Rasmussen invariant and Milnor conjecture

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THE RASMUSSEN INVARIANT AND THE MILNOR CONJECTURE

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Abstract. These notes were written for a serie of lectures on the Rasmussen invariant and the Milnor conjecture, given at Winter Braids IV in February 2014.

Introduction

A torus knot is a knot in $\mathbb{R}^3$ which can be drawn without crossing on the surface of a trivially embedded solid torus. Up to mirror image, non trivial torus knots are classified by pairs $\{p, q\}$ of coprime non negative integers. By convention, the knot $T_{p,q}$ corresponds to the line with slope $\frac{p}{q}$ on the torus seen as $\mathbb{R}^2$ modulo the action of the integer lattice. In other words, $T_{p,q}$ winds p times around a circle which bounds a disc inside the solid torus and q times around a circle which bounds a disc outside the solid torus. As shown in Figure 1, $T_{p,q}$ can also be described as the braid closure of q strands twisted p times. Torus knots were intensively studied since they arise naturally in algebraic geometry as the intersection of a complex plane curve with the boundary of a sphere centered at some isolated singularity.

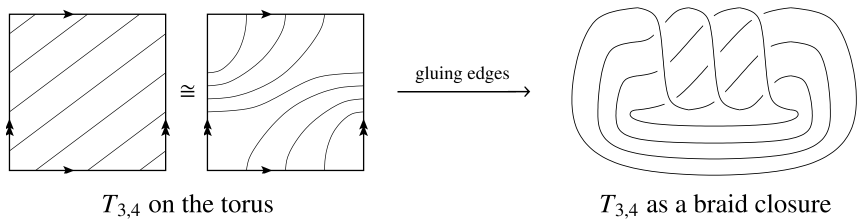

Figure 1: Torus knots descriptions

In [Mil68], John Milnor conjectured that the unknotting number — that is the minimal number of times a knot has to cross itself to unknot — of $T_{p,q}$ is $n_{p,q} := \frac{(p-1)(q-1)}{2}$. As noted in the introduction of [VM12], $n_{p,q}$ crossing changes are sufficient to transform the closed braid diagram of $T_{p,q}$ into a decreasing, and hence trivial, diagram.[1] On the other hand, it is known that the slice genus — that is the minimum genus of a surface in B^4 which bounds the knot seen as in $\mathbb{R}^3 \subset S^3 = \partial B^4$ — is a lower bound for the unknotting number [Mur65, Th. 10.2]. Indeed, as shown in Figure 2, a crossing change can be realized in B^4 with two saddles and two Reidemeister I moves. After capping off the final unknot, an unknotting sequence of length u produces hence a surface in B^4 with Euler characteristic $1 - 2u$, that is genus u, which bounds the knot. To prove the Milnor conjecture, it is hence sufficient to prove that the slice genus of $T_{p,q}$ is $n_{p,q}$. The first proof of that was given by Peter Kronheimer and Tomasz Mrowka in [KM93], but it relied on some involved Gauge theory.

Jacob Rasmussen gave in [Ras10] an alternative combinatorial proof. It uses Khovanov homology, a graded link invariant of homological nature which categorifies the Jones polynomial. Unlike some other known knot invariant categorification, such as knot Floer homology, its construction is combinatorial. Rasmussen's proof relies more exactly on a variation due to Eun Soo Lee which is not graded but filtred. For knots, this variation is always 2–dimensional and located in homological degree 0. Moreover, Lee gave an

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[1] see [BW83] for another argument based on the topology of some associated complex singularity

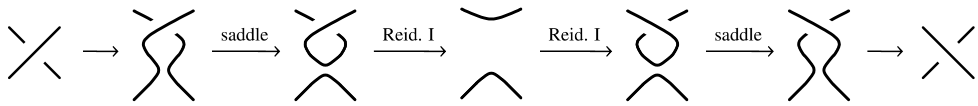


Figure 2: 4–dimensional realization of a crossing change

explicit description of the generators. At first sight, this may look a little bit disapointing for an invariant, but J. Rasmussen showed that looking at the filtration on this homology leads to a numerical knot invariant which enables a control of the slice genus. Indeed, for any (decomposition of) cobordism between two knots, Rasmussen defined an isomorphism between the Lee homologies of the knots whose behavior with regard to the filtration depends only on the genus of the cobordism.

These notes have been written on the occasion of a mini-course given by the author at WinterBraids IV, a winterschool organized in Dijon in February 2014. They aim at giving the most elementary proof of the Milnor conjecture. However, some digressions are made on the way, so it can be read as a gentle introduction to Khovanov homology theory. For instance, we shall address Khovanov's original graded construction, whereas only the filtred Lee version is actually needed to prove the Milnor conjecture.

The notes are organized in three parts, one for each lecture.
The first lecture recalls some standard material of homological algebra. It emphasizes the algebraic definition, without referring to their topological origins. No proof is given there but most of them are elementary. However, the interested reader may refer, for instance, to [Wei94] for further details. It ends with a definition and some examples of categorification.
The second lecture deals with Khovanov homology. Besides the construction, the outlines of its invariance under Reidemeister moves are sketched and the fact that it categorifies the unnormalized Jones polynomial is proved. The approach adopted there is rather close to Viro's reformulation in [Vir04]. Of course, the interested reader can refer to Khovanov seminal paper [Kho00]. Another fruitful point of view is given in [Bar02]. For a more detailled overview, the author also recommands Paul Turner's notes [Tur06] and [Tur14].
The third lecture begins with the modifications needed to define Lee's variation and with the explicit description of its generators. On this basis, we address Rasmussen's invariant. We omit some details which can be found in [Ras10]. Then, to any cobordism between two knots, we associate a filtred isomorphism between the Lee homologies of these knots. The proof of the Milnor conjecture follows then by considering the variation of filtration level. All the material of this section comes from [Lee05] and [Ras10].

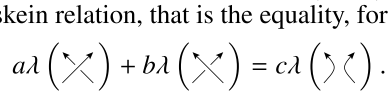
Acknowledgments. The author is grateful to Vincent Florens, Paolo Bellingeri, Jean-Baptiste Meilhan and Emmanuel Wagner for organizing Winter Braids IV, and to the winterschool audience as well as to the anonymous referee for their comments and feedbacks on the lectures.

1. First lecture: Categorification

The lectures assume some familiarity of the audience with knot theory. However, in order to clarify notation, we briefly recall that a link is, up to ambient isotopy, a smooth embedding in S^3 of a finite number of disjoint circles. It can be described as a diagram, that is, up to Reidemeister moves, a disjoint union of crossings in $\mathbb{R}^2$ connected by arcs. A crossing can be positive or negative, depending on whether the basis of $\mathbb{R}^2$ made of the tangent vectors of the highest and lowest strands, in this order, at a the double points is positive or negative. Repeteadly, we will represent only pieces of diagrams; it should be understood then that they stand for a whole diagram with a non represented part which is identical for all diagrams involved in the considered equality. A knot is a link with a single connected component.

1.1. Polynomial invariants.

Definition 1.1. A polynomial invariant of links is a map $\lambda\colon \{\text{links}\} \longrightarrow A$, where A is some Laurent polynomial ring, which satisfies a skein relation, that is the equality, for some given $a, b, c \in A$:



Remark 1.2. The map λ is often defined for some combinatorial description of a link, such as diagrams, and proved to be invariant under the relevant moves, Reidemeister moves in the case of diagrams. This motivates the terminology "invariant".

Remark 1.3. If c divides $a+b$, then if follows from $c\lambda\left(\ \right) = a\lambda\left(\ \right) + b\lambda\left(\ \right)$ that

$$\lambda(L \sqcup U) = \frac{a+b}{c}\lambda(L),$$

where U is the unknot. Moreover, if a and b are furthermore invertible, then the skein relation and the value on the unknot determine the whole invariant λ since they give an algorithmical computation based on an unknotting process. For instance:

$$\begin{aligned}\lambda\left(\ \right) &= \frac{c}{a}\lambda\left(\ \right) - \frac{b}{a}\lambda\left(\ \right)\\ &= \frac{c}{a}\lambda(U) - \frac{(a+b)b}{ac}\lambda(U) = \frac{c^2-ab-b^2}{ac}\lambda(U).\end{aligned}$$

Examples 1.4.

