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# EIGENVALUES OF THE SUB-LAPLACIAN AND DEFORMATIONS OF CONTACT STRUCTURES ON A COMPACT CR MANIFOLD

AMINE ARIBI, SORIN DRAGOMIR, AND AHMAD EL SOUFI

**ABSTRACT.** Given a compact strictly pseudoconvex CR manifold  $M$ , we study the differentiability of the eigenvalues of the sub-Laplacian  $\Delta_{b,\theta}$  associated with a compatible contact form (i.e. a pseudo-Hermitian structure)  $\theta$  on  $M$ , under conformal deformations of  $\theta$ . As a first application, we show that the property of having only simple eigenvalues is generic with respect to  $\theta$ , i.e. the set of structures  $\theta$  such that all the eigenvalues of  $\Delta_{b,\theta}$  are simple, is residual (and hence dense) in the set of all compatible positively oriented contact forms on  $M$ . In the last part of the paper, we introduce a natural notion of critical pseudo-Hermitian structure of the functional  $\theta \mapsto \lambda_k(\theta)$ , where  $\lambda_k(\theta)$  is the  $k$ -th eigenvalue of the sub-Laplacian  $\Delta_{b,\theta}$ , and obtain necessary and sufficient conditions for a pseudo-Hermitian structure to be critical.

## 1. INTRODUCTION

Let  $M$  be a compact strictly pseudoconvex CR manifold of real dimension  $2n + 1$ . A pseudo-Hermitian structure on  $M$  is a contact form  $\theta \in \Gamma(T^*M)$  whose kernel coincides with the horizontal distribution of  $M$ . The strict pseudoconvexity of  $M$  means that the Levi form associated to such a contact form is either positive definite or negative definite. We denote by  $\mathcal{P}_+(M)$  the set of all pseudo-Hermitian structures with positive definite Levi form on  $M$ .

To every pseudo-Hermitian structure  $\theta \in \mathcal{P}_+(M)$  we associate its sub-Laplacian  $\Delta_{b,\theta}$  (or simply  $\Delta_b$  if there is no risk of confusion) which is a sub-elliptic operator of order  $1/2$ , and denote by

$$0 = \lambda_0(\theta) < \lambda_1(\theta) \leq \lambda_2(\theta) \leq \dots \leq \lambda_k(\theta) \leq \dots \rightarrow \infty$$

the nondecreasing sequence of eigenvalues of  $\Delta_{b,\theta}$ .

Several works published in recent years are devoted to the study of the sub-Laplacian and the investigation of its spectral properties, see for instance [3, 5, 4, 6, 8, 9, 10, 14, 18, 19, 21, 25, 26, 27, 28, 30]. The aim of most of them is to extend to the CR context some of the spectral geometric results established in the Riemannian setting for the Laplace-Beltrami operator.

In our previous paper [5], we discussed the continuity of the eigenvalues  $\lambda_k(\theta)$ , as functions on the set  $\mathcal{P}_+(M)$  that we have endowed with a natural

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metric topology. In the present paper, we start by studying the differentiability of the spectrum of the sub-Laplacian  $\Delta_{b,\theta}$  under one-parameter deformations of the contact structure  $\theta$ . We apply classical perturbation theory of selfadjoint operators to get a differentiability result (Theorem 3.2). Moreover, we prove that if  $\theta(t) \in \mathcal{P}_+(M)$  is an analytic deformation of a contact structure  $\theta$ , then the function  $t \mapsto \lambda_k(\theta(t))$ , which is not differentiable if  $\lambda_k(\theta)$  is not simple, admits left-sided and right-sided derivatives at  $t = 0$ , and relate these derivatives to the eigenvalues of an explicit symmetric operator acting on the  $\lambda_k(\theta)$ -eigenspace (Theorem 3.3).

In the second part of the paper we use these facts to show that the property of having only simple eigenvalues is generic for the sub-Laplacians on a given compact strictly pseudoconvex CR manifold  $M$ . Indeed, we prove that the set of contact structures  $\theta \in \mathcal{P}_+(M)$  such that all the eigenvalues of  $\Delta_{b,\theta}$  are simple, is a residual set in the complete metric space  $\mathcal{P}_+(M)$  (see Theorem 4.1). Our proof relies on an eigenvalue splitting technique (Proposition 4.1) used by many authors in the Riemannian setting (see [1, 7, 13]; see also [31] for a different approach).

The last section is devoted to the notion of critical pseudo-Hermitian structure. Despite the lack of differentiability of the eigenvalues  $\lambda_k(\theta)$  upon analytic deformations  $\theta(t) \in \mathcal{P}_+(M)$  of the pseudo-Hermitian structure, a natural notion of criticality can be defined using the existence of left-sided and right-sided derivatives of  $\lambda_k(\theta(t))$  at  $t = 0$  (see Definition 5.1). Since  $\lambda_k(\theta)$  is not invariant under scaling of the pseudo-Hermitian structure, we restrict ourselves to the deformations that preserve the global volume  $\text{vol}(\theta) = \int_M \theta \wedge (d\theta)^n$ . We give necessary and sufficient conditions for a pseudo-Hermitian structure to be a critical point of the functional  $\theta \in \mathcal{P}_+(M) \mapsto \lambda_k(\theta)$ , under the volume-preserving constraint. In particular, we will see that the criticality condition is strongly related to the existence of a finite family of  $\lambda_k(\theta)$ -eigenfunctions  $v_1, \dots, v_d$ , satisfying  $v_1^2 + \dots + v_d^2 = 1$  (Corollary 5.1). This last condition is satisfied for instance by the first positive eigenvalue of the standard CR sphere  $\mathbb{S}^{2n+1}$  (see [30, Proposition 4.4]).

## 2. PRELIMINARIES

Let  $M$  be a compact connected orientable CR manifold of CR dimension  $n$  (and real dimension  $2n + 1$ ). Such a manifold  $M$  is equipped with a pair  $(H, J)$ , where  $H$  is a sub-bundle of the tangent bundle  $TM$  of real rank  $2n$  (often called Levi distribution) and  $J$  is an integrable complex structure on  $H$  which means that,  $\forall X, Y \in \Gamma(H)$ ,

$$[X, Y] - [JX, JY] \in \Gamma(H)$$

and

$$[JX, Y] + [X, JY] = J([X, Y] - [JX, JY]).$$

Since  $M$  is orientable, there exists a nonzero 1-form  $\theta \in \Gamma(T^*M)$  whose kernel coincides with  $H$ . Such a 1-form, called *pseudo-Hermitian* structure on  $M$ , is of course not unique. Actually, the set of pseudo-Hermitian structures on  $M$  consists in all the forms  $\pm e^u \theta$ ,  $u \in C^\infty(M)$ .

To each pseudo-Hermitian structure  $\theta$  we associate its *Levi form*  $G_\theta$  defined on  $H$  by

$$G_\theta(X, Y) = -d\theta(JX, Y) = \theta([JX, Y]).$$

The integrability of  $J$  implies that  $G_\theta$  is symmetric and  $J$ -invariant. The CR manifold  $M$  is said to be *strictly pseudoconvex* if the Levi form  $G_\theta$  of a pseudo-Hermitian structure  $\theta$  is either positive definite or negative definite. Of course, this condition does not depend on the choice of  $\theta$ . In all the sequel, we assume that  $M$  is strictly pseudoconvex and denote by  $\mathcal{P}_+(M)$  the set of all pseudo-Hermitian structures with positive definite Levi form on  $M$ . Every  $\theta \in \mathcal{P}_+(M)$  is in fact a contact form which induces on  $M$  the following volume form

$$\psi_\theta = \frac{1}{2^n n!} \theta \wedge (d\theta)^n.$$

The associated divergence  $\text{div}_\theta$  is defined, for every smooth vector field  $Z$  on  $M$ , by

$$\mathcal{L}_Z \psi_\theta = \text{div}_\theta(Z) \psi_\theta.$$

We denote by  $L^2(M)$  the set of squared integrable functions on  $M$  with respect to  $\psi_\theta$ . A function  $u \in L^2(M)$  is *weakly differentiable* (w.d.) along  $H$  if there is  $Y_u \in \Gamma(H)$  such that  $|Y_u|_{G_\theta} = G_\theta(Y_u, Y_u)^{\frac{1}{2}} \in L^1_{loc}(M)$  and

$$\int_M G_\theta(Y_u, X) \psi_\theta = - \int_M u \text{div}_\theta(X) \psi_\theta$$

for every  $X \in \Gamma^\infty(H)$ . Such  $Y_u$  is unique up to a set of measure zero and is denoted by  $Y_u = \nabla^H u$  and called *weak horizontal gradient* of  $u$ . It is easy to check that if  $u$  is differentiable, then  $\forall X \in \Gamma^\infty(H)$ ,  $du(X) = G_\theta(X, \nabla^H u)$ . Let

