



A combinatorial theory of random matrices III: random walks on $S(N)$, ramified coverings and the $S(\infty)$ Yang-Mills measure

Franck Gabriel

► To cite this version:

Franck Gabriel. A combinatorial theory of random matrices III: random walks on $S(N)$, ramified coverings and the $S(\infty)$ Yang-Mills measure. 2015. hal-01211302v1

HAL Id: hal-01211302

<https://hal.science/hal-01211302v1>

Preprint submitted on 4 Oct 2015 (v1), last revised 28 Oct 2016 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A COMBINATORIAL THEORY OF RANDOM MATRICES III: RANDOM WALKS ON $\mathfrak{S}(N)$, RAMIFIED COVERINGS AND THE $\mathfrak{S}(\infty)$ YANG-MILLS MEASURE

by

Franck Gabriel

E-mail: franck.gabriel@normalesup.org

Abstract. — The aim of this article is to study some asymptotics of a natural model of random ramified coverings on the disk of degree N . We prove that the monodromy field, called also the holonomy field, converges in probability to a non-random field as N goes to infinity. In order to do so, we use the fact that the monodromy field of random uniform labelled simple ramified coverings on the disk of degree N has the same law as the $\mathfrak{S}(N)$ -Yang-Mills measure associated with the random walk by transposition on $\mathfrak{S}(N)$.

This allows us to restrict our study to random walks on $\mathfrak{S}(N)$: we prove theorems about asymptotics of random walks on $\mathfrak{S}(N)$ in a new framework based on the geometric study of partitions and the Schur-Weyl-Jones's dualities. In particular, given a sequence of conjugacy classes $(\lambda_N \subset \mathfrak{S}(N))_{N \in \mathbb{N}}$, we define a notion of convergence for $(\lambda_N)_{N \in \mathbb{N}}$ which implies the convergence in non-commutative distribution and in \mathcal{P} -expectation of the λ_N -random walk to a \mathcal{P} -free multiplicative Lévy process. This limiting process is shown not to be a free multiplicative Lévy process and we compute its log-cumulant functional. We give also a criterion on $(\lambda_N)_{N \in \mathbb{N}}$ in order to know if the limit is random or not.

Contents

1. Introduction.....	1
2. Convergence of random walks on $\mathfrak{S}(N)$	3
3. Convergence of $YM(\mathfrak{S}(N))$	21
4. Random ramified coverings.....	29
References.....	33

1. Introduction

Yang-Mills theory was introduced by Yang and Mills, in 1954, in [26] as a theory of random connections on a principal bundle with gauge symmetry. In two dimensions, it has been defined by mathematicians ([3], [4], [5], [6], [8], [9], [14], [15], [17], [19], [24], [25]) and it has become well understood that it was a theory of random multiplicative functions from the set of paths of a two dimensional surface to a compact group G . In [13], the author proved that an axiomatic formulation of planar Yang-Mills measures,

similar to the axioms for Lévy processes, could be set: this allowed to prove a correspondence between Yang-Mills measures and a set of Lévy processes on G . In the following, by Yang-Mills measure, we consider the one given by choosing a Brownian motion on G .

When the structure group G is a discrete group, T. Lévy proved in [19] that the Yang-Mills measure could be seen as the random monodromy field of a random ramified G -bundles. Since ramified $\mathfrak{S}(N)$ -bundles are in bijection with ramified coverings with N sheets, one recovers the link explained by A. D’Adda and P. Provero in [1] and [2] between $\mathfrak{S}(N)$ -Yang Mills measure and random branched $\mathfrak{S}(N)$ coverings. It has to be noticed that this link is different from the $U(N)$ -Yang Mills measure/ramified coverings partly explained in [18] and also known as the Yang-Mills/String duality. The theory of random ramified coverings has also some interesting and challenging links with quantum gravity ([27]).

In this article, we study the asymptotic of the theory of random ramified coverings coming from the $\mathfrak{S}(N)$ -Yang Mills measure as N goes to infinity: we construct the $\mathfrak{S}(\infty)$ -master field. The rigorous study of the asymptotics of Yang-Mills measures driven by the Brownian motion on the unitary group begun with M. Anshelevich and A.N. Sengupta in [7] where the convergence was proved for a weak Yang-Mills measure and T. Lévy in [16] where asymptotics and Makeenko-Migdal equations were proved for the full Yang-Mills measure. In this last article, the unitary, orthogonal and symplectic groups were considered, and the convergence of the Yang-Mills measure driven by the different Brownian motions, as the dimension of the group goes to infinity, was proved by using estimates for the speed of convergence in non-commutative distribution of arbitrary words in independent Brownian motions. In the article [10] in preparation, the author and his co-author show how to prove asymptotics of Yang-Mills measures driven by Lévy processes on the unitary and orthogonal groups without using any estimates for the speed of convergence: the asymptotic of Yang-Mills measure is a consequence of the convergence in non-commutative distribution of the Lévy processes considered and a kind of two-dimensional Kolmogorov’s continuity theorem proved by T. Lévy in [19].

Using similar arguments, we prove convergence of Yang-Mills measure driven by random walks on the symmetric groups by proving the convergence in non-commutative distribution of some continuous-time random walks on the symmetric groups. For sake of simplicity, for any integer N , we only consider random walks which jump by an element which is drawn uniformly from a conjugacy class $\lambda(N)$ of $\mathfrak{S}(N)$. If the conjugacy class $\lambda(N)$ converges in some sense, then the random walks will converge in non-commutative distribution. In particular, the eigenvalue distribution will converge when N goes to infinity. When λ_N is the set of transpositions, this result was shown using representation theory [22]. Besides, it seems possible that some of these results could be deduced from the proofs of articles like [20], [21] where the distance from the identity was proved to converge. The study of asymptotics linked with random walks was also one of the concern of the article [23]. In these last articles, the heuristic idea was to consider the symmetric group as a “Lie group” whose “Lie algebra” would be in some sense $\mathbb{Z}[C]$ where $C = \cup_{k=2}^{\infty} ([1, N]^k / \sim)$ where $(i_1, \dots, i_k) \sim (j_1, \dots, j_k)$ if the second one is obtained by a cyclic permutation of the first one. In this picture, the exponential of $c \in C$ would just be the permutation which has c as a single non-trivial cycle. The interesting fact about this is that, one can link easily the Brownian motion on the “Lie algebra” which

drives the Brownian motion on the symmetric group (the random walk by transposition) with some Erdős-Renyi random graph process. Using the natural coupling between the two processes, one can then transfer results from Erdős-Renyi random graph processes to the study of random walks on the symmetric graph.

In this article, we use a generalization of the non-commutative probability ideas, defined in [11] and [12], in order to prove asymptotics and phase transitions for the random walks on the symmetric group without using the coupling with the Erdős-Renyi random graph processes. This allows us to show that asymptotics of random walks on the symmetric groups can be studied with the same tools than the one used for the study of multiplicative unitary Lévy processes (Section 10 of [12]). This method is a generalization of the method used by T. Lévy in [18] or [16] in order to study the large N asymptotics of the Brownian motions on $U(N)$, $O(N)$ and $Sp(N)$. In particular, we do not use any theory of representations, as opposed to [22] where some results were given for the random walk by transpositions. We prove results in a more general setting, in particular we do not ask that the elements of λ_N have bounded support when N goes to infinity. This allows us to show that there exist two behaviors for the eigenvalues empirical distribution: if the size of the support is $o(N)$ then it converges in probability to a non random probability measure we are able to compute explicitly, and if the support is growing like αN , the eigenvalues empirical distribution converges in law to a random measure. As an application of asymptotic \mathcal{P} -freeness of matrices which are invariant by the symmetric group, we get that the whole random walk converges in distribution in non-commutative distribution to a process whose increments are not free but \mathcal{P} -free. This gives the first non-trivial example of \mathcal{P} -free multiplicative Lévy process which log cumulant functional is computed.

1.1. Layout. — The results we present in this article are based on the study of the asymptotic of random walks on the symmetric group (Section 2). The theorems about convergence of random walks on the symmetric group $\mathfrak{S}(N)$ are presented in Section 2. After some preliminary results explained in Section 2.2, we give the proofs of these theorems in Section 2.3. The log-cumulant functional for the limit of random walks on the symmetric group is computed in Section 2.4.

A short presentation of Yang-Mills measure with $\mathfrak{S}(N)$ -gauge group is explained in Section 3. In the same section, we prove that the Wilson loops in $\mathfrak{S}(N)$ -Yang-Mills measure converge in probability as N goes to infinity to a non-random field: the $\mathfrak{S}(\infty)$ -master field.

Based on the results of T. Lévy in [19], we explain in Section 4 how to link the study of random coverings of the disk and the study of $\mathfrak{S}(N)$ -Yang-Mills measure. This allows us to prove that the monodromy field of a model of random simple ramified labelled covering of the unit disk with N sheets converges in probability to a non-random field as N goes to infinity.

2. Convergence of random walks on $\mathfrak{S}(N)$

2.1. General theorems of convergence. — In this section, we state the general theorems about convergence of random walks on the symmetric group. The proofs will

be given in Section 2.3. Let N be a positive integer, let us consider λ_N a conjugacy class of $\mathfrak{S}(N)$, the symmetric group on N elements. We denote by $\#\lambda_N$ the size of the conjugacy class λ_N . Let σ be in λ_N and let i be in $\{1, \dots, N\}$. For any $k \in \{1, \dots, N\}$, the period of k in σ is the smallest positive integer n such that $\sigma^n(k) = k$. We denote by $\lambda_N(i)$ the number of elements in $\{1, \dots, N\}$ which period in σ is equal to i : this number does not depend on the choice of σ . Thus, we can see λ_N as a partition of N :

$$\lambda = (\lambda_N(i))_{i=1}^{\infty} \text{ and } \sum_{i=1}^{\infty} \lambda_N(i) = N.$$

Definition 2.1. — We define the λ_N -random walk on $\mathfrak{S}(N)$, denoted by $(S_t^N)_{t \geq 0}$, as the Markov process on $\mathfrak{S}(N)$ such that $S_0^N = \text{id}_N$ and whose generator is given by:

$$\forall f \in \mathbb{R}^{\mathfrak{S}(N)}, \forall \sigma_0 \in \mathfrak{S}(N), H_N f(\sigma_0) = \frac{N}{\lambda_N(1^c)} \frac{1}{\#\lambda_N} \sum_{\sigma \in \lambda_N} [f(\sigma \sigma_0) - f(\sigma_0)],$$

where we used the following notation:

$$\lambda_N(1^c) = N - \lambda_N(1).$$

This random walk is invariant by conjugation by $\mathfrak{S}(N)$.

Lemma 2.1. — Let σ be in $\mathfrak{S}(N)$, let $(S_t^N)_{t \geq 0}$ be a λ_N -random walk on $\mathfrak{S}(N)$. We have the equality in law: $(\sigma S_t^N \sigma^{-1})_{t \geq 0} = (S_t^N)_{t \geq 0}$.

Proof. — It is a consequence of the fact that the generator of $(S_t^N)_{t \geq 0}$ is invariant by conjugation by $\mathfrak{S}(N)$. \square

For any positive integer N , let λ_N be a conjugacy class of $\mathfrak{S}(N)$.

Definition 2.2. — The sequence $(\lambda_N)_{N \in \mathbb{N}}$ converges if and only if there exists:

$$(\lambda(i))_{i \geq 2} \in \left\{ (a_i)_{i \in \mathbb{N} \setminus \{0,1\}} \mid \forall i \geq 2, a_i \geq 0, \sum_{i=2}^{\infty} a_i \leq 1 \right\}$$

such that for any integer $i \geq 2$:

$$\frac{\lambda_N(i)}{\lambda_N(1^c)} \xrightarrow{N \rightarrow \infty} \lambda(i),$$

and there exists $\alpha \in [0, 1]$ such that:

$$\frac{\lambda_N(1)}{N} \xrightarrow{N \rightarrow \infty} 1 - \alpha.$$

The sequence $(\lambda_N)_{N \in \mathbb{N}}$ is evanescent if $\alpha = 0$ and it is macroscopic if $\alpha > 0$.

Moreover we say that $(\lambda_N)_{N \in \mathbb{N}}$ converges at any order of fluctuations if one can find $((\lambda^{(m)}(i))_{m \in \mathbb{N}})_{i \geq 2}$ and $(\beta^{(m)})_{m \in \mathbb{N}}$ such that for any $k \geq 0$

$$\frac{\lambda_N(i)}{\lambda_N(1^c)} - \sum_{m=0}^k \frac{\lambda^{(m)}(i)}{N^m} = o\left(\frac{1}{N^k}\right),$$

$$\frac{\lambda_N(1)}{N} - \sum_{m=0}^k \frac{\beta^{(m)}}{N^m} = o\left(\frac{1}{N^k}\right).$$

For any positive integer N , let us consider $(S_t^N)_{t \geq 0}$ a λ_N -random walk on $\mathfrak{S}(N)$. Recall Definition 5.1 of [12]. Let us suppose that the sequence $(\lambda_N)_{N \in \mathbb{N}}$ converges.

Theorem 2.1. — *For any $t \geq 0$, the mean empirical eigenvalues distribution of $(S_t^N)_{N \in \mathbb{N}}$ converges as N goes to infinity to a probability measure $\bar{\mu}_t^\lambda$ which has the form:*

$$\bar{\mu}_t^\lambda = \sum_{n \in \mathbb{N}^*} \sum_{k=0}^{n-1} \frac{m_{n^c}(t)}{n} \delta_{e^{\frac{2ik\pi}{n}}} + m_{\infty^c}(t) \lambda_{\mathbb{U}},$$

with $m_{\infty^c}(t) = 1 - \sum_{k=0}^{\infty} m_{n^c}(t) \geq 0$ and $m_{n^c}(t) \geq 0$ for any integer n .

Recall that $[\sigma]$ is the conjugacy class of σ which can be seen as a partition of N . Let us take the convention that for any positive integer k : $\frac{((1-0)^k - 1)}{0} = -k$. Let us consider the unique solution $((m_{\sigma^c}(t))_{\sigma \in \cup_k \mathfrak{S}_k})_{t \geq 0}$ of the system of differential equations: $\forall k \in \mathbb{N}^*$, $\forall \sigma_0 \in \mathfrak{S}_k$, $\forall t_0 \geq 0$:

$$\begin{aligned} \frac{d}{dt} m_{\sigma_0^c}(t) &= \frac{((1-\alpha)^k - 1)}{\alpha} m_{\sigma_0^c}(t_0) \\ &+ \sum_{\sigma \in \mathfrak{S}_k \setminus \{id_k\}, \sigma \leq \sigma_0} \alpha^{nc(\sigma \vee id) - [\sigma](1) - 1} \left(\prod_{i=2}^k (\lambda(i))^{\frac{[\sigma](i)}{i}} \right) (1-\alpha)^{[\sigma](1)} m_{(\sigma \sigma_0)^c}(t_0), \end{aligned}$$

with the initial conditions: $\forall k \in \mathbb{N}^*$, $\forall \sigma \in \mathfrak{S}_k$:

$$m_{\sigma^c}(0) = \delta_{\sigma=id_k}.$$

Then for any positive integer n , for any real $t \geq 0$:

$$m_{n^c}(t) = m_{(1, \dots, n)^c}(t),$$

where $(1, \dots, n) \in \mathfrak{S}_n$ is a n cycle. Besides for any $t \geq 0$ and any $\sigma \in \mathfrak{S}_k$:

$$m_{\sigma^c}(t) = \lim_{N \rightarrow \infty} \mathbb{E} m_{\sigma^c} \left[(S_t^N)^{\otimes k} \right].$$

This theorem can be extended to the whole process $(S_t^N)_{t \geq 0}$. Recall the notion of convergence in \mathcal{P} -expectation defined in Definition 2.2 of [12]. Recall also the Definition 7.5 in [12].