(1) $\Delta\left(\ \right) - \Delta\left(\ \right) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\Delta\left(\ \right)$ and $\Delta(U) = 1$ defines the Alexander polynomial $\Delta(L) \in \mathbb{Z}[t^{\pm\frac{1}{2}}]$;

(2) $t^{-1}V\left(\ \right) - tV\left(\ \right) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V\left(\ \right)$ and $V(U) = 1$ defines the normalized Jones polynomial $V(L) \in \mathbb{Z}[t^{\pm\frac{1}{2}}]$;

(3) $t^{-1}\widetilde{V}\left(\ \right) - t\widetilde{V}\left(\ \right) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\widetilde{V}\left(\ \right)$ and $\widetilde{V}(U) = -t^{\frac{1}{2}} - t^{-\frac{1}{2}}$ defines the unnormalized Jones polynomial $\widetilde{V}(L) \in \mathbb{Z}[t^{\pm\frac{1}{2}}]$. Note that $\widetilde{V}(L) = (-t^{\frac{1}{2}} - t^{-\frac{1}{2}})V(L)$;

(4) $\ell^{-1}P\left(\ \right) - \ell P\left(\ \right) = mP\left(\ \right)$ and $P(U) = 1$ defines the HOMFLY–PT polynomial $P(L) \in \mathbb{Z}[\ell^{\pm1}, m^{\pm1}]$. Note that evaluating $P(L)$ at $\ell = 1$ and $m = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$ gives $\Delta(L)$ and evaluating it at $\ell = t$ and $m = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$ gives $V(L)$.

1.2. **Some homological algebra.**

1.2.1. *Chain complexes & their homologies.*

Definition 1.5. A (ascending) chain complex C is a sequence $(C_i)_{i\in\mathbb{Z}}$ of $\mathbb{Q}$–vector spaces together with linear boundary maps $(\partial_i : C_i \to C_{i+1})_{i\in\mathbb{Z}}$ s.t. $\partial_i \circ \partial_{i-1} = 0$, that is $\mathrm{Im}(\partial_{i-1}) \subset \mathrm{Ker}(\partial_i)$, for all $i \in \mathbb{Z}$.
The homology $H_*(C)$ of C is defined as the sequence $\big(H_i(C)\big)_{i\in\mathbb{Z}} := \big(\mathrm{Ker}(\partial_i)/\mathrm{Im}(\partial_{i-1})\big)_{i\in\mathbb{Z}}$.
For any $x \in \mathrm{Ker}(\partial_i)$, we denote by $[x]$ its image in $H_*(C)$.

Definition 1.6. A decreasing chain complex $\mathcal{D}$ is a sequence $(D_i)_{i\in\mathbb{Z}}$ of $\mathbb{Q}$–vector spaces together with linear boundary maps $(\partial_i : D_i \to D_{i-1})_{i\in\mathbb{Z}}$ s.t. $\partial_{i-1} \circ \partial_i = 0$, that is $\mathrm{Im}(\partial_i) \subset \mathrm{Ker}(\partial_{i-1})$, for all $i \in \mathbb{Z}$.
To any chain complex C, one can associate a dual decreasing chain complex $C^\vee := (C_i^\vee)_{i\in\mathbb{Z}}$ defined by $C_i^\vee := \mathrm{Hom}(C_i, \mathbb{Q})$ and $\partial_i^\vee(f) := f \circ \partial_{i-1}$.
The cohomology $H^*(C)$ of C is defined as the sequence $\big(H^i(C)\big)_{i\in\mathbb{Z}} := \Big(\mathrm{Ker}(\partial_i^\vee)/\mathrm{Im}(\partial_{i+1}^\vee)\Big)_{i\in\mathbb{Z}}$.

Remark 1.7. In the litterature, decreasing chain complexes are often called chain complexes, and ascending ones cochain complexes. This is inherited from the seminal example of chain complexes coming from cellular decompositions of topological spaces, which are naturally descending whereas their duals are ascending. See also Remark 1.31. But since Khovanov's construction is historically ascending without being cofunctorial, we adopt the present non standard terminology.

Since we are working over $\mathbb{Q}$ which is a field, the following result holds:

Proposition 1.8. *For every chain complex C with finite total rank, $H^*(C^\vee) \cong H_*(C)$.*

Remark 1.9. This proposition is not a Poincaré duality-like result but a general fact coming from

- the fact that, over a field, homology groups are determined by their ranks;

- the fact that, if $(e_1, \cdots, e_{n_i})$ is a basis of $\mathrm{Im}(\partial_{i-1})$ completed into a basis of C_i, then $e_j^\vee \in \mathrm{Ker}(\partial_{i-1}^\vee)$ iff $e_i \notin \mathrm{Im}(\partial_{i-1})$, where $e_j^\vee \in C_i^\vee$ is the dual map of e_j;
- the rank–nullity theorem.

Notation 1.10. A chain complex C can be represented as $\cdots \xrightarrow{\partial_{i-1}} C_i \xrightarrow{\partial_i} C_{i+1} \xrightarrow{\partial_{i+1}} \cdots$.

Remark 1.11. For any chain complex C, $H_*(C)$ and $H^*(C)$ can be seen as chain complexes with trivial boundary maps.

Definition 1.12. An exact sequence is a chain complex with homology equal to zero, *i.e.* with $\mathrm{Im}(\partial_{i-1}) = \mathrm{Ker}(\partial_i)$ for all $i \in \mathbb{Z}$. We also say that the chain complex is acyclic.

Example 1.13. $0 \longrightarrow C_0 \xrightarrow{f} C_1 \longrightarrow 0$ is exact $\Leftrightarrow f : C_0 \to C_1$ is an isomorphism.

Definition 1.14. For a chain complex C whose total rank $\displaystyle\sum_{i\in\mathbb{Z}} \mathrm{rk}(C_i)$ is finite, the Euler charcateristic is defined as $\chi(C) := \displaystyle\sum_{i\in\mathbb{Z}} (-1)^i \mathrm{rk}(C_i)$.

The following is a direct consequence of the rank–nullity theorem:

Proposition 1.15. *For any chain complex C, $\chi\big(H_*(C)\big) = \chi(C)$.*

Notation 1.16. For any chain complex C and any integer $k \in \mathbb{Z}$, we define $C[k] := \big(\partial[k]_i : C[k]_i \to C[k]_{i+1}\big)_{i\in\mathbb{Z}}$ by $C[k]_i := C_{i-k}$ and $\partial[k]_i := (-1)^k \partial_{i-k}$, that is the chain complex obtained by shifting downward the homological grading of C by k and, when k is even, adding a minus sign to the boundary map. Same notation is used for decreasing chain complexes.

Lemma 1.17. *For any chain complex C and any integer $k \in \mathbb{Z}$, $\chi\big(C[k]\big) = (-1)^k \chi(C)$.*

Notation 1.18. For any chain complex C, we define $C_\dagger := \big(\partial_{\dagger i} : C_{\dagger i} \to C_{\dagger i-1}\big)_{i\in\mathbb{Z}}$ by $C_{\dagger i} := C_{-i}$ and $\partial_{\dagger i} := \partial_{-i}$, that is the decreasing chain complex obtained by reversing the homological grading of C.

Remark 1.19. For any chain complex C and any integer $k \in \mathbb{Z}$, $C_\dagger[k] = C[-k]_\dagger$ is the decreasing chain complex obtained by reversing the homological grading of C around k, that is $C_\dagger[k]_i = C_{k-i}$.

1.2.2. *Gradings, filtrations & their spectral sequences.*

Definition 1.20.
An internal grading on a chain complex C is a decomposition $C_i = \underset{j\in\mathbb{Z}}{\oplus} C_i^j$ for each $i \in \mathbb{Z}$. Moreover,

- C is said graded iff, for every $i, j \in \mathbb{Z}$, $\partial_i : C_i^j \to C_{i+1}^j$;
- C is said (ascendingly) filtred iff, for every $i, j \in \mathbb{Z}$, $\partial_i : C_i^j \to \underset{j'\geq j}{\oplus} C_{i+1}^{j'}$.

For each $j \in \mathbb{Z}$, we denote by $C^j = \oplus_{i\in\mathbb{Z}} C_i^j$ the subspace spanned by elements with internal grading j. Note that if C is graded, the boundary maps endows C^j with a chain complex structure C^j; then C splits into $\oplus_{j\in\mathbb{Z}} C^j$.

Definition 1.21. If C is a graded chain complex with finite total rank, then the graded Euler characteristic is defined as $\chi_{\mathrm{gr}}(C) := \displaystyle\sum_{j\in\mathbb{Z}} \chi(C^j) q^j = \sum_{i,j\in\mathbb{Z}} (-1)^i \mathrm{rk}(C_i^j) q^j \in \mathbb{Z}[q^{\pm 1}]$.

Notation 1.22. For any chain complex C given with an internal grading and for any integer $k \in \mathbb{Z}$, we set $C\{k\} := \underset{j\in\mathbb{Z}}{\oplus} C\{k\}^j$ the internal grading on C defined by $C\{k\}^j := C^{j-k}$, that is by shifting downward the internal grading of C by k.

Lemma 1.23. *For any graded chain complex C with finite total rank and any integer $k \in \mathbb{Z}$, $\chi_{gr}\big(C\{k\}\big) = q^k \chi(C)$.*

Notation 1.24. For any graded chain complex C, we define $C_\ddagger := \oplus_{j\in\mathbb{Z}} C_\ddagger^j$ by $C_\ddagger^j := C^{-j}$, that is the graded chain complex obtained by reversing the internal grading of C. By $C_\ddagger$, we denote $(C_\dagger)_\ddagger$ the decreasing chain complex obtained by reversing both the homological and the internal gradings.

