathcal{R}(\alpha+V(w_m))}\frac{e^{V(w_m)}}{\sum_{0<j\le m}e^{V(w_m)}}\mathbb{1}_{\Upsilon_{w_n}>m_0,\overline{V}(w_m)\ge b\sqrt{n}+b_n,\max_{k\le n}(\overline{V}(w_k)-V(w_k))\le a\sqrt{n}+a_n}\middle| V(w_0)=u\right]$$

which by Markov property and (3.3) is less than

$$\frac{n}{\mathcal{R}(\alpha+u)}\mathbf{E}\left[\frac{e^{S_m}}{\sum_{1\le j\le m}e^{S_m}}; \max_{i\le m}(\overline{S}_i - S_i) \le a\sqrt{n}+a_n, \Upsilon_S > m_0, \overline{S}_m \ge b\sqrt{n}+b_n-u, \underline{S}_m \ge -\alpha-u\right].$$

By considering $v_S = \inf\{j\le m : S_j = \underline{S}_m\}$ the first hitting time of $\underline{S}_m$, we get that

$$\begin{aligned} RHS_2(u) \le \frac{n}{\mathcal{R}(\alpha+u)}\sum_{k=m_0+1}^{m}\sum_{l=0}^{m}\mathbf{E}\Big[&\frac{e^{S_m}}{\sum_{1\le j\le m}e^{S_m}}; \Upsilon_S = k, v_S = l, \\ &\max_{i\le m}(\overline{S}_i - S_i) \le a\sqrt{n}+a_n, \overline{S}_m \ge b\sqrt{n}+b_n, \underline{S}_m \ge -\alpha-u\Big]. \end{aligned}$$

We decompose the above double sum into three parts,

$$\sum_{k=m_0+1}^{m}\sum_{l=0}^{m}\cdots=\sum_{k=m_0+1}^{m}\sum_{l=0}^{k_n^6-1}\cdots+\sum_{k=m_0+1}^{m}\sum_{l=k_n^6}^{k-1}\cdots+\sum_{k=m_0+1}^{m}\sum_{l=k}^{m}\cdots=:\Xi_1(u)+\Xi_2(u)+\Xi_3(u),$$

and deal with them separately. For Ξ_2, observe that

$$\Xi_2(u)\le\frac{n}{\mathcal{R}(\alpha+u)}\sum_{k=m_0+1}^{m}\sum_{l=k_n^6}^{k-1}\mathbf{E}\Big[e^{S_m-S_k};\mathrm{Y}_S=k,v_S=l,\underline{S}_m\ge-\alpha-u\Big],$$

which by Markov property at k then at l is less than

$$\frac{n}{\mathcal{R}(\alpha+u)}\sum_{k=m_0+1}^{m}\sum_{l=k_n^6}^{k-1}\mathbf{P}(S_l=\underline{S}_l\ge-\alpha-u)\mathbf{P}(\underline{S}_{k-l}\ge0,S_{k-l}=\overline{S}_{k-l})\mathbf{E}[e^{S_{m-k}};\overline{S}_{m-k}\le0].$$

By (A.8), (A.4) and (A.14), we obtain that for any $u\in[k_n^{1/3},k_n]$,

$$\Xi_2(u)\le cn(1+\alpha+u)\sum_{k=m_0+1}^{m}\sum_{l=k_n^6}^{k-1}\frac{1}{l^{3/2}(k-l)(m-k+1)^{3/2}}\le\frac{c}{k_n^2}.$$

For the third sum Ξ_3, by Markov property at $\mathrm{Y}=k$ then at $v_S=l$,

$$\begin{aligned}\Xi_3(u)\le&\frac{n}{\mathcal{R}(\alpha+u)}\sum_{k=m_0+1}^{m}\sum_{l=k}^{m}\mathbf{P}\Big[\mathrm{Y}_S=k,v_S=l,\underline{S}_m\ge-\alpha-u,\overline{S}_m\ge b\sqrt{n}+b_n\Big]\\\le&\frac{n}{\mathcal{R}(\alpha+u)}\sum_{k=m_0+1}^{m}\sum_{l=k+1}^{m}\mathbf{P}\Big(\underline{S}_k\ge-\alpha-u,S_k=\overline{S}_k\Big)\mathbf{P}\Big(S_{l-k}\le-b\sqrt{n}-b_n\Big)\mathbf{P}\Big(\underline{S}_{m-l}\ge0\Big),\end{aligned}$$

so by (A.18) and (A.2),

$$\Xi_3(u)\le\frac{n}{\mathcal{R}(\alpha+u)}\sum_{k=m_0+1}^{m}\sum_{l=k}^{m}\frac{c(1+\alpha+u)}{k(m-l+1)^{1/2}}e^{-cn^{1/2}}=o_n(1).$$

From the upper bounds of Ξ_2 and Ξ_3, we deduce that

$$\sup_{u\in[k_n^{1/3},k_n]}RHS_2(u)\le\sup_{u\in[k_n^{1/3},k_n]}\Xi_1(u)+o_n(1),$$

and we now treat the first term Ξ_1. By Markov property at time $v_S=l$,

$$\begin{aligned}\Xi_1(u)\le&\frac{n}{\mathcal{R}(\alpha+u)}\sum_{l=0}^{k_n^6-1}\mathbf{P}(S_l<\underline{S}_{l-1},S_l\ge-\alpha-u)\times\\&\sum_{k=m_0+1}^{m}\mathbf{E}\Big[\frac{e^{S_{m-l}}}{\sum_{1\le j\le m-l}e^{S_j}};\max_{i\le m-l}(\overline{S}_i-S_i)\le a\sqrt{n}+a_n,\mathrm{Y}_S>m_0-l,\overline{S}_{m-l}\ge b\sqrt{m-l}+b_n-k_n,\underline{S}_{m-l}\ge0\Big].\end{aligned}$$

Following the same arguments for $\sum_k\sigma_k$, one obtains that for all $u\in[k_n^{1/3},k_n]$,

$$\Xi_1(u)\le\frac{\sum_{l=0}^{k_n^6-1}\mathbf{P}(\underline{S}_{\ell-1}>S_l\ge-\alpha-u)}{\mathcal{R}(\alpha+u)}(\mathscr{C}_{a,b}+o_n(1))\le\mathscr{C}_{a,b}+o_n(1),$$

which completes the proof of (3.18) and conclude (3.16).

3.3 Proof of Corollary 1.9

In this subsection, we show that as $\beta \to \infty$,

$$\sum_{m=1}^{\infty} \sum_{|z|=m} \frac{1}{\sum_{\phi<y\le z} e^{V(y)}} \mathbb{1}_{\max_{\phi<y\le z(\overline{V}(y)-V(y))\le\beta, \overline{V}(z)\ge\beta\pm O(\log\beta)}} \xrightarrow{\text{in } \mathbf{P}^* \text{ probability}} \Lambda D_\infty. \tag{3.20}$$

Proof. Note that

$$W_m^*(\beta) = \sum_{|z|=m} \frac{1}{\sum_{\phi<y\le z} e^{V(y)}} \mathbb{1}_{\max_{\phi<y\le z(\overline{V}(y)-V(y))\le\beta, \overline{V}(z)\ge\beta}} = W_m(F_{\beta,\beta})/\sqrt{m}.$$

In fact, only those m that are comparable to β^2 really contribute to the sum. First, because of (2.12), we claim that for any $\eta > 0$,

$$\lim_{b\to 0} \limsup_{n\to\infty} \mathbf{P}\left(\sum_{m\le b\beta^2} W_m^*(\beta) \ge \eta \right) = 0 \tag{3.21}$$

$$\lim_{B\to\infty} \limsup_{n\to\infty} \mathbf{P}\left(\sum_{m\ge B\beta^2} W_m^*(\beta) \ge \eta \right) = 0. \tag{3.22}$$

For any $B > b > 0$ fixed, by (1.26), as $\beta \to \infty$,

$$\begin{aligned}\sum_{b\beta^2\le m\le B\beta^2} W_m^*(\beta) =& \sum_{b\beta^2\le m\le B\beta^2} \frac{W_m(F_{\beta,\beta})}{\sqrt{m}} \\ =& \mathbf{c}_0 D_\infty \sum_{b\beta^2\le m\le B\beta^2} \frac{1}{m} \mathscr{C}_{\beta/\sqrt{m},\beta/\sqrt{m}} + o_{\mathbf{P}^*}(1),\end{aligned}$$

where $o_{\mathbf{P}^*}(1)$ denotes a term such that $\lim_{\beta\to\infty} o_{\mathbf{P}^*}(1) = 0$ in $\mathbf{P}^*$-probability. On the other hand, by change of variables $m = \gamma\beta^2$,

$$\int_{b\beta^2}^{B\beta^2} \mathscr{C}_{\beta/\sqrt{m},\beta/\sqrt{m}} \frac{dm}{m} = \int_b^B \mathscr{C}_{\gamma^{-1/2},\gamma^{-1/2}} \frac{d\gamma}{\gamma}.$$

As $\mathscr{C}_{a,b}$ is continuous and monotone, we get that

$$\sum_{b\beta^2\le m\le B\beta^2} \frac{1}{m} \mathscr{C}_{\beta/\sqrt{m},\beta/\sqrt{m}} = \int_b^B \mathscr{C}_{\gamma^{-1/2},\gamma^{-1/2}} \frac{d\gamma}{\gamma} + o_\beta(1).$$

When $B \to \infty$ and $b \to 0$, $\int_b^B \mathscr{C}_{\gamma^{-1/2},\gamma^{-1/2}} \frac{d\gamma}{\gamma} \to \Lambda/\mathbf{c}_0$ because of Lemma A.1. In view of (3.21) and (3.22), we conclude that in $\mathbf{P}^*$-probability,

$$\lim_{\beta\to\infty} \sum_{m=1}^{\infty} W_m^*(\beta) = \Lambda D_\infty.$$

Note that if we replace $W_m(F_{\beta,\beta})$ by $W_m(F_{\beta,(1\pm\varepsilon)\beta})$ with $\beta \in (0,1)$, these arguments still work. By monotonicity of F, we have

$$\sum_{m=1}^{\infty} W_m(F_{\beta,(1-\varepsilon)\beta}) \le \sum_{m=1}^{\infty} \sum_{|z|=m} \frac{1}{\sum_{\phi<y\le z} e^{V(y)}} \mathbb{1}_{\max_{\phi<y\le z}(\overline{V}(y)-V(y))\le\beta, \overline{V}(z)\ge\beta\pm O(\log\beta)} \le \sum_{m=1}^{\infty} W_m(F_{\beta,(1+\varepsilon)\beta}).$$

By integrability and continuity of $\mathscr{C}$, as $\beta\to\infty$,

$$\sum_{m=1}^{\infty} W_m(F_{\beta,(1-\varepsilon)\beta}) = (\Lambda + o_\varepsilon(1))D_\infty + o_{\mathbf{P}^*}(1).$$

Consequently, the convergence (3.20) holds. □

4 Mean, variance of K_n and secondary results

In this section, we complete the proof of the main theorems by proving Lemmata 2.1, 2.2, 2.4 and Proposition 2.3.

Recall that the quenched expectation of K_n is

$$\mathcal{K}_n(\ell) = \mathbb{E}^{\mathcal{E}}[K_n(\ell)] = \sum_{|z|=\ell} \mathbb{P}^{\mathcal{E}}(T_z < T_\phi^n) = \sum_{|z|=\ell} (1-(1-a_z)^n) \le \sum_{|z|=\ell} (na_z)\wedge 1,$$

where $a_z = \frac{p(\phi,\overleftarrow{\phi})}{\sum_{\phi<x\le z} e^{V(x)}} \le e^{-\overline{V}(z)}$. Obviously, for any measurable event A with respect to the environment $\mathcal{E}$,

$$\mathcal{K}_n^A(\ell) \le \sum_{|z|=\ell} (na_z\wedge 1)\mathbb{1}_{z\in A} \le \sum_{|z|=\ell} (ne^{-\overline{V}(z)}\wedge 1)\mathbb{1}_{z\in A}. \tag{4.1}$$

Also observe that for any $z\in B_3 = \{z\in\mathbb{T} : \overline{V}(z)\ge \log n + \log\log n\}$, $1-(1-a_z)^n \ge na_z(1-1/\log n)$. So,

$$(1-\frac{1}{\log n}) \sum_{|z|=\ell} na_z\mathbb{1}_{z\in A\cap B_3} \le \mathcal{K}_n^A(\ell) \le \sum_{|z|=\ell} na_z\mathbb{1}_{z\in A\cap B_3} + \sum_{|z|=\ell} (na_z\wedge 1)\mathbb{1}_{z\in A\cap B_3^c}. \tag{4.2}$$

4.1 Annealed expectation of $K_n(\ell)$

We first study the annealed expectation of K_n which turns out to be of order $n/\log n$.

Lemma 4.1. *For $\ell\sim\gamma(\log n)^2$, there exists a constant $\kappa(\gamma)>0$ such that*

$$\lim_{n\to\infty} \frac{\log n}{n} \mathbb{E}[K_n(\ell)] = \kappa(\gamma). \tag{4.3}$$

In fact, applying (4.2) to $\mathcal{K}_n(\ell) = \mathbb{E}^{\mathcal{E}}[K_n(\ell)]$ gives that

$$\left(1-\frac{1}{\log n}\right) \sum_{|z|=\ell} na_z\mathbb{1}_{z\in B_3} \le \mathcal{K}_n(\ell) \le \sum_{|z|=\ell} na_z\mathbb{1}_{z\in B_3} + \sum_{|z|=\ell} (na_z\wedge 1)\mathbb{1}_{z\in B_3^c}.$$

As a consequence, Lemma 4.1 follows from the following lemma.

Lemma 4.2.

$$\mathbf{E}\Big[\sum_{|z|=\ell}(na_z\wedge 1)\mathbb{1}_{z\in B_3^c}\Big]=o(\frac{n}{\log n}). \tag{4.4}$$

$$\lim_{n\to\infty}\frac{\log n}{n}\mathbf{E}\Big[\sum_{|z|=\ell}na_z\mathbb{1}_{z\in B_3}\Big]=\kappa(\gamma). \tag{4.5}$$

Proof. We first prove the convergence in (4.5). Recall that $\{z\in B_3\}=\{\overline{V}(z)\geq \log n+\log\log n\}$.