$$\mathcal{D}(\nabla^H) = \left\{ u \in L^2(M) : u \text{ is (w.d.) along } H \text{ and } \nabla^H u \in L^2(H) \right\},$$

where  $L^2(H)$  stands for the set of squared integrable sections of  $H$  with respect to the inner product  $G_\theta$  and the volume element  $\psi_\theta$ . Then we may regard the weak horizontal gradient as a linear operator

$$\nabla^H : \mathcal{D}(\nabla^H) \subset L^2(M) \rightarrow L^2(H).$$

As  $C^\infty(M) \subset \mathcal{D}(\nabla^H)$  it follows that  $\mathcal{D}(\nabla^H)$  is a dense subspace of  $L^2(M)$ . Let

$$(\nabla^H)^* : \mathcal{D}[(\nabla^H)^*] \subset L^2(H) \rightarrow L^2(M)$$

be the adjoint of  $\nabla^H$ . Then  $\Gamma^\infty(H) \subset \mathcal{D}[(\nabla^H)^*]$  and, for all  $X \in \Gamma^\infty(H)$ , one has

$$(\nabla^H)^* X = -\text{div}_\theta(X).$$

In particular,  $(\nabla^H)^*$  is densely defined in  $L^2(H)$ . The sub-Laplacian  $\Delta_b$ , or  $\Delta_{b,\theta}$  if it is necessary to avoid confusion, is given by

$$\begin{aligned}\mathcal{D}(\Delta_b) &= \{u \in \mathcal{D}(\nabla^H) : \nabla^H u \in \mathcal{D}[(\nabla^H)^*]\}, \\ \Delta_b &= (\nabla^H)^* \circ \nabla^H = -\operatorname{div}_\theta \circ \nabla^H.\end{aligned}$$

Note that

$$(\Delta_b u, u)_{L^2(M)} = \|\nabla^H u\|_{L^2(H)}^2 \geq 0$$

for any  $u \in \mathcal{D}(\Delta_b)$ . Moreover, the sub-Laplacian is symmetric, i.e.

- (1)  $\mathcal{D}(\Delta_b)$  is dense in  $L^2(M)$ .
- (2)  $\mathcal{D}(\Delta_b) \subset \mathcal{D}(\Delta_b^*)$  and  $(\Delta_b u, v)_{L^2(M)} = (u, \Delta_b v)_{L^2(M)} \quad \forall u, v \in \mathcal{D}(\Delta_b)$ .

The operator  $\Delta_b$  is also known to be subelliptic of order  $\varepsilon = 1/2$ . Indeed, one has (cf. [15, Theorem 2.1]) for any  $u \in C^\infty(M)$ ,

$$\|u\|_{H^{1/2}(M)}^2 \leq C \left( (\Delta_b u, u)_{L^2(M)} + \|u\|_{L^2(M)}^2 \right), \quad (2.1)$$

for some constant  $C$  independent of  $u$ . It is worth noticing that  $\Delta_b$  can be seen as the real part of the Kohn Laplacian acting on functions  $\square_b = \bar{\partial}_b^* \bar{\partial}_b$ , where  $\bar{\partial}_b u$  is the projection of  $du$  onto  $T_{(0,1)}^* M$ . Indeed, we have (cf. [24, Theorem 2.3])  $\square_b = \Delta_b + i n T$ , where  $T$  is the unique vector field satisfying  $T \lrcorner \theta = 1$  and  $T \lrcorner d\theta = 0$ .

**Lemma 2.1.** *The space  $H^{1/2}(M) = W^{1/2,2}(M)$  admits a compact embedding into  $L^2(M)$ .*

The proof of this Lemma uses standard arguments (see [4]).

**Lemma 2.2.** *The operator  $(\Delta_b + I)^{-1} : \mathcal{D}((\Delta_b + I)^{-1}) \subset L^2(M) \rightarrow L^2(M)$  is compact.*

*Proof.* Based on the estimate (2.1) one has  $\operatorname{Ker}(\Delta_b + I) = \{0\}$ . Consequently,

$$\Delta_b + I : C^\infty(M) \rightarrow \mathcal{R}(\Delta_b + I) \subset C^\infty(M)$$

is invertible, where  $\mathcal{R}(A)$  denotes the range of the operator  $A$ . Therefore, we may consider the inverse

$$(\Delta_b + I)^{-1} : \mathcal{D}((\Delta_b + I)^{-1}) = \mathcal{R}(\Delta_b + I) \subset L^2(M) \rightarrow H^{1/2}(M).$$

Let  $v \in \mathcal{D}((\Delta_b + I)^{-1})$  and let us apply (2.1) to the function  $u = (\Delta_b + I)^{-1}(v)$  followed by the Cauchy-Schwartz inequality

$$\|(\Delta_b + I)^{-1} v\|_{H^{1/2}(M)}^2 \leq C \left( v, (\Delta_b + I)^{-1} v \right)_{L^2(M)} \leq C \|v\|_{L^2(M)} \|(\Delta_b + I)^{-1} v\|_{L^2(M)}.$$

Moreover, there is a continuous embedding  $H^{1/2}(M) \rightarrow L^2(M)$  so that

$$\|u\|_{L^2(M)} \leq C' \|u\|_{H^{1/2}(M)}, \quad u \in H^{1/2}(M),$$

for some constant  $C' > 0$  independent of  $u$ . Thus,

$$\|(\Delta_b + I)^{-1} v\|_{H^{1/2}(M)}^2 \leq C'' \|v\|_{L^2(M)} \|(\Delta_b + I)^{-1} v\|_{H^{1/2}(M)}$$

(with  $C'' = CC'$ ) or

$$\|(\Delta_b + I)^{-1} v\|_{H^{1/2}(M)} \leq C'' \|v\|_{L^2(M)}$$

which proves the continuity of the operator  $(\Delta_b + I)^{-1}$ . Finally, by Lemma 2.1, the embedding  $H^{1/2}(M) \rightarrow L^2(M)$  is compact. Hence,  $(\Delta_b + I)^{-1} : \mathcal{D}((\Delta_b + I)^{-1}) \subset L^2(M) \rightarrow L^2(M)$  is compact (as the composition of a compact operator with a continuous operator).  $\square$

**Corollary 2.1.** *The spectrum  $\sigma(\Delta_b)$  of the sub-Laplacian is discrete and consists of eigenvalues of finite multiplicity.*

### 3. DIFFERENTIABILITY OF EIGENVALUES WITH RESPECT TO 1-PARAMETER DEFORMATIONS OF THE PSEUDO-HERMITIAN STRUCTURE

We start by recalling the needed notions of functional analysis, cf. e.g. A. Kriegel & P.W. Michor [22, 23] and T. Kato [20]. Let  $\mathcal{H}$  be a Hilbert space and  $\{A(t)\}_{t \in \mathbb{R}}$  a family of linear operators  $A(t) : \mathcal{D}(A(t)) \subset \mathcal{H} \rightarrow \mathcal{H}$ . We say that  $A(t)$  is a *real analytic* (respectively  $C^\infty$ , or  $C^{k,\alpha}$ ) family of selfadjoint operators if there is a dense subspace  $V \subset \mathcal{H}$  such that

- i)  $\mathcal{D}(A(t)) = V$  and  $A(t)$  is selfadjoint for any  $t \in \mathbb{R}$  and
- ii) the function  $t \in \mathbb{R} \mapsto (A(t)u, v)_{\mathcal{H}} \in \mathbb{C}$  is real analytic (respectively  $C^\infty$ , or  $C^{k,\alpha}$ ) for every  $u \in V$  and  $v \in \mathcal{H}$ .

If this is the case then (by a result in [22]) the (vector valued) function

$$t \in \mathbb{R} \mapsto A(t)u \in \mathcal{H},$$

is of the same class for every  $u \in V$ .

A sequence  $\{\lambda_\nu\}_{\nu \geq 1}$  of scalar functions  $\lambda_\nu : \mathbb{R} \rightarrow \mathbb{C}$  is said to *parameterize the eigenvalues* of  $\{A(t)\}_{t \in \mathbb{R}}$  if for any  $t \in \mathbb{R}$  and any  $\lambda \in \sigma(A(t))$ , the cardinality of the set  $\{\nu \geq 1 : \lambda_\nu(t) = \lambda\}$  equals the multiplicity of  $\lambda$ .

We shall make use of the following result, which is referred hereafter as the Rellich-Alekseevsky-Kriegel-Losik-Michor theorem (cf. F. Rellich [29] for statement (i), D. Alekseevski & A. Kriegel & M. Losik & P.W. Michor [2] for statement (ii), and A. Kriegel & P.W. Michor [23] for statements (iii)-(iv)).