Theorem 2.2. — *The family $(S_t^N)_{t \geq 0}$ converges in \mathcal{P} -expectation as N goes to infinity. The process $\left((S_t^N)_{N \in \mathbb{N}}\right)_{t \geq 0}$ is a matricial \mathcal{P} -free multiplicative Lévy process. It is not a \mathfrak{S} -free multiplicative Lévy process. In particular, if $(\lambda_N)_{N \in \mathbb{N}}$ is evanescent, it is not a free multiplicative Lévy process in the sense of Voiculescu: the multiplicative increments are not asymptotically free.*

Besides, if $(\lambda_N)_{N \in \mathbb{N}}$ converges at any order of fluctuations, then the family $(S_t^N)_{t \geq 0}$ converges in \mathcal{P} -expectation up to any order of fluctuations as N goes to infinity.

In fact, the proof of Theorem 2.1 will show that the empirical eigenvalues distribution of $(S_t^N)_{N \in \mathbb{N}}$ converges in law as N goes to infinity to a random measure: depending on the behavior of $(\lambda_N)_{N \in \mathbb{N}}$, one can know if the limit is or is not non-random.

Theorem 2.3. — *Let t be a positive real. The empirical eigenvalues distribution $\mu_t^{\lambda_N}$ of S_t^N converges in law to a random measure on \mathbb{U} , denoted by μ_t^λ . Two behaviors are possible:*

1. *if the sequence $(\lambda_N)_{N \in \mathbb{N}}$ is evanescent then the limiting measure is a non-random measure on \mathbb{U} , $\mu_t^\lambda = \bar{\mu}_t^\lambda$, and the convergence holds in probability. The family $(S_t^N)_{N \in \mathbb{N}}$ satisfies the asymptotic \mathcal{P} -factorization and thus it converges in probability in \mathcal{P} -normalized moments,*
2. *if the sequence $(\lambda_N)_{N \in \mathbb{N}}$ is macroscopic, then the limiting measure is not a non-random measure on \mathbb{U} and the family $(S_t^N)_{N \in \mathbb{N}}$ does not satisfy the asymptotic \mathcal{P} -factorization.*

In the case where the sequence $(\lambda_N)_{N \in \mathbb{N}}$ is evanescent, we can compute explicitly the measure μ_t^λ . Given Theorem 2.1, we only need to compute its moments or equivalently $m_{n^c}(t)$.

Theorem 2.4. — *Let us suppose that $(\lambda_N)_{N \in \mathbb{N}}$ is evanescent. Let us use the same notations as in Theorem 2.1 and in Definition 2.2. Let n be a positive integer and let $t \geq 0$. We have:*

$$(1) \quad m_{n^c}(t) = e^{-nt} \sum_{k=0}^{n-1} t^k \frac{n^{k-1}}{k!} \sum_{(i_1, \dots, i_k) \in \mathbb{N}^*, \sum_{j=1}^k i_j = n-1} \prod_{j=1}^k \lambda(i_j + 1).$$

We used the usual conventions for the products and the sums, thus $m_{1^c}(t) = e^{-t}$. In particular for any positive integer n , and any $t \geq 0$:

$$m_n(t) := \int_{z \in \mathbb{U}} z^n d\bar{\mu}_t^\lambda(z) = \sum_{d|n} e^{-dt} \sum_{k=0}^{d-1} t^k \frac{d^{k-1}}{k!} \sum_{(i_1, \dots, i_k) \in \mathbb{N}^*, \sum_{j=1}^k i_j = d-1} \prod_{j=1}^k \lambda(i_j + 1).$$

In particular, let us consider a positive integer k , let us suppose that $\lambda(k) = 1$ and for any positive integer $l \neq k$, $\lambda(l) = 0$. This means that we are considering a random walk which jumps by multiplication by a uniform k -cycle. Let t be a non-negative real. If there does not exist any positive integer u such that $n = u(k-1) + 1$, then $m_{n^c}(t) = 0$.

Besides, for any $u \in \mathbb{N}$:

$$(2) \quad m_{(u(k-1)+1)^c}(t) = e^{-(u(k-1)+1)t} t^u \frac{(u(k-1)+1)^{u-1}}{u!}.$$

Let us suppose for this discussion that $(\lambda_N)_{N \in \mathbb{N}}$ is evanescent. In Theorem 2.1, we saw that the measure μ_t^λ is the sum of an atomic part and $m_{\infty^c}(t)$ times the Lebesgue measure on the unit circle. In fact there exists a real $t_c^\lambda \geq 0$ such that μ_t^λ is purely atomic for $t < t_c^\lambda$ and for any $t > t_c^\lambda$, μ_t^λ is a sum of a purely atomic and a multiple of the Lebesgue measure. This critical time is the same critical as found by N. Berestycki in [21] for the phase transition for the distance to the identity.

Theorem 2.5. — *Let us suppose that $(\lambda_N)_{N \in \mathbb{N}}$ is evanescent. The function $t \mapsto m_{\infty^c}(t)$ is continuous and converges to 1 as t goes to infinity. Besides, if we define:*

$$(3) \quad t_c^\lambda = \delta_{\sum_{j=2}^\infty \lambda(j)=1} \frac{1}{\sum_{j=2}^\infty (j-1)\lambda(j)},$$

for any $0 \leq t \leq t_c^\lambda$, $m_{\infty^c}(t) = 0$ and for any $t > t_c^\lambda$, $m_{\infty^c}(t) > 0$.

Using the theorems already explained, we recover Theorem 3 of [20], Theorem 4 of [22]. We recommend the reader to have also a look at Theorem 3 of [21].

Corollary 2.1. — *Let us suppose that $(\lambda_N)_{N \in \mathbb{N}}$ converges and is evanescent. For any positive integer N , let $(S_t^N)_{t \geq 0}$ be a λ_N -random walk on the symmetric group. For any permutations σ and σ' in $\mathfrak{S}(N)$, let $d_{\mathfrak{S}(N)}(\sigma, \sigma')$ be the distance in $\mathfrak{S}(N)$ between σ and σ' defined as $N - \text{nc}(\sigma \vee \sigma')$, where nc is the function which gives the number of cycles. Then for any $t \geq 0$, $\frac{1}{N} d_{\mathfrak{S}(N)}(id_N, S_t^N)$ converges in probability when N goes to infinity to the non-random continuous function:*

$$d^\lambda(t) = 1 - \sum_{k=1}^\infty \frac{1}{k} m_{k^c}(t),$$

where $m_{k^c}(t)$ is given by Equation (1).

Using the Equation (2), we recover Equation (5) of [21], yet this expression of $d^\lambda(t)$ for general λ seems to be new. The function $d^\lambda(t)$ was studied in [21], when $(\lambda_N(1^c))_{N \in \mathbb{N}}$ is constant and equal to a positive integer a : using t_c^λ defined before, it was shown that $d^\lambda(t)$ is C^∞ on a subset of the form $\mathbb{R}^+ \subset I$, with I a bounded interval of $]t_0, \infty[$, for any $t < t_c^\lambda$, $d^\lambda(t) = \frac{t}{a}$ and $(d^\lambda)''((t_c^\lambda)^+) = -\infty$. Using the Stirling's formula, it is easy to see that for the random walk which only jumps by multiplication by a k -cycle, the set I is empty.

In the second part of the article, in order to construct the Yang-Mills \mathfrak{S}_∞ -field, we will need the following result whose proof will not be given since it is an easy consequence of Theorems 2.2, 2.3, Lemma 2.1 of this article and the Theorem 7.8 and Proposition 2.1 of [12].

Let n be a positive integer and for any positive integer N , let us consider $(S_t^{1,N})_{t \geq 0}$, $\dots, (S_t^{n,N})_{t \geq 0}$, n independent λ_N -random walks on $\mathfrak{S}(N)$. Recall the Equation (4) of [12] and Section 2.2 of [11].

Theorem 2.6. — *The family of random matrices:*

$$\mathbf{F} = \left(\left(S_t^{k,N} \right)_{N \in \mathbb{N}}, \left(\left(S_t^{k,N} \right)^{-1} \right)_{N \in \mathbb{N}} \right)_{t \geq 0, k \in \{1, \dots, n\}}$$

converges in \mathcal{P} -expectation as N goes to infinity. Besides, if $(\lambda_N)_{N \in \mathbb{N}}$ is evanescent, then \mathbf{F} satisfies the asymptotic \mathcal{P} -factorization property as N goes to infinity. This implies that \mathbf{F} converges in probability in \mathcal{P} -normalized moment: for any positive integer k , for any $p \in \mathcal{P}_k$, for any k -uple $\left((B_1^N)_{N \in \mathbb{N}}, \dots, (B_k^N)_{N \in \mathbb{N}} \right)$ of elements of $\mathcal{A}(\mathbf{F})$:

$$\frac{1}{N^{\text{nc}(p \vee \text{id})}} \text{Tr}^k \left((B_1^N \otimes \dots \otimes B_k^N) \rho_N^{\mathcal{P}_k}(p) \right)$$

converges in probability to the limit of its expectation as N goes to infinity.

Besides, for any $i \in \{1, \dots, k\}$, $\left(S_t^{i,N} \right)_{t \geq 0} \cup \left(\left(S_t^{i,N} \right)^{-1} \right)_{t \geq 0}$ is asymptotically \mathcal{P} -free but not asymptotically free in the sense of Voiculescu from

$$\left(\left(S_t^{k,N} \right)_{N \in \mathbb{N}}, \left(\left(S_t^{k,N} \right)^{-1} \right)_{N \in \mathbb{N}} \right)_{t \geq 0, k \in \{1, \dots, n\} \setminus \{i\}}.$$

2.2. Preliminary results. — Before we prove the theorems we just stated, we need some preliminary results. We could have written them in [12], yet since most of them are specific to permutation matrices we preferred to explain them in this article.

2.2.1. Exclusive moments for permutation matrices. — Let k be a positive integer. Recall the notions of irreducible partition and of extraction defined in Definitions 2.9 and 2.12 of [11]. Let p be a partition, we recall that $l(p)$ is the length of p : it is the unique integer such that $p \in \mathcal{P}_{l(p)}$.

Definition 2.3. — Let $I = \{b_1, \dots, b_s\}$ be a partition of $\{1, \dots, k\}$, let σ be an irreducible permutation of $\{1, \dots, s\}$. For any integer $l \in \{1, \dots, s\}$, we denote by b'_l the set $\{j \in \{1', \dots, k'\}, \exists i \in b_l, j = i'\}$. The partition:

$$p = \left\{ b_l \cup b'_{\sigma(l)}, l \in \{1, \dots, s\} \right\}$$

is called the necklace associated with (I, σ) . The true-length of p , denoted $|p|$, is equal to s .

Let p be an irreducible partition in \mathcal{P}_k : it is a chain if and only if there exist two blocks s and r of p such that the partition $\tilde{p} = p \setminus \{r, s\} \cup \{r \cup s\}$ is a necklace. The true-length of p , denoted $|p|$, is equal to the true-length of \tilde{p} .

A partition p in \mathcal{P}_k is a parure if for any cycle c of p the extraction of p on c is either a chain or a necklace. The true-length of p is defined by:

$$|p| = \sum_{c \text{ cycle of } p} |p_c|.$$

Let N be a positive integer, let S be a permutation in $\mathfrak{S}(N)$. Recall Sections 2.2 and 2.3 of [11]. For sake of clarity, in the following we will denote by $m_{p^c}(S)$ the number $\frac{1}{N^{\text{nc}(p \vee \text{id})}} \text{Tr}(S^{\otimes k} \circ \rho_N^{\mathcal{P}_k}(p^c))$.

Proposition 2.1. — *Let S be a permutation in $\mathfrak{S}(N)$, let p be a partition in \mathcal{P}_k . If p is not a parure then $m_{p^c}(S) = 0$. If p is a necklace, then $m_{p^c}(S) = m_{(1, \dots, |p|)^c}(S)$, and if p is a chain, then:*

$$m_{p^c}(S) = 1 - \sum_{l=1}^{|p|} m_{(1, \dots, l)^c}(S).$$

Proof. — Let S be a permutation in $\mathfrak{S}(N)$. Let p be a partition in \mathcal{P}_k . Let us consider a block b of p . Let u and v (resp. u' and v') be in $\{1, \dots, k\}$ (resp. $\{1', \dots, k'\}$) which are in the same block of p . Then u' and v' (resp. u and v) must be in the same block of p if one wants $m_{p^c}(S)$ not to be equal to zero. This is a consequence of the fact that for any $i, j, l \in \{1, \dots, N\}$,

$$(4) \quad S_i^j S_i^l = S_i^j \delta_{l=j} \text{ and } S_j^i S_l^i = S_j^i \delta_{l=j}.$$

Yet, if p is not a parure, these conditions on the blocks of p are not satisfied, thus $m_{p^c}(S) = 0$.

The assertion about the exclusive moments when p is a necklace is a consequence of the Equations 4. Let us suppose that p is a chain. Using the Equations (4), we get that $m_{p^c}(S)$ is the fraction of elements of $\{1, \dots, N\}$ which period in S is strictly greater than $|p|$. Thus it is equal to one minus the fraction of elements of $\{1, \dots, N\}$ which period in S is less than $|p|$. Since for any positive integer l , $m_{(1, \dots, l)^c}(S)$ is the fraction of elements of $\{1, \dots, N\}$ which period in S is equal to l , we get the following equality:

$$m_{p^c}(S) = 1 - \sum_{l \geq 1, l \leq |p|} m_{(1, \dots, l)^c}(S),$$

which is the equality we had to prove. \square

2.2.2. Link between moments and exclusive moments for permutation matrices. — When one considers permutation matrices, an interesting link occurs between normalized moments and normalized exclusive moments. Let N be a positive integer and let S be a permutation in $\mathfrak{S}(N)$. For any positive integer l , $(1, \dots, l)$ is the l -cycle in \mathfrak{S}_l .

Proposition 2.2. — *For any positive integer k ,*

$$Tr \left[S^{\otimes k} \circ \rho_N^{\mathcal{P}_k}((1, \dots, k)) \right] = \sum_{d \in \mathbb{N}^*, d|k} Tr \left[S^{\otimes d} \circ \rho_N^{\mathcal{P}_k}((1, \dots, d)^c) \right].$$

Proof. — This is due to the fact that for any positive integer k ,

$$Tr \left[S^{\otimes k} \circ \rho_N^{\mathcal{P}_k}((1, \dots, k)) \right] = \frac{1}{N} Tr \left(S^k \right)$$

is equal to the fraction of elements $i \in \{1, \dots, N\}$ whose period divides k and $Tr \left[S^{\otimes d} \circ \rho_N^{\mathcal{P}_k}((1, \dots, d)^c) \right]$ is equal to the fraction of elements $i \in \{1, \dots, N\}$ whose period is equal to d . \square

2.2.3. Measures. — The following lemma is a special and easy case of the problem of moments.

Lemma 2.2. — *Let $(\kappa_n)_{n \in \mathbb{N}}$ be a sequence of positive numbers such that $\sum_{i=1}^{\infty} \kappa_i \leq \kappa_0$. There exists a unique measure μ on \mathbb{U} whose weight is equal to κ_0 such that:*

$$\forall n \in \mathbb{N}^*, \int_{\mathbb{U}} z^n d\mu(z) = \sum_{d \in \mathbb{N}^*, d|n} \kappa_d.$$

Besides μ is equal to:

$$\sum_{n \in \mathbb{N}^*} \sum_{k=0}^{n-1} \frac{\kappa_n}{n} \delta_{e^{\frac{2ik\pi}{n}}} + \left[\kappa_0 - \sum_{i=1}^{\infty} \kappa_i \right] \lambda_{\mathbb{U}}.$$

Proof. — Any measure on the unit circle \mathbb{U} is characterized by its non-negative moments as we saw in the proof of Theorem 2.1 of [12]. Let $(\kappa_n)_{n \in \mathbb{N}}$ be a sequence of positive numbers such that $\sum_{i=1}^{\infty} \kappa_i \leq \kappa_0$. It is enough to see that the moments of $\sum_{n \in \mathbb{N}^*} \sum_{k=0}^{n-1} \frac{\kappa_n}{n} \delta_{e^{\frac{2ik\pi}{n}}} + [\kappa_0 - \sum_{i=1}^{\infty} \kappa_i] \lambda_{\mathbb{U}}$ are equal to the ones expected. \square

Let $(\kappa_n)_{n \in \mathbb{N}}$ be a sequence of positive numbers such that $\sum_{i=1}^{\infty} \kappa_i \leq \kappa_0$. Let μ be the unique measure associated to $(\kappa_n)_{n \in \mathbb{N}}$ which is given by Lemma 2.2. By looking at the form of μ , one can see that the weight of the purely atomic part of μ is equal to $\sum_{n \geq 1} \kappa_n$.