- **Proof of** (4.5) Recall that $\mathrm{Y}_z=\inf\{k\leq \ell : V(z_k)=\overline{V}(z)\}$. With the same arguments used to show (3.11) for h_1, we can say that for $l_0=\lfloor \ell-\ell^{1/3}\rfloor$,

$$\mathbf{E}\Big[\sum_{|z|=\ell}na_z\mathbb{1}_{\overline{V}(z)\geq \log n+\log\log n,\mathrm{Y}_z\leq l_0}\Big]=o(\frac{n}{\log n}). \tag{4.6}$$

We deduce from (4.6) and the expression of a_z that

$$\begin{aligned}\mathbf{E}\Big(\sum_{|z|=\ell}na_z1_{\{\overline{V}(z)\geq \log n+\log\log n\}}\Big)=&\mathbf{E}\Big(\sum_{|z|=\ell}n\frac{p^{\mathcal{E}}(\phi,\overleftarrow{\phi})}{\sum_{k=1}^{|z|}e^{V(z_k)}}1_{\{\mathrm{Y}_z>l_0,\ \overline{V}(z)\geq \log n+\log\log n\}}\Big)+o(\frac{n}{\log n})\\ =&n\mathbf{E}\Big(\sum_{|u|=1}\frac{1}{1+\sum_{|v|=1}e^{-V(v)}}\kappa(V(u),\ell)\Big)+o(\frac{n}{\log n}),\end{aligned} \tag{4.7}$$

where

$$\kappa(x,\ell):=\mathbf{E}_x\Big(\sum_{|z|=\ell-1}\frac{1}{\sum_{k=0}^{\ell-1}e^{V(z_k)}}1_{\{\mathrm{Y}_z>l_0,\overline{V}(z)\geq \log n+\log\log n\}}\Big).$$

and recall that $\mathbf{E}_x$ means that $V(\phi)=x$ a.s. By (1.15),

$$\kappa(x,\ell)=e^{-x}\mathbf{E}\left(\frac{e^{S_{\ell-1}}}{\sum_{0\leq k\leq \ell-1}e^{S_k}};\mathrm{Y}_S>l_0,\overline{S}_{\ell-1}\geq \log n+\log\log n-x\right), \tag{4.8}$$

and recall $\mathrm{Y}_S=\inf\{k\leq \ell : S_k=\overline{S}_\ell\}$. By Markov property at time $\mathrm{Y}_S=j$,

$$\begin{aligned}e^x\kappa(x,\ell)=&\sum_{j=l_0+1}^{\ell-1}\mathbf{E}\Big[\mathcal{G}_{\ell-1-j}(\sum_{0\leq k\leq j}e^{S_k-S_j});\overline{S}_{j-1}<S_j,\overline{S}_j\geq \log n+\log\log n-x\Big]\\ =&\sum_{j=l_0+1}^{\ell-1}\mathbf{E}\Big[\mathcal{G}_{\ell-1-j}(\sum_{0\leq k\leq j}e^{-S_k});\underline{S}_j>0,S_j\geq \log n+\log\log n-x\Big]\end{aligned}$$

where the second equality follows from time-reversing, and $\mathcal{G}_\cdot$ is defined in (1.20). By the joint convergence (1.17), we get the following estimation similar to that of σ'_k defined just above (3.12). Uniformly on $j\in[l_0,\ell-1]$,

$$\mathbf{E}\Big[\mathcal{G}_{\ell-1-j}(\sum_{0\leq k\leq j}e^{-S_k});\underline{S}_j>0,S_j\geq \log n+\log\log n-x\Big]=\frac{\mathbf{E}[\mathcal{G}_{\ell-1-j}(\mathcal{H}_\infty)]\mathbf{P}(\mathfrak{m}_1\geq\frac{1}{\sqrt{\gamma}})}{\sqrt{\ell}}+\frac{g_{\ell-1-j}o_\ell(1)}{\sqrt{\ell}}$$

where for any $m \geq 0$, $g_m := (m+1)^{-3/2}$.

Therefore, for any fixed x,

$$e^x \kappa(x,\ell) = \frac{\mathbf{P}(\mathfrak{m}_1 \geq \frac{1}{\sqrt{\gamma}})}{\sqrt{\ell}} \sum_{j=l_0+1}^{\ell-1} \mathbf{E}[\mathcal{G}_{\ell-1-j}(\mathcal{H}_\infty)] + \sum_{j=l_0+1}^{\ell-1} g_{\ell-1-j} \frac{o_\ell(1)}{\sqrt{\ell}}$$
$$= \frac{\mathbf{P}(\mathfrak{m}_1 \geq \frac{1}{\sqrt{\gamma}})}{\sqrt{\ell}} \sum_{j\geq 0} \mathbf{E}[\mathcal{G}_j(\mathcal{H}_\infty)] + \frac{o_\ell(1)}{\sqrt{\ell}},$$

Moving back to (4.7), we deduce that

$$\mathbf{E}\Big(\sum_{|z|=l} n a_z 1_{\{\overline{V}(z) \geq \log n + \log\log n\}} \Big) = \frac{n\mathbf{P}(\mathfrak{m}_1 \geq \frac{1}{\sqrt{\gamma}})}{\sqrt{\ell}} \sum_{j\geq 0} \mathbf{E}[\mathcal{G}_j(\mathcal{H}_\infty)] \mathbf{E}\Big(\frac{\sum_{|u|=1} e^{-V(u)}}{1+\sum_{|u|=1} e^{-V(u)}} \Big) + o(\frac{n}{\log n}).$$

As $\ell \sim \gamma(\log n)^2$, let $\kappa(\gamma) := \frac{\mathbf{P}(\mathfrak{m}_1 \geq \frac{1}{\sqrt{\gamma}})}{\sqrt{\gamma}} \sum_{j\geq 0} \mathbf{E}[\mathcal{G}_j(\mathcal{H}_\infty)] \mathbf{E}\Big(\frac{\sum_{|u|=1} e^{-V(u)}}{1+\sum_{|u|=1} e^{-V(u)}} \Big)$, we then obtain (4.5).

- **Proof of** (4.4) Observe that

$$\sum_{|z|=\ell} (n a_z \wedge 1) \mathbb{1}_{z\in B_3^c} \leq \sum_{|z|=\ell} n a_z \mathbb{1}_{\log n \leq \overline{V}(z) \leq \log n + \log\log n} + \sum_{|z|=\ell} \mathbb{1}_{\overline{V}(z) \leq \log n}.$$

On the one hand, by (1.15) ,

$$\mathbf{E}\Big[\sum_{|z|=\ell} \mathbb{1}_{\overline{V}(z) < \log n} \Big] = \mathbf{E}\Big[e^{S_\ell}; \overline{S}_\ell \leq \log n \Big]$$

which by (A.15) is bounded by $c\frac{e^{\log n}}{\ell} = o(\frac{n}{\log n})$. On the other hand, as $a_z \leq e^{-\overline{V}(z)}$,

$$\mathbf{E}\Big[\sum_{|z|=\ell} n a_z \mathbb{1}_{\log n \leq \overline{V}(z) \leq \log n + \log\log n} \Big] \leq n\mathbf{E}\Big[\sum_{|z|=\ell} e^{-V(z)} e^{V(z)-\overline{V}(z)} \mathbb{1}_{\log n \leq \overline{V}(z) \leq \log n + \log\log n} \Big],$$

which by (1.15) is equal to

$$n\mathbf{E}\Big[e^{S_\ell - \overline{S}_\ell}, \log n \leq \overline{S}_\ell \leq \log n + \log\log n \Big].$$

By applying Markov property at the first hitting time $\overline{S}_\ell$, one sees that

$$\mathbf{E}\Big[\sum_{|z|=\ell} n a_z \mathbb{1}_{\log n \leq \overline{V}(z) \leq \log n + \log\log n} \Big] \leq n \sum_{j=1}^{\ell} \mathbf{E}\Big[e^{S_\ell - S_j}; \overline{S}_{j-1} < S_j, S_j = \overline{S}_\ell \in [\log n, \log n + \log\log n] \Big]$$
$$= n \sum_{j=1}^{\ell} \mathbf{P}\Big(\overline{S}_{j-1} < S_j, S_j \in [\log n, \log n + \log\log n] \Big) \mathbf{E}\Big[e^{S_{\ell-j}}; \overline{S}_{\ell-j} \leq 0 \Big]$$

which by time-reversing equals $n\sum_{j=1}^{\ell} \mathbf{P}\Big(\underline{S}_j > 0, S_j \in [\log n, \log n + \log\log n] \Big) \mathbf{E}\Big[e^{S_{\ell-j}}; \overline{S}_{\ell-j} \leq 0 \Big]$. By (A.3) and (A.14), we obtain that

$$\mathbf{E}\Big[\sum_{|z|=\ell} n a_z \mathbb{1}_{\log n \leq \overline{V}(z) \leq \log n + \log\log n} \Big] \leq n \sum_{j=1}^{\ell} c \frac{(\log n)(\log\log n)}{j^{3/2}(\ell - j + 1)^{3/2}} \leq cn \frac{\log\log n}{(\log n)^2}, \tag{4.9}$$

since $\ell \sim \gamma(\log n)^2$. This completes the proof.

□

4.2 Quenched variance of $K_n^{C^\delta}(\ell)$ and Proof of Proposition 2.3

Recall the definition of C^δ in Section 2.1, in this section we focus on the mean of the quenched variance of $K_n^{C^\delta}(\ell)$ which is a key step in the proof of Proposition 2.3.

4.2.1 Quenched expression for the variance

Lemma 4.3. *Recall that $a_z = \mathbb{P}^{\mathcal{E}}(T_z < T_\phi)$ and let $a_{v,z} := \mathbb{P}^{\mathcal{E}}(T_v \wedge T_z < T_\phi)$. For every event A measurable with respect to $\mathcal{E}$, denote the quenched variance of $K_n^A(\ell)$ as follows:*

$$\mathbb{V}^{\mathcal{E}}ar(K_n^A(\ell)) := \mathbb{E}^{\mathcal{E}}\left[\left(K_n^A(\ell) - \mathcal{K}_n^A(\ell)\right)^2\right],$$

then

$$\mathbb{V}^{\mathcal{E}}ar(K_n^A(\ell)) = \sum_{|z|=\ell,|v|=\ell,z\neq v} \left[(1-a_{v,z})^n - (1-a_z)^n(1-a_v)^n\right] \mathbb{1}_{z\in A}\mathbb{1}_{v\in A} \tag{4.10}$$

$$+ \sum_{|z|=\ell} \left[(1-a_z)^n - (1-a_z)^{2n}\right] \mathbb{1}_{z\in A}. \tag{4.11}$$

Proof. Note that

$$\begin{aligned} K_n^A(\ell) - \mathcal{K}_n^A(\ell) &= \sum_{|z|=\ell} \left(\mathbb{1}_{T_z<T_\phi^n} - (1-(1-a_z)^n)\right) \mathbb{1}_{z\in A} \\ &= \sum_{|z|=\ell} \left((1-a_z)^n - \mathbb{1}_{T_z\geq T_\phi^n}\right) \mathbb{1}_{z\in A}. \end{aligned}$$

So the lemma comes directly. □

A corollary of this Lemma is the following result, which gives a simple upper bound of the quenched variance when $A = C^\delta$:

Lemma 4.4. *Recall the definition of C^δ in Section 2.1, we have :*

$$\begin{aligned} \mathbb{V}^{\mathcal{E}}ar\left(K_n^{C^\delta}(\ell)\right) \leq &\sum_{|z|=|v|=\ell} na_z \mathbb{P}^{\mathcal{E}}_{v\wedge z}(T_v < T_\phi)\mathbb{1}_{\{z\in C^\delta, v\in C^\delta\}} + \sum_{|z|=|v|=\ell} na_v \mathbb{P}^{\mathcal{E}}_{v\wedge z}(T_z < T_\phi)\mathbb{1}_{\{z\in C^\delta, v\in C^\delta\}} \\ &+ \sum_{|z|=\ell} na_z \mathbb{1}_{\{z\in C^\delta\}}, \end{aligned} \tag{4.12}$$

where $v\wedge z$ is the latest common ancestor of v and z in the tree $\mathbb{T}$, and $\mathbb{P}_y^{\mathcal{E}}$ is the quenched probability of the random walk started from y.

Proof. This upper bound is actually true for every truncated version of $K_n(\ell)$, however it is optimized here for events included in B_3 so in particular for C^δ. For $a_{v,z}$ one sees that

$$a_{v,z} = \mathbb{P}^{\mathcal{E}}(T_v < T_z \wedge T_\phi) + \mathbb{P}^{\mathcal{E}}(T_z < T_v \wedge T_\phi) =: d_{v,z} + d_{z,v}.$$

We have,

$$(1-a_{v,z})^n - (1-a_z)^n(1-a_v)^n \leq (1-d_{v,z}-d_{z,v})^n - (1-a_z-a_v)^n \leq n(a_z - d_{z,v} + a_v - d_{v,z}).$$

Observe that

$$\begin{aligned} a_z - d_{z,v} + a_v - d_{v,z} =& \mathbb{P}^{\mathcal{E}}(T_v \vee T_z < T_\phi) \leq \mathbb{P}^{\mathcal{E}}(T_z < T_\phi)\mathbb{P}^{\mathcal{E}}_{z\wedge v}(T_v < T_\phi) + \mathbb{P}^{\mathcal{E}}(T_v < T_\phi)\mathbb{P}^{\mathcal{E}}_{z\wedge v}(T_z < T_\phi) \\ =& a_z \mathbb{P}^{\mathcal{E}}_{z\wedge v}(T_v < T_\phi) + a_v \mathbb{P}^{\mathcal{E}}_{z\wedge v}(T_z < T_\phi). \end{aligned}$$

This together with Lemma 4.3 yields that

$$\sum_{|z|=\ell,|v|=\ell,z\neq v} [(1-a_{v,z})^n - (1-a_z)^n(1-a_v)^n]\, \mathbb{1}_{z\in C^\delta}\mathbb{1}_{v\in C^\delta} \leq$$
$$\sum_{|z|=|v|=\ell} n a_z \mathbb{P}^{\mathcal{E}}_{v\wedge z}(T_v < T_\phi)\mathbb{1}_{\{z\in C^\delta, v\in C^\delta\}} + \sum_{|z|=|v|=\ell} n a_v \mathbb{P}^{\mathcal{E}}_{v\wedge z}(T_z < T_\phi)\mathbb{1}_{\{z\in C^\delta, v\in C^\delta\}}$$

Moreover, we have $(1-a_z)^n - (1-a_z)^{2n} \leq n a_z$. This leads to (4.12). □

4.2.2 Upper bound for the mean of the quenched variance

In this section we obtain an upper bound of the mean $\mathbf{E}\left(\mathbb{V}\mathcal{E}ar\left(K_n^{C^\delta}(\ell)\right)\right)$.

Lemma 4.5. *For $\ell \sim \gamma(\log n)^2$, every $\delta > 0$ and n large enough,*

$$\mathbf{E}\left(\mathbb{V}\mathcal{E}ar\left(K_n^{C^\delta}(\ell)\right)\right) \leq c\frac{n^2}{(\log n)^\delta}.$$

Proof. Because of (4.12), we only have to bound the mean of

$$t_n := \sum_{\substack{|z|=|v|=\ell \\ z\neq v}} a_z \mathbb{P}^{\mathcal{E}}_{v\wedge z}(T_v < T_\phi)\mathbb{1}_{\{z\in C^\delta, v\in C^\delta\}},$$

since the second term on the RHS of (4.12) is its symmetric and the last term has been considered before (see Lemma 4.1).