Remark 1.25. For any chain complex C and any integer $k \in \mathbb{Z}$, $C_{\ddagger}[k]\{k\} = C[-k]\{-k\}_{\ddagger}$ is the decreasing chain complex obtained by reversing both the homological and the internal grading of C around k, that is $C_{\ddagger}[k]\{k\}_i^j = C_{k-i}^{k-j}$.

If C is only filtred, then we have to deal with sums of elements in different gradings. We can however extend the grading to such sums.

Definition 1.26. For element x of a filtred chain complex, we define $j(x) := \max\{j \in \mathbb{Z} | x \in \oplus_{j' \geq j} C^{j'}\}$ if $x \neq 0$ and $j(0) = +\infty$.

The following won't be used in our context, but for the sake of completeness, it is worthwhile mentioning it. See [McC01] or [Cho06] for the definition of a spectral sequence.

Proposition 1.27. *If C is a filtred chain complex, then $\overline{C} := (\overline{\partial}_i : C_i \to C_{i+1})_{i\in\mathbb{Z}}$ defined, for each $i, j \in \mathbb{Z}$, by*

$$\overline{\partial}_i : C_i^j \xrightarrow{\partial_i} \bigoplus_{j' \geq j} C_{i+1}^{j'} \xrightarrow{\pi} C_{i+1}^j,$$

that is by composing $\partial_{|C^j}$ with the projection to C^j, is a graded chain complex.

Theorem 1.28. *If C is a filtred chain complex with finite total rank, then there is a spectral sequence which starts at $H_*(\overline{C})$ and converges to $H_*(C)$.*

1.2.3. *Chain maps & their cones.*

Definition 1.29. A chain map $f : C^1 \to C^2$ between two chain complexes $C^1 := (\partial_i^1 : C_i^1 \to C_{i+1}^1)_{i\in\mathbb{Z}}$ and $C^2 := (\partial_i^2 : C_i^2 \to C_{i+1}^2)_{i\in\mathbb{Z}}$ is a sequence of maps $(f_i : C_i^1 \to C_i^2)_{i\in\mathbb{Z}}$ s.t. $f_{i+1} \circ \partial_i^1 = \partial_i^2 \circ f_i$ for every $i \in \mathbb{Z}$, *i.e.*

$$\begin{tikzcd} C_i^1 \arrow[r, "\partial_i^1"] \arrow[d, "f_i"'] \arrow[dr, phantom, "\circlearrowleft"] & C_{i+1}^1 \arrow[d, "f_{i+1}"] \\ C_i^2 \arrow[r, "\partial_i^2"'] & C_{i+1}^2 \end{tikzcd} .$$

It is graded if C^1 and C^2 are graded and $f_i : C_i^{1,j} \to C_i^{2,j}$ for every $i, j \in \mathbb{Z}$.
It is filtred if C^1 and C^2 are filtred and $j\big(f(x)\big) \geq j(x)$ for every $x \in C^1$.

Proposition 1.30. *A chain map $f : C^1 \to C^2$ induces a well defined chain map $f^* : H_*(C^1) \to H_*(C^2)$ at the level of homologies.*

Remark 1.31. Chain complexes and chain maps form a category, and the operation which takes a chain complex to its homology and a chain map to its induced map is a functor to the category of graded abelian groups. A chain map also induces a map at the level of cohomologies, but the operation is then a cofunctor.

Definition 1.32. If $f : C^1 \to C^2$ is a (graded, filtred) chain map, then $\mathrm{Cone}(f)$ is the (graded, filtred) chain complex defined as $C^1 \oplus C^2[1]$ with boundary maps

$$\partial_i : \begin{tikzcd} C_i^1 \arrow[r, "\partial_i^1"] \arrow[dr, "f_i"] & C_{i+1}^1 \\ \oplus & \oplus \\ C_{i-1}^2 \arrow[r, "-\partial_{i-1}^2"'] & C_i^2 \end{tikzcd}$$

for every $i \in \mathbb{Z}$.

Lemma 1.33. *For any chain map $f : C^1 \to C^2$, $\chi_{(gr)}\big(\mathrm{Cone}(f)\big) = \chi_{(gr)}(C^1) - \chi_{(gr)}(C^2)$.*

Proposition 1.34. *For any chain map $f : C^1 \to C^2$, there is an exact sequence*

$$\cdots \longrightarrow H_{i-1}(C^2) \xrightarrow{\iota^*} H_i\big(\mathrm{Cone}(f)\big) \xrightarrow{\pi^*} H_i(C^1) \xrightarrow{f^*} H_i(C^2) \longrightarrow \cdots$$

where ι^*, π^* and f^* are the maps induced in homology by the chain injection $\iota : C^2 \to \mathrm{Cone}(f)$, the chain surjection $\pi : \mathrm{Cone}(f) \to C^1$ and the chain map f.

Corollary 1.35. *The map* $f^* : H_*(C^1) \to H_*(C^2)$ *is an isomorphism if and only if* $\mathrm{Cone}(f)$ *is acyclic.*

Example 1.36. If f is already an isomorphism at the level of chain complexes, then it induces an isomorphism at the level of homologies and $\mathrm{Cone}(f)$ is acyclic.

1.3. **Categorification.** Categorifying a polynomial invariant λ means associating a graded chain complex $C(D)$ to any link diagram D (or any combinatorial representation of a link) s.t.

(1) each $H_i\big(C^j(D)\big)$ is invariant under Reidemeister moves;
(2) $\chi_{\mathrm{gr}}\big(C(D)\big) = \lambda$, at least up to some change of variable.

Examples 1.37.

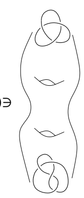
- Heegaard–Floer homology $\widehat{\mathrm{HF}}$ categorifies the Alexander polynomial Δ [OS04b, Ras03];
- Khovanov homology Kh categorifies the unnormalized Jones polynomial $\widetilde{V}$ [Kho00].

Categorifying is worthwhile since

(1) It detects more knots:
   - $\widehat{\mathrm{HF}}(K_{11n34}) \neq \widehat{\mathrm{HF}}(K_{11n42})$ while $\Delta(K_{11n34}) = \Delta(K_{11n42})$ [BG12];
   - $\mathrm{Kh}(10_{132}) \neq \mathrm{Kh}(5_1)$ while $V(10_{132}) = V(5_1)$ [Bar02].

   However, there are some distincts knots with same Heegaard–Floer or Khovanov [Wat07] homology.
(2) It is stronger at detecting geometrical properties:
   - Δ gives a lower bound for the genus of knots;
     $\longrightarrow$ $\widehat{\mathrm{HF}}$ detects the genus of knots [OS04a]
   - Δ gives a necessary condition for a knot to be fibered;
     $\longrightarrow$ $\widehat{\mathrm{HF}}$ gives a necessary and sufficient condition [Ghi08, Ni07, Ni09]
   - $\widehat{\mathrm{HF}}$ [OS04a] and Kh [KM11] detects the unknot, while Δ doesn't and it is still an open question to know whether V does.
(3) It is (expectedly) functorial:
   links can be seen as the objects of the *Cob* category whose morphisms are oriented surfaces bordered by the source and the target links; the *Comp* category has chain complexes as objects and chain maps as morphisms. One can hope to associate chain map to surfaces such that the whole picture is functorial:

$$\begin{tikzcd} \mathrm{Obj}(Cob) \ni L_1 \arrow[rr] \arrow[d, "\mathrm{Mor}(Cob)\ni"'] \arrow[rr, phantom, shift right=8ex, dotted] & & C(L_1) \in \mathrm{Obj}(Comp) \arrow[d, "f\in\mathrm{Mor}(Comp)"] \\ L_2 \arrow[rr] & & C(L_2) \end{tikzcd}$$

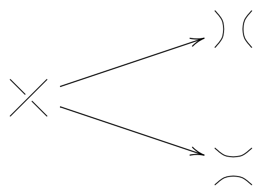


2. Second lecture: Khovanov homology

2.1. **Definitions.** Let D be a link diagram, we want to associate a graded chain complex

$$\widehat{C}(D) := \big(\partial_D : C_i^j(D) \to C_{i+1}^j(D)\big)_{i\in\mathbb{Z}}.$$

2.1.1. *Generators.* A crossing can be considered as a singularity, and there are two ways to smooth it:



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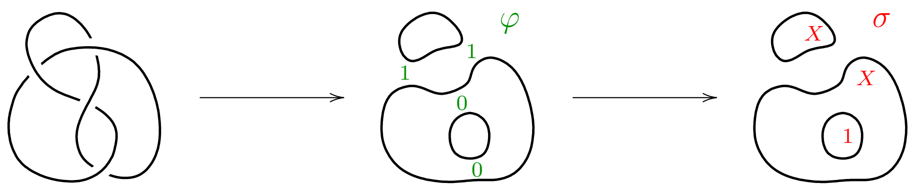


D $\qquad$ D_φ $\qquad$ D_φ^σ

Figure 3: Example of enhanced resolution

A resolution of D is a map $\varphi : \{\text{crossings of } D\} \longrightarrow \{0,1\}$. It specifies a smoothing for each crossing, so it corresponds to a diagram D_φ where all crossings have been resolved. See Figure 3 for an example. It is hence a disjoint union of closed curves, called circles. Note that these resolved diagrams are not considered up to isotopy, in particular $\varphi_1 \neq \varphi_2 \Rightarrow D_{\varphi_1} \neq D_{\varphi_2}$.