2.2.4. Criterion for non \mathfrak{S} -freeness. — In the article [12], we saw that the Voiculescu asymptotic freeness and the asymptotic \mathcal{P} -freeness are not the same notions. Let us state some consequence of Proposition 7.1 of [12] when one considers random matrices whose entries are equal either to zero or one. Recall that 0_2 is the partition $\{\{1, 2, 1', 2'\}\}$ in \mathcal{P}_2 . Recall that the notion of $\mathcal{L}^{\infty-} \otimes \mathcal{M}(\mathbb{C})$ was defined at the beginning of Section 2.1 of [12]. At last, recall Definition 2.1 and Notation 2.1 of [12].

Lemma 2.3. — *Let $S = (S^N)_{N \in \mathbb{N}}$ be an element of $L^{\infty-} \otimes \mathcal{M}(\{0, 1\})$. Let us suppose that S converges in expectation in \mathcal{P} -moments then:*

$$\mathbb{E} \kappa_{\mathcal{P}}^{0_2}[S, S] = \mathbb{E} m_{id_1}[S] - \mathbb{E} m_{id_2}[S, S].$$

If the asymptotic \mathfrak{S} -factorization property holds for S , then:

$$\mathbb{E} \kappa_{\mathcal{P}}^{0_2}[S, S] = \mathbb{E} m_{id_1}[S] (1 - \mathbb{E} m_{id_1}[S]).$$

Proof. — Indeed, we have $\mathbb{E} \kappa_{\mathcal{P}}^{0_2}[S, S] = \mathbb{E} m_{0_2}[S, S] - \mathbb{E} m_{id_2}[S, S]$. Yet, one does not forget that for any integer N , S^N is a matrix of zeros and ones, thus for any positive integer N , $\mathbb{E} m_{0_2}[S^N \otimes S^N] = \mathbb{E} m_{id_1}[S^N]$. This implies that:

$$\mathbb{E} \kappa_{\mathcal{P}}^{0_2}[S, S] = \mathbb{E} m_{id_1}[S] - \mathbb{E} m_{id_2}[S, S].$$

The second assertion is a direct consequence of the \mathfrak{S} -factorization property. \square

Proposition 2.3. — *Let S_1 and S_2 be two elements of $L^{\infty-} \otimes \mathcal{M}(\{0, 1\})$ which converge in expectation in \mathcal{P} -moments and which are asymptotically \mathcal{P} -free. If for $i \in \{1, 2\}$, $\mathbb{E} m_{id_2}[S_i, S_i] \neq \mathbb{E} m_{id_1}[S_i]$, which is equivalent to say that $\frac{1}{N} \text{Tr}(S_i^N)$ does not converge*

in law to a random variable which takes value in $\{0, 1\}$, then S_1 and S_2 are not asymptotically \mathfrak{S} -free.

In particular, if the asymptotic \mathcal{P} -factorization property holds for both of them, if $\mathbb{E}m_{id}[S_1] \notin \{0, 1\}$ and $\mathbb{E}m_{id}[S_2] \notin \{0, 1\}$, then S_1 and S_2 are not asymptotically free in the sense of Voiculescu.

Proof. — This is a consequence of Proposition 7.1 of [12], of Lemma 2.3 of this article and also of the fact that under the hypothesis of asymptotic factorization the \mathfrak{S} -freeness is equivalent to the usual freeness as explained in Section 7.1.3 of [12]. \square

2.3. Proofs of the theorems of Section 2.1. — We can now prove the theorems of Section 2.1.

Proof of Theorem 2.1 and Theorem 2.2. — For any positive integer N , let us consider λ_N a conjugacy class of $\mathfrak{S}(N)$. Let us suppose that $(\lambda_N)_{N \in \mathbb{N}}$ converges as N goes to infinity. For any positive integer N , let us consider $(S_t^N)_{t \geq 0}$ a λ_N -random walk on $\mathfrak{S}(N)$. Let N and k be two positive integers and let us define:

$$G_k^N = \frac{d}{dt} \Big|_{t=0} \mathbb{E} \left[(S_t^N)^{\otimes k} \right] = \frac{N}{\lambda_N(1^c)} \frac{1}{\#\lambda_N} \sum_{\sigma \in \lambda_N} [\sigma^{\otimes k} - id^{\otimes k}].$$

Let p be a partition in \mathcal{P}_k . Let $\sigma_N \in \lambda_N$, let us remark that:

$$m_{p^c}(G_k^N) = \frac{N}{\lambda_N(1^c)} \left[m_{p^c}(\sigma_N^{\otimes k}) - m_{p^c}(id^{\otimes k}) \right],$$

thus, using Proposition 2.1, if p is not a parure, then $m_{p^c}(G_k^N) = 0$.

Let us suppose that p is an irreducible parure, then it is either a necklace or a chain. Yet, using Proposition 2.1, it is enough to consider the case when p is a necklace, and thus when it is a cycle. Let us suppose that $p = (1, \dots, k)$, then:

$$m_{p^c}(G_k^N) = \frac{N}{\lambda_N(1^c)} \left[m_{(1, \dots, k)^c}(\sigma_N^{\otimes k}) - m_{(1, \dots, k)^c}(id^{\otimes k}) \right].$$

If $k = 1$, then:

$$m_{p^c}(G_k^N) = \frac{N}{\lambda_N(1^c)} \left[\frac{\lambda_N(1)}{N} - \frac{N}{N} \right] = -1.$$

If $k \neq 1$, then:

$$m_{p^c}(G_k^N) = \frac{N}{\lambda_N(1^c)} \left[\frac{\lambda_N(k)}{N} \right] = \frac{\lambda_N(k)}{\lambda_N(1^c)} \xrightarrow{N \rightarrow \infty} \lambda(k).$$

If p is a chain, let us remark that, using again Proposition 2.1:

$$\begin{aligned} m_{p^c}(G_k^N) &= \frac{N}{\lambda_N(1^c)} m_{p^c}(\sigma_N^{\otimes k}) \\ &= \frac{N}{\lambda_N(1^c)} \left[1 - \sum_{l=1}^{|p|} m_{(1, \dots, l)^c}(\sigma_N^{\otimes l}) \right] \xrightarrow{N \rightarrow \infty} 1 - \sum_{l=2}^{|p|} \lambda(l). \end{aligned}$$

Thus, for any irreducible partition, $m_{p^c}(G_k^N)$ converges as N goes to infinity. Yet, if p is irreducible, any partition p' which is coarser than p is also irreducible. This implies that, for any irreducible partition, $m_p(G_k^N)$ converges as N goes to infinity.

Let us remark that for any partition p in \mathcal{P}_k :

$$(5) \quad m_p(G_k^N) = \frac{N}{\lambda_N(1^c)} \left[m_p(\sigma_N^{\otimes k}) - 1 \right].$$

Thus, we have proved that for any irreducible partition p , $m_p(\sigma_N^{\otimes k})$ converges as N goes to infinity, and besides, if $(\lambda_N)_{N \in \mathbb{N}}$ is evanescent, then $\lim_{N \rightarrow \infty} m_p(\sigma_N^{\otimes k}) = 1$. Let r be a positive integer, let us consider r irreducible partitions p_1, \dots, p_r , we have:

$$\begin{aligned} m_{p_1 \otimes \dots \otimes p_r}(G_k^N) &= \frac{N}{\lambda_N(1^c)} \left[m_{p_1 \otimes \dots \otimes p_r}(\sigma_N^{\otimes k}) - 1 \right] \\ &= \frac{N}{\lambda_N(1^c)} \left[\prod_{i=1}^r m_{p_i}(\sigma_N^{\otimes k}) - 1 \right] \\ &= \sum_{i=1}^r \left(\frac{N}{\lambda_N(1^c)} \left[m_{p_i}(\sigma_N^{\otimes k}) - 1 \right] \prod_{l=i+1}^r m_{p_l}(\sigma_N^{\otimes k}) \right) \\ &= \sum_{i=1}^r \left(m_{p_i}(G_k^N) \prod_{l=i+1}^r m_{p_l}(\sigma_N^{\otimes k}) \right). \end{aligned}$$

This proves that $m_{p_1 \otimes \dots \otimes p_r}(G_k^N)$ converges as N goes to infinity. Thus for any positive integer k , for any partition $p \in \mathcal{P}_k$, $m_p(G_k^N)$ converges as N goes to infinity.

Using Theorem 10.1 of [12], the family $(S_t^N)_{t \geq 0}$ converges in \mathcal{P} -expectation and $((S_t^N)_{N \in \mathbb{N}})_{t \geq 0}$ is a \mathcal{P} -matricial free Lévy process. It is easy to see that the last assertion of Theorem 2.2 holds.

In particular, for any $t \geq 0$, $(S_t^N)_{N \in \mathbb{N}}$ converges in \mathcal{P} -expectation as N goes to infinity. Using Theorem 2.1 of [12], we deduce that the mean empirical eigenvalues distribution of $(S_t^N)_{N \in \mathbb{N}}$ converges as N goes to infinity to a probability measure $\bar{\mu}_t^\lambda$ defined on the circle \mathbb{U} . Besides, the measure $\bar{\mu}_t^\lambda$ is characterized by the fact that for any positive integer n :

$$\int_{\mathbb{U}} z^n d\bar{\mu}_t^\lambda = \lim_{N \rightarrow \infty} \mathbb{E} m_{(1, \dots, n)} \left[(S_t^N)^{\otimes n} \right].$$

Using Proposition 2.2, we get that:

$$\int_{\mathbb{U}} z^n d\bar{\mu}_t^\lambda = \sum_{d \in \mathbb{N}^*, d|n} \lim_{N \rightarrow \infty} \mathbb{E} m_{(1, \dots, d)^c} \left[(S_t^N)^{\otimes d} \right].$$

We are in the setting of Lemma 2.2 thus $\bar{\mu}_t^\lambda$ is equal to:

$$\bar{\mu}_t^\lambda = \sum_{n \in \mathbb{N}^*} \sum_{k=0}^{n-1} \frac{m_{n^c}(t)}{n} \delta_{e^{\frac{2ik\pi}{n}}} + m_{\infty^c}(t) \lambda_{\mathbb{U}},$$

where for any integer $n \in \mathbb{N}^*$, $m_{n^c}(t) = \lim_{N \rightarrow \infty} \mathbb{E} m_{(1, \dots, n)^c} \left[(S_t^N)^{\otimes n} \right]$.

For any positive integer k , any $\sigma \in \mathfrak{S}_k$ and any $t \geq 0$, let us denote by $m_{\sigma^c}(t)$ the limit $\lim_{N \rightarrow \infty} \mathbb{E} m_{\sigma^c} \left[(S_t^N)^{\otimes k} \right]$. Using Theorem 5.2 of [12], we know that:

$$m_{\sigma^c}(t) = \mathbb{E} \kappa_{\mathcal{P}}^{\sigma} [S_t, \dots, S_t].$$

Using Theorem 10.2 of [12], and using the same notations as for this theorem, we get that $m_{\sigma^c}(t)$ satisfies the system of equations, $\forall t_0 \geq 0, \forall k \in \mathbb{N}^*, \forall \sigma_0 \in \mathfrak{S}_k$:

$$\frac{d}{dt} \Big|_{t=t_0} m_{\sigma_0^c}(t) = \sum_{p_1 \in \mathcal{P}_k, p_2 \in \mathcal{P}_k | p_1 \circ p_2 = \sigma_0, p_1 \prec \sigma_0} (\mathcal{R}_{\mathcal{P}}[G])_{p_1} (\mathcal{R}_{\mathcal{P}}[S_t])_{p_2}.$$

Yet, we saw, in Lemma 6.3 of [11], that if p_1 and p_2 are two partitions such that $p_1 \circ p_2 = \sigma_0$ and $p_1 \prec \sigma_0$, then p_1 and p_2 are two permutations and $p_1 \in [id, \sigma_0]_{\mathfrak{S}_k}$. Thus $\forall t_0 \geq 0, \forall k \in \mathbb{N}^*, \forall \sigma_0 \in \mathfrak{S}_k$:

$$\frac{d}{dt} \Big|_{t=t_0} m_{\sigma_0^c}(t) = \sum_{\sigma \in \mathfrak{S}_k | \sigma \leq \sigma_0} (\mathcal{R}_{\mathcal{P}}[G])_{\sigma} m_{t_{\sigma\sigma_0}}(t_0).$$

Using again Theorem 5.2 of [12] and thus the fact that for any permutation σ , $(\mathcal{R}_{\mathcal{P}}[G])_{\sigma} = \lim_{N \rightarrow \infty} m_{\sigma^c}(G_k^N)$, we see that we only need to compute $\lim_{N \rightarrow \infty} m_{\sigma^c}(G_k^N)$ for any permutation in \mathfrak{S}_k in order to finish the proof of Theorem 2.1.

Let σ_0 be a permutation in \mathfrak{S}_k and let us compute $m_{\sigma_0^c}(G_k^N)$. If σ_0 is equal to id_k , then:

$$\begin{aligned} m_{id_k^c}(G_k^N) &= \frac{N}{\lambda_N(1^c)} \frac{1}{\#\lambda_N} \sum_{\sigma \in \lambda_N} \left[m_{id_k^c}(\sigma^{\otimes k}) - m_{id_k^c}(id^{\otimes k}) \right] \\ &= \frac{N}{\lambda_N(1^c)} \left[m_{id_k^c}(\sigma^{\otimes k}) - m_{id_k^c}(id^{\otimes k}) \right], \end{aligned}$$

where σ is any permutation in λ_N .

In the following, we will use the following convention: for any $n \in \mathbb{N}$ for any $m \in \mathbb{N}$:

$$\frac{n!}{(n-m)!} = \prod_{n-m+1 \leq i \leq n} i,$$

with of course the fact that $\prod_{i \in \emptyset} i = 1$. This is not of course a convention if $n-m+1 > 0$.