The main idea of this proof, is to decompose the double sum $\sum_{|z|=|v|=\ell}$ according to the latest common ancestor $z\wedge v$.

Define $\Sigma_1 := \sum_{\phi<s\leq v\wedge z} e^{V(s)-V(z\wedge v)}$, $\Sigma_2 := \sum_{v\wedge z<s\leq z} e^{V(s)-V(z\wedge v)}$ and $\Sigma_3 := \sum_{v\wedge z<s\leq v} e^{V(s)-V(z\wedge v)}$. We then have

$$a_z = \frac{p^{\mathcal{E}}(\phi, \overleftarrow{\phi})e^{-V(v\wedge z)}}{\Sigma_1+\Sigma_2} \text{ and } \mathbb{P}^{\mathcal{E}}_{z\wedge v}(T_v < T_\phi) = \frac{\Sigma_1}{\Sigma_1+\Sigma_3}.$$

By comparing $\Sigma_1,\ \Sigma_2,\ \Sigma_3$, we get

$$t_n \le t_n^1 + t_n^2 + t_n^3 + t_n^4,$$

with

$$t_n^1 := \sum_{\substack{|z|=|v|=\ell \\ z\neq v}} \frac{e^{-V(z\wedge v)}}{\Sigma_1} \mathbb{1}_{\{z\in C^\delta, v\in C^\delta, \Sigma_1\ge \Sigma_2\vee\Sigma_3\}},$$

$$t_n^2 := \sum_{\substack{|z|=|v|=\ell \\ z\neq v}} \frac{e^{-V(z\wedge v)}}{\Sigma_3} 1_{\{z\in C^\delta, v\in C^\delta, \Sigma_2\le\Sigma_1\le\Sigma_3\}}, t_n^3 := \sum_{\substack{|z|=|v|=\ell \\ z\neq v}} \frac{e^{-V(z\wedge v)}}{\Sigma_2} 1_{\{z\in C^\delta, v\in C^\delta, \Sigma_3\le\Sigma_1\le\Sigma_2\}},$$

$$t_n^4 := \sum_{\substack{|z|=|v|=\ell \\ z\neq v}} e^{-V(z\wedge v)} \frac{\Sigma_1}{\Sigma_2 * \Sigma_3} \mathbb{1}_{\{z\in C^\delta, v\in C^\delta, \Sigma_1\le\Sigma_2\wedge\Sigma_3\}}.$$

We treat each term separately. Notice that by symmetry $\mathbf{E}(t_n^2) = \mathbf{E}(t_n^3)$, so we only estimate $\mathbf{E}(t_n^1)$, $\mathbf{E}(t_n^2)$ and $\mathbf{E}(t_n^4)$.

Recall that for every $z \in C^\delta$, $\overline{V}(z) \ge \log n + \log\log n$ and $a_z \le (n\log n)^{-1}$. In particular, if $\{\sum_i \ge \sum_j\}$ with $i \neq j$, $\{\sum_i + \sum_j \ge n\log n e^{-V(z\wedge v)}\} \subset \{\sum_i \ge n\log n e^{-V(z\wedge v)}/2\}$. We start with $\mathbf{E}(t_n^1)$:

$*$ *Upper bound for* $\mathbf{E}(t_n^1)$, as Σ_1 is the largest term here using the above remark we have $\{B_3, \Sigma_1 \ge \Sigma_2\vee\Sigma_3\} \subset \{\overline{V}(z\wedge v) > \log n + \log\log n - \log 2\ell =: m_n\}$, also as $z \in B_2^\delta$, $\Sigma_1 \le s_n = n/(\log n)^{1+\delta}$, so

$$\begin{aligned} t_n^1 &\le \sum_{\substack{|z|=|v|=\ell \\ z\neq v}} \frac{e^{-V(z\wedge v)}}{\Sigma_1} \mathbb{1}_{\overline{V}(z\wedge v)>m_n, \Sigma_1\le s_n, \underline{V}(z)\wedge\underline{V}(v)\ge -\alpha} \\ &\le \sum_{j=0}^{\ell-1} \sum_{|u|=j} \frac{e^{-V(u)}}{\Sigma_1^u} \mathbb{1}_{\overline{V}(u)>m_n, \Sigma_1^u\le s_n, \underline{V}(u)\ge -\alpha} \sum_{\substack{\overleftarrow{x}=u=\overleftarrow{y} \\ x\neq y}} \sum_{|z|=\ell, z\ge x} \mathbb{1}_{\sum_2^{x,z} e^{V(x)-V(u)} \le \Sigma_1^u} \sum_{|v|=\ell, v\ge y} \mathbb{1}_{\sum_2^{y,v} e^{V(y)-V(u)}\le\Sigma_1^u}, \end{aligned}$$

where $\Sigma_1^u := \sum_{\phi<s\le u} e^{V(s)-V(u)}$ and $\Sigma_2^{x,z} := \sum_{x\le s\le z} e^{V(s)-V(x)}$.

Applying Markov property at time $|u|+1$ and then (1.15) yields that

$$\mathbf{E}[t_n^1] \le \sum_{j=0}^{\ell-1} \mathbf{E}\left[\sum_{|u|=j} \frac{e^{-V(u)}}{\Sigma_1^u} \mathbb{1}_{\overline{V}(u)>m_n, \Sigma_1^u\le s_n, \underline{V}(u)\ge -\alpha} \sum_{\substack{\overleftarrow{x}=u=\overleftarrow{y} \\ x\neq y}} f_{j,\ell}(\Sigma_1^u e^{V(u)-V(x)}) f_{j,\ell}(\Sigma_1^u e^{V(u)-V(y)}) \right],$$

where

$$f_{j,\ell}(t) := \mathbf{E}\left[e^{S_{\ell-1-j}}; \sum_{i=0}^{\ell-1-j} e^{S_i} \le t \right].$$

By (A.15), $f_{j,\ell}(t) \le \mathbf{E}(e^{S_{\ell-1-j}}; \overline{S}_{\ell-1-j} \le \log_+ t) \le c(\log_+ t + 1)t/(\ell-j)^{3/2}$. Plugging this into the

previous inequality yields

$$\mathbf{E}[t_n^1] \le \sum_{j=0}^{\ell-1} \mathbf{E}\left[\sum_{|u|=j} e^{-V(u)}\Sigma_1^u(1+\log\Sigma_1^u)^2 \mathbb{1}_{\overline{V}(u)>m_n,\Sigma_1^u\le s_n,\underline{V}(u)\ge-\alpha}\right.$$

$$\left. \times \frac{c}{(\ell-j)^3} \sum_{\substack{\overleftarrow{x}=u=\overleftarrow{y}\\ x\neq y}} [1+(V(u)-V(x))_+]e^{V(x)-V(u)}[1+(V(u)-V(y))_+]e^{V(y)-V(u)}\right]$$

$$\le \sum_{j=0}^{\ell-1} \frac{c}{(\ell-j)^3}\mathbf{E}\left[\sum_{|u|=j} e^{-V(u)}\Sigma_1^u(1+\log\Sigma_1^u)^2\mathbb{1}_{\overline{V}(u)>m_n,\Sigma_1^u\le s_n,\underline{V}(u)\ge-\alpha}\right]\mathbf{E}\left[\Big(\sum_{|x|=1}[1+(-V(x))_+]e^{-V(x)}\Big)^2\right].$$

By Many-to-one Lemma and hypothesis (1.3), we get

$$\mathbf{E}(t_n^1) \le \sum_{j=0}^{\ell-1} \frac{c}{(\ell-j)^3} s_n(1+\log_+ s_n)^2 \mathbf{P}\left[\overline{S}_j > m_n,\ \Sigma_1^S \le s_n,\ \underline{S}_j \ge -\alpha\right],$$

with $\Sigma_1^S := \sum_{i=1}^j e^{S_i-S_j}$. Also by (A.2), $\mathbf{P}\left[\underline{S}_j \ge -\alpha\right] \le c(1+\alpha)j^{-1/2}$, so

$$\mathbf{E}(t_n^1) \le \sum_{j=0}^{\ell-1} \frac{c}{(\ell-j)^3} s_n(1+\log_+ s_n)^2 \frac{(1+\alpha)}{(j+1)^{1/2}} \le \frac{c(1+\alpha)n}{(\log n)^\delta}.$$

$*$ *Upper bound for* $\mathbf{E}(t_n^2)$, with the same ideas than for the upper bound of t_n^1, we have

$$t_n^2 \le \sum_{\substack{|z|=|v|=\ell\\ z\neq v}} \frac{e^{-V(z\wedge v)}}{\Sigma_3} 1_{\overline{V}(z\wedge v)>m_n,\Sigma_1\le s_n,\Sigma_2\le\Sigma_1,\underline{V}(z\wedge v)\ge-\alpha}$$

$$\le \sum_{j=0}^{\ell-1}\sum_{|u|=j} e^{-V(u)}\mathbb{1}_{\overline{V}(u)\ge m_n,\Sigma_1^u\le s_n,\underline{V}(u)\ge-\alpha} \sum_{\substack{\overleftarrow{x}=u=\overleftarrow{y}\\ x\neq y}} \sum_{|z|=\ell,z\ge x} \mathbb{1}_{\Sigma_2^{x,z}e^{V(x)-V(u)}\le s_n} \sum_{|v|=\ell,v\ge y} \frac{1}{\Sigma_2^{y,v}e^{V(y)-V(u)}},$$

By Markov property then by (1.15), it follows that

$$\mathbf{E}(t_n^2) \le \sum_{j=0}^{\ell-1} \mathbf{E}\left[\sum_{|u|=j} e^{-V(u)}\mathbb{1}_{\underline{V}(u)\ge-\alpha} \sum_{\substack{\overleftarrow{x}=u=\overleftarrow{y}\\ x\neq y}} e^{V(u)-V(y)}\mathbf{E}\left[\frac{e^{S_{\ell-j-1}}}{\sum_{i=0}^{\ell-j-1}e^{S_i}}\right] f_{j,\ell}\left(s_n e^{V(u)-V(x)}\right)\right],$$

which by (A.15) and (A.16), is less than

$$\sum_{j=0}^{\ell-1} \frac{cs_n(1+\log_+ s_n)}{(\ell-j)^2}\mathbf{E}\left[\sum_{|u|=j} e^{-V(u)}\mathbb{1}_{\underline{V}(u)\ge-\alpha} \sum_{\substack{\overleftarrow{x}=u=\overleftarrow{y}\\ x\neq y}} e^{V(u)-V(y)}[1+(V(u)-V(x))_+]e^{V(u)-V(x)}\right]$$

$$\le \sum_{j=0}^{\ell-1} \frac{cs_n(1+\log_+ s_n)}{(\ell-j)^2}\mathbf{E}\left[\sum_{|u|=j} e^{-V(u)}\mathbb{1}_{\underline{V}(u)\ge-\alpha}\right]\mathbf{E}\left[\Big(\sum_{|x|=1}[1+(-V(x))_+]e^{-V(x)}\Big)^2\right].$$

Applying again (1.15) and (1.3), we have,

$$\mathbf{E}(t_n^2) \le \sum_{j=0}^{\ell-1} \frac{cs_n(1+\log_+ s_n)}{(\ell-j)^2} \mathbf{P}\left(\underline{S}_j \ge -\alpha\right).$$

By (A.2),

$$\mathbf{E}(t_n^2) \le \sum_{j=0}^{\ell-1} \frac{c(1+\alpha)s_n(1+\log_+ s_n)}{(\ell-j)^2(j+1)^{1/2}} \le \frac{c(1+\alpha)n}{(\log n)^{1+\delta}}.$$

$*$ *Upper bound for* $\mathbf{E}(t_n^4)$, we have :

$$\begin{aligned} t_n^4 &\le \sum_{\substack{|z|=|v|=\ell \\ z\neq v}} e^{-V(z\wedge v)} \frac{\sum_1}{\sum_2 * \sum_3} \mathbb{1}_{\{\underline{V}(z\wedge v)\ge -\alpha, \sum_1 \le s_n\}} \\ &\le \sum_{j=0}^{\ell-1} \sum_{|u|=j} e^{-V(u)} s_n \mathbb{1}_{\underline{V}(u)\ge -\alpha} \sum_{\substack{\overleftarrow{x}=u=\overleftarrow{y} \\ x\neq y}} \sum_{|z|=\ell, z\ge x} \frac{1}{\Sigma_2^{x,z} e^{V(x)-V(u)}} \sum_{|v|=\ell, v\ge y} \frac{1}{\Sigma_2^{y,v} e^{V(y)-V(u)}}. \end{aligned}$$

With the same arguments as above, one sees that

$$\mathbf{E}(t_n^4) \le \sum_{j=0}^{\ell-1} \mathbf{E}\left[\sum_{|u|=j} e^{-V(u)} s_n \mathbb{1}_{\underline{V}(u)\ge -\alpha} \sum_{\substack{\overleftarrow{x}=u=\overleftarrow{y} \\ x\neq y}} e^{2V(u)-V(x)-V(y)} \mathbf{E}\left[\frac{e^{S_{\ell-j-1}}}{\sum_{i=0}^{\ell-j-1} e^{S_i}} \right]^2 \right],$$

which by (A.16) and (1.3) is less than

$$\sum_{j=0}^{\ell-1} \frac{cs_n}{(\ell-j)} \mathbf{E}\left[\sum_{|u|=j} e^{-V(u)} \mathbb{1}_{\underline{V}(u)\ge -\alpha} \right].$$

Once again by (1.15) and (A.2), we end up with

$$\mathbf{E}(t_n^4) \le \sum_{j=0}^{\ell-1} \frac{cs_n}{(\ell-j)} \mathbf{P}(\underline{S}_j \ge -\alpha) \le \sum_{j=0}^{\ell-1} \frac{cs_n(1+\alpha)}{(\ell-j)(j+1)^{1/2}} \le \frac{c(1+\alpha)n(\log\log n)}{(\log n)^{2+\delta}}.$$

Consequently, we have

$$t_n \le c\frac{n}{(\log n)^{\delta}},$$

which concludes the proof. □

4.3 Complementary arguments: Proofs of Lemmas 2.1, 2.2, 2.4 and Proposition 2.3

4.3.1 Proof of Lemma 2.1

The computations are essentially the same as in the proof of (4.4). We obtain from (4.2) that

$$\begin{aligned} \mathbb{E}[K_n^{B\setminus B_3}(\ell)] &\le \mathbf{E}\Big(\sum_{|z|=\ell} (na_z \wedge 1) \mathbb{1}_{\overline{V}(z)\le \log n+\log\log n, \underline{V}(z)\ge -\alpha} \Big) \\ &\le \mathbf{E}\Big(\sum_{|z|=\ell} na_z \mathbb{1}_{\log n-3\log\log n\le \overline{V}(z)\le \log n+\log\log n, \underline{V}(z)\ge -\alpha} \Big) + \mathbf{E}\Big(\sum_{|z|=\ell} \mathbb{1}_{\overline{V}(z)\le \log n-3\log\log n} \Big). \end{aligned}$$

It follows immediately from (1.15) that

$$\mathbf{E}\Big(\sum_{|z|=\ell}\mathbb{1}_{\overline{V}(z)\le \log n-3\log\log n}\Big)=\mathbf{E}\left(e^{S_\ell}\mathbb{1}_{\overline{S}_\ell\le \log n-3\log\log n}\right)\le e^{\log n-3\log\log n}=\frac{n}{(\log n)^3}=o(\frac{n}{(\log n)^2}).$$

For the other term, similarly to the proof of (4.9), by applying (A.5) instead of (A.3), one obtains that

$$\mathbf{E}\Big(\sum_{|z|=\ell}na_z\mathbb{1}_{\log n-3\log\log n\le\overline{V}(z)\le\log n+\log\log n,\underline{V}(z)\ge-\alpha}\Big)\le c\frac{n\log\log n}{(\log n)^3}=o(\frac{n}{(\log n)^2}),$$

which completes the proof.