Example 2.1. For any oriented diagram, the Seifert resolution is the unique resolution which respects the orientation. This means it sends both crossings and to . In case of knots, both choice of orientation lead to the same Seifert resolution so it is even defined for unoriented diagrams.

Now, a resolution φ of D is said enhanced if it is given a labelling map $\sigma : \{\text{circles of } D_\varphi\} \longrightarrow \{1, X\}$. Such an enhanced resolution of D will be denoted by D_φ^σ. It shall be convenient to see the set $\{1, X\}$ as a subset of $\mathbb{Q}[X]/_{X^2}$.

Definition 2.2. For every $i, j \in \mathbb{Z}$, $C_i^j(D)$ is spanned over $\mathbb{Q}$ by $\left\{D_\varphi^\sigma \,\middle|\, \#\varphi^{-1}(1) = i, \#\sigma^{-1}(1) - \#\sigma^{-1}(X) = j - i\right\}$.

Note that, as a $\mathbb{Q}$–vector space, $\widehat{C}(D)$ is spanned by all enhanced resolutions of D.

Notation 2.3. The i and the j–gradings are respectively called the *homological* and the *Khovanov gradings*. In the forthcoming chain complex, they will respectively play the role of the homological and internal gradings.

2.1.2. *Boundary map.* Let D_φ^σ be a generator of $\widehat{C}(D)$ and c a crossing of D such that $\varphi(c) = 0$. Then D_φ and $D_{\varphi+\delta_c}$, where δ_c is the Kronecker map which is 1 for c and 0 for anything else, differ from the merging of two circles or the splitting of one circle. So $D_{\varphi+\delta_c}$ inherits an enhancing σ_c from σ everywhere except on the (one or two) circles involved. On these circles, we determine σ_c as shown in Figure 4, using the multiplication in $\mathbb{Q}[X]/_{X^2}$. In these pictures, we assume multi-linearity of the enhancing. In particular, a 0–label just means no contribution. As a matter of fact, in the second rule, the case $\alpha = 1$ leads to two summands, with exchanged labels 1 and X, whereas the case $\alpha = X$ leads to a single summand, with two labels X. See also Figure 7. We set $\partial_c(D_\varphi^\sigma) := D_{\varphi+\delta_c}^{\sigma_c}$.

To continue, we need a global order $c_1 < c_2 < \cdots < c_n$ on the crossings of D. For every $E \subset \{c_1, \cdots, c_n\}$ and every crossing c, we denote by $o(c, E) := \#\{c' \in E | c' < c\}$ the number of crossing in E which are lower than c.

Definition 2.4. For any generator $D_\varphi^\sigma \in \widehat{C}(D)$, $\partial_D(D_\varphi^\sigma) := \displaystyle\sum_{c \in \varphi^{-1}(0)} (-1)^{o(c, \varphi^{-1}(1))} \partial_c(D_\varphi^\sigma)$.

Proposition 2.5. *For every $i, j \in \mathbb{Z}$,*

- $\partial_D : C_i^j(D) \longrightarrow C_{i+1}^j(D)$*;*
- $\partial_D^2 : C_i^j(D) \longrightarrow C_{i+2}^j(D)$ *is the zero map.*

Proof. The first assertion states that the boundary map ∂_D increases the homological grading and preserves the Khovanov grading. It is quite immediate by definition of the maps ∂_c.

The second assertion states that ∂_D is a boundary map and hence that $\widehat{C}(D)$ is a chain complex. It is a consequence of the equality $\partial_{c_1} \circ \partial_{c_2} = \partial_{c_2} \circ \partial_{c_1}$, where c_1 and c_2 are two distinct crossings, which can be

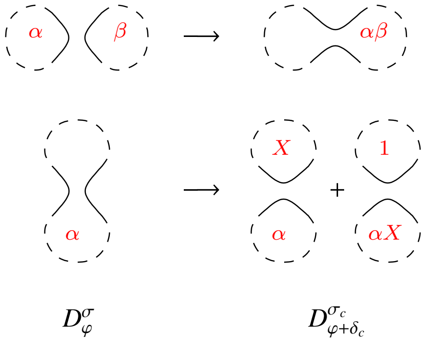


Figure 4: Enhancing rules

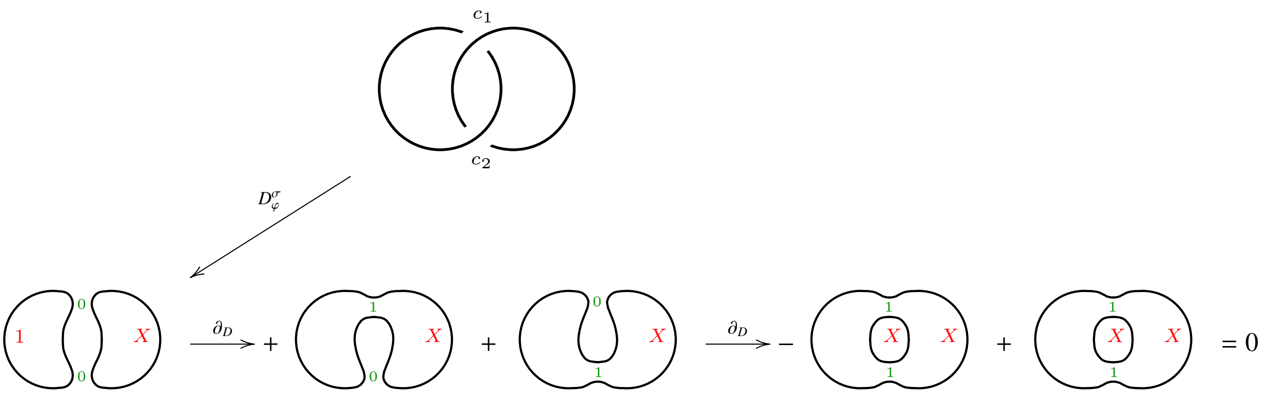


Figure 5: Illustration of the boundary map

checked by hand through a case by case process on the generator it is evaluated on. Each case depends on how c_1 and c_2 connect circles and the labels of these circles. Then, one can notice that $\partial_{c_1} \circ \partial_{c_2}$ and $\partial_{c_2} \circ \partial_{c_1}$ arise with opposite signs in ∂^2_D. See Figure 5 for an example. □

The mirror image of a diagram, that is the diagram obtained by reversing the sign of each crossing, is a natural operation on diagrams. Khovanov homology has a controlled behavior with regards to it.

Proposition 2.6. *For every diagram D, $\widehat{C}(D!) \cong \widehat{C}(D)^{\vee}_{\ddagger}[n]\{n\}$ where $D!$ is the mirror image of D and n is the number of crossings in D.*

Proof. Any resolution $D!_{\varphi}$ of $D!$ can be seen as the resolution $D_{1-\varphi}$ of D. We define then the one-to-one map $\varphi_m : \widehat{C}(D!) \to \widehat{C}(D)^{\vee}_{\ddagger}[n]\{n\}$ by $\varphi_m(D!^{\sigma}_{\varphi}) := D^{-\sigma}_{1-\varphi}$ where, compared to σ, $-\sigma$ switches the labels 1 and X. It can be checked by hand that both homological and Khovanov gradings are preserved and that, for every crossing c of D and every generator D^{σ}_{φ} of $\widehat{C}(D!)$, $\varphi_m \circ \partial_c(D^{\sigma}_{\varphi}) = \partial_c \circ \varphi_m(D^{\sigma}_{\varphi})$. It follows that φ_m is a graded chain isomorphism. □

2.2. **Invariance.** For each Reidemeister move, we define an explicit chain map between the chain complexes associated to the diagrams on each sides of the move and prove that it induces an isomorphism at the level of homologies. We shall consider the case of Reidemeister move II only, the others being similar. So let's consider two diagrams D_1 and D_2 which differ from a Reidemeister move II only. They are represented