**Theorem 3.1.** *Let  $t \in \mathbb{R} \mapsto A(t)$  be a curve of unbounded selfadjoint operators in a Hilbert space  $\mathcal{H}$ , with common domain of definition and compact resolvent. Then*

- (i) *If  $A(t)$  is real analytic in  $t \in \mathbb{R}$ , then the eigenvalues and the eigenvectors of  $A(t)$  may be parameterized real analytically in  $t$ .*
- (ii) *If  $A(t)$  is  $C^\infty$  in  $t \in \mathbb{R}$  and if no two unequal continuously parameterized eigenvalues meet of infinite order at any  $t \in \mathbb{R}$ , then the eigenvalues and eigenvectors can be parameterized  $C^\infty$  in  $t$  on the whole parameter domain.*
- (iii) *If  $A(t)$  is  $C^\infty$  in  $t \in \mathbb{R}$ , then the eigenvalues of  $A(t)$  may be parameterized  $C^2$  in  $t$ .*
- (iv) *If  $A(t)$  is  $C^{k,\alpha}$  in  $t \in \mathbb{R}$  for some  $\alpha > 0$ , then the eigenvalues of  $A(t)$  may be parameterized  $C^1$  in  $t$ .*

Among the applications to statements (i) and (iii) in Theorem 3.1 as proposed in [23], one may consider a compact manifold  $M$  and a smooth curve

$t \mapsto g_t$  of smooth Riemannian metrics on  $M$ . If moreover  $t \mapsto \Delta_{g_t}$  is the corresponding smooth curve of Laplace-Beltrami operators on  $L^2(M)$  then (by (iii) in Theorem 3.1) the eigenvalues may be parameterized  $C^2$  in  $t$ . This was exploited by A. El Soufi & S. Ilias, [16]-[17], who discussed an array of related questions such as critical points of the functional  $g \in \mathcal{M}(M) \mapsto \lambda_k(g)$ , or suitable deformations of  $g \in \mathcal{M}(M)$  producing quantitative variations of  $\lambda_k$ . Here  $\mathcal{M}(M)$  is the set of all Riemannian metrics on  $M$ .

Let  $M$  be a compact strictly pseudoconvex CR manifold and let  $\theta$  be a pseudo-Hermitian structure on  $M$  with positive definite Levi form. Let

$$\theta(t) = e^{u_t} \theta, \quad t \in \mathbb{R},$$

be an *analytic deformation* of  $\theta$ , i.e.  $\{u_t\}_{t \in \mathbb{R}}$  is a family of real valued  $C^\infty$  functions which is analytic with respect to  $t$  and  $u_0 = 0$ . Here  $C^\infty(M, \mathbb{R})$  is thought of as organized as a real Fréchet space and the vector valued function

$$u : \mathbb{R} \rightarrow C^\infty(M, \mathbb{R}), \quad u(t) = u_t, \quad t \in \mathbb{R},$$

is assumed to be of class  $C^\omega$ . For a theory of power series in Fréchet spaces we shall use Appendix B in [11].

Let  $\Delta_b$  be the sub-Laplacian on  $M$  associated with  $\theta$  and denote for each  $t$ , by  $\Delta_{b,t}$  the sub-Laplacian associated with  $\theta(t)$ .

**Theorem 3.2.** *Let  $\theta(t) = e^{u_t} \theta$  be an analytic deformation of  $\theta$  and let  $\lambda \in \sigma(\Delta_b)$  be an eigenvalue of multiplicity  $m$ . There exist a positive real number  $\varepsilon$ , a family of  $m$  real analytic functions  $\{\Lambda_i\}_{1 \leq i \leq m} \subset C^\omega((-\varepsilon, \varepsilon), \mathbb{R})$ , and  $m$  families of  $C^\infty$  functions  $\{v_i(t)\}_{|t| < \varepsilon} \in C^\infty(M, \mathbb{R})$ ,  $1 \leq i \leq m$ , such that each  $v_i : (-\varepsilon, \varepsilon) \rightarrow C^\infty(M, \mathbb{R})$  is real analytic in  $t$  and*

- (1)  $\Lambda_i(0) = \lambda$ ,  $1 \leq i \leq m$ ,
- (2)  $\Delta_{b,t} v_i(t) = \Lambda_i(t) v_i(t)$ ,  $1 \leq i \leq m$ ,  $t \in (-\varepsilon, \varepsilon)$
- (3)  $\{v_i(t) : 1 \leq i \leq m\}$  is orthonormal in  $L^2(M, \psi_{\theta(t)})$ ,  $t \in (-\varepsilon, \varepsilon)$ .

*Proof.* The proof relies on the Rellich-Alekseevsky-Kriegl-Losik-Michor theorem (cf. Theorem 3.1 above). To this end we introduce the family of operators  $U_t : L^2(M, \psi_\theta) \rightarrow L^2(M, \psi_{\theta(t)})$ ,

$$U_t v = e^{-(n+1)u_t/2} v, \quad v \in L^2(M, \psi_\theta).$$

The family  $\{U_t\}_{t \in \mathbb{R}}$  is a real analytic family of unitary operators, i.e.

$$\|U_t v\|_{L^2(M, \psi_{\theta(t)})} = \|v\|_{L^2(M, \psi_\theta)},$$

and  $U_t^{-1} v = e^{(n+1)u_t/2} v$ . Moreover, let  $A(t)$  be the family of operators

$$A(t) = U_t^{-1} \circ \Delta_{b,t} \circ U_t : L^2(M, \psi_\theta) \rightarrow L^2(M, \psi_\theta).$$

Then

$$\Delta_{b,t} v(t) = \lambda v(t) \iff A(t) (U_t^{-1} v(t)) = \lambda U_t^{-1} v(t).$$

In particular, the spectrum of  $\Delta_{b,t}$  coincides with that of  $A(t)$ . Let us show that the family  $\{A(t)\}_{t \in \mathbb{R}}$  is analytic in  $t$ . Indeed, the dense subspace  $\mathcal{D}(\Delta_b) \subset L^2(M, \psi_\theta)$  is the domain of  $A(t)$  and, as we shall check in a moment,  $A(t) \subset$



$A(t)^*$ . Indeed, the sub-Laplacians  $\Delta_b$  and  $\Delta_{b,t} = \Delta_{b,\theta(t)}$  are related by (see [8, Proposition 5] or [30, Lemma 1.8])

$$\Delta_{b,t}v = e^{-u_t} \left( \Delta_b v - n G_\theta(\nabla^H u_t, \nabla^H v) \right), \quad v \in C^2(M). \quad (3.1)$$

Then, for each  $v \in \mathcal{D}(\Delta_b)$ ,

$$\begin{aligned} A(t)v &= (U_t^{-1} \circ \Delta_{b,t} \circ U_t)v = \dots = \\ &= e^{-u_t} \left[ \Delta_b v + G_\theta(\nabla^H u_t, \nabla^H v) - \frac{n+1}{2} \left( \Delta_b u_t - \frac{(n-1)}{2} |\nabla^H u_t|^2 \right) v \right]. \end{aligned}$$

Finally, the family  $\{A(t)\}_{t \in \mathbb{R}}$  is an analytic curve of self-adjoint operators in  $L^2(M, \psi_\theta)$  with common domain of definition and with compact resolvent. Therefore, we can apply Theorem 3.1 (i) to deduce that the eigenvalues and the eigenvectors of  $A(t)$  depend analytically in  $t$ , i.e., there exists  $m$  analytic families of vectors  $v_i(t)$  and  $m$  real analytic valued functions  $\Lambda_i(t)$  in  $t$  satisfying (1), (2) and (3) of Theorem 3.2.  $\square$

For any  $\theta \in \mathcal{P}_+(M)$ , the set of all pseudo-Hermitian structures with positive definite Levi form on  $M$ , let

$$0 = \lambda_0(\theta) < \lambda_1(\theta) \leq \lambda_2(\theta) \leq \dots \leq \lambda_k(\theta) \leq \dots$$

be the spectrum of the sub-Laplacian  $\Delta_b = \Delta_{b,\theta}$  of  $(M, \theta)$ . For every  $k \in \mathbb{N}$ , let

$$E_k(\theta) = \text{Ker} (\Delta_b - \lambda_k(\theta)I)$$

be the eigenspace of  $\Delta_b$  corresponding to the eigenvalue  $\lambda_k(\theta)$ . Also let  $\Pi_k : L^2(M, \psi_\theta) \rightarrow E_k(\theta)$  be the orthogonal projection on  $E_k(\theta)$ . Let us fix  $k \in \mathbb{N}$  and consider the functional  $\theta \in \mathcal{P}_+(M) \mapsto \lambda_k(\theta) \in \mathbb{R}$ . This functional is continuous (with respect to an appropriate metric topology on  $\mathcal{P}_+(M)$ , as shown in [5]) but not differentiable in general. However, one has the following

**Theorem 3.3.** *Let  $M$  be a compact strictly pseudoconvex CR manifold and let  $\theta \in \mathcal{P}_+(M)$ . Let  $\theta(t) = e^{u_t} \theta$ ,  $t \in (-\varepsilon, \varepsilon)$ , be an analytic deformation of  $\theta$ . Then, for every positive  $k \in \mathbb{N}$ ,*

(1) *The function  $t \in (-\varepsilon, \varepsilon) \mapsto \lambda_k(\theta(t))$  admits left and right derivatives at  $t = 0$ .*

(2) *The derivatives  $\left. \frac{d}{dt} \lambda_k(\theta(t)) \right|_{t=0^-}$  and  $\left. \frac{d}{dt} \lambda_k(\theta(t)) \right|_{t=0^+}$  are eigenvalues of the operator  $\Pi_k \circ \Delta'_b : E_k(\theta) \rightarrow E_k(\theta)$  where,  $\forall v \in C^\infty(M)$ ,*

$$\Delta'_b v = \frac{d}{dt} \Delta_{b,t} v \Big|_{t=0} = -f \Delta_b v - n G_\theta(\nabla^H f, \nabla^H v)$$

with  $f = \frac{d}{dt} u_t \Big|_{t=0}$ .