Now, if σ is a permutation in λ_N :

$$m_{id_k^c}(\sigma^{\otimes k}) = \frac{1}{N^k} \frac{\lambda_N(1)!}{(\lambda_N(1) - k)!}.$$

Besides, $m_{id_k^c}(id^{\otimes k}) = \frac{N!}{(N-k)!}$. Thus:

$$\begin{aligned} m_{id_k^c}(G_k^N) &= \frac{N}{\lambda_N(1^c)} \frac{1}{N^k} \left[\frac{\lambda_N(1)!}{(\lambda_N(1) - k)!} - \frac{N!}{(N-k)!} \right] \\ &= \frac{N}{\lambda_N(1^c)} \left[\prod_{i=0}^{k-1} \left(1 - \frac{\lambda_N(1^c) + i}{N} \right) - \prod_{i=0}^{k-1} \left(1 - \frac{i}{N} \right) \right]. \end{aligned}$$

Let us denote by α the limit of $\frac{\lambda_N(1^c)}{N}$ as N goes to infinity. We get:

$$\lim_{N \rightarrow \infty} m_{id_k^c}(G_k^N) = \begin{cases} -k & \text{if } (\lambda_N)_{N \in \mathbb{N}} \text{ is evanescent,} \\ \frac{1}{\alpha}((1-\alpha)^k - 1) & \text{if } (\lambda_N)_{N \in \mathbb{N}} \text{ is macroscopic.} \end{cases}$$

Now, let us suppose that σ_0 is not equal to id_k . Let σ be in λ_N , since $m_{\sigma_0}(id^{\otimes k}) = 0$, we get that:

$$m_{\sigma_0^c}(G_k^N) = \frac{N}{\lambda_N(1^c)} m_{\sigma_0^c}(\sigma^{\otimes k}).$$

Yet, by denoting by $[\sigma_0]$ the conjugacy class of σ_0 , it is easy to see that:

$$m_{\sigma_0^c}(\sigma^{\otimes k}) = \frac{1}{N^{\text{nc}(\sigma_0 \vee id)}} \prod_{i=1}^k \frac{\left(\frac{\lambda_N(i)}{i}\right)!}{\left(\frac{\lambda_N(i)}{i} - \frac{[\sigma_0](i)}{i}\right)!} i^{\frac{[\sigma_0](i)}{i}},$$

thus:

$$m_{\sigma_0^c}(G_k^N) = \frac{N}{\lambda_N(1^c)} \frac{1}{N^{\text{nc}(\sigma_0 \vee id)}} \prod_{i=1}^k \frac{\left(\frac{\lambda_N(i)}{i}\right)!}{\left(\frac{\lambda_N(i)}{i} - \frac{[\sigma_0](i)}{i}\right)!} i^{\frac{[\sigma_0](i)}{i}},$$

Let us notice that $\text{nc}(\sigma_0 \vee id) = \sum_{i=1}^k \frac{[\sigma_0](i)}{i}$. Thus:

$$m_{\sigma_0^c}(G_k^N) = \frac{N}{\lambda_N(1^c)} \left(\prod_{i=2}^k \frac{1}{N^{\frac{[\sigma_0](i)}{i}}} \frac{\left(\frac{\lambda_N(i)}{i}\right)!}{\left(\frac{\lambda_N(i)}{i} - \frac{[\sigma_0](i)}{i}\right)!} i^{\frac{[\sigma_0](i)}{i}} \right) \frac{1}{N^{[\sigma_0](1)}} \frac{\lambda_N(1)!}{(\lambda_N(1) - [\sigma_0](1))!}.$$

We recall that for any $i \geq 2$, there exists $\lambda(i)$ such that $\frac{\lambda_N(i)}{\lambda_N(1^c)}$ converges to $\lambda(i)$ when N goes to infinity, and $\lim_{N \rightarrow \infty} \frac{\lambda_N(1)}{N} \rightarrow 1 - \alpha$. Thus, when N goes to infinity, $m_{\sigma^c}(G_k^N)$ has the same limit as:

$$\frac{N}{\lambda_N(1^c)} \left(\prod_{i=2}^k \left(\frac{\lambda_N(i)}{\lambda_N(1^c)} \frac{\lambda_N(1^c)}{N} \right)^{\frac{[\sigma_0](i)}{i}} \right) \left(\frac{\lambda_N(1)}{N} \right)^{[\sigma_0](1)},$$

or the same limit as:

$$\left(\frac{\lambda_N(1^c)}{N} \right)^{\text{nc}(\sigma_0 \vee id) - [\sigma_0](1) - 1} \prod_{i=2}^k \left(\frac{\lambda_N(i)}{\lambda_N(1^c)} \right)^{\frac{[\sigma_0](i)}{i}} \left(\frac{\lambda_N(1)}{N} \right)^{[\sigma_0](1)}.$$

This implies that:

$$\lim_{N \rightarrow \infty} m_{\sigma_0^c}(G_k^N) = \alpha^{\text{nc}(\sigma_0 \vee id) - [\sigma_0](1) - 1} \left(\prod_{i=2}^k (\lambda(i))^{\frac{[\sigma_0](i)}{i}} \right) (1 - \alpha)^{[\sigma_0](1)}.$$

Let us remark that, since $\sigma_0 \neq id_k$, $\text{nc}(\sigma_0 \vee id) - [\sigma_0](1) - 1$ is always non negative. So the following formula has a meaning even if $\alpha = 0$. Using these calculations, we recover the system of differential equations stated in Theorem 2.1.

At last, let us prove that $(S_t^N)_{t \geq 0}$ is not a \mathfrak{S} -free multiplicative Lévy process. In order to do so, we will prove that the increments of $(S_t^N)_{t \geq 0}$ are not asymptotically \mathfrak{S} -free as N goes to infinity. Let t_1 and t_2 be two positive reals. For any positive integer N ,

let $S_{t_2}'^N$ be a random variable which has the same law as $S_{t_2}^N$ and which is independent with $S_{t_1}^N$. Since $(S_t^N)_{t \geq 0}$ is a Lévy process, it is enough to prove that $S_{t_1}^N$ and $S_{t_2}'^N$ are not asymptotically free as N goes to infinity. We already know that $S_{t_1}^N$ and $S_{t_2}^N$ are asymptotically \mathcal{P} -free. Besides, using the relation between moments and exclusive moments, we know that for any real $t \geq 0$, $\mathbb{E}m_{id_1}[S_t] = \mathbb{E}m_{id_1^c}[S_t]$ and $\mathbb{E}m_{id_2}[S_t, S_t] = \mathbb{E}m_{id_2^c}[S_t, S_t]$. Using the differential system of equations proved in Theorem 2.1, we know that for any $t_0 \geq 0$,

$$\begin{aligned} \frac{d}{dt} \Big|_{t=t_0} \mathbb{E}m_{id_1}[S_t] &= -\mathbb{E}m_{id_1}[S_{t_0}] \\ \frac{d}{dt} \Big|_{t=t_0} \mathbb{E}m_{id_2}[S_t, S_t] &= (-2 + \alpha) \mathbb{E}m_{id_2}[S_{t_0}, S_{t_0}], \end{aligned}$$

and $\mathbb{E}m_{id_1}[S_0] = \mathbb{E}m_{id_2}[S_0, S_0] = 1$. Since $\alpha \in [0, 1]$, this implies that for any positive real t , $\mathbb{E}m_{id_1}[S_t] \neq \mathbb{E}m_{id_2}[S_t, S_t]$. An application of Proposition 2.3 allows to conclude that $S_{t_1}^N$ and $S_{t_2}'^N$ are not asymptotically \mathfrak{S} -free. Moreover, we will see that if $(\lambda_N)_{N \in \mathbb{N}}$ is evanescent, then the asymptotic \mathcal{P} -factorization property holds for $(S_{t_1}^N)_{N \in \mathbb{N}}$ and $(S_{t_2}'^N)_{N \in \mathbb{N}}$. Using again Proposition 2.3, this shows that if $(\lambda_N)_{N \in \mathbb{N}}$ is evanescent then the process $\left((S_t^N)_{N \in \mathbb{N}}\right)_{t \geq 0}$ is not a free multiplicative Lévy process in the sense of Voiculescu. \square

Now we have proved the convergence in \mathcal{P} -expectation, let us understand when the convergence holds in probability or not, and let us consider the consequences for the empirical eigenvalues distribution.

Proof of Theorem 2.3. — Let us remark that if (λ_N) is macroscopic, we already saw that $m_{id_k^c}(G_k)$, which is also the limit of the coordinate numbers of G_k^N on id_k , is equal to $\frac{(1-\alpha)^k - 1}{\alpha} \neq k \cdot m_{id_1^c}(G_k)$: the generator does not condensate weakly, thus the family $(S_t^N)_{N \in \mathbb{N}}$ does not satisfy the asymptotic \mathcal{P} -factorization property. Besides, by looking at the system of differential equations satisfied by the limits of the observable, this implies actually that for any $t > 0$:

$$\lim_{N \rightarrow \infty} \mathbb{E}m_{id_2}[S_t^N \otimes S_t^N] \neq \left(\lim_{N \rightarrow \infty} \mathbb{E}m_{id_1}[S_t^N] \right)^2.$$

Thus, if (λ_N) is macroscopic, for any positive real t the \mathfrak{S} -asymptotic factorization property does not hold for $(S_t^N)_{N \in \mathbb{N}}$.

Let p be a partition in \mathcal{P}_k ; we can suppose, up to a permutation of the columns, that there exist r irreducible partitions p_1, \dots, p_r such that $p = p_1 \otimes \dots \otimes p_r$. We saw in the proof of Theorem 2.1 that for any integer N :

$$m_p(G_k^N) = \sum_{i=1}^r \left(m_{p_i}(G_k^N) \prod_{l=i+1}^r m_{p_l}(\sigma_N^{\otimes k}) \right),$$

where $\sigma_N \in \lambda_N$. Besides, using Equation (5), $\lim_{N \rightarrow \infty} m_p(\sigma_N^{\otimes k}) = 1 + \alpha \left(\lim_{N \rightarrow \infty} m_p(G_k^N) \right)$, where we recall that $\alpha = \lim_{N \rightarrow \infty} \frac{\lambda_N(1^c)}{N}$. Thus, denoting by $m_p(G_k)$ the limit of $m_p(G_k^N)$:

$$m_p(G_k) = \sum_{i=1}^r \left(m_{p_i}(G_k) \prod_{l=i+1}^r [1 + \alpha m_{p_l}(G_k)] \right).$$

Recall Theorem 10.3 of [11]: this last equation implies that if (λ_N) is evanescent then $((G_k^N)_k)_N$ weakly condensates. Thus, if (λ_N) is evanescent, then by Theorem 10.3 of [12], we know that $((S_t^N)_{t \geq 0})_{N \in \mathbb{N}}$ satisfies the asymptotic \mathcal{P} -factorization property.

Using Theorem 2.2 of [12], this implies that, in this case, $((S_t^N)_{t \geq 0})_{N \in \mathbb{N}}$ converges in probability in \mathcal{P} -moments.

Let us translate these results for the empirical eigenvalues distribution. Let t be a positive real. Let us denote by $\mu_t^{\lambda_N}$ the random empirical eigenvalues distribution of S_t^N . Let r be a positive integer, let (n_1, \dots, n_r) be in \mathbb{Z}^r and let σ be a permutation which has r cycles, the i^{th} having size $|n_i|$. Then using the definitions and the fact that any permutation matrix is real valued, we get:

$$(6) \quad \mathbb{E} \left[\prod_{i=1}^r \left(\int_{\mathbb{U}} z^{n_i} \mu_t^{\lambda_N}(dz) \right) \overline{\prod_{i=1}^r \left(\int_{\mathbb{U}} z^{n'_i} \mu_t^{\lambda_N}(dz) \right)} \right] = \mathbb{E} m_{\sigma} \left[(S_t^N)^{\otimes r} \right].$$

Since $(S_t^N)_{N \in \mathbb{N}}$ converges in \mathfrak{S} -expectation, we get that the measures $\mu_t^{\lambda_N}$ converge in law to a random measure on \mathbb{U} , denoted by μ_t^{λ} .

The measure μ_t^{λ} is not random if and only if for any positive integer r , and any (n_1, \dots, n_r) in \mathbb{Z}^r :

$$\mathbb{E} \left[\prod_{i=1}^r \left(\int_{\mathbb{U}} z^{n_i} \mu_t^{\lambda}(dz) \right) \overline{\prod_{i=1}^r \left(\int_{\mathbb{U}} z^{n'_i} \mu_t^{\lambda}(dz) \right)} \right] = \prod_{i=1}^r \mathbb{E} \left[\int_{\mathbb{U}} z^{n_i} \mu_t^{\lambda}(dz) \right] \overline{\prod_{i=1}^r \mathbb{E} \left[\int_{\mathbb{U}} z^{n'_i} \mu_t^{\lambda}(dz) \right]}.$$

Using the Equation (6), this means that the measure μ_t^{λ} is not random if and only if the asymptotic \mathfrak{S} -factorization holds for $(S_t^N)_{N \in \mathbb{N}}$. Using the results previously obtained, we get that measure μ_t^{λ} is not random if and only if $(\lambda_N)_{N \in \mathbb{N}}$ is evanescent. \square

From now on, we will suppose that $(\lambda_N)_{N \in \mathbb{N}}$ is evanescent: the limiting empirical eigenvalues distribution is not-random. Let us compute this limiting measure.

Proof of Theorem 2.4. — We recall that in the proof of Theorem 2.1, we used the following notation: for any positive integer k , any $\sigma \in \mathfrak{S}_k$ and any $t \geq 0$, we denote by $m_{\sigma^c}(t)$ the limit $\lim_{N \rightarrow \infty} \mathbb{E} m_{\sigma^c} \left[(S_t^N)^{\otimes k} \right]$. Besides, we proved that the family $(m_{\sigma^c}(t))_{t, \sigma}$ satisfies the system of differential equations stated in Theorem 2.1. Since we suppose that $(\lambda_N)_{N \in \mathbb{N}}$ is evanescent, for any $t_0 \geq 0$ and any $\sigma_0 \in \mathfrak{S}_k$, one has:

$$\frac{d}{dt} \Big|_{t=t_0} m_{\sigma_0^c}(t) = -k m_{\sigma_0^c}(t_0) + \sum_{\sigma \in \mathfrak{S}_k \setminus \{id_k\}, \sigma \leq \sigma_0} 0^{\text{nc}(\sigma \vee id) - [\sigma](1) - 1} \left(\prod_{i=2}^k (\lambda(i))^{\frac{[\sigma](i)}{i}} \right) m_{(t \sigma \sigma_0)^c}(t_0),$$

yet $\text{nc}(\sigma \vee id) - [\sigma](1) - 1 = 0$ if and only if σ is weakly irreducible, thus if and only if this is a cycle. Thus, if we set $m_{n^c}(t) = m_{(1, \dots, n)^c}(t)$, for any $t_0 \geq 0$ and any positive integer n :

$$(7) \quad \frac{d}{dt}|_{t=t_0} m_{n^c}(t) = -nm_{\sigma_0^c}(t_0) + \sum_{k=2}^n \sum_{\sigma \in \mathfrak{S}_n \setminus \{id_k\}, \sigma \text{ is a } k \text{ cycle}, \sigma \leq (1, \dots, n)} \lambda(k) m_{(t \sigma \sigma_0)^c}(t_0).$$

Yet, using Theorem 2.3, the \mathfrak{S} -asymptotic factorization holds when $(\lambda_N)_{N \in \mathbb{N}}$ is evanescent. This implies that we can write Equation (7) only in terms of $(m_{n^c}(t))_{n,t}$. For any positive integer n , any $t_0 \geq 0$:

$$(8) \quad \frac{d}{dt}|_{t=t_0} m_{n^c}(t) = -nm_{n^c}(t) + \sum_{k=2}^n \lambda(k) \frac{n}{k} \sum_{(n_1, \dots, n_k) \in (\mathbb{N}^*)^k | \sum_{i=1}^k n_i = n} \prod_{i=1}^k m_{n_i^c}.$$

Let us introduce the generating formal series of $(e^{nt} m_{n^c}(t))_{n \geq 1}$:

$$R(t, z) = \sum_{n \geq 1} e^{nt} m_{n^c}(t) z^n.$$

Let us remark that $R(0, z) = z$. The Equation (8) can be written as:

$$(9) \quad \partial_t R(t, z) = z \partial R(t, z) \text{LS}(R)(t, z),$$

where we defined:

$$\text{LS}(z) = \sum_{n \geq 1} \lambda(n+1) z^n.$$

Let us define $S(t, z)$ the reciprocal formal series such that for any $t \geq 0$:

$$S(t, R(t, z)) = z.$$

Let us remark that $S(0, z) = z$. The Equation (9) implies an equation on S :

$$\partial_t S(t, z) = -\text{LS}(z) S(t, z).$$

Thus $S(t, z)$ is given by $S(t, z) = z e^{-t \text{LS}(z)}$. Let $t \geq 0$ and let n be a positive integer. Using the usual notations, since $e^{nt} m_{n^c}(t) = [z^n] R(t, \bullet)$, we can compute $e^{nt} m_{n^c}(t)$ by using the Lagrange inversion. This implies that:

$$[z^n] R(t, \bullet) = \frac{1}{n} [z^{n-1}] e^{tn \text{LS}(z)},$$

thus $m_{1^c}(t) = e^{-t}$ and for $n > 1$:

$$m_{n^c}(t) = e^{-nt} \sum_{k=1}^{n-1} t^k \frac{n^{k-1}}{k!} \sum_{(i_1, \dots, i_k) \in (\mathbb{N}^*)^k, i_1 + \dots + i_k = n-1} \prod_{j=1}^k \lambda(i_j + 1),$$

hence the assertions in Theorem 2.4. \square

Let us prove the assertion on the existence of a phase transition for the random walks on the symmetric group.