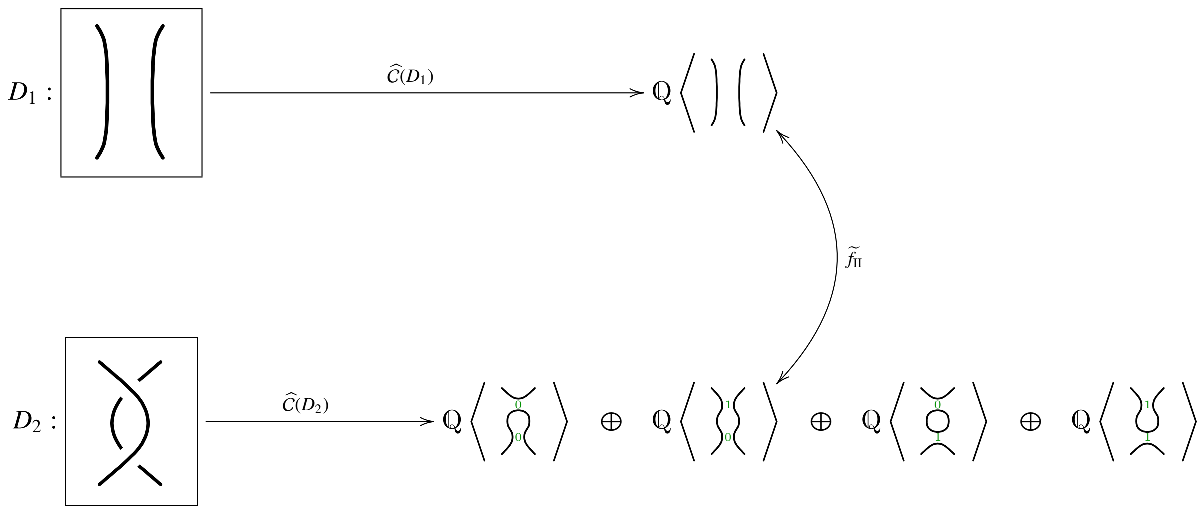


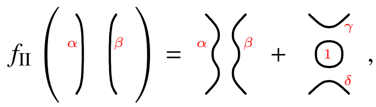
Figure 6: Sketch of invariance under Reidemeister move II

in Figure 6, together with their associated chain complexes — omitting the boundary map — and an obvious one-to-one correspondance $\widetilde{f}_{\mathrm{II}}$ between generators of $\widehat{C}(D_1)$ and a subset of the generators of $\widehat{C}(D_2)$.

We fix an order $c_1 < \cdots < c_n$ on the crossings of D_2 such that c_1 and c_2 are respectively the bottom and top crossings represented in Figure 6. It induces an order $c_3 < \cdots < c_n$ on the crossings of D_1.

Problem 1: The map $\widetilde{f}_{\mathrm{II}}$ is not graded. Indeed, if D^σ_φ is a generator of $\widehat{C}(D_1)$ with homological degree i and Khovanov degree j, then $\widetilde{f}_{\mathrm{II}}(D^\sigma_\varphi)$ is a generator of $\widehat{C}(D_1)$ which has one 1–smoothed crossing more than D^σ_φ. It follows that it has homological degree $i+1$ and Khovanov degree $j+1$. For $\widetilde{f}_{\mathrm{II}}$ to be graded, its source should be shifted into $\widehat{C}(D_1)[1]\{1\}$.

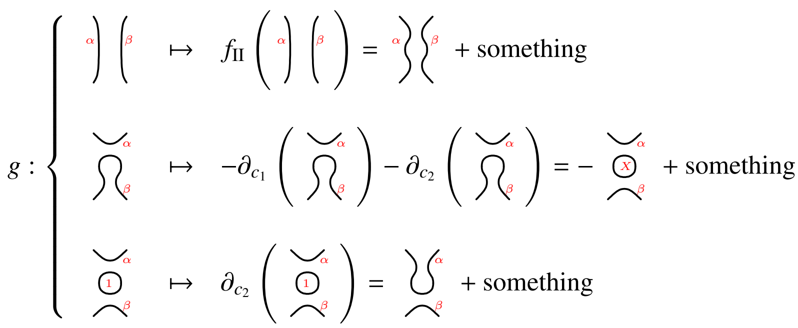
Problem 2: The map $\widetilde{f}_{\mathrm{II}}$ is not a chain map since the partial boundary map $\partial_{c_{n+1}}$ may produce terms in $\partial_{D_2} \circ \widetilde{f}_{\mathrm{II}}$ which are not in $\widetilde{f}_{\mathrm{II}} \circ \partial_{D_1}$. This can be fixed by deforming $\widetilde{f}_{\mathrm{II}}$ into f_{II} defined by $f_{\mathrm{II}}(x) = \widetilde{f}_{\mathrm{II}}(x) + M\big(\partial_{c_1}(x)\big)$ where M is the map which switches back to 0 the smoothing of c_2 and label by 1 the circle which appears then. Graphically,



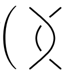
where γ and δ are labels in $\mathbb{Q}[X]/X^2$ which depend on α, β and how the two pieces of circle are connected outside the represented part.

Proposition 2.7. Cone(f_{II}) *is acyclic.*

Proof. The cone of f_{II} is combinatorially equal to the cone of the chain map

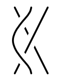
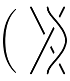


where the boundary map on each side is the signed sum over the 0–smoothed crossings among $c_3, \cdots, c_n$, obtained by just ignoring c_1 and c_2, and with an extra minus sign for the first two lines (but not for the third). Since g is easily seen to be an isomorphism, its cone is acyclic and so is the one of f_{II}. □

Corollary 2.8. $H_*\left(\widehat{C}\left(\right.\right.$  $\left.\left.\right)\right) \cong H_*\left(\widehat{C}\left(\right.\right.$  $\left.\left.\right)[1]\{1\}\right)$.

Similarly, one can define maps f_{I^+}, f_{I^-} and f_{III} whose cones are acyclic and prove

Corollary 2.9.

- $H_*\left(\widehat{C}\left(\right.\right.$  $\left.\left.\right)\right) \cong H_*\left(\widehat{C}\left(\right.\right.$  $\left.\left.\right)\{-1\}\right)$;
- $H_*\left(\widehat{C}\left(\right.\right.$  $\left.\left.\right)\right) \cong H_*\left(\widehat{C}\left(\right.\right.$  $\left.\left.\right)[1]\{2\}\right)$;
- $H_*\left(\widehat{C}\left(\right.\right.$  $\left.\left.\right)\right) \cong H_*\left(\widehat{C}\left(\right.\right.$  $\left.\left.\right)\right)$.

Definition 2.10. For every diagram D, we define

$C(D) := \widehat{C}(D)[-\#$  $]\{\#$  $-2.\#$  $\}$,

where $\#$  and $\#$  denote, respectively, the number of positive and negative crossings in D.

Contrary to $\widehat{C}(D)$, the gradings on $C(D)$ do depend on the choice of an orientation when D has more than one connected component.

Theorem 2.11. *The isomorphism class of $H_*\big(C(D)\big)$, as a bigraded abelian group, is invariant under Reidemeister moves and under the choice of order on the crossings.*

Proof. The first assertion is a corollary of Corollaries 2.8 and 2.9. To prove the second, it is sufficient to deal with the swap of two adjacent crossings c_1 and c_2. In this case, the map which sends D^σ_φ to $(-1)^{\varphi(c_1)\varphi(c_2)}D^\sigma_\varphi$ is a grading-preserving isomorphism which is a chain map. □

Definition 2.12. For any link L, we define $\mathrm{Kh}(L)$, the Khovanov homology of L, as the graded homology of $C(D)$, with D any diagram of L.

Proposition 2.13. *For every link L, $\mathrm{Kh}(L!) \cong \mathrm{Kh}(L)_\ddagger$ where $L!$ is the mirror image of L.*

Proof. Let D be a diagram for L. We denote by, respectively, n_+ and n_- the number of positive and negative crossings in D and by $n := n_+ + n_-$ the total number of crossings. The diagram $D!$ has hence n_- positive and n_+ negative crossings. Using Proposition 2.6 and the definition of $C(D)$ and $C(D!)$, we obtain

$$\begin{array}{rcl} C(D!) & = & \widehat{C}(D!)[-n_+]\{n_- - 2n_+\} \\ & \cong & \widehat{C}(D)^\vee_\ddagger[-n_+ + n]\{n_- - 2n_+ + n\} \\ & = & \widehat{C}(D)^\vee_\ddagger[-n_+ + n_+ + n_-]\{n_- - 2n_+ + n_+ + n_-\} \\ & = & \widehat{C}(D)^\vee_\ddagger[n_-]\{2n_- - n_+\} \\ & = & \widehat{C}(D)^\vee[-n_-]\{n_+ - 2n_-\}_\ddagger \;=\; C(D)^\vee_\ddagger. \end{array}$$

The result follows then from Proposition 1.8. □

2.3. **Categorification of the Jones polynomial.** Let D be a diagram given with an order on its crossings, c its lowest crossing and D^σ_φ any generator of $\widehat{C}(D)$. We denote by D_0 and D_1 the diagrams obtained by, respectively, 0 and 1–smoothing c in D. If $\varphi(c) = 1$, then D^σ_φ can be seen as a generator of $\widehat{C}(D_1)[1]\{1\}$ and since c is not anymore considered in ∂_D but when counting $o(\ .\ , \varphi^{-1}(1))$, we have $\partial_D(D^\sigma_\varphi) = -\partial_{D_1}(D^\sigma_\varphi)$. On the opposite, if $\varphi(c) = 0$, then D^σ_φ can be seen as a generator of $\widehat{C}(D_0)$ and $\partial_D(D^\sigma_\varphi) = \partial_{D_0}(D^\sigma_\varphi) + \partial_c(D^\sigma_\varphi)$. As a consequence:

Proposition 2.14. $\widehat{C}(D) \cong \mathrm{Cone}\big(\partial_c : \widehat{C}(D_0) \to \widehat{C}(D_1)\{1\}\big)$.