(3) *If  $\lambda_k(\theta) > \lambda_{k-1}(\theta)$ , then  $\left. \frac{d}{dt} \lambda_k(\theta(t)) \right|_{t=0^-}$  and  $\left. \frac{d}{dt} \lambda_k(\theta(t)) \right|_{t=0^+}$  are the greatest and the least eigenvalues of  $\Pi_k \circ \Delta'_b$  on  $E_k(\theta)$ , respectively.*

(4) *If  $\lambda_k(\theta) < \lambda_{k+1}(\theta)$  then  $\left. \frac{d}{dt} \lambda_k(\theta(t)) \right|_{t=0^-}$  and  $\left. \frac{d}{dt} \lambda_k(\theta(t)) \right|_{t=0^+} \in \mathbb{R}$  are the smallest and the greatest eigenvalue of  $\Pi_k \circ \Delta'_b$  on  $E_k(\theta)$ , respectively.*



*Proof.* Let us denote by  $m$  the dimension of  $E_k(\theta)$ . We apply Theorem 3.2 with  $\lambda = \lambda_k(\theta)$  to derive the existence of  $m$  real analytic functions  $\{\Lambda_i\}_{1 \leq i \leq m} \subset C^\omega((-\varepsilon, \varepsilon), \mathbb{R})$  and  $m$  analytic families of functions  $\{v_i(t)\}_{|t| < \varepsilon} \in C^\infty(M, \mathbb{R})$ ,  $1 \leq i \leq m$ , satisfying (1)-(3) of Theorem 3.2. Since  $t \mapsto \lambda_k(\theta(t))$  and  $t \mapsto \Lambda_i(t)$  are continuous and  $\Lambda_1(0) = \dots = \Lambda_m(0) = \lambda_k(\theta)$ , one deduces that  $\lambda_k(\theta(t)) \in \{\Lambda_1(t), \dots, \Lambda_m(t)\}$  for sufficiently small  $t$ . Since, moreover,  $\forall i \leq m$ ,  $t \mapsto \Lambda_i(t)$  is analytic, there exist  $\delta > 0$  and two integers  $p, q \leq m$  such that

$$\lambda_k(\theta(t)) = \begin{cases} \Lambda_p(t) & \text{for } t \in (-\delta, 0) \\ \Lambda_q(t) & \text{for } t \in (0, \delta). \end{cases}$$

Therefore, the function  $t \mapsto \lambda_k(\theta(t))$  admits left and right derivatives at  $t = 0$  with

$$\frac{d}{dt} \lambda_k(\theta(t)) \Big|_{t=0^-} = \Lambda'_p(0) \quad \text{and} \quad \frac{d}{dt} \lambda_k(\theta(t)) \Big|_{t=0^+} = \Lambda'_q(0).$$

Now, one has for all  $i \leq m$  and  $t \in (-\delta, \delta)$ ,  $\Delta_{b,t} v_i(t) = \Lambda_i(t) v_i(t)$ . Differentiating at  $t = 0$ , we get

$$\Delta'_b v_i + \Delta_b v'_i = \Lambda'_i(0) v_i + \lambda_k(\theta) v'_i \quad (3.2)$$

where  $v_i = v_i(0)$  and  $v'_i = \frac{d}{dt} v_i(t) \Big|_{t=0}$ . Multiplication by  $v_j$  and integration by parts yield

$$\int_M v_j \Delta'_b v_i \psi_\theta = \begin{cases} \Lambda'_i(0) & \text{if } j = i \\ 0 & \text{otherwise.} \end{cases}$$

Since  $\{v_1, \dots, v_m\}$  is an orthonormal basis of  $E_k(\theta)$  with respect to the inner product of  $L^2(M, \psi_\theta)$ , we deduce that

$$(\Pi_k \circ \Delta'_b) v_i = \Lambda'_i(0) v_i.$$

That is  $\Lambda'_1(0), \dots, \Lambda'_m(0)$  are the eigenvalues of  $\Pi_k \circ \Delta'_b : E_k(\theta) \rightarrow E_k(\theta)$ . Differentiating the identity (3.1) at  $t = 0$  we get

$$\Delta'_b v = -f \Delta_b v - n G_\theta (\nabla^H v, \nabla^H f).$$

Assume now  $\lambda_k(\theta) > \lambda_{k-1}(\theta)$ . For any  $i \leq m$ , one then has  $\Lambda_i(0) = \lambda_k(\theta) > \lambda_{k-1}(\theta)$ . By continuity, we necessarily have  $\Lambda_i(t) > \lambda_{k-1}(\theta(t))$  for sufficiently small  $t$ . Hence, there exists  $\eta > 0$  such that,  $\forall |t| < \eta$  and  $\forall i \leq m$ ,  $\Lambda_i(t) \geq \lambda_k(\theta(t))$ , which means that  $\lambda_k(\theta(t)) = \min \{\Lambda_1(t), \dots, \Lambda_m(t)\}$ . This implies that

$$\frac{d}{dt} \lambda_k(\theta(t)) \Big|_{t=0^-} = \max \{\Lambda'_1(0), \dots, \Lambda'_m(0)\}$$

and

$$\frac{d}{dt} \lambda_k(\theta(t)) \Big|_{t=0^+} = \min \{\Lambda'_1(0), \dots, \Lambda'_m(0)\}$$

which proves (3).

Similarly, if  $\lambda_k(\theta) < \lambda_{k+1}(\theta)$ , one has, for sufficiently small  $t$ ,  $\Lambda_i(t) \leq \lambda_k(\theta(t))$  which means that  $\lambda_k(\theta(t)) = \max \{\Lambda_1(t), \dots, \Lambda_m(t)\}$  and, then,

$$\frac{d}{dt} \lambda_k(\theta(t)) \Big|_{t=0^+} = \max \{\Lambda'_1(0), \dots, \Lambda'_m(0)\}$$

and

$$\frac{d}{dt}\lambda_k(\theta(t))\Big|_{t=0^-} = \min\{\Lambda'_1(0), \dots, \Lambda'_m(0)\}.$$

□

**Corollary 3.1.** *Let  $M$  be a compact strictly pseudoconvex CR manifold and let  $\theta \in \mathcal{P}_+(M)$ . Let  $\theta(t) = e^{t\theta}$ ,  $t \in (-\varepsilon, \varepsilon)$ , be an analytic deformation of  $\theta$  and set  $f = \frac{d}{dt}u_i\Big|_{t=0}$ . For every positive integer  $k$ , let  $Q_{f,k} : E_k(\theta) \rightarrow \mathbb{R}$  be the quadratic form given by*

$$Q_{f,k}(v) = - \int_M \left( \lambda_k(\theta)v^2 + \frac{n}{2}\Delta_b v^2 \right) f \psi_\theta.$$

(1) *If  $Q_{f,k}$  is positive definite on  $E_k(\theta)$ , then there exists  $\varepsilon > 0$  such that  $\lambda_k(\theta(-t)) < \lambda_k(\theta) < \lambda_k(\theta(t))$  for all  $t \in (0, \varepsilon)$ .*

(2) *Assume that  $\lambda_k(\theta) > \lambda_{k-1}(\theta)$ . If  $Q_{f,k}$  takes negative values somewhere in  $E_k(\theta)$ , then  $\lambda_k(\theta(t)) < \lambda_k(\theta)$  for all  $t \in (0, \varepsilon)$ , for some  $\varepsilon > 0$ .*