Proof of Theorem 2.5. — Let us suppose that the sequence $(\lambda_N)_{N \in \mathbb{N}}$ is evanescent. Let us show that the function $f(t) = \sum_{n=1}^{\infty} m_{n^c}(t)$, which is equal to $1 - m_{\infty^c}(t)$ is continuous, converges to 0 as t goes to infinity. Indeed, we have:

$$f(t) = \sum_{k,n=0}^{\infty} \frac{1}{n} e^{-nt} t^k \frac{n^k}{k!} p(k, n),$$

where $p(k, n) = \sum_{(i_1, \dots, i_k) \in \mathbb{N}^*, \sum_{j=1}^k i_j = n-1} \prod_{j=1}^k \lambda(i_j + 1)$. For any k and n in \mathbb{N} , $f_{k,n}(t) = \frac{1}{n} e^{-nt} t^k \frac{n^k}{k!} p(k, n)$ is continuous and goes to zero as t goes to infinity, besides $f_{k,n}$ is non-negative and maximal at $t_{k,n} = \frac{k}{n}$ and using Stirling's formula, there exists a constant C such that $f_{k,n}(t_{k,n}) = \frac{1}{n} e^{-k} \frac{k^k}{k!} p(k, n) \leq C \frac{1}{k^{3/2}} p(k, n)$. In order to finish, one has to remark that:

$$\sum_{n \in \mathbb{N}} p(k, n) = \sum_{(i_1, \dots, i_k) \in \mathbb{N}^*} \prod_{j=1}^k \lambda(i_j + 1) = \left(\sum_{i \in \mathbb{N}^*} \lambda(i + 1) \right)^k \leq 1,$$

thus $\sum_{k,n} f_{k,n}(t_{k,n}) < \infty$. This allows to apply the dominated convergence theorem, thus f is a continuous function and converges to zero as t goes to infinity.

Recall the definition of t_c^λ given by Equation (3). Let us prove that $f(t) = 1$ for any $t \leq t_c^\lambda$ and $f(t) < 1$ for any $t > t_c^\lambda$. Using the generating function $R(t, \bullet)$ of $e^{nt} m_{n^c}(t)$, we know that for any real $t \geq 0$:

$$f(t) = R(t, e^{-t}).$$

Using the fact that $S(t, R(t, e^{-t})) = e^{-t}$, and given that $S(t, z) = z e^{-t \text{LS}(z)}$, we get that:

$$R(t, e^{-t}) e^{-t \text{LS}(R(t, e^{-t}))} = e^{-t}.$$

Thus for any $t \geq 0$, $f(t)$ is a solution in $[0, 1]$ of $\Phi_t(z) = z e^{-t(\text{LS}(z)-1)} = 1$. The function Φ_t is log-concave on $[0, 1]$, $\Phi_t(0) = 0$ and $\Phi_t(1) = e^{-t(\text{LS}(1)-1)}$. If $\text{LS}(1) = \sum_{i=2}^{\infty} \lambda(i)$ is not equal to one it must be strictly smaller than 1, thus in this case for any $t > 0$, $\Phi_t(1) > 1$ and thus there exists a unique solution of $\Phi_t(z) = 1$ in $[0, 1]$ which is in fact in $]0, 1[$. Thus we recover the delta function in Equation (3). Let us suppose now that $\sum_{i=2}^{\infty} \lambda(i) = 1$. Then $\text{LS}(1) = 1$: thus, since Φ_t is log-concave, there exists a solution ν_t (which is unique) of $\Phi_t(z) = 1$ on $]0, 1[$ if and only if $\Phi'_t(1) < 0$. Since $\Phi'_t(1) = 1 - t \text{LS}'(1)$, we get that the critical time after which one observes a solution in $[0, 1]$ which is different from the trivial solution 1 is equal to $\frac{1}{\text{LS}'(1)}$ which is the value of t_c given by Equation (3). Since $f(t)$ is a continuous function which must converge to zero as t goes to infinity, it must be equal to 1 if $t \leq t_c$ and then it must be equal to $\nu(t)$ if $t > t_c$. \square

Let us finish with the proof of Corollary 2.1.

Proof of Corollary 2.1. — Let t be a non-negative real number, let N be a positive integer, we have to understand:

$$\frac{1}{N} d_{\mathfrak{S}(N)}(id_N, S_t^N) = 1 - \frac{\text{nc}(S_t^N \vee id_N)}{N}.$$

Recall that:

$$\frac{1}{N} \text{nc} (S_t^N \vee id_N) = \sum_{k \geq 1} \frac{1}{k} m_{(1, \dots, k)^c} \left((S_t^N)^{\otimes k} \right)$$

since $m_{(1, \dots, k)^c} \left((S_t^N)^{\otimes k} \right)$ is the fraction of integers in $\{1, \dots, N\}$ whose period in S_t^N is equal to k . It remains to see if one can interchange the limit and the sum. For any positive integer N , for any $\sigma \in \mathfrak{S}(N)$, if $c_k(\sigma)$ is the numbers of cycles of size k in σ , we have for any $K \in \mathbb{N}^*$:

$$\sum_{k \geq K} \frac{1}{k} m_{(1, \dots, k)^c} \left(\sigma^{\otimes k} \right) = \frac{1}{N} \sum_{k \geq K} c_k(\sigma) \leq \frac{1}{N} \frac{N}{K} = \frac{1}{K},$$

since there can not be more than $\frac{N}{K}$ cycles in σ of size bigger than K . Thus:

$$\sup_N \sum_{k \geq K} \frac{1}{k} m_{(1, \dots, k)^c} \left((S_t^N)^{\otimes k} \right) \xrightarrow{N \rightarrow \infty} 0,$$

almost surely. Thus we can interchange limits and thus, since for any integer $k \geq 1$, $m_{(1, \dots, k)^c} \left((S_t^N)^{\otimes k} \right)$ converges in probability to $m_k^c(t)$, then we have the convergence in probability:

$$\lim_{N \rightarrow \infty} \frac{1}{N} \text{nc} (S_t^N \vee id_N) = \sum_{k \geq 1} \lim_{N \rightarrow \infty} \frac{1}{k} m_{(1, \dots, k)^c} \left((S_t^N)^{\otimes k} \right) = \sum_{k=1}^{\infty} \frac{1}{k} m_k^c(t).$$

This allows to conclude the proof. \square

2.4. Log-cumulant calculations. — In the article [12], we studied the log-cumulant invariant of a free multiplicative infinitely divisible measure. The log-cumulant was defined in Definition 10.18. We can generalize the definition of log-cumulant in the setting of \mathcal{P} -free multiplicative Lévy processes. Recall the notion of matricial \mathcal{P} -free multiplicative Lévy process defined in Definition 7.4 of [12].

Definition 2.4. — Let $(S_t)_{t \geq 0}$ be a matricial \mathcal{P} -free multiplicative Lévy process. The log-cumulant invariant of $(S_t)_{t \geq 0}$ is the unique element $\mathcal{LR}((S_t)_{t \geq 0}) \in \mathfrak{C}[\mathcal{P}]$ such that:

$$\forall t \geq 0, \quad \frac{d}{dt} \Big|_{t=t_0} \mathcal{R}(S_t) = \mathcal{LR} \left((S_t)_{t \geq 0} \right) \boxtimes \mathcal{R}(S_{t_0}).$$

Let $(S_t)_{t \geq 0} = \left((S_t^N)_{N \geq 1} \right)_{t \geq 0}$ be a matricial \mathcal{P} -free multiplicative Lévy process. For any positive integer k and N , we consider $G_k^N = \frac{d}{dt} \Big|_{t=0} \mathbb{E} \left[(S_t^N)^{\otimes k} \right]$. Recall that, using the same notation as in Definition 10.3 of [11], $\mathcal{LR} \left((S_t)_{t \geq 0} \right) = \mathcal{R}_{\mathcal{P}}[G]$.

In the setting of \mathcal{P} -free multiplicative Lévy processes which are invariant by conjugation by the unitary group, the log-cumulant invariant is an important tool in order to characterize them. We hope that this could also be the case for more general \mathcal{P} -free multiplicative Lévy processes. We are thus interested in computing the log-cumulant

invariant for some examples of \mathcal{P} -free multiplicative Lévy processes: the following theorem is the first computation of the log-cumulants of \mathcal{P} -free multiplicative Lévy processes which are not free Lévy processes in the sense of Voiculescu.

For any positive integer N , let λ_N be a conjugacy of $\mathfrak{S}(N)$ and let us consider $(S_t^N)_{N \in \mathbb{N}}$ a λ_N -random walk on $\mathfrak{S}(N)$. For any positive integer t , let us denote by S_t^λ the family $(S_t^N)_{N \in \mathbb{N}}$. Let us suppose that $(\lambda_N)_{N \in \mathbb{N}}$ converges as N goes to infinity and that it is evanescent. As we already did, for any $i \geq 2$, we set $\lambda(i) = \lim_{N \rightarrow \infty} \frac{\lambda_N(i)}{\lambda_N(1^c)}$.

We need to define the notion of ears.

Definition 2.5. — Let k be a positive integer, let i be an element of $\{1, \dots, k\}$ and let $p \in \mathcal{P}_k$. We say that $\{i, i'\}$ is an ear of p if $\{i, i'\}$ are in the same block of p . The set of ears of p is denoted by $E(p)$. The head of p , denoted by $H(p)$, is the extraction of p to $\{1, \dots, k, 1', \dots, k'\} \setminus \cup_{i, i' \in E(p)} \{i, i'\}$.

Let us state the main result about the log-cumulant functional. Recall the notion of true-length that we defined in Definition 2.3.

Theorem 2.7. — The log-cumulant invariant of $(S_t^\lambda)_{t \geq 0}$, denoted by \mathcal{LR}^λ , is characterized by:

1. $\mathcal{LR}^\lambda \in \mathfrak{mc}_{\boxtimes}[\mathcal{P}]$,
2. for any positive integer k , for any irreducible partition $p \in \mathcal{P}_k$, if $H(p)$ is not a parure then $(\mathcal{LR}^\lambda)_p = 0$,
3. for any positive integer k , for any irreducible partition $p \in \mathcal{P}_k$, if $H(p)$ is a necklace then:

$$(\mathcal{LR}^\lambda)_p = (-1)^{\#E(p)} \lambda(|H(p)|),$$

with the convention that $|\emptyset| = 0$ and $\lambda(0) = 1$, and if $H(p)$ is a chain then:

$$(\mathcal{LR}^\lambda)_p = (-1)^{\#E(p)} \left(1 - \sum_{i=2}^{|H(p)|} \lambda(i) \right).$$

Proof. — Let us denote $E = \mathcal{LR}^\lambda$. Let us consider $F \in \mathfrak{mc}_{\boxtimes}[\mathcal{P}]$ which satisfies the conditions 2. and 3. of the theorem. Since we supposed in this section that $(\lambda_N)_{N \in \mathbb{N}}$ converges and is evanescent, by the proof of Theorem 2.3 we know that the generator of $(S_t^N)_{N \in \mathbb{N}}$ weakly condensates as N goes to infinity. By definition, this implies that $E \in \mathfrak{mc}_{\boxtimes}[\mathcal{P}]$. Besides, we have computed the exclusive moments of the generator in Section 2.3. Using the notations of [11], we know $\mathcal{M}^{\rightarrow c}(E)$: the goal is to invert the transformation $\mathcal{M}^{\rightarrow c}$.

It remains to show that for any weakly irreducible partition p , $E_p = F_p$. Let us denote by E^c the element $\mathcal{M}^{\rightarrow c}(E)$. Recall that $|p|$ is the true-length of a partition p . Using the calculations in the proof of Theorem 2.1, we know for any irreducible partition p , if p is not a parure then $E_p^c = 0$ and if p is a necklace of true-length equal to 1 then $E_p^c = -1$, if it is a necklace of true-length greater than 1 then $E_p^c = \lambda(|p|)$, and if p is a chain then $E_p^c = 1 - \sum_{k=2}^{|p|} \lambda(k)$. Recall Definition 3.11 of [11]. Since $E \in \mathfrak{mc}_{\boxtimes}[\mathcal{P}]$, E is

characterized by the fact that for any irreducible partition p , $E_p^c = \sum_{p' \sqsupset p} E_{p'}$. Let p be an irreducible partition in \mathcal{P}_k , it remains to prove that:

$$E_p^c = \sum_{p' \sqsupset p} F_{p'}.$$

Let us recall that $F_{p'} = 0$ if p' is not weakly irreducible. Yet if $p' \sqsupset p$ and p' is weakly irreducible, this means that one can get p' by choosing a certain number of ears of p and by cutting each of them in p . Let us consider the two possible cases $p = 0_k$ or $p \neq 0_k$. If $p = 0_k$, then:

$$\begin{aligned} \sum_{p' \sqsupset 0_k} F_{p'} &= \left(\sum_{I \subset \{1, \dots, k\}, \#I > 1} (-1)^{\#I-1} \right) (-1) - k \\ &= - \left(\left(\sum_{l=0}^k (-1)^{l-1} \frac{k!}{l!(k-l)!} \right) + 1 \right) = -1 = E_{0_k}^c, \end{aligned}$$

the last equality coming from the fact that 0_k is a necklace of true-length equal to 1.

If $p \neq 0_k$, then:

$$\sum_{p' \sqsupset p} F_{p'} = \sum_{I \in \mathbf{E}(p)} (-1)^{\#\mathbf{E}(p) - \#I} F_{\mathbf{T}(p)} = \delta_{\#\mathbf{O}(p)=0} F_{\mathbf{T}(p)} = \delta_{\#\mathbf{O}(p)=0} F_p = \delta_{\#\mathbf{O}(p)=0} E_p^c = E_p^c,$$

since the only irreducible parure in \mathcal{P}_k which has ears is 0_k . \square

Let us remark that, since we saw that $\mathcal{LR} \left((S_t)_{t \geq 0} \right) = \mathcal{RP}[G]$, we could have try to prove the last theorem by computing the coordinate numbers of G_k^N for any positive integer k and N . For example, if one considers the random walk by transposition, if \mathcal{T}_N is the set of transpositions in $\mathfrak{S}(N)$:

$$\begin{aligned} G_k^N &= \frac{1}{N-1} \sum_{\tau \in \mathcal{T}_N} \left(\tau^{\otimes k} - Id_k \right) \\ &= \frac{1}{2(N-1)} \sum_{i,j=1}^N \left(\left(Id_N - E_i^i - E_j^j + E_i^j + E_j^i \right)^{\otimes k} - Id^{\otimes k} \right), \end{aligned}$$

where E_i^j is the usual elementary matrix which sends e_j on e_i and where (e_1, \dots, e_N) is the canonical basis of \mathbb{C}^N . Then one can develop the tensor product and compute the coordinate numbers and their limits. Yet, one can see that it becomes less tractable as soon as one considers general random walks on the symmetric group.