Proof. As an exercice for the reader, we let to check that all degrees and signs coincide. $\square$

Corollary 2.15. $\chi_{gr}\Big(\widehat{C}\,(\slashoverback)\Big) = \chi_{gr}\Big(\widehat{C}\,(\smoothing)\Big) - q\chi_{gr}\Big(\widehat{C}\,(\hsmoothing)\Big)$.

Corollary 2.16. $q^{-2}\chi_{gr}\Big(C\,(\positivecrossing)\Big) - q^2\chi_{gr}\Big(C\,(\negativecrossing)\Big) = (q^{-1}-q)\chi_{gr}\Big(C\,(\orientedsmoothing)\Big)$.

Proof. Let $\positivecrossing$, $\negativecrossing$ and $\orientedsmoothing$ be three oriented diagrams which are identical except inside a small disk where they each correspond to their picture. Let $\slashoverback$, $\backoverslash$, $\smoothing$ and $\hsmoothing$ the corresponding non oriented diagrams. Now we denote by, respectively, m and ℓ the numbers of positive and negative crossings in $\orientedsmoothing$. Then $\positivecrossing$ has respectively $m+1$ and ℓ positive and negative crossings and $\negativecrossing$ m and $\ell+1$ positive and negative crossings. Applying several times Corollary 2.15, we obtain:

$$\begin{aligned}
(1)\quad \chi_{\mathrm{gr}}\Big(C\,(\positivecrossing)\Big) &= (-1)^{-\ell}q^{m+1-2\ell}\chi_{\mathrm{gr}}\Big(\widehat{C}\,(\slashoverback)\Big)\\
&= (-1)^{-\ell}q^{m+1-2\ell}\chi_{\mathrm{gr}}\Big(\widehat{C}\,(\smoothing)\Big) + (-1)^{-\ell-1}q^{m+2-2\ell}\chi_{\mathrm{gr}}\Big(\widehat{C}\,(\hsmoothing)\Big)\\
&= q\chi_{\mathrm{gr}}\Big(C\,(\orientedsmoothing)\Big) + (-1)^{-\ell-1}q^{m+2-2\ell}\chi_{\mathrm{gr}}\Big(\widehat{C}\,(\hsmoothing)\Big)\\
(2)\quad \chi_{\mathrm{gr}}\Big(C\,(\negativecrossing)\Big) &= (-1)^{-\ell-1}q^{m-2\ell-2}\chi_{\mathrm{gr}}\Big(\widehat{C}\,(\backoverslash)\Big)\\
&= (-1)^{-\ell-1}q^{m-2\ell-2}\chi_{\mathrm{gr}}\Big(\widehat{C}\,(\hsmoothing)\Big) + (-1)^{-\ell}q^{m-2\ell-1}\chi_{\mathrm{gr}}\Big(\widehat{C}\,(\smoothing)\Big)\\
&= (-1)^{-\ell-1}q^{m-2\ell-2}\chi_{\mathrm{gr}}\Big(\widehat{C}\,(\hsmoothing)\Big) + q^{-1}\chi_{\mathrm{gr}}\Big(C\,(\orientedsmoothing)\Big).
\end{aligned}$$

Then, by substracting q^{-2}(1)-q^2(2), we obtain

$$q^{-2}\chi_{\mathrm{gr}}\Big(C\,(\positivecrossing)\Big) - q^2\chi_{\mathrm{gr}}\Big(C\,(\negativecrossing)\Big) = (q^{-1}-q)\chi_{\mathrm{gr}}\Big(C\,(\orientedsmoothing)\Big)$$

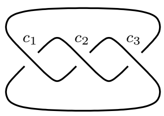
$\square$

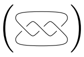
Up to the change of variable $q = -t^{\frac{1}{2}}$, the graded Euler characteristic of Khovanov homology satisfies hence the skein relation of the Jones polynomial. Since it can be directly computed that $\chi_{\mathrm{gr}}\big(C(U)\big) = q + q^{-1} = -t^{\frac{1}{2}} - t^{-\frac{1}{2}}$, it follows that:

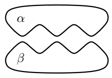
Theorem 2.17. $\mathrm{Kh}(L)$ *is a categorification of the unnormalized Jones polynomial of L.*

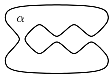
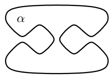
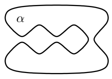
Remark 2.18. In the case of knots, there is a reduced version of Khovanov homology which categorifies the normalized Jones polynomial. The categorification also holds for links, but then it depends on the choice of a connected component.

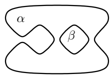
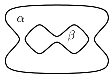
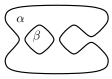
Example 2.19 (Computation for the Khovanov homology of the positive trefoil).

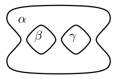
Diagram of the positive trefoil: 

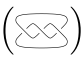
Generators for C ():

$D^{000}_{\alpha\beta} :=$ 

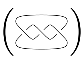
$D^{100}_{\alpha} :=$  $D^{010}_{\alpha} :=$  $D^{001}_{\alpha} :=$ 

$D^{011}_{\alpha\beta} :=$  $D^{101}_{\alpha\beta} :=$  $D^{110}_{\alpha\beta} :=$ 

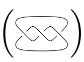
$D^{111}_{\alpha\beta\gamma} :=$ 

Gradings on C ():

$j \backslash i$	0	1	2	3
9				D^{111}_{111}
7			D^{011}_{11} D^{101}_{11} D^{110}_{11}	D^{111}_{11X} D^{111}_{1X1} D^{111}_{X11}
5	D^{000}_{11}	D^{001}_{1} D^{010}_{1} D^{100}_{1}	D^{011}_{1X} D^{011}_{X1} D^{101}_{1X} D^{101}_{X1} D^{110}_{1X} D^{110}_{X1}	D^{111}_{1XX} D^{111}_{X1X} D^{111}_{XX1}
3	D^{000}_{1X} D^{000}_{X1}	D^{001}_{X} D^{010}_{X} D^{100}_{X}	D^{011}_{XX} D^{101}_{XX} D^{110}_{XX}	D^{111}_{XXX}
1	D^{000}_{XX}			

Boundary map on C ():

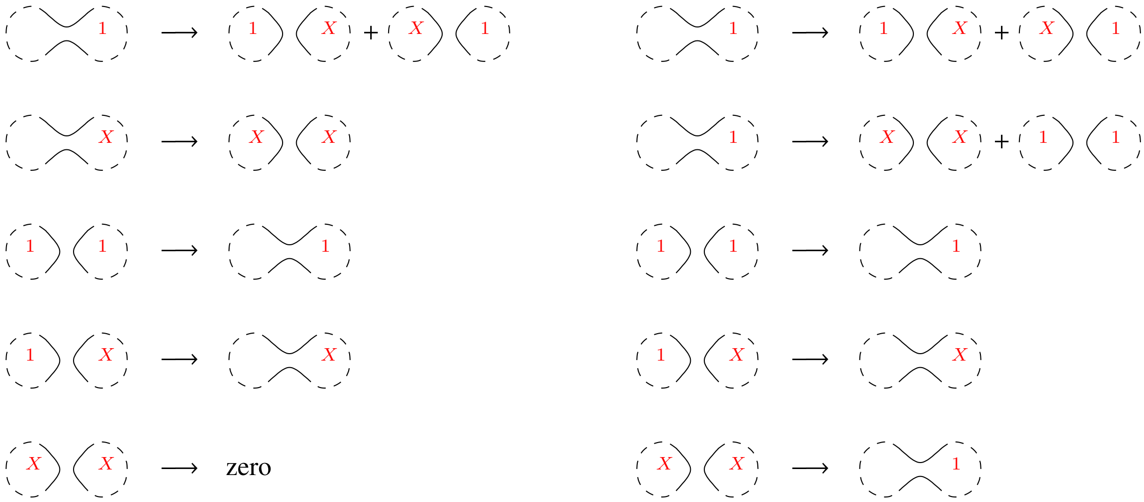
$$\left\{\begin{array}{l}
\partial(D^{111}_{111}) = 0;\\[1em]
\partial(D^{011}_{11}) = D^{111}_{1X1} + D^{111}_{X11};\ \partial(D^{101}_{11}) = -D^{111}_{11X} - D^{111}_{1X1};\ \partial(D^{110}_{11}) = D^{111}_{11X} + D^{111}_{X11};\\[1em]
\partial(D^{000}_{11}) = D^{001}_{1} + D^{010}_{1} + D^{100}_{1};\\
\partial(D^{001}_{1}) = D^{011}_{1X} + D^{011}_{X1} + D^{101}_{1X} + D^{101}_{X1};\ \partial(D^{010}_{1}) = D^{110}_{1X} + D^{110}_{X1} - D^{011}_{1X} - D^{011}_{X1};\\
\partial(D^{011}_{X1}) = D^{111}_{XX1};\ \partial(D^{101}_{1X}) = -D^{111}_{1XX};\ \partial(D^{110}_{X1}) = D^{111}_{X1X};\ \partial(D^{011}_{1X} + D^{101}_{1X} - D^{110}_{X1}) = 0;\\[1em]
\partial(D^{000}_{1X}) = \partial(D^{000}_{X1}) = D^{001}_{X} + D^{010}_{X} + D^{100}_{X};\\
\partial(D^{001}_{X}) = D^{011}_{XX} + D^{101}_{XX};\ \partial(D^{010}_{X}) = D^{110}_{XX} - D^{011}_{XX};\\
\partial(D^{011}_{XX}) = D^{111}_{XXX};\\[1em]
\partial(D^{000}_{XX}) = 0.
\end{array}\right.$$