(3) *Assume that  $\lambda_k(\theta) < \lambda_{k+1}(\theta)$ . If  $Q_{f,k}$  takes positive values somewhere in  $E_k(\theta)$ , then  $\lambda_k(\theta(t)) > \lambda_k(\theta)$  for all  $t \in (0, \varepsilon)$ , for some  $\varepsilon > 0$ .*

*Proof.* First, we have with the notations of Theorem 3.3,  $\forall v \in E_k(\theta)$ ,

$$Q_{f,k}(v) = \int_M v \Delta'_b v \psi_\theta. \quad (3.3)$$

Indeed,  $\forall v \in E_k(\theta)$ ,

$$\begin{aligned} \int_M v \Delta'_b v \psi_\theta &= - \int_M v \left( f \Delta_b v + n G_\theta(\nabla^H v, \nabla^H f) \right) \psi_\theta \\ &= - \int_M \left( f \lambda_k(\theta)v^2 + \frac{n}{2} G_\theta(\nabla^H v^2, \nabla^H f) \right) \psi_\theta \\ &= - \int_M \left( \lambda_k(\theta)v^2 + \frac{n}{2} \Delta_b v^2 \right) f \psi_\theta = Q_{f,k}(v). \end{aligned}$$

Now, if  $Q_{f,k}$  is positive definite on  $E_k(\theta)$ , then, thanks to (3.3), all the eigenvalues of the operator  $\Pi_k \circ \Delta'_b : E_k(\theta) \rightarrow E_k(\theta)$  are positive. Applying Theorem 3.3 (2), it follows that both  $\frac{d}{dt}\lambda_k(\theta(t))\Big|_{t=0^+}$  and  $\frac{d}{dt}\lambda_k(\theta(t))\Big|_{t=0^-}$  are positive and that there exists  $\varepsilon > 0$  such that  $\lambda_k(\theta(-t)) < \lambda_k(\theta) < \lambda_k(\theta(t))$  for all  $t \in (0, \varepsilon)$ .

Assume that  $\lambda_k(\theta) > \lambda_{k-1}(\theta)$  and that there exists  $v \in E_k(\theta)$  such that  $Q_{f,k}(v) < 0$ . This implies that the operator  $\Pi_k \circ \Delta'_b : E_k(\theta) \rightarrow E_k(\theta)$  has at least one negative eigenvalue. Applying Theorem 3.3 (3), we deduce that  $\frac{d}{dt}\lambda_k(\theta(t))\Big|_{t=0^+}$  is negative and that there exists  $\varepsilon > 0$  such that  $\lambda_k(\theta(t)) < \lambda_k(\theta)$  for all  $t \in (0, \varepsilon)$ .

The last part of the corollary can be proved using similar arguments.

□

## 4. GENERIC SIMPLICITY OF SUB-LAPLACIAN EIGENVALUES

Let  $M$  be a compact strictly pseudoconvex CR manifold and denote by  $\mathcal{P}_+(M)$  the set of all pseudo-Hermitian structures with positive definite Levi form on  $M$ . In [5], we defined a complete distance on  $\mathcal{P}_+(M)$  so that the eigenvalues of the sub-Laplacian  $\theta \in \mathcal{P}_+(M) \mapsto \lambda_k(\theta)$  are continuous. This distance is defined as follows : We fix a form  $\theta \in \mathcal{P}_+(M)$ . Given  $\theta_1 = e^{u_1}\theta$  and  $\theta_2 = e^{u_2}\theta$  in  $\mathcal{P}_+(M)$ , we set

$$d(\theta_1, \theta_2) = d_{C^\infty}(u_1, u_2) + \rho(G_{\theta_1}, G_{\theta_2})$$

where  $d_{C^\infty}$  is the distance function associated with the canonical Frechet structure of  $C^\infty(M)$  and

$$\rho(G_{\theta_1}, G_{\theta_2}) = \inf\{\delta > 0 : e^{-\delta}G_{\theta_1}(X, X) \leq G_{\theta_2}(X, X) \leq e^\delta G_{\theta_1}(X, X), \forall X \in H\}.$$

In [5], we proved that  $(\mathcal{P}_+(M), d)$  is a complete metric space and that if  $\rho(G_{\theta_1}, G_{\theta_2}) < \varepsilon$ , then,  $\forall k \geq 1$ ,

$$e^{-\varepsilon} \leq \frac{\lambda_k(\theta_1)}{\lambda_k(\theta_2)} \leq e^\varepsilon.$$

In the sequel, we denote by  $\mathcal{J}$  the set of all elements  $\theta \in \mathcal{P}_+(M)$  such that all the eigenvalues of the sub-Laplacian  $\Delta_{b,\theta}$  have multiplicity one, that is,

$$\mathcal{J} = \{\theta \in \mathcal{P}_+(M) : 0 < \lambda_1(\theta) < \lambda_2(\theta) < \dots < \lambda_k(\theta) < \dots\}$$

Our main aim in this section is to prove the following

**Theorem 4.1.** *The set  $\mathcal{J}$  is a residual set in  $(\mathcal{P}_+(M), d)$ , i.e., a countable intersection of open dense subsets. In particular,  $\mathcal{J}$  is dense in  $(\mathcal{P}_+(M), d)$ .*

The proof of this theorem relies on the following proposition which is a consequence of Theorem 3.3.

**Proposition 4.1.** *Let  $M$  be a compact strictly pseudoconvex CR manifold and let  $\theta \in \mathcal{P}_+(M)$ . Let  $\lambda \in \sigma(\Delta_{b,\theta})$  be an eigenvalue of multiplicity  $m \geq 2$  and let  $k \in \mathbb{N}$  be such that*

$$\lambda = \lambda_k(\theta) = \lambda_{k+1}(\theta) = \dots = \lambda_{k+m-1}(\theta).$$

*There exist  $f \in C^\infty(M)$  and  $\varepsilon > 0$  such that  $\theta(t) = e^{tf}\theta$  satisfies for all  $t \in (0, \varepsilon)$ ,*

$$\lambda_k(\theta(t)) < \lambda_{k+m-1}(\theta(t)).$$

*Proof.* Let  $E = E_k(\theta) = E_{k+m-1}(\theta)$  be the eigenspace of  $\Delta_{b,\theta}$  corresponding to the eigenvalue  $\lambda$  and let  $\Pi : L^2(M) \rightarrow E$  be the orthogonal projection on  $E$ . For every  $f \in C^\infty(M)$ , we denote by  $L_f : E \rightarrow E$  the operator defined by

$$L_f v = \Pi \circ \Delta'_b v = -\Pi \left[ \lambda f v + n G_\theta \left( \nabla^H v, \nabla^H f \right) \right].$$

From the definition of the integers  $k$  and  $m$ , one has  $\lambda_k(\theta) > \lambda_{k-1}(\theta)$  and  $\lambda_{k+m-1}(\theta) < \lambda_{k+m}(\theta)$ . Therefore, Theorem 3.3 tells us that, for any  $f \in C^\infty(M)$ ,  $\frac{d}{dt}\lambda_k(e^{tf}\theta)|_{t=0^+}$  and  $\frac{d}{dt}\lambda_{k+m-1}(e^{tf}\theta)|_{t=0^+}$  represent the smallest and the largest eigenvalues of the operator  $L_f : E \rightarrow E$ , respectively.

Therefore, it suffices to prove the existence of a function  $f \in C^\infty(M)$  so that the operator  $L_f$  has at least two distinct eigenvalues (i.e.  $L_f$  is not proportional to the identity of  $E$ ). Indeed, in this case, we would have  $\frac{d}{dt}\lambda_k(e^{tf}\theta)|_{t=0^+} < \frac{d}{dt}\lambda_{k+m-1}(e^{tf}\theta)|_{t=0^+}$  which implies the conclusion of the proposition.

Thanks to (3.3), one has,  $\forall v, w \in E$

$$(L_f v, w)_{L^2(M)} = \cdots = - \int_M \left( \frac{n}{2} \Delta_b(vw) + \lambda vw \right) f \psi_\theta.$$

Let  $\{u_1, u_2\} \subset E$  be a pair of functions with  $\|u_1\|_{L^2(M)} = \|u_2\|_{L^2(M)}$  and  $u_1^2 \neq u_2^2$  (recall that  $E$  is of dimension at least 2) and set  $v = u_1 - u_2$  and  $w = u_1 + u_2$  so that  $(v, w)_{L^2(M)} = 0$ . The function

$$f_0 = \frac{n}{2} \Delta_b(vw) + \lambda vw = \frac{n}{2} \Delta_b(u_1^2 - u_2^2) + \lambda(u_1^2 - u_2^2) \quad (4.1)$$

is such that

$$(L_{f_0} v, w)_{L^2(M)} = - \int_M \left( \frac{n}{2} \Delta_b(u_1^2 - u_2^2) + \lambda(u_1^2 - u_2^2) \right)^2 \psi_\theta$$

which does not vanish since  $\Delta_b$  has no negative eigenvalues. Thus,  $L_{f_0}$  cannot be proportional to the identity of  $E$ .  $\square$

*Proof of Theorem 4.1.* For every positive integer  $k$ , let  $\mathcal{J}_k$  be the subset of  $\mathcal{P}_+(M)$  defined by

$$\mathcal{J}_k = \{ \theta \in \mathcal{P}_+(M) : 0 < \lambda_1(\theta) < \lambda_2(\theta) < \cdots < \lambda_k(\theta) \}.$$

We have  $\mathcal{P}_+(M) = \mathcal{J}_1 \supset \mathcal{J}_2 \supset \cdots \supset \mathcal{J}_k \supset \cdots$  and

$$\mathcal{J} = \bigcap_{k=1}^{\infty} \mathcal{J}_k.$$

According to Baire's category theorem, it suffices to prove that each  $\mathcal{J}_k$  is an open dense subset of  $\mathcal{P}_+(M)$ .