3. Convergence of $YM(\mathfrak{S}(N))$

We will not go into all the details of the theory of planar Yang-Mills fields, one can read [13] and [19] to have an introduction on planar Yang-Mills fields and planar Markovian holonomy fields. Yet, our presentation will be adequate so that the reader does not have to read other articles in order to understand the main result of this section, namely Theorem 3.2. The general ideas are all taken from the article [10] where asymptotics of unitary Yang-Mills measures are proved. In this article, the Yang-Mills measure with

$\mathfrak{S}(N)$ gauge group will denote for us the planar Markovian holonomy field associated with the \mathcal{T}_N -random walk, where \mathcal{T}_N is the set of transposition in $\mathfrak{S}(N)$. Yet, this section can easily be generalized to planar Markovian holonomy fields associated with any λ_N -random walk.

Definition 3.1. — *The set of paths P in the plane is the set of rectifiable oriented curves drawn in \mathbb{R}^2 up to increasing reparametrization. The set of loops based at 0, denoted by L_0 , is the set of paths l such that the two endpoints of l are 0. A loop is simple if it does not intersect with itself, except at the endpoints.*

We will consider Aff and Aff_0 respectively the set of piecewise affine paths in \mathbb{R}^2 and the set of piecewise affine loops based at 0.

We can define two operations on P : the concatenation and the inversion. Given two paths p_1 and p_2 such that the starting point of p_2 is the arrival point of p_1 , it is natural to concatenate p_1 and p_2 by gluing them at the arrival point of p_1 : it defines a new path $p_1 p_2$. The inversion of p_1 , denoted by p_1^{-1} , is defined by changing the orientation of p_1 . T. Lévy defined in [19], the notion of convergence with fixed endpoints. For any $p \in P$, \underline{p} denotes the starting point of p and \bar{p} denotes the arrival point of p . Let $(p_n)_{n \in \mathbb{N}}$ be a sequence of paths. The sequence $(p_n)_{n \in \mathbb{N}}$ converges with fixed endpoints if and only if there exists a path p such that for any integer n , p_n and p have the same endpoints and:

$$|\mathbf{l}(p_n) - \mathbf{l}(p)| + \inf_{t \in [0,1]} \sup_{t \in [0,1]} |p_n(t) - p(t)| \xrightarrow{n \rightarrow \infty} 0,$$

where the infimum is taken on the parametrization of the paths p_n and p and where $\mathbf{l}(p)$ is the length of p .

Let J be a subset of P , let G be a group. The set of multiplicative functions $\text{Mult}(J, G)$ from J to G is the subset of functions f in G^J such that for any $p_1, p_2, p_3 \in J$ such that $p_1 p_2 \in J$ and $p_3^{-1} \in J$, one has:

$$\begin{aligned} f(p_1 p_2) &= f(p_2) f(p_1), \\ f(p_3^{-1}) &= f(p_3)^{-1}. \end{aligned}$$

For any $p \in P$, we define h_p or $h(p)$ as the evaluation on p :

$$\begin{aligned} h_p : \text{Mult}(J, G) &\rightarrow G \\ h &\mapsto h(p). \end{aligned}$$

Let N be a positive integer. We are going to define a gauge-invariant measure on the set of multiplicative functions from P to $\mathfrak{S}(N)$. Thus we endow $\text{Mult}(J, \mathfrak{S}(N))$ with the cylinder σ -field \mathcal{B} which is the trace on $\text{Mult}(J, G)$ of the cylinder σ -field on $\mathfrak{S}(N)^J$. Let us denote by V the set $\{x \in \mathbb{R}^2, \exists p \in J, x = \underline{p} \text{ or } x = \bar{p}\}$. For any function $j : V \rightarrow \mathfrak{S}(N)$ and any $h \in \text{Mult}(J, \mathfrak{S}(N))$, we define $j \bullet h \in \text{Mult}(J, \mathfrak{S}(N))$ such that:

$$\forall c \in J, (j \bullet h)(c) = j_{\bar{c}}^{-1} h(c) j_{\underline{c}}.$$

A measure μ on $\text{Mult}(J, \mathfrak{S}(N))$ is gauge-invariant if for any continuous function f from $(\text{Mult}(J, \mathfrak{S}(N)), \mathcal{B})$ to \mathbb{R} , for any function $j : V \rightarrow \mathfrak{S}(N)$:

$$\int_{\text{Mult}(P, G)} f(j \bullet h) d\mu(h) = \int_{\text{Mult}(P, G)} f(h) d\mu(h).$$

In the up-coming paper [10], the author and his co-authors proved the following theorem which is a slight generalization of Theorem 3.3.1 proved by T. Lévy in [19]. The original formulation by T. Lévy of this theorem is the first part of Theorem 3.1.

Theorem 3.1. — *Let us denote by dx the Lebesgue measure on \mathbb{R}^2 . Let $(\Gamma_N, d_N)_{N \in \mathbb{N}}$ be a sequence of complete metric groups such that for any $N \in \mathbb{N}$, translations and inversion are isometries on Γ_N . For any integer N , let $H_N \in \text{Mult}(\text{Aff}, \Gamma_N)$ be a multiplicative function. Assume that there exists $K_N \geq 0$ such that for any $N \in \mathbb{N}$, for all simple loop $l \in \text{Aff}$ bounding a disk D and such that $\mathbf{l}(l) \leq K_N^{-1}$, the inequality:*

$$(10) \quad d_N(1, H_N(l)) \leq K_N \sqrt{dx(D)}$$

holds.

Then for each integer N , the function H_N admits a unique extension as an element of $\text{Mult}(P, G)$, also denoted by H_N , which is continuous for the convergence with fixed endpoints.

Let (E, d) be a metric space. For any integer N , let $\psi_N : \Gamma_N \rightarrow E$ be a Lipchitz function of Lipchitz norm $\|\psi_N\|_{\text{Lip}}$. Let us assume that the three following conditions hold:

1. *for any $l \in \text{Aff}_0$, $\psi_N(H_N(l))$ converges to a limit when N goes to infinity,*
2. $\sup_{N \in \mathbb{N}} \|\psi_N\|_{\text{Lip}} \leq \infty,$
3. $\sup_{N \in \mathbb{N}} K_N < \infty,$

then for any $l \in L_0$, $(\psi_N(H_N(l)))_{N \in \mathbb{N}}$ converges to a limit $\phi(l)$. Besides, the function:

$$\begin{aligned} \phi : L_0 &\rightarrow E \\ l &\mapsto \phi(l) \end{aligned}$$

is continuous for the convergence with fixed endpoints.

Recall that \mathcal{T}_N is the set of transpositions in $\mathfrak{S}(N)$. Let $(S_t^N)_{t \geq 0}$ be the \mathcal{T}_N random walk on $\mathfrak{S}(N)$. Let us explain how the first part of Theorem 3.1 allows us to construct the Yang-Mills field associated with $(S_t^N)_{t \geq 0}$. In order to do so, we need the notion of finite planar graph: it will be the usual notion, except that we ask that the bounded faces are homeomorphic to an open disk. Let \mathbb{G} be a finite planar graph: the set of bounded faces of \mathbb{G} is denoted by \mathbb{F} . For any finite planar graph \mathbb{G} , we define $P(\mathbb{G})$ as the set of paths that one can draw by concatenating edges of \mathbb{G} . Let us define also $\mathcal{G}(\text{Aff})$ the set of finite planar graphs \mathbb{G} whose edges are piecewise affine.

In order to construct a measure on $(\text{Mult}(P, \mathfrak{S}(N)), \mathcal{B})$, first we construct for any finite planar graph $\mathbb{G} \in \mathcal{G}(\text{Aff})$ an associated measure $\mu_{\mathbb{G}}$ on $(\text{Mult}(P(\mathbb{G}), \mathfrak{S}(N)), \mathcal{B})$. We will give the construction given by the author in [13], but one can have a look at [19] where a different formulation is given.

We need to introduce the loop paradigm for two dimensional Yang-Mills fields. Let us consider a finite planar graph \mathbb{G} in $\mathcal{G}(\text{Aff})$, let us consider v_0 a vertex of \mathbb{G} and T a covering tree of \mathbb{G} . Let us consider for any bounded face F of \mathbb{G} a loop $c_F \in P(\mathbb{G})$ which represents ∂F . For any vertex v of \mathbb{G} , we denote by $[v_0, v]_T$ the unique injective path in

T which goes from v_0 to v . Let $L_{v_0}(\mathbb{G})$ be the set of loops l in $P(\mathbb{G})$ such that $\underline{l} = v_0$. We define the facial lasso $l_F \in L_{v_0}(\mathbb{G})$ by:

$$l_F = [v_0, v]_T c_F [v_0, v]_T^{-1}.$$

It was proved in Proposition 6.1 that the application:

$$\begin{aligned} \Phi_{T, (c_F)_{F \in \mathbb{F}}} : \text{Mult}(L_{v_0}(\mathbb{G}), \mathfrak{S}(N)) &\rightarrow (\mathfrak{S}(N))^{\mathbb{F}} \\ h &\mapsto (h(l_F))_{F \in \mathbb{F}}, \end{aligned}$$

is a bijection and for any loop $l \in L_{v_0}(\mathbb{G})$, there exists a word $w \left((l_F)_{F \in \mathbb{F}}, (l_F^{-1})_{F \in \mathbb{F}} \right)$ in the letters $(l_F)_{F \in \mathbb{F}}$ and $(l_F^{-1})_{F \in \mathbb{F}}$ such that $h_l = w \left((h_{l_F})_{F \in \mathbb{F}}, (h_{l_F^{-1}})_{F \in \mathbb{F}} \right)$.

Using the fact that for any $\sigma \in \mathfrak{S}(N)$, σ^{-1} is in the same conjugacy class as σ , we know that for any non-negative real t , S_t^N has the same law as $(S_t^N)^{-1}$. This implies, with Proposition 8.1 proved by the author in [13], the following proposition.

Proposition 3.1. — *There exists a unique gauge-invariant measure $\mu_{\mathbb{G}}^{v_0, T, (c_F)_{F \in \mathbb{F}}}$ on $\text{Mult}(P(\mathbb{G}), \mathfrak{S}(N))$ such that under this measure:*

1. *the random variables $(h(l_F))_{F \in \mathbb{F}}$ are independent,*
2. *for any $F \in \mathbb{F}$, $h(l_F)$ has the same law as $S_{dx(F)}^N$.*

This measure does not depend neither on the choice of v_0 nor T nor on the choice of $(c_F)_{F \in \mathbb{F}}$, we denote it by $\mu_{\mathbb{G}}$.

Let \mathbb{G} and \mathbb{G}' be two finite planar graphs in $\mathcal{G}(\text{Aff})$ such that \mathbb{G}' is coarser than \mathbb{G} . Any function in $\text{Mult}(P(\mathbb{G}), \mathfrak{S}(N))$ allows us to define, by restriction, an element of $\text{Mult}(P(\mathbb{G}'), \mathfrak{S}(N))$. The measures $(\mu_{\mathbb{G}})_{\mathbb{G}}$ are compatible with the applications of restriction we have just described. The family $(\text{Mult}(P(\mathbb{G}), \mathfrak{S}(N)), \mu_{\mathbb{G}})_{\mathbb{G} \in \mathcal{G}(\text{Aff})}$ is thus a projective system and, as explained in Proposition 2.1 of [13] and in [19], we can take the projective limit.

Definition 3.2. — *The affine Yang-Mills measure associated to $(S_{N,t})_{t \geq 0}$ is the projective limit of:*

$$(\text{Mult}(P(\mathbb{G}), \mathfrak{S}(N)), \mu_{\mathbb{G}})_{\mathbb{G} \in \mathcal{G}(\text{Aff})}.$$

The affine Yang-Mills measure $YM_{\text{Aff}}^{\mathfrak{S}(N)}$ is a gauge-invariant measure on $\text{Mult}(\text{Aff}, \mathfrak{S}(N))$.

Let us consider a simple loop l in Aff and let \mathbb{G}^l be the finite planar graph in $\mathcal{G}(\text{Aff})$ which has l as unique edge. In this case, $\text{Mult}(P(\mathbb{G}^l, \mathfrak{S}(N))) \simeq \mathfrak{S}(N)$ and for any continuous function $f : \mathfrak{S}(N) \rightarrow \mathbb{R}$:

$$(11) \quad YM_{\text{Aff}}^{\mathfrak{S}(N)}[f(h_l)] = \mathbb{E} \left[f \left(S_{dx(\text{Int}(l))}^N \right) \right],$$

where $\text{Int}(l)$ is the bounded component of $\mathbb{R}^2 \setminus l$. This last equality shows that under $YM_{\text{Aff}}^{\mathfrak{S}(N)}$, h_l has the same law as $S_{dx(\text{Int}(l))}^N$. This will allow us to use the first part of Theorem 3.1 in order to construct the Yang-Mills measure, as it was done by T. Lévy in [19] and then by the author in [13]. Before doing so, we need some estimates on the walk $(S_t^N)_{t \geq 0}$: in order to do so, let us define a distance on $\mathfrak{S}(N)$.

Definition 3.3. — We will consider any element of $\mathfrak{S}(N)$ as a permutation matrix of size N . Let us consider for any $\sigma, \sigma' \in \mathfrak{S}(N)$:

$$d_N(\sigma, \sigma') = \left[2 \left(1 - \frac{1}{N} \text{Tr}(\sigma \sigma'^{-1}) \right) \right]^{\frac{1}{2}},$$

where we recall that Tr is the usual trace on $\mathcal{M}_N(\mathbb{C})$ which satisfies $\text{Tr}(Id_N) = N$.

Since the permutation matrices are orthogonal, for any σ and σ' in $\mathfrak{S}(N)$:

$$d_N(\sigma, \sigma') = \left[\frac{1}{N} \text{Tr}((\sigma - \sigma') \cdot {}^t(\sigma - \sigma')) \right]^{\frac{1}{2}}.$$

This shows that d_N is a distance on $\mathfrak{S}(N)$. Let us control the distance of $(S_t^N)_{t \geq 0}$ to the identity.

Lemma 3.1. — For any real $t \geq 0$, $\mathbb{E}[d_N(id, S_t^N)] \leq \sqrt{2t}$.

Proof. — Let t be a non-negative real. By definition:

$$\mathbb{E}[d_N(id, S_t^N)^2] = 2 \left[1 - \mathbb{E} \left[\frac{1}{N} \text{Tr}(S_t^N) \right] \right].$$

A simple calculation allows us to write that:

$$\frac{1}{N} \sum_{\tau \in \mathcal{T}_N} (\tau - Id) = \rho_N^{\mathcal{P}_k} \left[\frac{1}{N} 0_1 - Id \right],$$

where we recall that 0_1 is the partition $\{\{1\}, \{1'\}\}$. This implies that for any $t_0 \geq 0$:

$$\frac{d}{dt} \Big|_{t=t_0} \mathbb{E}[S_t^N] = \rho_N^{\mathcal{P}_k} \left(\frac{1}{N} 0_1 - Id \right) \mathbb{E}[S_{t_0}^N].$$

Thus, by linearity:

$$\frac{d}{dt} \Big|_{t=t_0} \mathbb{E} \left[\frac{1}{N} \text{Tr}(S_t^N) \right] = \frac{1}{N} \mathbb{E} \left[\frac{1}{N} \text{Tr}(\rho_N^{\mathcal{P}_k}(0_1) S_{t_0}^N) \right] - \mathbb{E} \left[\frac{1}{N} \text{Tr}(S_{t_0}^N) \right],$$

and, using the fact that $\frac{1}{N} \text{Tr}(\rho_N^{\mathcal{P}_k}(0_1)\sigma) = 1$ for any $\sigma \in \mathfrak{S}(N)$, we get the differential equation:

$$\begin{aligned} \frac{d}{dt} \Big|_{t=t_0} \mathbb{E} \left[\frac{1}{N} \text{Tr}(S_{t_0}^N) \right] &= \frac{1}{N} - \mathbb{E} \left[\frac{1}{N} \text{Tr}(S_{t_0}^N) \right], \\ \mathbb{E} \left[\frac{1}{N} \text{Tr}(S_0^N) \right] &= 1. \end{aligned}$$

The solution is given by the function $t \mapsto \frac{1}{N} + (1 - \frac{1}{N}) e^{-t}$. Thus for any real $t \geq 0$:

$$(12) \quad \mathbb{E} \left[\frac{1}{N} \text{Tr}(S_t^N) \right] = \frac{1}{N} + \left(1 - \frac{1}{N} \right) e^{-t},$$

and thus:

$$\mathbb{E}[d_N(id, S_t^N)^2] = 2 \left[1 - \frac{1}{N} \right] [1 - e^{-t}].$$

This implies that for any $t \geq 0$, and any positive integer N ,

$$\left(\mathbb{E} \left[d_N(id, S_t^N) \right] \right)^2 \leq \mathbb{E} \left[d_N(id, S_t^N)^2 \right] \leq 2t.$$

This allows us to finish the proof. \square

Using Lemma 3.1, we can prove the following proposition.