4.3.2 proof of Lemma 2.2

The quenched mean of $K_n^C(\ell)-K_n^{C^\delta}(\ell)$ satisfies that

$$0\le\mathbb{E}^{\mathcal{E}}[K_n^C(\ell)-K_n^{C^\delta}(\ell)]=\mathbb{E}^{\mathcal{E}}\left[K_n^{C\setminus C^\delta}(\ell)\right]=\mathcal{K}_n^{C\setminus C^\delta}(\ell)\le\widetilde{\mathcal{K}}_n^{C\setminus C^\delta}(\ell)=\sum_{|z|=\ell}na_z\mathbb{1}_{z\in C\setminus C^\delta}.$$

Similarly to (2.7), we have

$$\begin{aligned}\widetilde{\mathcal{K}}_n^{C\setminus C^\delta}(\ell)&\le\sum_{|z|=\ell}\frac{n}{\sum_{\phi<y\le z}e^{V(y)}}\mathbb{1}_{\log s_n-\log\ell<\max_{\phi<y\le z}(\overline{V}(y)-V(y))\le\log n,\overline{V}(z)\ge\log n+\log\log n,\underline{V}(z)\ge-\alpha}\\&=\frac{n}{\sqrt{\ell}}\Big(W_\ell^{(\alpha)}(F_{\log n,\log n+\log\log n})-W_\ell^{(\alpha)}(F_{\log s_n-\log\ell,\log n+\log\log n})\Big),\end{aligned}$$

with recall $s_n=n/(\log n)^{1+\delta}$. Taking expectation and using change of measures (3.1) yields that

$$\begin{aligned}\mathbf{E}\left[\widetilde{\mathcal{K}}_n^{C\setminus C^\delta}(\ell)\right]&\le\frac{n}{\sqrt{\ell}}\left(\mathbf{E}\left[W_\ell^{(\alpha)}(F_{\log n,\log n+\log\log n})\right]-\mathbf{E}\left[W_\ell^{(\alpha)}(F_{\log s_n-\log\ell,\log n+\log\log n})\right]\right)\\&=\frac{n}{\sqrt{\ell}}\mathcal{R}(\alpha)\left(\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\frac{W_\ell^{(\alpha)}(F_{\log n,\log n+\log\log n})}{D_\ell^{(\alpha)}}\right]-\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\frac{W_\ell^{(\alpha)}(F_{\log s_n-\log\ell,\log n+\log\log n})}{D_\ell^{(\alpha)}}\right]\right)\\&=\frac{n}{\ell}\mathcal{R}(\alpha)\left(\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\frac{\sqrt{\ell}W_\ell^{(\alpha)}(F_{\log n,\log n+\log\log n})}{D_\ell^{(\alpha)}}\right]-\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\frac{\sqrt{\ell}W_\ell^{(\alpha)}(F_{\log s_n-\log\ell,\log n+\log\log n})}{D_\ell^{(\alpha)}}\right]\right)\end{aligned}\tag{4.13}$$

In view of (3.5), as $\ell\sim\gamma(\log n)^2$, we have

$$\lim_{n\to\infty}\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\frac{\sqrt{\ell}W_\ell^{(\alpha)}(F_{\log n,\log n+\log\log n})}{D_\ell^{(\alpha)}}\right]=\lim_{n\to\infty}\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\frac{\sqrt{\ell}W_\ell^{(\alpha)}(F_{\log s_n-\log\ell,\log n+\log\log n})}{D_\ell^{(\alpha)}}\right]=\mathscr{C}_{\gamma^{-1/2},\gamma^{-1/2}}.$$

So,

$$\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\frac{\sqrt{\ell}W_\ell^{(\alpha)}(F_{\log n,\log n+\log\log n})}{D_\ell^{(\alpha)}}\right]-\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\frac{\sqrt{\ell}W_\ell^{(\alpha)}(F_{\log s_n-\log\ell,\log n+\log\log n})}{D_\ell^{(\alpha)}}\right]=o_n(1),$$

and $\frac{n}{\ell}=O(\frac{n}{(\log n)^2})$. This implies that

$$\mathbb{E}[K_n^C(\ell)-K_n^{C^\delta}(\ell)]\le\mathbf{E}\left[\widetilde{\mathcal{K}}_n^{C\setminus C^\delta}(\ell)\right]=o\Big(\frac{n}{(\log n)^2}\Big).$$

This ends the proof of Lemma 2.2. □

4.3.3 Proof of Proposition 2.3

Observe that

$$\mathbb{P}\left(|K_n^{C^\delta}(\ell)-\widetilde{\mathcal{K}}_n^{C^\delta}(\ell)|\geq \eta\frac{n}{(\log n)^2}\right) \leq \mathbb{P}\left(|K_n^{C^\delta}(\ell)-\mathcal{K}_n^{C^\delta}(\ell)|\geq \eta\frac{n}{2(\log n)^2}\right)+\mathbf{P}\left(|\mathcal{K}_n^{C^\delta}(\ell)-\widetilde{\mathcal{K}}_n^{C^\delta}(\ell)|\geq \eta\frac{n}{2(\log n)^2}\right). \tag{4.14}$$

For the second term on the right hand side, by Markov inequality and (4.2), we have

$$\mathbf{P}\left(|\mathcal{K}_n^{C^\delta}(\ell)-\widetilde{\mathcal{K}}_n^{C^\delta}(\ell)|\geq \eta\frac{n}{2(\log n)^2}\right)\leq \frac{2(\log n)^2}{\eta n}\mathbf{E}\left[\widetilde{\mathcal{K}}_n^{C^\delta}(\ell)-\mathcal{K}_n^{C^\delta}(\ell)\right]\leq \frac{2\log n}{\eta n}\mathbf{E}\left[\widetilde{\mathcal{K}}_n^{C^\delta}(\ell)\right].$$

In view of (2.7), this implies that

$$\begin{aligned}\mathbf{P}\left(|\mathcal{K}_n^{C^\delta}(\ell)-\widetilde{\mathcal{K}}_n^{C^\delta}(\ell)|\geq \eta\frac{n}{2(\log n)^2}\right)\leq&\frac{2\log n}{\eta n}\times\frac{n}{\sqrt{\ell}}\mathbf{E}\left[W_\ell^{(\alpha)}(F_{\log s_n,\log n+\log\log n})\right]\\ =&\frac{2c}{\eta}\mathcal{R}(\alpha)\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\frac{W_\ell^{(\alpha)}(F)}{D_\ell^{(\alpha)}}\right]=O(\frac{1}{\sqrt{\ell}}),\end{aligned}$$

where the last equalities come from the change of measures (3.1) and (3.5).

For the first term on the right hand side of (4.14), using Tchebychev inequality on the quenched probability yields that

$$\mathbb{P}^{\mathcal{E}}\left(|K_n^{C_\delta}(\ell)-\mathcal{K}_n^{C_\delta}(\ell)|\geq \eta\frac{n}{2(\log n)^2}\right)\leq \frac{4(\log n)^4}{\eta^2 n^2}\mathbb{V}\mathcal{E}ar\left(K_n^{C^\delta}(\ell)\right).$$

So,

$$\mathbb{P}\left(|K_n^{C^\delta}(\ell)-\mathcal{K}_n^{C^\delta}(\ell)|\geq \eta\frac{n}{2(\log n)^2}\right)\leq \frac{4(\log n)^4}{\eta^2 n^2}\mathbf{E}\left(\mathbb{V}\mathcal{E}ar\left(K_n^{C^\delta}(\ell)\right)\right)$$

Using Lemma 4.5 with $\delta=5$ gives what we need. □

4.3.4 Proof of Lemma 2.4

Proof of (2.12). Let us show that

$$\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty}\frac{1}{n}\mathbb{E}\Big[\sum_{\ell\geq(\log n)^2/\varepsilon}\sum_{|z|=\ell}K_n^B(\ell)\Big]=0, \tag{4.15}$$

$$\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty}\frac{1}{n}\mathbb{E}\Big[\sum_{\ell\leq\varepsilon(\log n)^2}\sum_{|z|=\ell}K_n^B(\ell)\Big]=0. \tag{4.16}$$

- *Proof of* (4.15). By (4.1),

$$\begin{aligned}LHS_{(4.15)}:=&\mathbb{E}\Big[\sum_{\ell\geq(\log n)^2/\varepsilon}\sum_{|z|=\ell}K_n^B(\ell)\Big]\leq\mathbf{E}\Big[\sum_{\ell\geq(\log n)^2/\varepsilon}\sum_{|z|=\ell}(ne^{-\overline{V}(z)}\wedge 1)\mathbb{1}_{z<\mathcal{L}_n,\underline{V}(z)\geq-\alpha}\Big]\\ \leq&\sum_{\ell\geq(\log n)^2/\varepsilon}n\mathbf{E}\left[\sum_{|z|=\ell}e^{-\overline{V}(z)}\mathbb{1}_{\overline{V}(z)\geq\log n,z\in B}\right]+\sum_{\ell\geq(\log n)^2/\varepsilon}\mathbf{E}\left[\sum_{|z|=\ell}\mathbb{1}_{\overline{V}(z)\leq\log n,\underline{V}(z)\geq-\alpha}\right]\\ =:&R_I+R_{II}.\end{aligned} \tag{4.17}$$

Applying (1.15) to R_{II} yields that

$$\begin{aligned} R_{II} &= \sum_{\ell \geq (\log n)^2/\varepsilon} \mathbf{E}\left[e^{S_\ell}; \overline{S}_\ell \leq \log n, \underline{S}_\ell \geq -\alpha\right] \\ &\leq \sum_{\ell \geq (\log n)^2/\varepsilon} \sum_{j=1}^{\ell} \mathbf{E}\left[e^{S_\ell}; \overline{S}_{j-1} < S_j = \overline{S}_\ell \leq \log n, \underline{S}_\ell \geq -\alpha\right], \end{aligned}$$

which by Markov property at time j, is bounded by

$$\begin{aligned} &\sum_{\ell \geq (\log n)^2/\varepsilon} \sum_{j=1}^{\ell} \mathbf{E}\left[e^{S_j}; S_j \leq \log n, \underline{S}_j \geq -\alpha\right] \mathbf{E}\left[e^{S_{\ell-j}} \mathbb{1}_{\overline{S}_{\ell-j} \leq 0}\right] \\ \leq &\sum_{\ell \geq (\log n)^2/\varepsilon} \sum_{j=1}^{\ell} \sum_{-\alpha \leq k \leq \log n} e^{k+1} \mathbf{P}\left[S_j \in [k, k+1], \underline{S}_j \geq -\alpha\right] \mathbf{E}\left[e^{S_{\ell-j}} \mathbb{1}_{\overline{S}_{\ell-j} \leq 0}\right]. \end{aligned}$$

By (A.14) and (A.3), we have

$$\begin{aligned} R_{II} &\leq \sum_{\ell \geq (\log n)^2/\varepsilon} \sum_{j=1}^{\ell} \sum_{-\alpha \leq k \leq \log n} e^{k+1} \frac{c(1+\alpha)(1+k+\alpha)}{j^{3/2}(\ell-j+1)^{3/2}} \\ &\leq \sum_{\ell \geq (\log n)^2/\varepsilon} \frac{cn(1+\alpha)(1+\log n+\alpha)}{\ell^{3/2}} \leq c \frac{n(1+\alpha)(1+\log n+\alpha)}{\sqrt{(\log n)^2/\varepsilon}} \end{aligned}$$

For any $\alpha > 0$ fixed, letting $\varepsilon \downarrow 0$ implies that

$$\lim_{\varepsilon \downarrow 0} \limsup_{n \to \infty} \frac{R_{II}}{n} = 0.$$

Also by (1.15), $R_I = \sum_{\ell \geq (\log n)^2/\varepsilon} n\mathbf{E}\left[\sum_{|z|=\ell} e^{-\overline{V}(z)} \mathbb{1}_{\overline{V}(z) \geq \log n, z < \mathcal{L}_n, \underline{V}(z) \geq -\alpha}\right]$ equals to

$$\sum_{\ell \geq (\log n)^2/\varepsilon} n\mathbf{E}\left[e^{S_\ell - \overline{S}_\ell} \mathbb{1}_{\overline{S}_\ell \geq \log n, \max_{1 \leq k \leq \ell} \sum_{i=1}^{k} e^{S_i - S_k} \leq n, \underline{S}_\ell \geq -\alpha}\right].$$

So,

$$\begin{aligned} R_I &\leq \sum_{\ell \geq (\log n)^2/\varepsilon} n\mathbf{E}\left[e^{S_\ell - \overline{S}_\ell}; \overline{S}_\ell \geq \log n, \max_{1 \leq k \leq \ell} (\overline{S}_k - S_k) \leq \log n, \underline{S}_\ell \geq -\alpha\right] \\ &\leq \sum_{\ell \geq (\log n)^2/\varepsilon} n \sum_{j=1}^{\ell} \mathbf{E}\left[e^{S_\ell - \overline{S}_\ell}; \overline{S}_{j-1} < S_j = \overline{S}_\ell, S_j \geq \log n, \max_{1 \leq k \leq \ell} (\overline{S}_k - S_k) \leq \log n, \underline{S}_\ell \geq -\alpha\right], \end{aligned}$$

which by Markov property at time j, is bounded by

$$\sum_{\ell \geq (\log n)^2/\varepsilon} n \sum_{j=1}^{\ell} \mathbf{P}\left[\overline{S}_{j-1} < S_j, S_j \geq \log n, \max_{1 \leq k \leq j} (\overline{S}_k - S_k) \leq \log n, \underline{S}_j \geq -\alpha\right] \mathbf{E}\left[e^{S_{\ell-j}} \mathbb{1}_{\overline{S}_{\ell-j} \leq 0}\right].$$

By (A.14),

$$R_I \le \sum_{\ell \ge (\log n)^2/\varepsilon} n \sum_{j=1}^{\ell} \mathbf{P}\left[\overline{S}_{j-1} < S_j, \max_{1\le k\le j}(\overline{S}_k - S_k) \le \log n, \underline{S}_j \ge -\alpha\right] \frac{c}{(\ell - j + 1)^{3/2}}$$

$$=: n \sum_{\ell \ge (\log n)^2/\varepsilon} \sum_{j=1}^{\ell} r_{\ell,j}. \tag{4.18}$$