$\rightsquigarrow$ Kh ():

$j \backslash i$	0	1	2	3
9				$\mathbb{Q}$
7				
5			$\mathbb{Q}$	
3	$\mathbb{Q}$			
1	$\mathbb{Q}$			

3. Third lecture: Milnor conjecture

3.1. **Lee variant and Rasmussen invariant.** Actually, the Rasmussen invariant is not extracted from usual Khovanov homology Kh but a variant Kh′ introduced by E. S. Lee. Basically, it is defined by replacing all occurence of $\mathbb{Q}[X]/X^2$ in the last lecture by $\mathbb{Q}[X]/X^2-1$. Essentially, this modifies the partial boundary map ∂_c into a map ∂'_c which satisfies the same enhancing rules presented in Figure 4. Differences between ∂_c and ∂'_c are given in Figure 7.



∂_c ∂'_c

Figure 7: Differences between ∂_c and ∂'_c

The chain complex is then not anymore graded but filtred since, for any diagram D, the new boundary map ∂'_D satisfies $\partial'_D : C_i^j(D) \to C_{i+1}^j(D) \oplus C_{i+1}^{j+4}$. One can moreover note that the graded part $\overline{\partial}'_D$ of ∂'_D is exactly ∂_D.

Proposition 3.1. *For every diagram D, there is a spectral sequence which starts at* $\mathrm{Kh}(D)$ *and converges to* $\mathrm{Kh}'(D)$*.*

Theorem 3.2 (Lee)**.** *For every knot K, the homology* $\mathrm{Kh}'(K)$ *is generated by two elements which are both signed sums of* $\pm D^\sigma_{\varphi_{Sei}}$ *over all labelling maps* σ *on* φ_{Sei}*, the Seifert resolution.*

Proof. We won't give a complete proof but sketch the outlines. The result of E. S. Lee is actually stated for any link and the generators are in one-to-one correspondence with all the possible orientations for this link. The description is explicit in the sense that a combinatorial rule is given for determining the sign affected to each $D^\sigma_{\varphi_{\mathrm{Sei}}}$. The result is obvious for unlinks and then the proof proceeds by induction on the number of crossings. Indeed, for any link diagram D and a crossing c of D, one can compute the dimension of $\mathrm{Kh}'(D)$ by chasing in the long exact sequence associated to Proposition 2.14 and then see each generator of $\mathrm{Kh}'(D)$ as the image or the preimage under an explicit map of a (explicitely known by induction hypothesis) generator of a diagram with one crossing less. □

Corollary 3.3. *For every knot K,* $\mathrm{Kh}'(K)$ *is zero but in homological degree 0.*

Proof. In the Seifert resolution of knot diagram D, every positive crossing is 0–smoothed and every negative crossing 1–smoothed. It follows that the generators described above are in degree $\#\times$ in $\widehat{C}'(D)$, and hence of degree 0 in $C'(D)$. □

Definition 3.4. For a knot K, we define

- $s_{\max}(K) := \max\left\{ j(\alpha) \middle| [\alpha] \in \mathrm{Kh}'(K) \setminus \{0\} \right\}$;
- $s_{\min}(K) := \min\left\{ j(\alpha) \middle| [\alpha] \in \mathrm{Kh}'(K) \setminus \{0\} \right\}$;

that is, respectively, the maximum and the minimum degree (induced by the filtration, see Definition 1.26) for a representative of a non trivial class in $\mathrm{Kh}'(K)$. It also corresponds to the degrees for which a class in $\mathrm{Kh}(K)$ survives the spectral sequence associated to the filtration.

Theorem 3.5 (Rasmussen)**.** *For any knot K,* $s_{\max}(K) = s_{\min}(K) + 2$.

Even though elementary, the proof needs a few intermediate results. Rather than copying it *in extenso*, we refer the reader to the original proof in [Ras10, Sec. 3.1].

Definition 3.6 (Rasmussen's invariant)**.** For every knot K, we define $s(K) = \frac{s_{\max}(K)+s_{\min}(K)}{2}$.

Example 3.7. For the unknot, we have $C(U) = \mathbb{Q}[1] \oplus \mathbb{Q}[X]$ and $\partial_U = \partial'_U \equiv 0$, so $s(U) = 0$.

Proposition 3.8. *For any knots K and K',*

- $s(K!) = -s(K)$;
- $s(K\#K') = s(K) + s(K')$ *where # denotes the connected sum.*

Only the first statement is necessary to prove the Milnor conjecture and it is a consequence of Proposition 2.13.

3.2. **Cobordisms.**

Definition 3.9. A cobordism between two links L_1 and L_2, eventually empty, is an embedded surface $S \subset \mathbb{R}^3 \times [0,1]$ such that $\partial S = L_1 \sqcup L_2$ with L_1 seen in $\mathbb{R}^3 \times \{0\}$ and L_2 in $\mathbb{R}^3 \times \{1\}$.

Definition 3.10. The slice genus $g_*(K)$ of a knot K is the minimum genus of a cobordism between K and $\emptyset$, that is the minimum genus of a surface embedded in B^4 which bounds K seen in $\mathbb{R}^3 \subset S^3 = \partial B^4$.

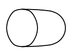
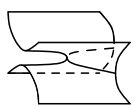
This can be compared with the genus $g(K)$ of K, that is the minimum genus of a surface embedded in $\mathbb{R}^3$ which bounds K. Obviously, $g_*(K) \leq g(K)$.

Theorem 3.11. [Rei72]+[KSS82, Lemma 2.5] *Any cobordism S can be continuously deformed so that each slice $S \cap \left(\mathbb{R}^3 \times \{t\}\right)$, with $t \in [0,1]$, projects to a classical link diagram, except for a finite number of times when the slice either*

(1) projects to a diagram with
  - *(a) an auto-tangency point:*  ;
  - *(b) a tangency point:*  ;
  - *(c) a triple point:*  ;

(2) or contains
  - *(a) an isolated point:* $\cdot$;
  - *(b) two transverse strands:*  .

Corollary 3.12. [CS93, Thm. 5.2] *Up to isotopy, every cobordism can be decomposed into a finite product of the following elementary cobordisms:*

(1) Reidemeister moves I, II or III performed through a time parameter;
(2) Morse moves:
  - *(a) death of a circle:*  ;
  - *(b) birth of a circle:*  ;
  - *(c) saddle:*  .

One can note that, since they are isotopic to the product of the considered link with $[0,1]$, elementary cobordisms corresponding to Reidemeister moves have Euler characteristic equal to 0. For their part, death and birth of circles have Euler characteristic equal to 1 and saddle equal to -1.

3.3. **(almost) Functoriality.** To a cobordism S between two link diagrams D_1 and D_2, we want to associate a chain map $f_S : C'(D_1) \to C'(D_2)$. By Corollary 3.12, it is sufficient to deal with elementary cobordisms:

(1) there are already maps f_{I^+}, f_{I^-}, f_{II} and f_{III} defined for Reidemeister moves;
(2) (a) we define a death map f_{death} by $f_{\text{death}}(D^\sigma_\varphi \sqcup 1) := 0$ and $f_{\text{death}}(D^\sigma_\varphi \sqcup X) := D^\sigma_\varphi$;
  (b) we define a birth map f_{birth} by $f_{\text{birth}}(D^\sigma_\varphi) := D^\sigma_\varphi \sqcup 1$;
  (c) we define a saddle map f_{saddle} by adding an extra crossing c between the merging strands and setting $f_{\text{saddle}} := \partial_c$

We already know that f_{I^+}, f_{I^-}, f_{II} and f_{III} preserve both homological and Khovanov gradings. Since death, birth and saddle cobordisms preserve the number of positive and negative crossings, the associated maps obviously preserve the homological grading, and it is directly checked that f_{death} and f_{birth} rise the Khovanov grading by one, while f_{saddle} reduces the associated filtration[2] by 1.

