



**HAL**  
open science

## Brief Announcement: On the Voting Time of the Deterministic Majority Process

Dominik Kaaser, Frederik Mallmann-Trenn, Emanuele Natale

► **To cite this version:**

Dominik Kaaser, Frederik Mallmann-Trenn, Emanuele Natale. Brief Announcement: On the Voting Time of the Deterministic Majority Process. DISC 2015, Toshimitsu Masuzawa; Koichi Wada, Oct 2015, Tokyo, Japan. hal-01207836

**HAL Id: hal-01207836**

**<https://hal.science/hal-01207836v1>**

Submitted on 1 Oct 2015

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Brief Announcement: On the Voting Time of the Deterministic Majority Process

Dominik Kaaser<sup>1</sup>, Frederik Mallmann-Trenn<sup>2,3</sup>, and Emanuele Natale<sup>4</sup>

<sup>1</sup> University of Salzburg

<sup>2</sup> École Normale Supérieure

<sup>3</sup> Simon Fraser University

<sup>4</sup> Sapienza Università di Roma

We study the *deterministic binary majority process* which is defined as follows. We are given a graph  $G = (V, E)$  where each node has one out of two opinions. The process runs in discrete rounds where in every round each node computes and adopts the majority opinion among all of its neighbors.

It was proved independently by Goles and Olivos [2], and Poljak and Sůra [3] with the same potential function argument that the process always converges to a two-periodic state. Their proof was popularized in the *Puzzled* columns of Communications of the ACM [6]. Let the *convergence time* of a given graph, for a given initial opinion assignment, be the time it takes until the two-periodic state is reached. In this work we give bounds on the *voting time*, which is the maximum convergence time over all possible initial opinion assignments. Frischknecht et al. [1] note that the potential argument by Goles et al. [2, 3, 6] can be used to prove an  $\mathcal{O}(|E|)$  upper bound on the voting time which is also shown to be tight.

Among its widespread applications, variants of the majority process have been used in the area of distributed community detection [4], where the voting time is essentially the convergence time of the proposed community-detection protocols. A lot of attention has been given to the two-periodic state to which the majority process converges to. However, besides the  $\mathcal{O}(|E|)$  upper bound that follows from the result by Goles et al. [2, 3, 6], no further upper bound on the voting time that holds for any initial opinion assignment has been proved. Still, one can observe that in many graphs the voting time is much smaller than  $\mathcal{O}(|E|)$ , e.g., the voting time of the complete graph is one.

We show that computing whether the voting time is greater than a given number is NP-hard. Unlike many generalizations of the majority process, this is the first NP-hardness proof that does not require any additional mechanisms besides the bare majority rule of the deterministic binary majority process.

**Theorem 1.** *For a given simple graph  $G$  and an integer  $k$  computing whether there exists an initial opinion assignment for which the voting time of  $G$  is at least  $k$  is NP-complete.*

A module of a graph is a subset of vertices  $S$  such that for each pair of nodes  $u, v \in S$  it holds that  $N(u) \setminus S = N(v) \setminus S$ . By carefully exploiting the structure of the potential function by Goles et al. and leveraging the particular behavior that certain modules of the graph exhibit in the majority process, we are able to prove that the voting time of a graph can be bounded by that of a smaller graph. This

graph can be constructed in linear time by contracting suitable vertices. Thus, we obtain a new upper bound that asymptotically improves on the previous  $\mathcal{O}(|E|)$  bound on graph classes characterized by a high number of modules that are either cliques or independent sets. Our bound relies on a well-known graph contraction technique, e.g., see the notion of *identical vertices* in [5]. Before we state our upper bound precisely, we need the following definitions.

**Definition 1.** *A set of nodes  $S$  is called a family if and only if for all pairs of nodes  $u, v \in S$  we have  $N(u) \setminus \{v\} = N(v) \setminus \{u\}$ . We say that a family  $S$  is proper if  $|S| > 1$ . Given a graph  $G = (V, E)$ , its asymmetric graph  $G^\Delta = (V^\Delta, E^\Delta)$  is the sub-graph of  $G$  induced by the subset  $V^\Delta \subseteq V$  constructed by replacing every family of odd-degree non-adjacent nodes with one node and replacing any other proper family with two nodes.*

The set of families of a graph forms a partition of the nodes into equivalence classes. We prove that the voting time of the majority process is bounded by that of the graph  $G^\Delta$ , obtained by contracting its families into one or two nodes, as stated in the following theorem.

**Theorem 2.** *Given any initial opinion assignment on a graph  $G = (V, E)$ , the voting time of the majority process is at most*

$$1 + \min\{|E^\Delta| - |V_{odd}^\Delta|/2, |E^\Delta|/2 + |V_{even}^\Delta|/4 + 7/4 \cdot |V^\Delta|\} .$$

*Furthermore, this bound can be computed in  $\mathcal{O}(|E|)$  time.*

For instance, for the convergence time of the Turán graph  $T(n, r)$  we obtain an  $\mathcal{O}(r^2)$  bound and for the convergence time of full  $d$ -ary trees we get an  $\mathcal{O}(|V|/d)$  bound, compared to the previously best known bounds of  $\mathcal{O}(n^2)$  and  $\mathcal{O}(|V|)$ , respectively, originating from the  $\mathcal{O}(|E|)$  bounds.

## References

1. Frischknecht, S., Keller, B., Wattenhofer, R.: Convergence in (Social) Influence Networks. In: Proc. DISC '13. pp. 433–446 (2013)
2. Goles, E., Olivos, J.: Periodic behaviour of generalized threshold functions. *Discrete Mathematics* 30(2), 187–189 (1980)
3. Poljak, S., Sůra, M.: On periodical behaviour in societies with symmetric influences. *Combinatorica* 3(1), 119–121 (1983)
4. Raghavan, U., Albert, R., Kumara, S.: Near linear time algorithm to detect community structures in large-scale networks. *Physical Review E* 76(3), 036106 (2007)
5. Saryüce, A., Saule, E., Kaya, K., Çatalyürek, U.: Shattering and Compressing Networks for Betweenness Centrality. In: Proc. SDM '13. pp. 686–694 (2013)
6. Winkler, P.: Puzzled: Delightful Graph Theory. *Comm. ACM* 51(8), 104 (2008)