The fact that  $\mathcal{J}_k$  is open follows immediately from the continuity of the eigenvalues  $\theta \in \mathcal{P}_+(M) \mapsto \lambda_i(\theta)$ ,  $i \leq k$ .

Let us prove that, for any  $k \geq 1$ ,  $\mathcal{J}_{k+1}$  is a dense subset of  $\mathcal{J}_k$ . An obvious recursion would then imply that each  $\mathcal{J}_k$  is a dense subset of  $\mathcal{P}_+(M)$ . So, let  $\theta \in \mathcal{J}_k \setminus \mathcal{J}_{k+1}$  and let  $\eta$  be any positive real number. Thus, one has

$$\lambda_1(\theta) < \lambda_2(\theta) < \cdots < \lambda_{k-1}(\theta) < \lambda_k(\theta) = \lambda_{k+1}(\theta) = \cdots = \lambda_{k+m-1}(\theta) < \lambda_{k+m}(\theta),$$

where  $m$  is the multiplicity of  $\lambda_k(\theta)$ . Using Proposition 4.1 and the continuity of the eigenvalues, one can find  $f \in C^\infty(M)$  and  $\varepsilon > 0$  such that the form  $\theta(t) = e^{tf}\theta$  satisfies, for every  $t \in (0, \varepsilon)$ ,

$$\lambda_1(\theta(t)) < \lambda_2(\theta(t)) < \cdots < \lambda_{k-1}(\theta(t)) < \lambda_k(\theta(t)) < \lambda_{k+m-1}(\theta(t))$$

which means that  $\theta(t)$  belongs to  $\mathcal{J}_k$  and the multiplicity of  $\lambda_k(\theta)$  is at most  $m-1$ . Choosing  $t_1 \in (0, \varepsilon)$  sufficiently small, one gets a form  $\theta_1 = \theta(t_1) \in \mathcal{J}_k$  such that the multiplicity of  $\lambda_k(\theta_1)$  is at most  $m-1$  and  $d(\theta_1, \theta) < \eta/m$ .

Repeating this argument at most  $m - 1$  times, we prove the existence of a 1-form  $\hat{\theta} \in \mathcal{J}_{k+1}$  such that  $d(\hat{\theta}, \theta) < \eta$ .

□

## 5. CRITICAL PSEUDO-HERMITIAN STRUCTURES

The content of this section is patterned after the article [17] by Ilias and the third author dealing with Laplacian eigenvalues in the Riemannian setting. For the sake of completeness, we shall give self-contained proofs of the results we obtain in the CR context.

Let  $M$  be a compact strictly pseudoconvex CR manifold. For every positive integer  $k$  we consider the map  $\theta \in \mathcal{P}_+(M) \mapsto \lambda_k(\theta) \in \mathbb{R}$ , where, as before,  $\mathcal{P}_+(M)$  denotes the set of all pseudo-Hermitian structures with positive definite Levi form on  $M$ , and  $\lambda_k(\theta)$  is the  $k$ -th eigenvalue of the sub-Laplacian associated to  $\theta$ . Since the eigenvalues are not invariant under scaling, we restrict  $\lambda_k$  to the subset

$$\mathcal{P}_{+,0}(M) = \{\theta \in \mathcal{P}_+(M) : \text{vol}(\theta) = 1\}$$

where  $\text{vol}(\theta) = \int_M \psi_\theta$  is the volume of  $M$  with respect to  $\psi_\theta$ .

Thanks to Theorem 3.3, one can introduce the following

**Definition 5.1.** *A pseudo-Hermitian structure  $\theta$  is said to be critical for the functional  $\lambda_k$  restricted to  $\mathcal{P}_{+,0}(M)$  if, for any analytic deformation  $\{\theta(t) = e^{tu}\theta\} \subset \mathcal{P}_{+,0}(M)$  of  $\theta$ , we have*

$$\frac{d}{dt} \lambda_k(\theta(t)) \Big|_{t=0^-} \times \frac{d}{dt} \lambda_k(\theta(t)) \Big|_{t=0^+} \leq 0.$$

It is easy to see that

$$\frac{d}{dt} \lambda_k(\theta(t)) \Big|_{t=0^+} \leq 0 \leq \frac{d}{dt} \lambda_k(\theta(t)) \Big|_{t=0^-} \iff \lambda_k(\theta(t)) \leq \lambda_k(\theta) + o(t) \text{ as } t \rightarrow 0$$

and

$$\frac{d}{dt} \lambda_k(\theta(t)) \Big|_{t=0^-} \leq 0 \leq \frac{d}{dt} \lambda_k(\theta(t)) \Big|_{t=0^+} \iff \lambda_k(\theta(t)) \geq \lambda_k(\theta) + o(t) \text{ as } t \rightarrow 0.$$

Of course, if  $\theta$  is a local maximizer or a local minimizer of  $\lambda_k$ , then  $\theta$  is critical in the sense of the previous definition. We set

$$\mathcal{A}_0(M, \theta) = \left\{ f \in C^\infty(M) : \int_M f \psi_\theta = 0 \right\}$$

and recall the definition of the quadratic form  $Q_{f,k} : E_k(\theta) \rightarrow \mathbb{R}$  associated to a pair  $(f, k) \in C^\infty(M) \times \mathbb{N}^*$  (see Corollary 3.1):

$$Q_{f,k}(v) = - \int_M \left( \lambda_k(\theta) v^2 + \frac{n}{2} \Delta_b v^2 \right) f \psi_\theta.$$

**Theorem 5.1.** *Let  $M$  be a compact strictly pseudoconvex CR manifold. Let  $\theta \in \mathcal{P}_{+,0}(M)$  be a pseudo-Hermitian structure and  $k \in \mathbb{N}^*$ .*

1) If  $\theta$  is a critical pseudo-Hermitian structure of the functional  $\lambda_k$  restricted to  $\mathcal{P}_{+,0}(M)$ , then,  $\forall f \in \mathcal{A}_0(M, \theta)$ , the quadratic form  $Q_{f,k}$  is indefinite on  $E_k(\theta)$ .

2) Assume that  $\lambda_k(\theta) > \lambda_{k-1}(\theta)$  or  $\lambda_k(\theta) < \lambda_{k+1}(\theta)$ . The pseudo-Hermitian structure  $\theta$  is critical for the functional  $\lambda_k$  restricted to  $\mathcal{P}_{+,0}(M)$  if and only if,  $\forall f \in \mathcal{A}_0(M, \theta)$ , the quadratic form  $Q_{f,k}$  is indefinite on  $E_k(\theta)$ .

*Proof.* Let  $f \in \mathcal{A}_0(M, \theta)$  and let  $\theta(t)$  be the analytic deformation of  $\theta$  given by

$$\theta(t) = \left( \frac{\text{vol}(\theta)}{\text{vol}(e^{tf}\theta)} \right)^{\frac{1}{n+1}} e^{tf}\theta = e^{u_t}\theta, \quad t \in \mathbb{R},$$

with  $u_t = tf - \frac{1}{n+1} \ln(\text{vol}(e^{tf}\theta))$ . Since

$$\psi_{e^{tf}\theta} = e^{(n+1)tf}\psi_\theta,$$

it is easy to check that  $\text{vol}(\theta(t)) = 1$ , that is  $\theta(t)$  belongs to  $\mathcal{P}_{+,0}(M)$  for every  $t \in \mathbb{R}$ . One has

$$\left. \frac{d}{dt} \text{vol}(e^{tf}\theta(t)) \right|_{t=0} = \left. \frac{d}{dt} \int_M e^{(n+1)tf}\psi_\theta \right|_{t=0} = (n+1) \int_M f\psi_\theta = 0.$$

Therefore,

$$\left. \frac{d}{dt} u_t \right|_{t=0} = \dots = f$$

and, then,

$$\Delta'_b v = \left. \frac{d}{dt} \Delta_{b,t} \right|_{t=0} = -f\Delta_b v - nG_\theta(\nabla^H v, \nabla^H f).$$

Thus, we have (see (3.3)),

$$Q_{f,k}(v) = \int_M v \Delta'_b v \psi_\theta. \quad (5.1)$$

Now, assuming that  $\theta$  is a critical pseudo-Hermitian structure of  $\lambda_k$  restricted to  $\mathcal{P}_{+,0}(M)$ , we obtain, using the definition of criticality and Theorem 3.3 (2), that the operator  $\Pi_k \circ \Delta'_b$  admits both nonnegative and nonpositive eigenvalues in  $E_k(\theta)$ , which means (thanks to (5.1) that the quadratic form  $Q_{f,k}$  is indefinite on  $E_k(\theta)$ ). This proves the first part of the theorem.