By composition, we obtain hence a filtred[3] map $f_{\mathcal{S}} : C'(D_1) \to C'(D_2)\big\{-\chi(S)\big\}$ where $\mathcal{S}$ denotes more specifically a given decomposition of S and $\chi(S)$ is the Euler characteristic of S.

In [Jac04], M. Jacobson proved that, for two decompositions $\mathcal{S}$ and $\mathcal{S}'$ of a same cobordism, the induced graded maps $f^*_{\mathcal{S}}, f^*_{\mathcal{S}'} : \mathrm{Kh}(D_1) \to \mathrm{Kh}(D_2)\big\{-\chi(S)\big\}$ are either equal or opposite. A similar result is most likely to hold in the filtred Lee case. Moreover, the sign issue can be fixed at the cost of a more involved construction; see [CMW09, Cap08, Bla10]. But anyway, this (up to sign) invariance of the induced maps is not necessary to prove the Milnor conjecture. On the contrary, we shall need the following fact which is proved by using the explicit description of the generators on both sides together with the explicit description of the elementary cobordism maps:

Proposition 3.13 (Rasmussen). *If $\mathcal{S}$ is a decomposition for a connected cobordism S between two knots K_1 and K_2, then $f^*_{\mathcal{S}} : \mathrm{Kh}'(K_1) \to \mathrm{Kh}'(K_2)\big\{-\chi(S)\big\}$ is an isomorphism.*

Corollary 3.14. *For every knot K, $|s(K)| \le 2g_*(K)$.*

Proof. Let S be a cobordism from K to $\emptyset$ with minimal genus $g_*(K)$. By removing a disk from it, we obtain a cobordism S' from K to the unknot with Euler characteristic $2 - 2g_*(K) - 2 = -2g_*(K)$. Considering a decomposition $\mathcal{S}'$ of S', we obtain an isomorphism $f^*_{\mathcal{S}'}$ between $\mathrm{Kh}'(K)$ and $\mathrm{Kh}'(U)\big\{2g_*(K)\big\}$. Now, we consider D a diagram for K and $\alpha \in \mathrm{Ker}(\partial'_D) \subset C'(D)$ such that $[\alpha] \neq 0$ and $j(\alpha) = s_{\max}(K)$ is maximal. The map $f^*_{\mathcal{S}'}$ is filtred so $j\big(f_{\mathcal{S}'}(\alpha)\big) \ge j(\alpha) = s(K) + 1$. On the other hand, $f^*_{\mathcal{S}'}$ is an isomorphism, so $\big[f_{\mathcal{S}'}(\alpha)\big] = f^*_{\mathcal{S}'}\big([\alpha]\big) \in \mathrm{Kh}'(U)\big\{2g_*(K)\big\}$ is non trivial and hence $j\big(f_{\mathcal{S}'}(\alpha)\big) \le s(U) + 1 + 2g_*(K) = 1 + 2g_*(K)$. It follows that $s(K) \le 2g_*(K)$.

Applying the same reasoning to $K!$ leads to $s(K!) \le 2g_*(K!)$, which becomes $s(K) \ge -2g_*(K)$ by Proposition 3.8. Finally, $-2g_*(K) \le s(K) \le 2g_*(K)$, that is $|s(K)| \le 2g_*(K)$. □

This has the following consequence. It won't be used for our purpose but it is an important feature about the Rasmussen invariant.

Corollary 3.15. *The Rasmussen invariant is a concordance invariant, that is if there is a genus zero cobordism between two knots K_1 and K_2, then $s(K_1) = s(K_2)$.*

Proof. A genus zero cobordism between K_1 and K_2 can be bended and punched into a genus zero cobordism between $K_1\#(K_2!)$ and the unknot. It follows that $|s(K_1) - s(K_2)| = |s\big(K_1\#(K_2!)\big)| \le 2g_*\big(K_1\#(K_2!)\big) = 0$. □

3.4. **Milnor conjecture.** The Rasmussen invariant is difficult to compute for a generic diagram D. Indeed, although Theorem 3.2 gives an explicit description of two independant generators α_+ and α_-, generic elements of $\mathrm{Ker}(\partial'_D)$ are of the form $k_+\alpha_+ + k_-\alpha_- + \partial'_D(\beta)$ where $k_+, k_- \in \mathbb{Q}$ and β is any generator in homological degree -1; the last term introduces an uncertainty which makes, in general, Khovanov degree hard to compute. However, under certain conditions, this difficulty can be avoided.

Proposition 3.16. *If a knot K has a diagram with no negative crossing, then $s(K) = 2g_*(K) = 2g(K)$.*

[2] in the graded original Khovanov construction, "filtration" should be replace by "grading"

[3] in the graded original Khovanov construction, "filtred" should be replace by "graded"

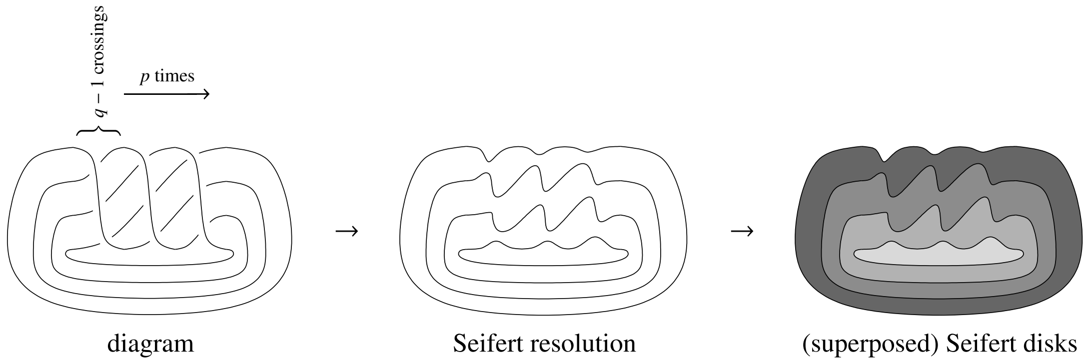


Figure 8: Computing the genus of $T_{3,4}$

Proof. Let us consider D a diagram for K with positive crossings only. In this case, all generators in $C'(D)$ have positive homological degrees. It follows that there is no non trivial element of the form $\partial'_D(\beta)$ in homological degree zero, so that elements which survive in $\mathrm{Kh}'(K)$ are of the form $k_+\alpha_+ + k_-\alpha_-$. Using the description of $a_\pm$ given in Theorem 3.2, it is easily seen that $s_{\min}$ corresponds to the Khovanov grading of the Seifert resolution enhanced with X–labels for all circles, that is $n-r$ circles, where n is the number of (positive) crossings in D and r the number of circles in the Seifert resolution of D. By Corollary 3.14, it follows then that $g_*(K) \geq \frac{1}{2}s(K) = \frac{s_{\min}(K)+1}{2} = \frac{1-r+n}{2}$.

On the other side, since a disc has Euler characteristic 1 and a band with two open sides Euler characteristic -1, the Seifert algorithm on D — that is considering the Seifert resolution of D, pasting a disc on each circle and adding a twisted band for each crossing — provides an oriented surface S bounded by D with Euler characteristic $r-n = 1-2g(S)$, that is $g(S) = \frac{1-r+n}{2}$. It follows that $g_*(K) \leq g(K) \leq g(S) = \frac{1-r+n}{2}$. □

A corollary of the proof is that, if D is a diagram for a knot K with no negative crossing, then the genus and the slice genus of K are computed by the Seifert algorithm.

Corollary 3.17 (modified Milnor conjecture)**.** *For every coprime integers $p,q \in \mathbb{N}^*$, $g_*(T_{p,q}) = g(T_{p,q}) = \frac{(p-1)(q-1)}{2}$.*

Proof. The knot $T_{p,q}$ can be seen as the braid closure D of q strands on which one has performed p times the operation which takes an extremal strand and pulls it to the other side. Since the moving strand crosses all the other strands, each operation produces $q-1$ positive crossings. The diagram D has hence $p(q-1)$ positive crossings. Moreover, the Seifert resolution is nothing but the q parallel strands, which close into q circles. The associated Seifert surface S has hence Euler characteristic $q-p(q-1) = p+q-pq = 1-2g(S)$ and genus $g(S) = \frac{pq-p-q+1}{2} = \frac{(p-1)(q-1)}{2}$. See Figure 8 for illustrations. □

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