Proposition 3.2. — *The measure $YM_{\text{Aff}}^{\mathfrak{S}(N)}$ can be extended by continuity to a measure on $\text{Mult}(P, \mathfrak{S}(N))$. This means that there exists a measure $YM^{\mathfrak{S}(N)}$ on $\text{Mult}(P, \mathfrak{S}(N))$ such that:*

1. *the restriction of $YM^{\mathfrak{S}(N)}$ on $\text{Mult}(\text{Aff}, \mathfrak{S}(N))$ is equal to $YM_{\text{Aff}}^{\mathfrak{S}(N)}$,*
2. *for any sequence of paths $(p_n)_{n \in \mathbb{N}}$ and any path $p \in P$ such that $(p_n)_{n \in \mathbb{N}}$ converges with fixed endpoints to p , we have:*

$$YM^{\mathfrak{S}(N)}[d_N(h_{p_n}, h_p)] \xrightarrow{n \rightarrow \infty} 0.$$

We only recall the proof given in [19].

Proof. — Let $(\Omega, \mathcal{A}, \mathbb{P})$ be equal to $(\text{Mult}(\text{Aff}, \mathfrak{S}(N)), \mathcal{B}, YM_{\text{Aff}}^{\mathfrak{S}(N)})$. For any $p \in \text{Aff}$, h_p is a function on $\text{Mult}(\text{Aff}, \mathfrak{S}(N))$ thus it can be seen as a G -valued random variable on $(\Omega, \mathcal{A}, \mathbb{P})$. For any positive integer N , let $\Gamma_N = L(\Omega, \mathcal{A}, \mathbb{P}; \mathfrak{S}(N))$ be the set of $\mathfrak{S}(N)$ -valued random variables defined on Ω : this is a group for the pointwise multiplication of random variables. We endow Γ_N with the distance:

$$d_N(X, Y) = \mathbb{E}[d_N(X, Y)].$$

It is a distance which is invariant by translations and inversion. Let us consider the mapping:

$$\begin{aligned} H_N : \text{Aff} &\rightarrow \Gamma_N \\ l &\rightarrow h_l. \end{aligned}$$

This is a multiplicative function. Besides, using Lemma 3.1 and Equality 11, we get that for any simple loop l : $d_N(1, h_l) \leq \sqrt{2} \sqrt{dx(\text{Int}(l))}$. We can apply Theorem 3.1: there exists an extension:

$$\begin{aligned} H_N : P &\rightarrow \Gamma_N \\ p &\mapsto H(p) \end{aligned}$$

which is continuous for the convergence with fixed endpoints. For any sequence of paths $(p_n)_{n \in \mathbb{N}}$ and any path $p \in P$ such that $(p_n)_{n \in \mathbb{N}}$ converges with fixed endpoints to p , we get:

$$d_N(H(p_n), H(p)) = YM_{\text{Aff}}^{\mathfrak{S}(N)}[d_N(H(p_n), H(p))] \xrightarrow{n \rightarrow \infty} 0.$$

Thus we have constructed a $\mathfrak{S}(N)$ -valued process $(H(p))_{p \in P}$ on Ω such that for any p and p' in P such that $\bar{p} = \underline{p}'$, almost surely $H(pp') = H(p')H(p)$, $H(p^{-1}) = H(p)^{-1}$. Using Proposition 2.1 in [13] this allows us to construct a measure on $\text{Mul}(P, \mathfrak{S}(N))$ called $YM^{\mathfrak{S}(N)}$, such that the process $(h_p)_{p \in P}$ has the same law under $YM^{\mathfrak{S}(N)}$ as the process $(H(p))_{p \in P}$ under $YM_{\text{Aff}}^{\mathfrak{S}(N)}$. The measure $YM^{\mathfrak{S}(N)}$ satisfies the desired properties. \square

Now that we have defined the Yang-Mills measure $YM^{\mathfrak{S}(N)}$ for any positive integer N , we are interested in the convergence of these measures as N goes to infinity. Let us define the notion of Wilson loops. Using the gauge-invariance of the Yang-Mills measure, we can restrict ourself to the study of Wilson loops based at 0 instead of studying all the Wilson loops.

Definition 3.4. — Let l_0 be a loop based at 0, the Wilson loop on l_0 is the function:

$$W_{l_0}^N : \mathcal{Mult}(P, \mathfrak{S}(N)) \rightarrow \mathbb{R}$$

$$(h_p)_{p \in P} \mapsto \frac{1}{N} \text{Tr}(h_{l_0}).$$

The main result about the limit of Yang-Mills measure on the symmetric group is given by the following result.

Theorem 3.2. — For any loop l based at 0 the Wilson loop on l converges in probability to a constant denoted by $\phi(l)$ when N goes to infinity. The function:

$$\phi : L_0 \rightarrow \mathbb{R}$$

$$l \mapsto \phi_l,$$

is continuous for the convergence with fixed endpoints.

The asymptotic factorization property holds: for any positive integer k , any k -tuple of loops l_1, \dots, l_k in L_0 :

$$YM^{\mathfrak{S}(N)} [W_{l_1}^N \dots W_{l_k}^N] \xrightarrow{N \rightarrow \infty} \phi(l_1) \dots \phi(l_k).$$

The function ϕ in Theorem 3.2 is called the $\mathfrak{S}(\infty)$ -master field. Let us prove Theorem 3.2 when one considers only piecewise affine loops.

Proposition 3.3. — For any loop l in Aff_0 the Wilson loop W_l^N converges in expectation and in probability as N tends to infinity to a constant $\phi(l)$.

Proof. — Let l_0 be a loop in Aff_0 . Let \mathbb{G} be a graph in $\mathcal{G}(\text{Aff})$ such that l_0 is a loop in \mathbb{G} . Let us consider T a covering tree of \mathbb{G} , let us consider for any bounded face F of \mathbb{G} a loop $c_F \in P(\mathbb{G})$ which represents ∂F and let us consider the facial lassos l_F associated with these choices of tree and loops. Let $w \left((l_F)_{F \in \mathbb{F}}, (l_F^{-1})_{F \in \mathbb{F}} \right)$ be a word in the letters $(l_F)_{F \in \mathbb{F}}$ and $(l_F^{-1})_{F \in \mathbb{F}}$ such that $h_{l_0} = w \left((h_{l_F})_{F \in \mathbb{F}}, (h_{l_F^{-1}})_{F \in \mathbb{F}} \right)$: the random variable h_{l_0} is a product of random variables of the form h_{l_F} or $h_{l_F^{-1}}$, with $F \in \mathbb{F}$.

Using Proposition 3.1, the random variables $(h_{l_F})_{F \in \mathbb{F}}$ on $(\mathcal{Mult}(L_0, \mathbb{G}), \mathcal{B}, YM_{L_0}^{\mathfrak{S}(N)})$ are independent and for any $F \in \mathbb{F}$, h_{l_F} has the same law as $S_{dx(F)}^N$. For all positive integer N , let $\left(S_{t,N}^{(1)} \right)_{t \geq 0}, \dots, \left(S_{t,N}^{(\#\mathbb{F})} \right)_{t \geq 0}$ be $\#\mathbb{F}$ independent random walks identically distributed as $(S_t^N)_{t \geq 0}$. The discussion we just had implies that there exist two sequences of non-negative integers $(k_i)_{i=1}^{\#\mathbb{F}}$ and $(k'_i)_{i=1}^{\#\mathbb{F}}$ such that the Wilson loop W_l^N is equal to:

$$\frac{1}{N^{\text{nc}(\sigma \vee id)}} \text{Tr}^K \left(\left(S_{t,N}^{(1)} \right)^{\otimes k_1} \otimes \left(\left(S_{t,N}^{(1)} \right)^{-1} \right)^{\otimes k'_1} \otimes \dots \otimes \left(S_{t,N}^{(\#\mathbb{F})} \right)^{\otimes k_{\#\mathbb{F}}} \otimes \left(\left(S_{t,N}^{(\#\mathbb{F})} \right)^{-1} \right)^{\otimes k'_{\#\mathbb{F}}} \circ \sigma \right),$$

where $K = \sum_{i=1}^{\#\mathbb{F}} (k_i + k'_i)$ and σ is a K -cycle. An application of Theorem 2.6, applied to the evanescent family $((N-2, 2, 0, \dots, 0))_{N \in \mathbb{N}}$ allows us to conclude. \square

In order to generalize Proposition 3.3 and to prove Theorem 3.2, we need to have an estimate on the Lipschitz norm of the function which gives the expectation of the fraction of fixed points of a random permutation. Recall Definition 3.3 where we defined a distance d_N on $\mathfrak{S}(N)$.

Lemma 3.2. — *For any positive integer N and any $\mathfrak{S}(N)$ -valued random variables S and S' , one has:*

$$\left| \mathbb{E} \left[\frac{1}{N} \text{Tr}(S) \right] - \mathbb{E} \left[\frac{1}{N} \text{Tr}(S') \right] \right| \leq \mathbb{E} [d_N(S, S')].$$

Proof. — It is a consequence of the Cauchy-Schwarz's inequality. Indeed, for any positive integer N and any $\sigma, \sigma' \in \mathfrak{S}(N)$:

$$\frac{1}{N} |\text{Tr}(\sigma) - \text{Tr}(\sigma')| = \frac{1}{N} |\text{Tr}(\sigma - \sigma')| \leq \left[\frac{1}{N} \text{Tr}((\sigma - \sigma')^t (\sigma - \sigma')) \right]^{\frac{1}{2}} = d_N(\sigma, \sigma'),$$

hence the result by taking the expectation. \square

We can finish the proof of Theorem 3.2.

Proof of Theorem 3.2. — We will use the second part of Theorem 3.1. For this, we consider $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space on which is defined for each positive integer N a process $(h_p^N)_{p \in P}$ whose law is the law of the canonical process $(h_p)_{p \in P}$ under the $\mathfrak{S}(N)$ -Yang-Mills measure associated with the \mathcal{T}_N -random walk on $\mathfrak{S}(N)$. Recall the notations defined in the proof of Proposition 3.2: we consider $\Gamma_N = L(\Omega, \mathcal{A}, \mathbb{P} : \mathfrak{S}(N))$ endowed with the distance d_N and we consider the mappings H_N defined from Aff to Γ_N .

Let us denote by E the space $L(\Omega, \mathcal{A}, \mathbb{P} : \mathbb{R})$ of real valued random variables defined on Ω . Let us endow E with the distance $d(X, Y) = \mathbb{E}[\|X - Y\|]$. For any positive integer, let:

$$\begin{aligned} \psi_N : \Gamma_N &\rightarrow E \\ S &\mapsto \frac{1}{N} \text{Tr}(S). \end{aligned}$$

Using Lemma 3.2, for any positive integer N , ψ_N is Lipschitz and $\sup_{N \in \mathbb{N}} \|\psi_N\|_{\text{Lip}} \leq 1$. Besides, using Proposition 3.3, and using the dominated convergence theorem, we know that for any $l \in \text{Aff}_0$, $\psi_N(H_N(l))$ converges in E to a limit which is the non-random variable $\phi(l)$. At last, the Lemma 3.1 shows that the constant K_N in (10) can be taken equal to $\sqrt{2}$ for any positive integer N . Thus we can apply the second part of Theorem 3.1: for any $l \in L_0$, $\psi_N(H_N(l)) = W_l^N$ converges to a limit $\phi(l)$ and the function:

$$\begin{aligned} \phi : L_0 &\rightarrow E \\ l &\mapsto \phi(l) \end{aligned}$$

is continuous for the convergence with fixed endpoints. Let l be a loop based at 0. One can approximate the loop $l \in L_0$ by a sequence of loops $(l_n)_{n \in \mathbb{N}}$ in Aff_0 . Since for any

positive integer n , $\phi(l_n)$ is almost surely constant then $\phi(l)$ is almost surely constant. The convergence in probability holds since we proved that:

$$\mathbb{E} [| W_l^N - \phi(l) |] \xrightarrow{N \rightarrow \infty} 0.$$

The asymptotic factorization property is a simple consequence of the dominated convergence theorem. \square

Remark 3.1. — *Theorem 3.2 is also true for more general sequence of Yang-Mills measures, the proof follows exactly the same steps. For any integer positive N , let us consider a conjugacy class λ_N of $\mathfrak{S}(N)$, and let $(S_t^N)_{t \geq 0}$ be a λ_N -random walk on $\mathfrak{S}(N)$. One can define for any positive integer N , a Yang-Mills measure $YM_{\lambda_N}^{\mathfrak{S}(N)}$ associated with $(S_t^N)_{t \geq 0}$. If the sequence $(\lambda_N)_{N \in \mathbb{N}}$ is evanescent, then Theorem 3.2 holds without any modification.*

If the sequence $(\lambda_N)_{N \in \mathbb{N}}$ is macroscopic, then Proposition 3.3 already does not hold: since the random walk does not satisfy the asymptotic \mathcal{P} -factorization property, for any loop $l \in \text{Aff}_0$, the Wilson loop W_l^N converges in law to a random variable $\phi(l)$ which is not almost surely constant. A slight modification of Theorem 3.1 in order to consider functions of several loops and not only one (in order to consider the functions $(l_1, \dots, l_k) \mapsto \mathbb{E} [W_{l_1}^N \dots W_{l_k}^N]$) allows to prove in a similar way that the process $(h(l))_{l \in L_0}$ under $YM_{\lambda_N}^{\mathfrak{S}(N)}$ converges in law to a real valued random process indexed by L_0 .

4. Random ramified coverings

In this section, we present a natural model of random ramified coverings on the unit disk \mathbb{D} . This model was first defined in [19], in Chapter 5 in the general setting of ramified G -bundles when G is a finite group. We translate the results for random ramified coverings without any conditions on the monodromy on the boundary. This needs some simple verifications which will not be further discussed here.

Let Y be a finite subset of $\mathbb{D} \setminus \partial\mathbb{D}$.

Definition 4.1. — *A ramified covering of the disk with ramification locus Y is a continuous mapping $\pi : R \rightarrow \mathbb{D}$ from a surface R such that the following conditions hold:*

1. *the restriction of π to $\pi^{-1}(\mathbb{D} \setminus Y)$ is a covering,*
2. *for all $y \in Y$ and any $p \in \pi^{-1}(y)$, one can find a neighborhood U of p and an integer $n \geq 1$ such that the mapping:*

$$\begin{aligned} \pi|_U : (U, p) &\rightarrow (\pi(U), y) \\ x &\mapsto \pi(x) \end{aligned}$$

is conjugated to the mapping $z \mapsto z^n : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$.

The integer n is the order of ramification of p and will be denoted by $\text{or}(p)$.

Let N be a positive integer. A ramified covering $\pi : R \rightarrow \mathbb{D}$ with ramification locus Y has degree N if the restriction π to $\pi^{-1}(\mathbb{D} \setminus Y)$ is a covering of degree N .

For sake of simplicity, in this paper, we will only consider simple ramified coverings but it is easy to extend the results to general ramified covering by using Chapter 5 of [19].