The last part of the theorem follows from Theorem 3.3 (3) and (4), and (5.1).  $\square$

**Proposition 5.1.** *Let  $M$  be a compact strictly pseudoconvex CR manifold and let  $\theta \in \mathcal{P}_{+,0}(M)$  be a pseudo-Hermitian structure. For any positive integer  $k$ , the two following conditions are equivalent:*

- (1) For all  $f \in \mathcal{A}_0(M, \theta)$ , the quadratic form  $Q_{f,k}$  is indefinite on  $E_k(\theta)$ .
- (2) There exists a finite family  $\{v_1, \dots, v_d\} \subset E_k(\theta)$  of eigenfunctions associated with  $\lambda_k(\theta)$  such that  $\sum_{i=1}^d v_i^2 = 1$ .

*Proof.* Assume first that there exist  $v_1, \dots, v_d \in E_k(\theta)$  such that  $\sum_i^d v_i^2 = 1$ . Therefore,  $\forall f \in \mathcal{A}_0(M, \theta)$ ,

$$\sum_{i=1}^d Q_{f,k}(v_i) = \dots = -\lambda_k(\theta) \int_M f \psi_\theta = 0$$

which implies that  $Q_{f,k}$  is indefinite on  $E_k(\theta)$ .

Conversely, assume that  $Q_{f,k}$  is indefinite on  $E_k(\theta)$  for all  $f \in \mathcal{A}_0(M, \theta)$  and consider the convex set

$$K = \left\{ \sum_{i \in J} \left[ \lambda_k(\theta) v_i^2 + \frac{n}{2} \Delta_b v_i^2 \right]; v_i \in E_k(\theta), J \subset \mathbb{N}, J \text{ finite} \right\} \subset L^2(M).$$

Let us prove that the constant function 1 belongs to  $K$ . Indeed, if  $1 \notin K$ , then, applying classical separation theorem in the finite dimensional subspace of  $L^2(M, \theta)$  generated by  $K$  and 1, we deduce the existence of  $h \in L^2(M)$  such that  $(h, 1)_{L^2(M)} = \int_M h \psi_\theta > 0$  and,  $\forall w \in K$ ,  $(h, w)_{L^2} = \int_M h w \psi_\theta \leq 0$ . Let  $f = h - \frac{1}{\text{vol}(\theta)} \int_M h \psi_\theta \in \mathcal{A}_0(M, \theta)$ . Then,  $\forall v \in E_k(\theta)$

$$\begin{aligned} Q_{f,k}(v) &= - \int_M \left( \lambda_k(\theta) v^2 + \frac{n}{2} \Delta_b v^2 \right) f \psi_\theta \\ &= - \int_M \left( \lambda_k(\theta) v^2 + \frac{n}{2} \Delta_b v^2 \right) h \psi_\theta + \frac{\int_M h \psi_\theta}{\text{vol}(\theta)} \lambda_k(\theta) \int_M v^2 \psi_\theta \end{aligned}$$

since  $\int_M \Delta_b v^2 \psi_\theta = 0$ . Moreover,  $\forall v \in E_k(\theta)$ , the function  $\left( \lambda_k(\theta) v^2 + \frac{n}{2} \Delta_b v^2 \right)$  belongs to  $K$  which implies that  $\int_M \left( \lambda_k(\theta) v^2 + \frac{n}{2} \Delta_b v^2 \right) h \psi_\theta \leq 0$  and, then

$$Q_{f,k}(v) \geq \frac{\lambda_k(\theta) \int_M h \psi_\theta}{\text{vol}(\theta)} \int_M v^2 \psi_\theta.$$

Therefore, the quadratic form  $Q_{f,k}$  is positive definite on  $E_k(\theta)$  which contradicts the assumptions.

Now, since  $1 \in K$ , there exist  $v_1, \dots, v_d \in E_k(\theta)$  such that

$$\sum_{i=1}^d \left( \lambda_k(\theta) v_i^2 + \frac{n}{2} \Delta_b v_i^2 \right) = \lambda_k(\theta) \quad (5.2)$$

which leads to

$$\Delta_b \left( \sum_{i \leq d} v_i^2 - 1 \right) = -\frac{2}{n} \lambda_k(\theta) \left( \sum_{i \leq d} v_i^2 - 1 \right)$$

This implies that  $\sum_{i \leq d} v_i^2 - 1 = 0$  since the sub-Laplacian admits no negative eigenvalues. □

Theorem 5.1 and Proposition 5.1 lead to the following



**Corollary 5.1.** *Let  $M$  be a compact strictly pseudoconvex CR manifold. Let  $\theta \in \mathcal{P}_{+,0}(M)$  be a pseudo-Hermitian structure and  $k \in \mathbb{N}^*$ .*

(1) *If  $\theta$  is a critical pseudo-Hermitian structure of the functional  $\lambda_k$  restricted to  $\mathcal{P}_{+,0}(M)$ , then there exists a finite family  $\{v_1, \dots, v_d\} \subset E_k(\theta)$  of eigenfunctions associated with  $\lambda_k(\theta)$  such that  $\sum_i^d v_i^2 = 1$ .*

(2) *Assume that  $\lambda_k(\theta) > \lambda_{k-1}(\theta)$  or  $\lambda_k(\theta) < \lambda_{k+1}(\theta)$ . Then,  $\theta$  is critical for the functional  $\lambda_k$  restricted to  $\mathcal{P}_{+,0}(M)$  if and only if there exists a finite family  $\{v_1, \dots, v_d\} \subset E_k(\theta)$  of eigenfunctions associated with  $\lambda_k(\theta)$  such that  $\sum_i^d v_i^2 = 1$ .*

According to [30, Proposition 4.4], the first positive eigenvalue of the standard CR sphere  $\mathbb{S}^{2n+1}$  is equal to  $2n$  and the corresponding eigenspace is generated by the restriction of coordinate functions, the sum of whose squares is 1 on  $\mathbb{S}^{2n+1}$ . Hence, the standard contact form of  $\mathbb{S}^{2n+1}$  is a critical pseudo-Hermitian structure of  $\lambda_1$  restricted to  $\mathcal{P}_{+,0}(\mathbb{S}^{2n+1})$ . On the other hand, the condition that there exists a finite family  $\{v_1, \dots, v_d\} \subset E_k(\theta)$  such that  $\sum_i^d v_i^2 = 1$ , is equivalent to the existence of a pseudo-harmonic map from  $(M, \theta)$  to the sphere  $\mathbb{S}^{d-1}$  (see [6, Lemma 6.1 ] and [12]).

An immediate consequence of Corollary 5.1 is the following:

**Corollary 5.2.** *Let  $M$  be a compact strictly pseudoconvex CR manifold. If  $\theta \in \mathcal{P}_{+,0}(M)$  is a critical metric of the functional  $\lambda_k$  restricted to  $\mathcal{P}_{+,0}(M)$ , then  $\lambda_k(\theta)$  is a degenerate eigenvalue, that is*

$$\dim E_k(\theta) \geq 2.$$

In the case when  $\theta$  is a local maximizer or a local minimizer, we have the following more precise result

**Proposition 5.2.** *Let  $M$  be a compact strictly pseudoconvex CR manifold.*

(1) *If  $\theta \in \mathcal{P}_{+,0}(M)$  is a local minimizer of the functional  $\lambda_k$  restricted to  $\mathcal{P}_{+,0}(M)$ , then  $\lambda_k(\theta) = \lambda_{k-1}(\theta)$ .*

(2) *If  $\theta \in \mathcal{P}_{+,0}(M)$  is a local maximizer of the functional  $\lambda_k$  restricted to  $\mathcal{P}_{+,0}(M)$ , then  $\lambda_k(\theta) = \lambda_{k+1}(\theta)$ .*