We split this sum into two parts: $\sum_{\ell \ge (\log n)^2/\varepsilon} \sum_{j=1}^{\ell - \ell^{1/3}}$ and $\sum_{\ell \ge (\log n)^2/\varepsilon} \sum_{\ell - \ell^{1/3} \le j \le \ell}$. For the first sum, by (A.4), one sees that

$$\begin{aligned} n \sum_{\ell \ge (\log n)^2/\varepsilon} \sum_{j=1}^{\ell - \ell^{1/3}} r_{\ell,j} &\le n \sum_{\ell \ge (\log n)^2/\varepsilon} \sum_{j=1}^{\ell - \ell^{1/3}} \mathbf{P}\left(\underline{S}_j \ge -\alpha, S_j = \overline{S}_j\right) \frac{c}{(\ell - j + 1)^{3/2}} \\ &\le n \sum_{\ell \ge (\log n)^2/\varepsilon} \sum_{j=1}^{\ell - \ell^{1/3}} \frac{c(1+\alpha)}{j(\ell - j + 1)^{3/2}} \le c \frac{n(1+\alpha)\varepsilon^{1/6}}{(\log n)^{1/3}} \end{aligned}$$

For the second sum, by Markov property at $j/3$ and $2j/3$ then by reversing time,

$$\begin{aligned} r_{\ell,j} \le& \mathbf{P}\left(\underline{S}_{j/3} \ge -\alpha\right) \mathbf{P}\left(\max_{k\le j/3}(\overline{S}_k - S_k) \le \log n\right) \mathbf{P}\left(S_{j/3} = \overline{S}_{j/3}\right) \frac{c}{(\ell - j + 1)^{3/2}} \\ =& \mathbf{P}\left(\underline{S}_{j/3} \ge -\alpha\right) \mathbf{P}\left(\max_{k\le j/3}(\overline{S}_k - S_k) \le \log n\right) \mathbf{P}\left(\underline{S}_{j/3} \ge 0\right) \frac{c}{(\ell - j + 1)^{3/2}}. \end{aligned}$$

It is known by [16] that for sufficiently large $\lambda > 0$, $\mathbf{P}(\max_{1\le k\le j}(\overline{S}_k - S_k) \le \lambda) \le e^{-c\lfloor \frac{j}{\lceil \lambda^2 \rceil} \rfloor}$. This together with (A.2) implies that for any n large enough,

$$\begin{aligned} n \sum_{\ell \ge (\log n)^2/\varepsilon} \sum_{\ell - \ell^{1/3} \le j \le \ell} r_{\ell,j} &\le n \sum_{\ell \ge (\log n)^2/\varepsilon} \sum_{\ell - \ell^{1/3} \le j \le \ell} \frac{c(1+\alpha)}{j(\ell - j + 1)^{3/2}} e^{-c\frac{j}{(\log n)^2}} \\ &\le n \sum_{\ell \ge (\log n)^2/\varepsilon} \frac{c(1+\alpha)}{\ell} e^{-c\frac{\ell}{(\log n)^2}} \\ &\le c(1+\alpha) n \int_{(\log n)^2/\varepsilon}^{\infty} e^{-c\frac{t}{(\log n)^2}} \frac{dt}{t}, \end{aligned}$$

which by change of variables implies that

$$n \sum_{\ell \ge (\log n)^2/\varepsilon} \sum_{\ell - \ell^{1/3} \le j \le \ell} r_{\ell,j} \le c(1+\alpha) n \int_{1/\varepsilon}^{\infty} e^{-cs} \frac{ds}{s} \le c\varepsilon(1+\alpha) n.$$

Consequently, one gets that

$$\lim_{\varepsilon \downarrow 0} \limsup_{n\to\infty} \frac{R_I}{n} = 0.$$

This leads to (4.17) and (4.15) follows immediately.

• *Proof of* (4.16). Again by (4.1),

$$LHS_{(4.16)} := \mathbb{E}\Big[\sum_{\ell\le\varepsilon(\log n)^2}\sum_{|z|=\ell} K_n^B(\ell)\Big]$$
$$\le \mathbf{E}\left[\sum_{\ell\le\varepsilon(\log n)^2}\sum_{|z|=\ell}(ne^{-\overline{V}(z)}\wedge 1)\mathbb{1}_{z<\mathcal{L}_n,\underline{V}(z)\ge-\alpha}\right] \le R'_I + R'_{II},$$

where

$$R'_I := \sum_{\ell\le\varepsilon(\log n)^2}\mathbf{E}\left[\sum_{|z|=\ell} ne^{-\overline{V}(z)}\mathbb{1}_{\overline{V}(z)\ge\log n/2,\underline{V}(z)\ge-\alpha}\right],\ R'_{II} := \sum_{\ell\le\varepsilon(\log n)^2}\mathbf{E}\left[\sum_{|z|=\ell}\mathbb{1}_{\overline{V}(z)\le\log n/2,\underline{V}(z)\ge-\alpha}\right].$$

Similarly as above, by (1.15),

$$R'_I = n\sum_{\ell\le\varepsilon(\log n)^2}\mathbf{E}\left[e^{S_\ell-\overline{S}_\ell};\overline{S}_\ell\ge\log n/2,\underline{S}_\ell\ge-\alpha\right]$$
$$= n\sum_{\ell\le\varepsilon(\log n)^2}\sum_{j=1}^{\ell}\mathbf{E}\left[e^{S_\ell-S_j};\overline{S}_{j-1}<S_j=\overline{S}_\ell,S_j\ge\log n/2,\underline{S}_\ell\ge-\alpha\right],$$

which by Markov property at time j is bounded by

$$n\sum_{\ell\le\varepsilon(\log n)^2}\sum_{j=1}^{\ell}\mathbf{P}\left(\underline{S}_j\ge-\alpha,\overline{S}_j=S_j\ge\log n/2\right)\mathbf{E}\left[e^{S_{\ell-j}}\mathbb{1}_{\overline{S}_{\ell-j}\le0}\right].$$

By (A.14),

$$R'_I \le n\sum_{\ell\le\varepsilon(\log n)^2}\sum_{j=1}^{\ell}\mathbf{P}\left(\underline{S}_j\ge-\alpha,\overline{S}_j=S_j\ge\log n/2\right)\frac{c}{(\ell-j+1)^{3/2}}. \tag{4.19}$$

then by (A.6),

$$R'_I \le n\sum_{\ell\le\varepsilon(\log n)^2}\sum_{j=1}^{\ell}\frac{c(1+\alpha)}{j^{1/2}(\ell-j+1)^{3/2}\log n} \le c(1+\alpha)n\sum_{\ell\le\varepsilon(\log n)^2}\frac{1}{\ell^{1/2}\log n} \le c\sqrt{\varepsilon}(1+\alpha)n.$$

Therefore,

$$\lim_{\varepsilon\downarrow0}\limsup_{n\to\infty}\frac{R'_I}{n} = 0.$$

It remains to bound $R'_{II} = \sum_{\ell\le\varepsilon(\log n)^2}\mathbf{E}\left[\sum_{|z|=\ell}\mathbb{1}_{\overline{V}(z)\le\log n/2,\underline{V}(z)\ge-\alpha}\right]$. By (1.15),

$$R'_{II} = \sum_{\ell\le\varepsilon(\log n)^2}\mathbf{E}\left[e^{S_\ell};\overline{S}_\ell\le\log n/2,\underline{S}_\ell\ge-\alpha\right] \le \sum_{\ell\le\varepsilon(\log n)^2}e^{\log n/2} \le \varepsilon(\log n)^2\sqrt{n},$$

so

$$\lim_{\varepsilon\downarrow0}\limsup_{n\to\infty}\frac{R'_{II}}{n} = 0.$$

This completes the proof of (4.16). □

Proof of (2.13). We now prove that for any $\varepsilon > 0$,

$$\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \mathbb{E}\left[K_n^{B\setminus B_3}(m)\right] = o(n), \tag{4.20}$$

$$\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \mathbb{E}\left[K_n^{C\setminus C^\delta}(m)\right] = o(n). \tag{4.21}$$

As shown in the proof of Lemma 2.1, for any $m \geq \varepsilon(\log n)^2$,

$$\mathbb{E}\left[K_n^{B\setminus B_3}(m)\right] \leq c(\varepsilon) n \frac{\log\log n}{(\log n)^3},$$

so (4.20) follows. It remains to show (4.21). It remains to show (4.21). Observe that

$$\mathbb{E}\left[K_n^{C\setminus C^\delta}(m)\right] = \mathbf{E}\left[\mathcal{K}_n^{C\setminus C^\delta}(m)\right] \leq \mathbf{E}\left[\widetilde{\mathcal{K}}_n^{C\setminus C^\delta}(m)\right] = \mathbf{E}\left[\sum_{|z|=m} n a_z \mathbb{1}_{z\in C\setminus C^\delta}\right]$$

Take $\beta = \log n + \log\log n$. Because of (4.13), for any $\varepsilon > 0$ fixed, there exists $c_1 > 0$ such that when $n \geq 10$, for any $m \in [\varepsilon\beta^2/2, \beta^2/\varepsilon] \cap \mathbb{Z}$,

$$\mathbf{E}\left[\widetilde{\mathcal{K}}_n^{C\setminus C^\delta}(m)\right] \leq \frac{n}{\sqrt{m}}\mathcal{R}(\alpha)\left(\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\frac{W_m^{(\alpha)}(F_{\beta,\beta})}{D_m^{(\alpha)}}\right] - \mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\frac{W_m^{(\alpha)}(F_{\beta-c_1\log\beta,\beta})}{D_m^{(\alpha)}}\right]\right).$$

It follows immediately that

$$\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \mathbb{E}\left[K_n^{C\setminus C^\delta}(m)\right] \leq n\mathcal{R}(\alpha) \sum_{m=\varepsilon\beta^2/2}^{\beta^2/\varepsilon} \frac{1}{\sqrt{m}}\left(\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\frac{W_m^{(\alpha)}(F_{\beta,\beta})}{D_m^{(\alpha)}}\right] - \mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\frac{W_m^{(\alpha)}(F_{\beta-c_1\log\beta,\beta})}{D_m^{(\alpha)}}\right]\right). \tag{4.22}$$

Similarly to (1.26), the convergence (3.5) holds uniformly. So following the arguments used to prove Corollary 1.9, we deduce that for any $0 < b < B < \infty$, as $\beta \to \infty$,

$$\sum_{b\beta^2\leq m\leq B\beta^2} \frac{1}{\sqrt{m}}\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\frac{W_m^{(\alpha)}(F_{\beta,\beta})}{D_m^{(\alpha)}}\right] = \int_b^B \mathscr{C}_{\gamma^{-1/2},\gamma^{-1/2}}\frac{d\gamma}{\gamma} + o_\beta(1).$$

Similarly, we also have

$$\sum_{b\beta^2\leq m\leq B\beta^2} \mathbf{E}_{\mathbf{Q}^{(\alpha)}}\frac{1}{\sqrt{m}}\left[\frac{W_m^{(\alpha)}(F_{\beta-c_1\log\beta,\beta})}{D_m^{(\alpha)}}\right] = \int_b^B \mathscr{C}_{\gamma^{-1/2},\gamma^{-1/2}}\frac{d\gamma}{\gamma} + o_\beta(1).$$

As a consequence, (4.22) becomes

$$\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \mathbb{E}\left[K_n^{C\setminus C^\delta}(m)\right] \leq o_n(1) n\mathcal{R}(\alpha),$$

which ends the proof. □

4.4 Proof of Proposition 1.6

Most of the arguments are already present in the proof of Theorem 1.5 in section 2.2. Indeed we have stressed on the fact that the main contribution of visited sites comes from the set of individuals of the tree truncated by C^δ.

Similarly to the proof of (2.13), the restriction $A_3 := \{z \in \mathbb{T} : \overline{V}(z) > \max_{|y|\le|z|-|z|^{1/3},\, y\le z} V(y)\}$ follows easily from (3.11). So it remains to consider $D := \left\{z \in \mathbb{T} : \max_{\phi<y\le z}(\overline{V}(y) - V(y)) \le \frac{\log n}{a_0}\right\}$, and $F := \left\{z \in \mathbb{T} : \overline{V}(z) \ge a_1 \log n\sqrt{\log\log n}\right\}$. We only need to show that

$$\lim_{a_0\to+\infty} \lim_{n\to+\infty} \mathbb{E}\left[n^{-1} \sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} K_n^{B\cap D\cap A_3}(m)\right] = 0, \tag{4.23}$$

and

$$\lim_{n\to+\infty} \mathbb{E}\left[n^{-1} \sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} K_n^{B\cap F}(m)\right] = 0. \tag{4.24}$$

For (4.23) we do as usual and get that the expectation is smaller than

$$\begin{aligned}
&\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \mathbf{E}\left[e^{S_m-\overline{S}_m}\mathbb{1}_{\max_{1\le i\le m}(\overline{S}_i-S_i)\le \log n/a_0,\ \underline{S}_m\ge-\alpha, Y_S>m-m^{1/3}}\right]\\
&\le \sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \sum_{j=m-m^{1/3}}^{m} \mathbf{E}\left[e^{S_{m-j}}\mathbb{1}_{\overline{S}_{m-j}\le 0}\right] \mathbf{P}\left[\max_{1\le i\le j}(\overline{S}_i - S_i) \le \log n/a_0,\ \overline{S}_{j-1} < S_j,\ \underline{S}_j \ge -\alpha\right].
\end{aligned}$$

Similarly to (4.18), the above sum is bounded by $\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \sum_{j=m-m^{1/3}}^{m} \frac{c(1+\alpha)}{(m-j+1)^{3/2}m} e^{-c'ma_0/(\log n)^2} \le -2(\log\varepsilon)e^{-c'\varepsilon a_0}$ which goes to zero as $a_0 \to \infty$.

Also for the expectation in (4.24) we can prove that it is smaller than

$$\begin{aligned}
&\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \mathbf{E}\left[e^{S_m-\overline{S}_m}\mathbb{1}_{\overline{S}_m\ge a_1\log n\sqrt{\log\log n},\ \underline{S}_m\ge-\alpha}\right]\\
&\le \sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \sum_{j=1}^{m} \mathbf{E}\left[e^{S_{m-j}}\mathbb{1}_{\overline{S}_{m-j}\le 0}\right] \mathbf{P}\left[\underline{S}_j \ge -\alpha,\ \overline{S}_j \ge a_1\log n\sqrt{\log\log n}\right]
\end{aligned}$$

which by (A.14) and (A.19) is bounded by $c\sum_{m=\varepsilon(\log n)^2}^{(\log n)^2/\varepsilon} \sum_{j=1}^{m} (m-j+1)^{-3/2} m^{-c'*a_1^2} = o_n(1)$ by choosing a_1 properly. □

A Appendix

A.1 Finiteness of Λ [see (1.23)]

Lemma A.1. *The function $\lambda : (0,\infty) \to (0,\infty)$ is integrable, i.e.,*

$$\Lambda = \int_0^\infty \lambda(\gamma) d\gamma = \mathbf{c}_0 \int_0^\infty \frac{\mathscr{C}_{\gamma^{-1/2},\gamma^{-1/2}}}{\gamma} d\gamma < \infty. \tag{A.1}$$

Further, for any $a, b > 0$, $\mathcal{C}_{a\gamma^{-1/2},b\gamma^{-1/2}}/\gamma$ is integrable.