Definition 4.2. — Let R be a ramified covering of the disk with ramification locus Y . Let $x \in Y$ be a ramification point of R . It is a simple ramification point if there exists $p_0 \in \pi^{-1}(x)$ such that $\text{or}(p) = 2$, and for any other $p \in \pi^{-1}(x)$, $\text{or}(p) = 1$. The ramified covering R is simple if for any $x \in Y$, x is a simple ramification point.

Often we will denote the covering $\pi : R \rightarrow \mathbb{D}$ just by R . The set of simple ramified covering of the disk is too big to be interesting. As one does for the theory of random maps, we will only work with the isomorphism classes of simple ramified coverings.

Definition 4.3. — Let $\pi : R \rightarrow \mathbb{D}$ and $\pi' : R' \rightarrow \mathbb{D}$ be two simple ramified coverings. They are isomorphic if there exists a homeomorphism $h : R \rightarrow R'$ such that $\pi' \circ h = \pi$.

Let N be a positive integer. We denote by $\mathcal{R}^N(Y)$ the set of isomorphism classes of simple ramified coverings of degree N of \mathbb{D} with ramification locus equal to Y . In fact, it is even easier to work with labelled simple ramified coverings since the set of automorphism of a labelled ramified covering is trivial.

Definition 4.4. — Let $\pi : R \rightarrow \mathbb{D}$ be a simple ramified covering of the disk of degree N with ramification locus equal to Y . Let x be in $\mathbb{D} \setminus Y$. A labelling l of R at the point x is a bijection from $\{1, \dots, n\}$ to $\pi^{-1}(x)$. The pair (R, l) is a labelled simple ramified covering based at x .

Let $(R, l), (R', l')$ be two labelled simple ramified coverings based at x . They are isomorphic if there exists an isomorphism of simple ramified coverings $h : R \rightarrow R'$ such that $h \circ l = l'$.

Let x be a point of $\mathbb{D} \setminus Y$. The set of isomorphism classes of labelled simple ramified coverings of \mathbb{D} with ramification locus Y based at x and with degree N is denoted by $\mathcal{R}_x^N(Y)$. In order to define a measure on $\mathcal{R}^N(Y)$ or $\mathcal{R}_x^N(Y)$, we need to define a σ -field. The σ -field we will consider will be a Borel σ -field.

Definition 4.5. — We consider on $\mathcal{R}_x^N(Y)$ the topology generated by:

$$\mathcal{V}((R, l), U) = \{R' \in \mathcal{R}_x^N(Y) \mid R|_{M \setminus U} \simeq R'|_{M \setminus U}\},$$

where U is any open subset such that $Y \subset U \subset \mathbb{D} \setminus x$.

Also, we consider on $\mathcal{R}^N(Y)$ the topology generated by:

$$\mathcal{V}(R, U) = \{R' \in \mathcal{R}^N(Y) \mid R|_{M \setminus U} \simeq R'|_{M \setminus U}\},$$

where U is any open subset such that $Y \subset U \subset \mathbb{D}$.

Let \mathcal{T}_N be the set of transpositions in $\mathfrak{S}(N)$. The set $\mathcal{R}_x^N(Y)$ is in bijection with $(\mathcal{T}_N)^{\#Y}$: this is a finite set, and thus we can consider the uniform measure on $\mathcal{R}_x^N(Y)$. When one wants to define a measure on a finite set of objects, it is common to take into account the size of the automorphism group: in case of labelled ramified coverings, the uniform measure is the natural one.

Definition 4.6. — The uniform measure on $\mathcal{R}_x^N(Y)$ is:

$$\mathbf{U}_{x,Y}^N = \frac{1}{(\#\mathcal{T}_n)^{\#Y}} \sum_{(R,l) \in \mathcal{R}_x^N(Y)} \delta_{(R,l)}.$$

The natural measure on $\mathcal{R}^N(Y)$ is:

$$\mathbf{U}_Y^N = \frac{1}{(\#\mathcal{T}_n)^{\#Y}} \sum_{(R,l) \in \mathcal{R}^N(Y)} \frac{n!}{\#\text{Aut}(R)} \delta_R.$$

Using the Equation (63) of [19], one gets the following lemma.

Lemma 4.1. — Let $\mathcal{F} : \mathcal{R}_x^N(Y) \rightarrow \mathcal{R}^N(Y)$ be the application where one forgets about the labelling. We have $\mathbf{U}_Y^N = \mathbf{U}_{x,Y}^N \circ \mathcal{F}^{-1}$.

Let $\mathcal{P}_N(dY)$ be a Poisson point process on \mathbb{D} of intensity equal to $\frac{N}{2}dx$. On the set of finite subsets of \mathbb{D} , $F(\mathbb{D})$, we will consider the topology which makes the bijection $F(\mathbb{D}) \simeq \cup_{k \geq 0} (\mathbb{D}^k \setminus \Delta_k) / \mathfrak{S}_k$ continuous. In [19], Proposition 5.3.3, Lévy showed that:

Lemma 4.2. — The application which sends Y , a finite subset of \mathbb{D} , on \mathbf{U}_Y^N and the one which sends Y , a finite subset of $\mathbb{D} \setminus \{x\}$, on $\mathbf{U}_{x,Y}^N$ are continuous.

Thus we can define the following measures on simple ramified coverings on the disk (labelled or not).

Definition 4.7. — We consider on $\mathcal{R}^N(Y)$ and $\mathcal{R}_x^N(Y)$ respectively the Borel measures:

$$\mathbf{U}^N = \int \mathbf{U}_Y^N \mathcal{P}_N(dY) \text{ and } \mathbf{U}_x^N = \int \mathbf{U}_{x,Y}^N \mathcal{P}_N(dY).$$

The main result in this article is that, in some sense, the measures \mathbf{U}^N or \mathbf{U}_x^N converge when N goes to infinity. This assertion has to be taken non-rigorously as the measures are not supported by the same space and the limiting object is not defined. What we will show instead is that the monodromies of the ramified coverings converge in probability. From now on, we will only consider the measure \mathbf{U}_x^N on labelled ramified coverings. The case of non labelled ramified coverings could be also studied, yet it would be necessary to be a little more careful on how we define the associated holonomy process thus, for sake of clarity, we preferred to present the results in the setting of labelled ramified coverings.

Let R be a ramified covering in $\mathcal{R}_x^N(Y)$ and let l be the labelling of the sheets of R at x . Let c be a rectifiable loop in \mathbb{D} based at x . We can transport the labelling l along the path c : it gives us an other labelling l' of the sheets above x . The unique element $\sigma \in \mathfrak{S}(N)$ such that $l' = l\sigma$ is called the monodromy of R along c with respect to l and is denoted by $\text{mon}_{R,l}(c)$. Suppose that we label R at x with $l \circ \eta$ where $\eta \in \mathfrak{S}(N)$, then c transports the labelling $l\eta$ on $l\text{mon}_{R,l}(c)\eta$: it shows that for any curve c_1 and c_2 based at x ,

$$(13) \quad \text{mon}_{R,l}(c_1) = \eta^{-1} \text{mon}_{R,l\circ\eta}(c_1) \eta,$$

$$(14) \quad \text{mon}_{R,l}(c_1^{-1}) = (\text{mon}_{R,l}(c_1))^{-1},$$

$$(15) \quad \text{mon}_{R,l}(c_1 c_2) = \text{mon}_{R,l}(c_2) \text{mon}_{R,l}(c_1),$$

where $c_1 c_2$ is the concatenation of c_1 with c_2 and c_1^{-1} is the curve c_1 with reversed orientation.

If c is a rectifiable curve, $\mathcal{P}_N(dY)$ -a.s. the range of c is inside $\mathbb{D} \setminus Y$. We can thus define the holonomy process associated with \mathbf{U}_x^N by using the monodromy along any rectifiable loop based at x . The set of rectifiable loops in \mathbb{D} based at x is denoted by $L_x(\mathbb{D})$.

Definition 4.8. — *The random holonomy field on $L_x(\mathbb{D})$ associated with \mathbf{U}_x^N is the process $(\mathbf{m}(c))_{c \in L_x(\mathbb{D})}$ defined on $(\mathcal{R}_x^N, \mathbf{U}_x^N)$ where:*

$$\begin{aligned} \mathbf{m}_N(c): \mathcal{R}_x^N &\rightarrow \mathfrak{S}(N) \\ (R, l) &\mapsto \text{mon}_{R,l}(c). \end{aligned}$$

For any c_1 and c_2 in $L_x(\mathbb{D})$, \mathbf{U}_x^N -a.s.,

$$\begin{aligned} \mathbf{m}_N(c_1 c_2) &= \mathbf{m}_N(c_2) \mathbf{m}_N(c_1), \\ \mathbf{m}_N(c_1^{-1}) &= \mathbf{m}_N(c_1)^{-1}. \end{aligned}$$

It is quite natural to wonder how a change of the base point x changes the random holonomy field: in order to do so, we need to consider the same index set for the random processes.

Definition 4.9. — *Let $c_{0 \rightarrow x}$ be a path from 0 to x . The random holonomy field on $L_0(\mathbb{D})$ associated with \mathbf{U}_x^N is the process $(\mathbf{m}_N(c_{0 \rightarrow x}^{-1} c c_{0 \rightarrow x}))_{c \in L_0(\mathbb{D})}$ defined on $(\mathcal{R}_x^N, \mathbf{U}_x^N)$.*

Its law does not depend on the choice of $c_{0 \rightarrow x}$ and it was also proved by Lévy that the laws of the random holonomy field on $L_0(\mathbb{D})$ associated with \mathbf{U}_x^N do not depend on the choice of x . From now on, we will only consider the random holonomy field on $L_0(\mathbb{D})$ associated with \mathbf{U}_0^N . Let us state a theorem which is a direct consequence of Proposition 5.4.4 of [19]. One can see also this theorem as a consequence of the results of [13].

Theorem 4.1. — *Let $(S_t^N)_{t \geq 0}$ be a \mathcal{T}_N -random walk on $\mathfrak{S}(N)$. The holonomy field on $L_0(\mathbb{D})$ associated with \mathbf{U}_0^N has the same law as the holonomy process $(h(l))_{l \in L_0(\mathbb{D})}$ under the $\mathfrak{S}(N)$ -valued Yang-Mills measure associated with $(S_t^N)_{t \geq 0}$.*

As already said, one can easily generalize this theorem in order to study any λ_N -random walk. In a nutshell we have the following “equality”.

Monodromy of random ramified coverings = Yang–Mills measure with $\mathfrak{S}(N)$ gauge group.

Using this equality and Theorem 3.2, we have proved in this article that the traces of the monodromies of random ramified coverings of the disk of degree N converge in probability when N goes to infinity.

Theorem 4.2. — *There exists an application:*

$$\begin{aligned} \phi: L_0(\mathbb{D}) &\rightarrow \mathbb{R} \\ l &\mapsto \phi(l), \end{aligned}$$

which is continuous for convergence with fixed endpoints such that for any $l \in L_0(\mathbb{D})$, $\frac{1}{N} \text{Tr}(\mathbf{m}_N(l))$ converges in probability to $\phi(l)$ as N goes to infinity.

References

- [1] A. D’Adda and P. Provero. Two-dimensional gauge theories of the symmetric group $\mathfrak{S}(n)$ and branched n -coverings of Riemann surfaces in the large- n limit. *Nucl.Phys.Proc.Suppl.*, 108:79–83, 2002.
- [2] A. D’Adda and P. Provero. Two-dimensional gauge theories of the symmetric group \mathfrak{S}_n in the large- n limit. *Commun.Math.Phys.*, 245:1–25, 2004.
- [3] S. Alberverio, R. Høegh-Krohn, and H. Holden. Random fields with values in Lie groups and Higgs fields. *Lecture notes in Physics*, 262:1–13, 1986.
- [4] S. Alberverio, R. Høegh-Krohn, and H. Holden. *Stochastic Lie group-valued measures and their relations to stochastic curve integrals, gauge fields and Markov cosurfaces*. Springer, 1986.
- [5] S. Alberverio, R. Høegh-Krohn, and H. Holden. Stochastic Multiplicative Measures, Generalized Markov Semigroups, and Group-Valued Stochastic Processes and Fields. *Journal of function analysis*, 78:154–184, 1988.
- [6] S. Alberverio, R. Høegh-Krohn, and H. Holden. Stochastic multiplicative measures, generalized Markov semigroups, and group-valued stochastic processes and fields. *J. Funct. Anal.*, 78(1):154–184, 1988.
- [7] M. Anshelevich and A.N. Sengupta. Quantum free Yang–Mills on the plane. *Journal of Geometry and Physics*, 62:330–343, 2012.
- [8] B.K. Driver. YM_2 : continuum expectations, lattice convergence, and lassos. *Comm. Math. Phys.*, 123(4):575–616, 1989.
- [9] B.K. Driver. Two-dimensional Euclidean quantized Yang-Mills fields. In *Probability models in mathematical physics*, pages 21–36. World Sci. Publ., 1991.
- [10] G. Cébron and A. Dahlqvist and F. Gabriel. The general Lévy Master Field. *Work in progress*.
- [11] F. Gabriel. Combinatorial theory of permutation-invariant random matrices I: Partitions, geometry and renormalization. <http://arxiv.org/pdf/1503.02792.pdf>, 2015.
- [12] F. Gabriel. Combinatorial theory of permutation-invariant random matrices II: Freeness and Lévy processes. *arxiv*, 2015.
- [13] F. Gabriel. Planar Markovian holonomy fields. A first step to the characterization of Markovian holonomy fields. <http://arxiv.org/abs/1501.05077>, 2015.
- [14] L. Gross. A Poincaré lemma for connection forms. *J. Funct. Anal.*, 63(1):1–46, 1985.
- [15] L. Gross. The Maxwell equations for Yang-Mills theory. In *CMS Conf. Proc.*, volume 9, pages 193–203. Amer. Math. Soc., 1988.
- [16] T. Lévy. The master field on the plane. arxiv.org/abs/1112.2452.
- [17] T. Lévy. *Yang-Mills Measure on compact surfaces*. Number 790. American Mathematical Society, 2003.
- [18] T. Lévy. Schur-Weyl duality and the heat kernel measure on the unitary group. *Advances in Mathematics*, 218, no. 2:537–575, 2008.
- [19] T. Lévy. *Two-dimensional Markovian holonomy fields*. Number 329. Astérisque, 2010.
- [20] N. Berestycki and R. Durrett. Phase transition in the random transposition random walk. *Probab. Theor. Rel. Fields*, 136:203–233, 2006.

- [21] Nathanael Berestycki. Emergence of giant cycles and slowdown transition in random transpositions and k -cycles. *Electr. J. Probab.*, 16:152–173, 2011.
- [22] Nathanael Berestycki and Gady Kozma. Cycle structure of the interchange process. *Bull. Soc. math. France.*, 143(2):265–280, 2015.
- [23] Oded Schramm. Composition of random transpositions. *Israel J. Math.*, 147:221–243, 2005.
- [24] A.N. Sengupta. The Yang-Mills measure for S^2 . *J. Funct. Anal.*, 108(2):231–273, 1992.
- [25] A.N. Sengupta. Gauge theory on compact surfaces. *Mem. Amer. Math. Soc.*, 126(600):viii+85, 1997.
- [26] C.N. Yang and R.L. Mills. Conservation of isotopic spin and isotopic gauge invariance. *Physical Rev.*, 96(2):191–195, 1954.
- [27] Zvonkine. Enumeration of ramified coverings of the sphere and 2-dimensional gravity. 2005.

FRANCK GABRIEL • *E-mail* : franck.gabriel@normalesup.org, • Université Pierre et Marie Curie (Paris 6), Laboratoire de Probabilités et Modèles Aléatoires, 4, Place Jussieu, F-75252 Paris Cedex 05. • Mathematics Institute, Zeeman Building, University of Warwick, Coventry CV4 7AL.