*Proof.* Let  $\theta \in \mathcal{P}_{+,0}(M)$  be a local minimizer of  $\lambda_k$ , that is  $\lambda_k(\hat{\theta}) \geq \lambda_k(\theta)$  for every  $\hat{\theta}$  in a neighborhood of  $\theta$  in  $\mathcal{P}_{+,0}(M)$ . Assume for a contradiction that  $\lambda_k(\theta) > \lambda_{k-1}(\theta)$ . Let  $f \in \mathcal{A}_0(M, \theta)$  and let  $\theta(t) = e^{u_t} \theta \in \mathcal{P}_{+,0}(M)$  with  $u_t = tf - \frac{1}{n+1} \ln(\text{vol}(e^{tf} \theta))$ . Then  $\theta(t)$  is a volume-preserving analytic deformation of  $\theta$  such that  $\frac{d}{dt} u_t \Big|_{t=0} = f$  (see the proof of Theorem 5.1). Denote by  $\Lambda_1(t), \dots, \Lambda_m(t)$  the associated family of eigenvalues of  $\Delta_{b,t}$ , depending analytically on  $t$  and such that  $\Lambda_1(0) = \dots = \Lambda_m(0) = \lambda_k(\theta)$  with  $m = \dim E_k(\theta)$  (see Theorem 3.3). For continuity reasons, we have, for sufficiently small  $t$  and all  $i \leq m$ ,

$$\Lambda_i(t) > \lambda_{k-1}(\theta(t)).$$

Hence,  $\forall i \leq m$  and  $\forall t$  sufficiently small,

$$\Lambda_i(t) \geq \lambda_k(\theta(t)) \geq \lambda_k(\theta) = \Lambda_i(0).$$

Consequently,  $\Lambda'_i(0) = 0$  for all  $i \leq m$ . Since (Theorem 3.3)  $\Lambda'_1(0), \dots, \Lambda'_m(0)$  are eigenvalues of the operator  $\Pi_k \circ \Delta'_b : E_k(\theta) \rightarrow E_k(\theta)$ , it follows that  $\Pi_k \circ \Delta'_b$  is identically zero on  $E_k(\theta)$ . Consequently, thanks to (3.3), for any  $f \in \mathcal{A}_0(M, \theta)$ , the quadratic form  $Q_{f,k}$  is identically zero on  $E_k(\theta)$  which implies that,  $\forall v \in E_k(\theta)$ ,

$$\lambda_k(\theta)v^2 + \frac{n}{2}\Delta_b v^2 = c$$

for some constant  $c \in \mathbb{R}$ . Therefore,

$$\Delta_b \left( v^2 - \frac{c}{\lambda_k(\theta)} \right) = -\frac{2}{n}\lambda_k(\theta) \left( v^2 - \frac{c}{\lambda_k(\theta)} \right)$$

which leads to a contradiction since the sub-Laplacian admits no negative eigenvalues.

A similar proof works for (2). □

#### REFERENCES

- [1] J. H. Albert. Genericity of simple eigenvalues for elliptic PDE's. *Proc. Amer. Math. Soc.*, 48:413–418, 1975.
- [2] Dmitri Alekseevsky, Andreas Kriegl, Peter W. Michor, and Mark Losik. Choosing roots of polynomials smoothly. *Israel J. Math.*, 105:203–233, 1998.
- [3] A. Aribi, S. Dragomir, and A. El Soufi. A lower bound on the spectrum of the sublaplacian. *preprint*, pages 1–43, 2012.
- [4] A. Aribi, S. Dragomir, and M. Magliaro. Dirichlet and Neumann eigenvalue problems on CR manifolds. *Preprint*, 2014.
- [5] Amine Aribi, Sorin Dragomir, and Ahmad El Soufi. On the continuity of the eigenvalues of a sublaplacian. *Canad. Math. Bull.*, 57(1):12–24, 2014.
- [6] Amine Aribi and Ahmad El Soufi. Inequalities and bounds for the eigenvalues of the sub-Laplacian on a strictly pseudoconvex CR manifold. *Calc. Var. Partial Differential Equations*, 47(3-4):437–463, 2013.
- [7] Shigetoshi Bando and Hajime Urakawa. Generic properties of the eigenvalue of the Laplacian for compact Riemannian manifolds. *Tôhoku Math. J. (2)*, 35(2):155–172, 1983.
- [8] E. Barletta and S. Dragomir. On the spectrum of a strictly pseudoconvex CR manifold. *Abh. Math. Sem. Univ. Hamburg*, 67:33–46, 1997.
- [9] Elisabetta Barletta. The Lichnerowicz theorem on CR manifolds. *Tsukuba J. Math.*, 31(1):77–97, 2007.
- [10] Elisabetta Barletta and Sorin Dragomir. Sublaplacians on CR manifolds. *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)*, 52(100)(1):3–32, 2009.
- [11] Elisabetta Barletta and Sorin Dragomir. Vector valued holomorphic and CR functions. In *Lecture Notes of Seminario Interdisciplinare di Matematica. Volume VIII*, volume 8 of *Lect. Notes Semin. Interdiscip. Mat.*, pages 69–100. Semin. Interdiscip. Mat. (S.I.M.), Potenza, 2009.
- [12] Elisabetta Barletta, Sorin Dragomir, and Hajime Urakawa. Pseudoharmonic maps from nondegenerate CR manifolds to Riemannian manifolds. *Indiana Univ. Math. J.*, 50(2):719–746, 2001.
- [13] David D. Bleeker and Leslie C. Wilson. Splitting the spectrum of a Riemannian manifold. *SIAM J. Math. Anal.*, 11(5):813–818, 1980.
- [14] Hung-Lin Chiu. The sharp lower bound for the first positive eigenvalue of the sublaplacian on a pseudohermitian 3-manifold. *Ann. Global Anal. Geom.*, 30(1):81–96, 2006.

- [15] Sorin Dragomir and Giuseppe Tomassini. *Differential geometry and analysis on CR manifolds*, volume 246 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 2006.
- [16] Ahmad El Soufi and Saïd Ilias. Riemannian manifolds admitting isometric immersions by their first eigenfunctions. *Pacific J. Math.*, 195(1):91–99, 2000.
- [17] Ahmad El Soufi and Saïd Ilias. Laplacian eigenvalue functionals and metric deformations on compact manifolds. *J. Geom. Phys.*, 58(1):89–104, 2008.
- [18] Allan Greenleaf. The first eigenvalue of a sub-Laplacian on a pseudo-Hermitian manifold. *Comm. Partial Differential Equations*, 10(2):191–217, 1985.
- [19] Gao Jia, Jianming Wang, and Ya Xiong. On Riesz mean inequalities for subelliptic Laplacian. *Appl. Math. (Irvine)*, 2(6):694–698, 2011.
- [20] Tosio Kato. *Perturbation theory for linear operators*. Springer-Verlag, Berlin, second edition, 1976. Grundlehren der Mathematischen Wissenschaften, Band 132.
- [21] Gerasim Kokarev. Sub-Laplacian eigenvalue bounds on CR manifolds. *Comm. Partial Differential Equations*, 38(11):1971–1984, 2013.
- [22] Andreas Kriegl and Peter W. Michor. *The convenient setting of global analysis*, volume 53 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997.
- [23] Andreas Kriegl and Peter W. Michor. Differentiable perturbation of unbounded operators. *Math. Ann.*, 327(1):191–201, 2003.
- [24] John M. Lee. The Fefferman metric and pseudo-Hermitian invariants. *Trans. Amer. Math. Soc.*, 296(1):411–429, 1986.
- [25] Song-Ying Li and Hing-Sun Luk. The sharp lower bound for the first positive eigenvalue of a sub-Laplacian on a pseudo-Hermitian manifold. *Proc. Amer. Math. Soc.*, 132(3):789–798 (electronic), 2004.
- [26] Pengcheng Niu and Huiqing Zhang. Payne-Polya-Weinberger type inequalities for eigenvalues of nonelliptic operators. *Pacific J. Math.*, 208(2):325–345, 2003.
- [27] Xiao-chun PENG and Wen-yi Chang CHEN. The spectrum of subelliptic operators on  $S^3$ . *J. of Math. (PRC)*, 29(3):297–299, 2009.
- [28] Raphaël S. Ponge. Heisenberg calculus and spectral theory of hypoelliptic operators on Heisenberg manifolds. *Mem. Amer. Math. Soc.*, 194(906):viii+ 134, 2008.
- [29] Franz Rellich. *Perturbation theory of eigenvalue problems*. Assisted by J. Berkowitz. With a preface by Jacob T. Schwartz. Gordon and Breach Science Publishers, New York, 1969.
- [30] Nancy K. Stanton. Spectral invariants of CR manifolds. *Michigan Math. J.*, 36(2):267–288, 1989.
- [31] K. Uhlenbeck. Generic properties of eigenfunctions. *Amer. J. Math.*, 98(4):1059–1078, 1976.

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