Proof. It suffices to show that $\frac{\mathcal{C}_{\gamma^{-1/2},\gamma^{-1/2}}}{\gamma}$ is integrable. Let $(\mathfrak{m}_s, s \in [0,1])$ and $(\widetilde{\mathfrak{m}}_s, s \in [0,1])$ be two independent Brownian meanders. Then

$$\mathcal{C}_{a,b} \leq \begin{cases} c\mathbf{P}\left(\sigma(\mathfrak{m}_1 + \widetilde{\mathfrak{m}}_1) \geq \sqrt{2}b\right) \\ c\mathbf{P}\left(\max_{s\in[0,1]} \sigma(\mathfrak{m}_s - \underline{\mathfrak{m}}_{[s,1]}) \leq \sqrt{2}a\right). \end{cases}$$

It follows from the first inequality that

$$\mathcal{C}_{a,b} \leq c\mathbf{P}(\mathfrak{m}_1 \geq \frac{\sqrt{2}b}{2\sigma}) = ce^{-b^2/4\sigma^2},$$

since the density of $\mathfrak{m}_1$ is $xe^{-x^2/2}\mathbb{1}_{x\geq 0}$. On the other hand, according to [7], with $(b_s, s \in [0,1])$ a Brownian bridge

$$\mathbf{P}\left(\max_{s\in[0,1]} \sigma(\mathfrak{m}_s - \underline{\mathfrak{m}}_{[s,1]}) \leq \sqrt{2}a\right) = \mathbf{P}\left(\max_{s\in[0,1]} |b_s| \leq \sqrt{2}a/\sigma\right) \leq \mathbf{P}\left(\max_{s\in[0,1]} b_s \leq \sqrt{2}a/\sigma\right).$$

This shows that

$$\mathcal{C}_{a,b} \leq c\mathbf{P}\left(\max_{s\in[0,1]} b_s \leq \sqrt{2}a/\sigma\right) = c\left(1 - \exp\left(-2\Big(\frac{\sqrt{2}a}{\sigma}\Big)^2\right)\right) \leq \frac{4c}{\sigma^2}a^2.$$

We are now ready to prove the integrability. Observe that

$$\begin{aligned} \int_0^\infty \frac{\mathcal{C}_{\gamma^{-1/2},\gamma^{-1/2}}}{\gamma} d\gamma &= \int_0^1 \frac{\mathcal{C}_{\gamma^{-1/2},\gamma^{-1/2}}}{\gamma} d\gamma + \int_1^\infty \frac{\mathcal{C}_{\gamma^{-1/2},\gamma^{-1/2}}}{\gamma} d\gamma \\ &\leq \int_0^1 ce^{-\frac{1}{4\sigma^2\gamma}} \frac{d\gamma}{\gamma} + \int_1^\infty \frac{4c}{\sigma^2}\frac{d\gamma}{\gamma^2} = \int_0^1 ce^{-\frac{1}{4\sigma^2\gamma}} \frac{d\gamma}{\gamma} + \frac{4c}{\sigma^2}. \end{aligned}$$

By change of variables $t = 1/\gamma$,

$$\int_0^1 ce^{-\frac{1}{4\sigma^2\gamma}} \frac{d\gamma}{\gamma} = \int_1^\infty ce^{-\frac{t}{4\sigma^2}} \frac{dt}{t} < \infty.$$

We hence conclude the integrability of $\frac{\mathcal{C}_{\gamma^{-1/2},\gamma^{-1/2}}}{\gamma}$, as well as $\frac{\mathcal{C}_{a\gamma^{-1/2},b\gamma^{-1/2}}}{\gamma}$ for any $a, b > 0$. □

A.2 Results on one-dimensional random walks

In this section we state technical inequalities that are used all along the paper. The sequence (S_k, k) which appears here is the one defined in (1.15). The proofs are postpone Section A.3.

We start with two well know inequalities (see [3] for instance) and some basic Facts. There exists constant $c > 0$ such that for any $n \geq 1$ and $u \geq 0$

$$\mathbf{P}(\underline{S}_n \geq -u) \leq \frac{c(1+u)}{\sqrt{n}} \text{ and } \mathbf{P}(\overline{S}_n \leq u) \leq \frac{c(1+u)}{\sqrt{n}}. \tag{A.2}$$

By Lemma 2.2 in [3], there exists some constant $c > 0$ such that for any $u \geq 0$, $b \geq a \geq -u$ and any $n \geq 1$,

$$\mathbf{P}(\underline{S}_n \geq -u, a \leq S_n \leq b) \leq \frac{c(1+u)(1+b+u)(1+b-a)}{n^{3/2}}. \tag{A.3}$$

Fact A.2. *1. For any $u, \alpha \geq 0$ and $\forall n \geq 1$,*

$$\mathbf{P}_u(\underline{S}_n \geq -\alpha, S_n = \overline{S}_n) \leq \frac{c(1+\alpha+u)}{n}. \tag{A.4}$$

2. (a) For any $n \geq 1$, $B > 0$ fixed, there exists $c(B) > 0$ such that for any $u \geq 0$, $-B\sqrt{n} \leq -\alpha \leq 0 < a < b \leq B\sqrt{n}$,

$$\mathbf{P}_u(\underline{S}_n \geq -\alpha, S_n = \overline{S}_n \in [a,b]) \leq \frac{c(B)(1+\alpha+u)(b-a)}{n^{3/2}}. \tag{A.5}$$

(b) For any $n \geq 1$, $A > 0$,

$$\mathbf{P}(\underline{S}_n \geq -\alpha, S_n = \overline{S}_n \geq A) \leq \frac{c(1+\alpha)}{An^{1/2}}. \tag{A.6}$$

3. For any $a, A, \alpha > 0$ and $\forall n > m \geq 1$,

$$\mathbf{P}(\underline{S}_n \geq -\alpha, S_n = \overline{S}_n, \overline{S}_m - S_n \geq -A, \overline{S}_m - S_m \leq a) \leq \frac{c(1+A)(1+a+A)(1+\alpha)}{m^{1/2}(n-m)^{3/2}} \tag{A.7}$$

We now state the following Lemma which is mostly a consequence of the above facts.

Lemma A.3. *For any $\alpha \geq 0$, $0 < a \leq b$ and $n \geq 1$, we have*

$$\mathbf{P}\Big(S_n = \underline{S}_n \geq -\alpha\Big) \leq \frac{c(1+\alpha)^2}{n^{3/2}}, \tag{A.8}$$

$$\mathbf{P}\Big(\sum_{1 \leq i \leq n} e^{S_i - S_n} \in [a,b], \underline{S}_n \geq -\alpha\Big) \leq c\frac{(1+\alpha)(1+\log b)(1+\log b - \log a + \log n)}{n}, \tag{A.9}$$

$$\mathbf{E}\Big(e^{S_n}; S_n \in [a,b], \underline{S}_n \geq -\alpha\Big) \leq \frac{ce^b(b+\alpha+1)(1+b-a)(1+\alpha)}{n^{3/2}}. \tag{A.10}$$

Following Lemma focus on asymptotic results.

Lemma A.4. *Let $a, b \geq 0$ fixed and $\lim_n \frac{a_n}{\sqrt{n}} = \lim_n \frac{b_n}{\sqrt{n}} = 0$. We have the following convergences.*

1. Let $\mathcal{C}_b := \mathbf{P}(\mathbb{m}_1 \geq b)\mathbf{c}_2^+$ with $(\mathbb{m}_s, s \in [0,1])$ a Brownian meander and $\mathbf{c}_2^+$ defined in (1.18). Then,

$$\lim_n \sqrt{n}\mathbf{P}\left(S_n > \overline{S}_{n-1}, S_n \geq b\sqrt{n} + b_n\right) = \mathcal{C}_b. \tag{A.11}$$

2. Moreover, for any $\alpha \geq 0$ fixed,

$$\lim_n n\mathbf{P}\left(\underline{S}_n \geq -\alpha, S_n > \overline{S}_{n-1}, \max_{1\leq i\leq n}(\overline{S}_i - S_i) \leq a\sqrt{n} + a_n, S_n \geq b\sqrt{n} + b_n\right) = \mathcal{C}_{a,b}\mathcal{R}(\alpha), \tag{A.12}$$

where $\mathcal{C}_{a,b} \in (0,\infty)$ is a constant depending on a and b.

3. Let $g : [1,\infty) \to \mathbb{R}_+$ be a uniformly continuous and bounded function. Then,

$$\lim_{n\to\infty} n\mathbf{E}\left[g\Big(\sum_{j=1}^n e^{S_j - S_n}\Big); \underline{S}_n \geq -\alpha, S_n > \overline{S}_{n-1}, \max_{1\leq i\leq n}(\overline{S}_i - S_i) \leq a\sqrt{n} + a_n, S_n \geq b\sqrt{n} + b_n\right] = \mathcal{C}_{a,b}\mathcal{R}(\alpha)\mathbf{E}[g(\mathcal{H}_\infty)] \tag{A.13}$$

Below we collect some more basic facts.

Fact A.5. *1. For any $n \geq 1$,*

$$\mathbf{E}[e^{S_n}; \overline{S}_n \leq 0] \leq \frac{c}{n^{3/2}}. \tag{A.14}$$

2. For any $A > 0$ and $n \geq 1$,

$$\mathbf{E}[e^{S_n}; \overline{S}_n \leq A] \leq \frac{c(1+A)e^A}{n^{3/2}}. \tag{A.15}$$

3. For any $n \geq 1$,

$$\mathbf{E}\Big[\frac{e^{S_n}}{\sum_{1\leq i\leq n} e^{S_n}}\Big] \leq \frac{c}{n^{1/2}}. \tag{A.16}$$

4. For any $A, \delta > 0$ and $\forall n \geq k_2 > k_1 \geq 1$,

$$\mathbf{P}(\underline{S}_{[k_1,k_2]} \leq A, \underline{S}_{(1+\delta)n} \geq 0) \leq \frac{c(1+A)}{\sqrt{\delta n k_1}}. \tag{A.17}$$

5. If $\mathbf{E}[e^{\pm\theta S_1}] < \infty$ for some $\theta > 0$ [see (1.3)], then for any $\delta > 0$ there exists $c(\delta,\theta) > 0$ such that for any $n \geq 1$,

$$\mathbf{P}(\overline{S}_n \geq n^{1+\delta}) \leq c(\delta,\theta)e^{-\theta n^{1+\delta}/2}. \tag{A.18}$$

6. Let $a > 0$. With the same hypothesis as above there exists $c_2 > 0$ such that

$$\mathbf{P}(\overline{S}_n \geq a\sqrt{n\log n}, \underline{S}_n \geq -\alpha) \leq \frac{c}{n^{a^2c_2}}. \tag{A.19}$$

The following corollary follows from above lemmas.

Corollary A.6. *Let $a \geq 0$, $b_n \geq 0$ and $\lim_n \frac{a_n}{\sqrt{n}} = \lim_n \frac{b_n}{\sqrt{n}} = 0$. Take $n_0 = n - n^{1/3}$, then*

$$\mathbf{P}\Big(\max_{1\leq i\leq n}(\overline{S}_i - S_i) \in [a\sqrt{n}+a_n, a\sqrt{n}+a_n+b_n], \underline{S}_n \geq -\alpha, \overline{S}_{n-1} < S_n\Big) = \frac{o_n(1)}{n}, \tag{A.20}$$

$$\mathbf{E}\Big[e^{S_n-\overline{S}_n}; \max_{1\leq i\leq n}(\overline{S}_i - S_i) \in [a\sqrt{n}+a_n, a\sqrt{n}+a_n+b_n], \underline{S}_n \geq -\alpha, \overline{S}_{n_0} < \overline{S}_n\Big] = \frac{o_n(1)}{n}, \tag{A.21}$$

$$\mathbf{P}\Big(\underline{S}_{[n/2,n]} \leq b_n, \underline{S}_n \geq -\alpha, S_n > \overline{S}_{n-1}\Big) = \frac{o_n(1)}{n}. \tag{A.22}$$

A.3 Proofs of (A.4)–(A.22)

We show these results one by one.

Proof of (A.4). Let $R_k := S_n - S_{n-k}$ for $0 \leq k \leq n$. Clearly, $(R_k, 0 \leq k \leq n/2)$ is an independent copy of $(S_k, 0 \leq k \leq n/2)$. Hence,

$$\mathbf{P}_u\Big(\underline{S}_n \geq -\alpha, S_n = \overline{S}_n\Big) \leq \mathbf{P}_u\Big(\underline{S}_{n/2} \geq -\alpha, \underline{R}_{n/2} \geq 0\Big) = \mathbf{P}\Big(\underline{S}_{n/2} \geq -\alpha - u\Big)\mathbf{P}\Big(\underline{R}_{n/2} \geq 0\Big).$$

Applying (A.2) to both $(S_\cdot)$ and $(R_\cdot)$ yields that

$$\mathbf{P}_u\Big(\underline{S}_n \geq -\alpha, S_n = \overline{S}_n\Big) \leq \frac{c(1+\alpha+u)}{n}, \tag{A.23}$$

which is exactly (A.4). □

Proof of (A.5)-(A.6). Using the same arguments as above, as $S_n = S_{n/2} + R_{n/2}$, we get that

$$\begin{aligned}\mathbf{P}_u\Big(\underline{S}_n \geq -\alpha, S_n = \overline{S}_n \in [a,b]\Big) =& \mathbf{P}\Big(\underline{S}_n \geq -\alpha - u, S_n = \overline{S}_n \in [a-u, b-u]\Big)\\ \leq& \mathbf{P}\Big(\underline{S}_{n/2} \geq -\alpha - u, \underline{R}_{n/2} \geq 0, S_{n/2} + R_{n/2} \in [a-u, b-u]\Big)\\ =& \mathbf{E}\Big(\psi(S_{n/2}); \underline{S}_{n/2} \geq -\alpha - u\Big),\end{aligned}$$

where $\psi(x) := \mathbf{P}(\underline{R}_{n/2} \geq 0, R_{n/2} \in [(a-u-x)_+, b-u-x])\mathbb{1}_{-\alpha-u\leq x\leq b-u}$. By (A.3),

$$\psi(x) \leq \frac{c(1+b-u-x)(1+b-a)}{n^{3/2}}\mathbb{1}_{-\alpha-u\leq x\leq b-u}.$$

It follows that

$$\begin{aligned}\mathbf{P}_u\Big(\underline{S}_n \geq -\alpha, S_n = \overline{S}_n \in [a,b]\Big) \leq& \frac{c(1+b-a)}{n^{3/2}}\mathbf{E}\Big((1+b-u-S_{n/2})_+; \underline{S}_{n/2} \geq -\alpha - u\Big)\\ \leq& \frac{c(1+b-a)}{n^{3/2}}(1+b+\alpha)\mathbf{P}\Big(\underline{S}_{n/2} \geq -\alpha - u\Big),\end{aligned} \tag{A.24}$$

which by (A.2) is less than $\frac{c(1+b-a)}{n^{3/2}}\frac{(1+\alpha+u)(1+b+\alpha)}{\sqrt{n}} = O(1)\frac{(1+b-a)(1+\alpha+u)}{n^{3/2}}$ since $b \vee \alpha \leq B\sqrt{n}$. This completes the proof of (A.5).

Similarly for (A.6), we have

$$\begin{aligned}\mathbf{P}\left(\underline{S}_n \geq -\alpha, \overline{S}_n = S_n \geq A\right) &\leq \mathbf{P}\left(\underline{S}_{n/2} \geq -\alpha, \underline{R}_{n/2} \geq 0, R_{n/2} + S_{n/2} \geq A\right) \\ &\leq \mathbf{P}\left(\underline{S}_{n/2} \geq -\alpha, \underline{R}_{n/2} \geq 0, R_{n/2} \geq A/2\right) + \mathbf{P}\left(\underline{S}_{n/2} \geq -\alpha, \underline{R}_{n/2} \geq 0, S_{n/2} \geq A/2\right),\end{aligned}$$

which by independence between $(S_i, i \leq n/2)$ and $(R_i, i \leq n/2)$ and (A.2), is bounded by

$$\frac{c(1+\alpha)\mathbf{P}\left(\underline{S}_{n/2} \geq 0, S_{n/2} \geq A/2\right)}{n^{1/2}} + \frac{c\mathbf{P}\left(\underline{S}_{n/2} \geq -\alpha, S_{n/2} \geq A/2\right)}{n^{1/2}}.$$

By Lemma 2.3 in [3], there exists a constant c such that for any $\alpha \geq 0$,

$$\sup_{n\geq 1} \mathbf{E}\left[|S_n| \mathbb{1}_{\underline{S}_n \geq -\alpha}\right] \leq c(\alpha+1).$$

It follows from this lemma and Markov's inequality that for any $\alpha \geq 0$,

$$\mathbf{P}\left(\underline{S}_{n/2} \geq -\alpha, S_{n/2} \geq A/2\right) \leq 2\mathbf{E}\left[\frac{|S_{n/2}|}{A}\mathbb{1}_{\underline{S}_{n/2}\geq -\alpha}\right] \leq \frac{c(1+\alpha)}{A}.$$

As a consequence,

$$\mathbf{P}\left(\underline{S}_n \geq -\alpha, \overline{S}_n = S_n \geq A\right) \leq \frac{c(1+\alpha)}{A\sqrt{n}}.$$

□

Proof of (A.7). To obtain (A.7), we consider the two independent random walks $(S_k, 0 \leq k \leq m)$ and $(R_k, 0 \leq k \leq n-m)$. As $S_n = R_{n-m} + S_m$, one immediately sees that

$$\begin{aligned}&\mathbf{P}\Big(\underline{S}_n \geq -\alpha, S_n = \overline{S}_n, \overline{S}_m - S_n \geq -A, \overline{S}_m - S_m \leq a\Big) \\ \leq &\mathbf{P}\Big(\underline{S}_m \geq -\alpha, \overline{S}_m - S_m \leq a, \underline{R}_{n-m} \geq 0, R_{n-m} \in [\overline{S}_m - S_m, \overline{S}_m - S_m + A]\Big) \\ \leq &\mathbf{E}\left[\mathbf{P}\left(\underline{R}_{n-m} \geq 0, R_{n-m} \in [\overline{S}_m - S_m, \overline{S}_m - S_m + A] \middle| (S_k, 0 \leq k \leq m)\right) \mathbb{1}_{\underline{S}_m \geq -\alpha, \overline{S}_m - S_m \leq a}\right].\end{aligned}$$

Applying (A.3) to this conditional probability implies that

$$\begin{aligned}\mathbf{P}\Big(\underline{S}_n \geq -\alpha, S_n = \overline{S}_n, \overline{S}_m - S_n \geq -A, \overline{S}_m - S_m \leq a\Big) \leq &\mathbf{E}\left[c\frac{(1+A)(1+A+\overline{S}_m - S_m)}{(n-m)^{3/2}}\mathbb{1}_{\underline{S}_m \geq -\alpha, \overline{S}_m - S_m \leq a}\right] \\ \leq &c\frac{(1+A)(1+A+a)}{(n-m)^{3/2}}\mathbf{P}\left(\underline{S}_m \geq -\alpha\right),\end{aligned}$$

which by (A.2) is bounded by

$$c\frac{(1+\alpha)(1+A)(1+A+a)}{m^{1/2}(n-m)^{3/2}}.$$

This ends the proof of (A.7). □

Proof of (A.8). Let $T_k := S_{n-k} - S_n = -R_k$. Then $(T_k, 0 \le k \le n)$ is a random walk distributed as $(-S_k, 0 \le k \le n)$. It follows from (A.3) that

$$\mathbf{P}\Big(S_n = \underline{S}_n \ge -\alpha\Big) \le \mathbf{P}\Big(\underline{T}_n \ge 0, T_n \le \alpha\Big) \le \frac{c(1+\alpha)^2}{n^{3/2}}.$$

□

Proof of (A.9). Observe that $e^{\overline{S}_n - S_n} \le \sum_{1\le i\le n} e^{S_i - S_n} \le n e^{\overline{S}_n - S_n}$, then

$$\Big\{ \sum_{1\le i\le n} e^{S_i - S_n} \in [a,b] \Big\} \subset \{\log a - \log n \le \overline{S}_n - S_n \le \log b\}.$$

We thus bound the left hand side of (A.9) as follows

$$\begin{aligned} LHS_{(A.9)} :=& \mathbf{P}\Big(\sum_{1\le i\le n} e^{S_i - S_n} \in [a,b], \underline{S}_n \ge -\alpha \Big) \le \mathbf{P}(\log a - \log n \le \overline{S}_n - S_n \le \log b, \underline{S}_n \ge -\alpha) \\ =& \sum_{k=1}^{n} \mathbf{P}\left(\overline{S}_{k-1} < S_k = \overline{S}_n, \log a - \log n \le \overline{S}_n - S_n \le \log b, \underline{S}_n \ge -\alpha\right). \end{aligned}$$

By Markov property at the first hitting time $\overline{S}_n$,

$$LHS_{(A.9)} \le \sum_{k=1}^{n} \mathbf{P}\left(\underline{S}_k \ge -\alpha, S_k = \overline{S}_k\right) \mathbf{P}\left(\overline{S}_{n-k} \le 0, S_{n-k} \in [-\log b, \log n - \log a]\right).$$

By (A.4) and (A.3), we deduce that

$$\begin{aligned} LHS_{(A.9)} &\le \sum_{k=1}^{n} \frac{c(1+\alpha)}{k} \frac{(1+\log b)(1+\log b - \log a + \log n)}{(n-k+1)^{3/2}} \\ &\le \frac{c(1+\alpha)(1+\log b)(1+\log b - \log a + \log n)}{n}, \end{aligned}$$

which ends the proof. □

Proof of (A.10). By (A.3), one sees that

$$\begin{aligned} LHS_{(A.10)} := \mathbf{E}\left(e^{S_n}; \underline{S}_n \ge -\alpha, S_n \in [a,b]\right) \le& e^b \mathbf{P}\left(\underline{S}_n \ge -\alpha, S_n \in [a,b]\right) \\ \le& c e^b \frac{(1+\alpha)(1+b-a)(b+\alpha+1)}{n^{3/2}}. \end{aligned}$$

□

Proof of (A.11). By considering $R_k = S_n - S_{n-k}$, one sees that

$$\begin{aligned} \mathbf{P}_{(A.11)} :=& \mathbf{P}\left(S_n > \overline{S}_{n-1}, S_n \ge b\sqrt{n} + b_n\right) = \mathbf{P}\left(\underline{R}_n > 0, R_n \ge b\sqrt{n} + b_n\right) \\ =& \mathbf{P}\left(R_n \ge b\sqrt{n} + b_n \middle| \underline{R}_n > 0\right) \mathbf{P}\left(\underline{R}_n > 0\right). \end{aligned}$$

By the theorem in [10] and (1.18), we get the following convergence

$$\lim_{n\to\infty} \sqrt{n} \mathbf{P}_{(A.11)} = \mathbf{P}(m_1 \ge b) \mathbf{c}_2^+ = \mathcal{C}_b.$$

□

Proof of (A.12). Consider the two independent random walks $(S_k, 0 \le k \le n)$ and $(R_k, 0 \le k \le n)$. One observes that

$$\max_{1\le i\le n}(\overline{S}_i - S_i) = \max\left\{\max_{1\le i\le n/2}(\overline{S}_i - S_i), \max_{1\le i\le n/2}(R_k - \underline{R}_{[k,n/2]}), \overline{S}_{n/2} - S_{n/2} + \overline{R}_{n/2} - R_{n/2}\right\},$$

and that

$$\{\underline{S}_n \ge -\alpha, S_n > \overline{S}_{n-1}\} = \{\underline{S}_{n/2} \ge -\alpha\} \cap \{\overline{R}_{n/2} - R_{n/2} \le S_{n/2} + \alpha\} \cap \{\underline{R}_{n/2} > 0\} \cap \{R_{n/2} > \overline{S}_{n/2} - S_{n/2}\}.$$

Let

$$\mathbf{P}_{(A.12)} := \mathbf{P}\left(\underline{S}_n \ge -\alpha, S_n > \overline{S}_{n-1}, \max_{1\le i\le i}(\overline{S}_i - S_i) \le a\sqrt{n} + a_n, S_n \ge b\sqrt{n} + b_n\right).$$

It is immediate that

$$\mathbf{P}_{(A.12)} = \mathbf{P}\Big(\underline{S}_{n/2} \ge -\alpha, \underline{R}_{n/2} > 0, \max_{1\le i\le n/2}(\overline{S}_i - S_i) \le a\sqrt{n} + a_n, R_{n/2} + S_{n/2} > (b\sqrt{n} + b_n) \vee \overline{S}_{n/2},$$
$$\overline{R}_{n/2} - R_{n/2} \le (a\sqrt{n} + a_n - \overline{S}_{n/2} + S_{n/2}) \wedge (S_{n/2} + \alpha), \max_{1\le i\le n/2}(R_k - \underline{R}_{[k,n/2]}) \le a\sqrt{n} + a_n\Big)$$

which equals to

$$\mathbf{E}\left(\Psi_n^{a,b}\left(\frac{S_{n/2}}{\sqrt{n/2}}, \frac{\overline{S}_{n/2} - S_{n/2}}{\sqrt{n/2}}\right); \underline{S}_{n/2} \ge -\alpha, \max_{1\le i\le n/2}(\overline{S}_i - S_i) \le a\sqrt{n} + a_n\right)$$

where

$$\Psi_n^{a,b}(x,h) := \mathbf{P}\Big(\underline{R}_{n/2} > 0\Big) \times \mathbf{P}\Big(\frac{R_{n/2}}{\sqrt{n/2}} > (\sqrt{2}b + \frac{b_n}{\sqrt{n/2}}) \vee (x+h) - x,$$
$$\frac{\overline{R}_{n/2} - R_{n/2}}{\sqrt{n/2}} \le (\sqrt{2}a - h + \frac{a_n}{\sqrt{n/2}}) \wedge (x + \frac{\alpha}{\sqrt{n/2}}), \max_{1\le i\le n/2}(R_k - \underline{R}_{[k,n/2]}) \le a\sqrt{n} + a_n \Big| \underline{R}_{n/2} > 0\Big).$$

Again by invariance principle and (1.18), as $n \to \infty$,

$$\text{(A.25)} \quad \sqrt{n/2}\Psi_n^{a,b}(x,h) \longrightarrow \Psi^{a,b}(x,h) =$$
$$\mathbf{c}_2^+ \mathbf{P}\left(\sigma m_1 > (\sqrt{2}b - x) \vee h, \sigma\overline{m}_1 - \sigma m_1 \le (\sqrt{2}a - h) \wedge x, \max_{0\le s\le 1}\sigma(m_s - \underline{m}_{[s,1]}) \le \sqrt{2}a\right).$$

Because $\Psi_n^{a,b}(x,h)$ is monotone for $x \ge 0$ and $h \ge 0$ and $\Psi^{a,b}$ is continuous, by Dini's theorem, we have uniformly for $(x,h) \in \mathbb{R}_+^2$,

$$\Psi_n^{a,b}(x,h) = \frac{\Psi^{a,b}(x,h) + o_n(1)}{\sqrt{n/2}}.$$

As a consequence,

$$\mathbf{P}_{(A.12)} = \mathbf{E}\left(\frac{\Psi^{a,b}(\frac{S_{n/2}}{\sqrt{n/2}}, \frac{\overline{S}_{n/2} - S_{n/2}}{\sqrt{n/2}}) + o_n(1)}{\sqrt{n/2}}; \underline{S}_{n/2} \ge -\alpha, \max_{1\le i\le n/2}(\overline{S}_i - S_i) \le a\sqrt{n} + a_n\right)$$
$$= \frac{1}{\sqrt{n/2}}\mathbf{E}\left(\Psi^{a,b}\left(\frac{S_{n/2}}{\sqrt{n/2}}, \frac{\overline{S}_{n/2} - S_{n/2}}{\sqrt{n/2}}\right) + o_n(1); \max_{1\le i\le n/2}(\overline{S}_i - S_i) \le a\sqrt{n} + a_n \Big| \underline{S}_{n/2} \ge -\alpha\right) \mathbf{P}\left(\underline{S}_{n/2} \ge -\alpha\right).$$

Once again by invariance principle and the fact that $\lim_{n\to\infty}\sqrt{n}\mathbf{P}(\underline{S}_n \geq -\alpha) = \mathcal{R}(\alpha)\mathbf{c}_1^+$,

$$\mathbf{P}_{(A.12)} = \frac{\mathcal{R}(\alpha)\mathcal{C}_{a,b}}{n} + \frac{o_n(1)}{n},$$

with $\mathcal{C}_{a,b}$ defined in (1.21). □

Proof of (A.13). We turn to consider

$$n\mathbf{E}\left[g\Big(\sum_{j=1}^{n} e^{S_j - S_n}\Big); \underline{S}_n \geq -\alpha, S_n > \overline{S}_{n-1}, \max_{1\leq i\leq n}(\overline{S}_i - S_i) \leq a\sqrt{n} + a_n, S_n \geq b\sqrt{n} + b_n\right].$$

First, we show that in this case with high probability, $\overline{S}_{n/2} \leq S_n - n^{-1/3}$. In fact,

$$\begin{aligned}
&\mathbf{E}\left[g\Big(\sum_{j=1}^{n} e^{S_j - S_n}\Big); \underline{S}_n \geq -\alpha, \overline{S}_{n/2} \geq S_n - n^{-1/3}, S_n > \overline{S}_{n-1}, \max_{1\leq i\leq n}(\overline{S}_i - S_i) \leq a\sqrt{n} + a_n, S_n \geq b\sqrt{n} + b_n\right] \\
&\leq ||g||_\infty \mathbf{P}\left(\underline{S}_n \geq -\alpha, \overline{S}_{n/2} \geq S_n - n^{-1/3}, S_n = \overline{S}_n, S_{n/2} - \overline{S}_{n/2} \leq a\sqrt{n} + a_n\right) \\
&\leq c||g||_\infty \frac{a(1+\alpha)}{n^{1+1/6}},
\end{aligned}$$

where the last inequality follows from (A.7). Now, given $\overline{S}_{n/2} \leq S_n - n^{-1/3}$, $\sum_{j=1}^{n} e^{S_j - S_n}$ can be replaced by $\sum_{n/2\leq j\leq n} e^{S_j - S_n}$ which is independent of $(S_k; 0 \leq k \leq n/2)$. Note that on $\{\overline{S}_{n/2} \leq S_n - n^{-1/3}\}$,

$$\sum_{n/2\leq j\leq n} e^{S_j - S_n} \leq \sum_{j=1}^{n} e^{S_j - S_n} \leq ne^{-n^{1/3}} + \sum_{n/2\leq j\leq n} e^{S_j - S_n}.$$

and that g is uniformly continuous. Hence,

$$\left|g\Big(\sum_{n/2\leq j\leq n} e^{S_j - S_n}\Big) - g\Big(\sum_{j=1}^{n} e^{S_j - S_n}\Big)\right| = o_n(1).$$

Therefore, we deduce that

$$\begin{aligned}
&\mathbf{E}\left[g\Big(\sum_{j=1}^{n} e^{S_j - S_n}\Big); \underline{S}_n \geq -\alpha, S_n > \overline{S}_{n-1}, \max_{1\leq i\leq n}(\overline{S}_i - S_i) \leq a\sqrt{n} + a_n, S_n \geq b\sqrt{n} + b_n\right] \\
=&\mathbf{E}\left[g\Big(\sum_{n/2\leq j\leq n} e^{S_j - S_n}\Big); \underline{S}_n \geq -\alpha, S_n > \overline{S}_{n-1}, \max_{1\leq i\leq n}(\overline{S}_i - S_i) \leq a\sqrt{n} + a_n, S_n \geq b\sqrt{n} + b_n\right] + \frac{o_n(1)}{n}
\end{aligned}$$

Now we use $(R_k, 0 \leq k \leq n/2)$ in replace of $(S_n - S_{n-k}, 0 \leq k \leq n/2)$ and recount on the same arguments as in the proof of (A.12). Thanks to (1.17), (A.13) follows immediately. □

Proof of (A.14). Let $\mathbb{S}_n := -S_n$. Observe that

$$\mathbf{E}[e^{S_n}; \overline{S}_n \leq 0] = \mathbf{E}[e^{-\mathbb{S}_n}; \underline{\mathbb{S}}_n \geq 0] \leq \sum_{k=0}^{\infty} e^{-k}\mathbf{P}\left[\underline{\mathbb{S}}_n \geq 0, \mathbb{S}_n \in [k, k+1)\right].$$

Applying (A.3) to S implies that

$$\mathbf{E}[e^{S_n}; \overline{S}_n \le 0] \le \sum_{k=0}^{\infty} e^{-k} \frac{c(1+k)}{n^{3/2}} \le \frac{c}{n^{3/2}},$$

since $\sum_{k\ge 0}(1+k)e^{-k} < \infty$. □

Proof of (A.15). By applying Markov property at the first hitting time $\overline{S}_n$, one sees that

$$\begin{aligned}
\mathbf{E}[e^{S_n}; \overline{S}_n \le A] &= \sum_{k=0}^{n} \mathbf{E}[e^{S_n}; \overline{S}_{k-1} < S_k \le A, S_k \ge \overline{S}_{[k,n]}] \\
&= \sum_{k=0}^{n} \mathbf{E}[e^{S_k}; \overline{S}_{k-1} < S_k \le A] \mathbf{E}[e^{S_{n-k}}; \overline{S}_{n-k} \le 0] \\
&= \sum_{k=0}^{n} \mathbf{E}[e^{S_k}; \underline{S}_k > 0, S_k \le A] \mathbf{E}[e^{S_{n-k}}; \overline{S}_{n-k} \le 0] \qquad \text{(A.26)}
\end{aligned}$$

where the last equality follows from time-reversing. Next, one observes that for any $k \ge 1$, by (A.3),

$$\begin{aligned}
\mathbf{E}[e^{S_k}; \underline{S}_k > 0, S_k \le A] &\le \sum_{j \in [0,A)\cap \mathbb{Z}} e^{j+1} \mathbf{P}\Big(\underline{S}_k > 0, S_k \in [j, j+1]\Big) \\
&\le \frac{c}{k^{3/2}} \sum_{j \in [0,A)\cap \mathbb{Z}} e^{j+1}(1+j) \le \frac{c(1+A)e^A}{k^{3/2}},
\end{aligned}$$

since $\sum_{j\in[0,A\sqrt{n})\cap\mathbb{Z}} e^{j+1}(1+j) \le c(A+1)e^A$. Plugging this inequality and (A.14) into (A.26) yields that

$$\mathbf{E}[e^{S_n}; \overline{S}_n \le A] \le \sum_{k=0}^{n} \frac{c(1+A)e^A}{(k+1)^{3/2}(n-k+1)^{3/2}} \le \frac{c(1+A)e^A}{n^{3/2}},$$

which is what we need. □

Proof of (A.16). We have,

$$\mathbf{E}\left[\frac{e^{S_n}}{\sum_{1\le i\le n} e^{S_i}}\right] \le \mathbf{E}\left[e^{S_n - \overline{S}_n}\right] = \sum_{k=0}^{n} \mathbf{E}\left[e^{S_n - S_k}; S_k > \overline{S}_{k-1}, S_k \ge \overline{S}_{[k,n]}\right],$$

then by Markov property and a time reversal for $(S_j, 0 \le j \le k)$, one gets that

$$\begin{aligned}
\mathbf{E}\left[\frac{e^{S_n}}{\sum_{1\le i\le n} e^{S_i}}\right] &\le \sum_{k=0}^{n} \mathbf{P}\left(S_k > \overline{S}_{k-1}\right) \mathbf{E}\left[e^{S_{n-k}}; \overline{S}_{n-k} \le 0\right] \\
&\le \sum_{k=0}^{n} \mathbf{P}\left(\underline{S}_k > 0\right) \mathbf{E}\left[e^{S_{n-k}}; \overline{S}_{n-k} \le 0\right].
\end{aligned}$$

By (A.2) and (A.14)

$$\mathbf{E}\left[\frac{e^{S_n}}{\sum_{1\le i\le n} e^{S_i}}\right] \le \sum_{k=0}^{n} \frac{c}{(k+1)^{1/2}(n-k+1)^{3/2}} \le \frac{c}{\sqrt{n}}.$$

□

(A.17) follows immediately from Lemma 3 in [18].

Proof of (A.18). For $\theta > 0$ such that $\varphi(\theta) := \log \mathbf{E}[e^{\theta S_1}] \in (-\infty, \infty)$, $\{e^{\theta S_n - n\varphi(\theta)}; n \geq 0\}$ is a non-negative martingale. The existence of θ comes from (1.3). Therefore, by Doob's inequality,

$$\begin{aligned}\mathbf{P}\left(\overline{S}_n \geq n^{1+\delta}\right) \leq & \mathbf{P}\left(\max_{0\leq k\leq n} e^{\theta S_k - k\varphi(\theta)} \geq e^{\theta n^{1+\delta} - n\varphi(\theta)}\right) \\ \leq & e^{-\theta n^{1+\delta} + n\varphi(\theta)} \mathbf{E}\left[e^{\theta S_n - n\varphi(\theta)}\right] = e^{-\theta n^{1+\delta} + n\varphi(\theta)}.\end{aligned}$$

For n large enough, $\theta n^{1+\delta} - n\varphi(\theta) \geq \theta n^{1+\delta}/2$. Hence, for any $n \geq 1$,

$$\mathbf{P}\left(\overline{S}_n \geq n^{1+\delta}\right) \leq c(\delta, \theta) e^{-\theta n^{1+\delta}/2}.$$

(A.19) can be treated similarly choosing θ properly as a, decreasing to zero, function of n. □

(A.20) is an immediate consequence of (A.12), and we are left to prove (A.21) and (A.22).

Proof of (A.21). Let

$$\mathbf{E}_{(A.21)} := \mathbf{E}\left[e^{S_n - \overline{S}_n}; \max_{1\leq i\leq n} (\overline{S}_i - S_i) \in [a\sqrt{n} + a_n, a\sqrt{n} + a_n + b_n], \underline{S}_n \geq -\alpha, \overline{S}_{n_0} < \overline{S}_n\right].$$

Recall that Y_S is the first hitting time of $\overline{S}_n$. So, $\max_{1\leq i\leq n}(\overline{S}_i - S_i) = \max\{\max_{1\leq i\leq \mathrm{Y}_S}(\overline{S}_i - S_i), S_{\mathrm{Y}_S} - \underline{S}_{[\mathrm{Y}_S, n]}\}$, it follows that

$$\begin{aligned}\text{(A.27)}\quad \mathbf{E}_{(A.21)} \leq & \mathbf{E}\left[e^{S_n - \overline{S}_n}; S_{\mathrm{Y}_S} - \underline{S}_{[\mathrm{Y}_S, n]} \geq a\sqrt{n} + a_n, \underline{S}_n \geq -\alpha, \mathrm{Y}_S > n_0\right] + \\ & \mathbf{E}\left[e^{S_n - \overline{S}_n}; \max_{1\leq i\leq \mathrm{Y}_S} (\overline{S}_i - S_i) \in [a\sqrt{n} + a_n, a\sqrt{n} + a_n + b_n], \underline{S}_n \geq -\alpha, \mathrm{Y}_S > n_0\right].\end{aligned}$$

On the one hand,

$$\begin{aligned}& \mathbf{E}\left[e^{S_n - \overline{S}_n}; S_{\mathrm{Y}_S} - \underline{S}_{[\mathrm{Y}_S, n]} \geq a\sqrt{n} + a_n, \underline{S}_n \geq -\alpha, \mathrm{Y}_S > n_0\right] \\ \leq & \sum_{k=n_0+1}^{n} \mathbf{P}\left(\mathrm{Y}_S = k, \underline{S}_k \geq -\alpha, S_k - \underline{S}_{[k,n]} \geq a\sqrt{n} + a_n, \overline{S}_{[k,n]} \leq S_k\right),\end{aligned}$$

which by Markov property is bounded by

$$\sum_{k=n_0+1}^{n} \mathbf{P}\left(\underline{S}_k \geq -\alpha, S_k = \overline{S}_k\right) \mathbf{P}\left(\underline{S}_{n-k} \leq -a\sqrt{n} - a_n\right).$$

By (A.4) and (A.18),

$$\mathbf{E}\left[e^{S_n - \overline{S}_n}; S_{\mathrm{Y}_S} - \underline{S}_{[\mathrm{Y}_S, n]} \geq a\sqrt{n} + a_n, \underline{S}_n \geq -\alpha, \overline{S}_{n_0} < \overline{S}_n\right] = o_n(1)/n.$$

On the other hand,

$$\mathbf{E}\Big[e^{S_n-\overline{S}_n}; \max_{1\le i\le \Upsilon_S}(\overline{S}_i - S_i) \in [a\sqrt{n}+a_n, a\sqrt{n}+a_n+b_n], \underline{S}_n \ge -\alpha, \Upsilon_S > n_0\Big]$$
$$\le \sum_{k=n_0+1}^{n} \mathbf{E}\Big[e^{S_n-S_k}, \Upsilon_S = k, \max_{1\le i\le k}(\overline{S}_i - S_i) \in [a\sqrt{n}+a_n, a\sqrt{n}+a_n+b_n], \underline{S}_k \ge -\alpha\Big]$$

which by Markov property at time k equals to

$$\sum_{k=n_0+1}^{n} \mathbf{E}\Big[e^{S_{n-k}}\mathbb{1}_{\overline{S}_{n-k}\le 0}\Big]\mathbf{P}\Big(\max_{1\le i\le k}(\overline{S}_i - S_i) \in [a\sqrt{n}+a_n, a\sqrt{n}+a_n+b_n], \underline{S}_k \ge -\alpha, S_k > \overline{S}_{k-1}\Big).$$

By (A.14) and (A.20), this sum is of order $o_n(1)/n$. We hence conclude (A.21). □

Proof of (A.22). Let

$$\mathbf{P}_{(A.22)} := \mathbf{P}\Big(\underline{S}_{[n/2,n]} \le b_n, \underline{S}_n \ge -\alpha, S_n > \overline{S}_{n-1}\Big)$$

Use again the notation $R_k = S_n - S_{n-k}$, we observe that

$$\mathbf{P}_{(A.22)} = \mathbf{P}\Big(\underline{S}_{n/2} \ge -\alpha, \overline{R}_{n/2} - R_{n/2} \in [(S_{n/2}-b_n)_+, S_{n/2}+\alpha], \underline{R}_{n/2} > 0, R_{n/2} > \overline{S}_{n/2} - S_{n/2}\Big)$$
$$\le \mathbf{E}\Big[\underline{S}_{n/2} \ge -\alpha, \hat{f}_n\big(\frac{S_{n/2}}{\sqrt{n/2}}\big)\Big],$$

where

$$\hat{f}_n(x) := \mathbf{P}\left(\frac{\overline{R}_{n/2} - R_{n/2}}{\sqrt{n/2}} \in \Big[(x - \frac{b_n}{\sqrt{n/2}}), x + \frac{\alpha}{\sqrt{n/2}}\Big], \underline{R}_{n/2} > 0\right).$$

By invariance principle, $\mathbf{P}(\frac{\overline{R}_{n/2}-R_{n/2}}{\sqrt{n/2}} \le x | \underline{R}_{n/2} > 0)$ converges to $\mathbf{P}(\overline{\mathfrak{m}}_1 - \mathfrak{m}_1 \le x)$ uniformly for $x \in \mathbb{R}_+$. Consequently,

$$\hat{f}_n(x) = \frac{o_n(1)}{\sqrt{n}}, \text{ uniformly for } x \in \mathbb{R}_+,$$

so

$$\mathbf{P}_{(A.22)} \le \frac{o_n(1)}{\sqrt{n}}\mathbf{P}\Big(\underline{S}_{n/2} \ge -\alpha\Big) = \frac{o_n(1)}{n}.$$

□

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