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► **To cite this version:**

Marie Du Roy de Chaumaray. Weighted least squares estimator for the squared radial Ornstein-Uhlenbeck process. 2015. hal-01207617v1

HAL Id: hal-01207617

<https://hal.science/hal-01207617v1>

Preprint submitted on 1 Oct 2015 (v1), last revised 25 Jan 2018 (v3)

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WEIGHTED LEAST SQUARES ESTIMATOR FOR THE SQUARED RADIAL ORNSTEIN-UHLENBECK PROCESS

MARIE DU ROY DE CHAUMARAY

ABSTRACT. We estimate simultaneously both dimensional and drift parameters of a squared radial Ornstein-Uhlenbeck process. We do not restrict ourself to the case where the process never reaches zero. In order to avoid the use of unmanageable stopping times and natural but intractable estimator, we propose to make use of a weighted least squares estimator. We establish strong consistency and asymptotic normality for this estimator.

1. INTRODUCTION

The generalized squared radial Ornstein-Uhlenbeck process, also known as the Cox-Ingersoll-Ross process, is the strong solution of the stochastic differential equation

$$(1.1) \quad dX_t = (a + bX_t) dt + 2\sqrt{X_t} dB_t$$

where the initial state $X_0 = x \geq 0$, the dimensional parameter $a > 0$, the drift coefficient $b \in \mathbb{R}$ and (B_t) is a standard Brownian motion. The behaviour of this process has been widely investigated and depends on the values of both coefficients a and b . On the one hand, the asymptotic behaviour relies on the value of b . We consider in this paper the geometrically ergodic case $b < 0$. Overbeck has shown in Lemma 3 of [10] that, for T going to infinity, X_T converges in distribution to a random variable X with Gamma $\Gamma(a/2, -b/2)$ distribution. On the other hand, the value of a governs the behaviour at zero: for $a > 2$, the process almost surely never reaches zero, whereas for $0 < a < 2$

$$(1.2) \quad \mathbb{P}(\tau_0 < \infty) = 1 \text{ where } \tau_0 = \inf \{T > 0 | X_T = 0\},$$

see for instance [8]. Our goal is to estimate parameters a and b at the same time using a trajectory of the process over the interval $[0, T]$. The maximum likelihood estimator (MLE) of (a, b) is given by:

$$(1.3) \quad (\tilde{a}_T, \tilde{b}_T) = \left(\frac{\int_0^T X_t dt \int_0^T \frac{1}{X_t} dX_t - TX_T}{\int_0^T X_t dt \int_0^T \frac{1}{X_t} dt - T^2}, \frac{X_T \int_0^T \frac{1}{X_t} dt - T \int_0^T \frac{1}{X_t} dX_t}{\int_0^T X_t dt \int_0^T \frac{1}{X_t} dt - T^2} \right).$$

The asymptotic behaviour of this estimator has been widely studied, see for example [10], [5] and [3]. For $a > 2$, the MLE converges a.s. to (a, b) and satisfies the following Central Limit Theorem (CLT)

$$\sqrt{T} \begin{pmatrix} \hat{a}_T - a \\ \hat{b}_T - b \end{pmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}(0, 4\Sigma^{-1}) \text{ where } \Sigma = \begin{pmatrix} \frac{-b}{a-2} & 1 \\ 1 & -\frac{a}{b} \end{pmatrix}.$$

A large deviations principle was established in [4], and in the case where one parameter is supposed to be known to estimate the other one, large deviations for both parameters can be found in [12] while moderate deviations are given in [6]. By contrast, in the case where $0 < a < 2$, (1.2) implies the non-integrability of X_T^{-1} for some values of T so that the MLE does not converge for T going to infinity. Thus this case has been less investigated. However, it is often of interest in finance, to compute long dated interest rates with the

Heston model for instance, as explained in [2]. Overbeck [10] used accurate stopping times to build a strongly consistent estimator based on the MLE:

$$(1.4) \quad \mathbb{1}_{T < \tau_0} \begin{pmatrix} \tilde{a}_T \\ \tilde{b}_T \end{pmatrix} + \mathbb{1}_{\tau_0 \leq T} \left(\left(\int_0^T X_s ds \right)^{-1} \begin{pmatrix} \lim_{t \rightarrow \tau_0} S_t \Sigma_t^{-1} \\ X_T - T \lim_{t \rightarrow \tau_0} S_t \Sigma_t^{-1} \end{pmatrix} \right)$$

where $S_t = \int_0^t X_s^{-1} dX_s$, $\Sigma_t = \int_0^t X_s^{-1} ds$ and τ_0 is given by (1.2). The aim of this paper is to investigate a new strongly consistent weighted least squares estimator for the couple of parameters (a, b) . The weighting allows us to avoid the explosion for X_T reaching zero and consequently shirk us the use of stopping times, which are not easy to handle in practice. Our results answer, by the way, the question of Ben Alaya and Kebaier in the conclusion of [3].

We consider $C_T = X_T + c$ where c is some positive constant. We define

$$(1.5) \quad A_T = \begin{pmatrix} \int_0^T \frac{1}{C_t} dt & \int_0^T \frac{X_t}{C_t} dt \\ \int_0^T \frac{X_t}{C_t} dt & \int_0^T \frac{X_t^2}{C_t} dt \end{pmatrix}$$

and a new couple of estimators

$$(1.6) \quad \begin{pmatrix} \hat{a}_T \\ \hat{b}_T \end{pmatrix} = A_T^{-1} \begin{pmatrix} \int_0^T \frac{1}{C_t} dX_t \\ \int_0^T \frac{X_t}{C_t} dX_t \end{pmatrix}.$$

In the particular case where $c = 0$, we notice that

$$A_T = \begin{pmatrix} \int_0^T \frac{1}{X_t} dt & T \\ T & \int_0^T X_t dt \end{pmatrix}$$

and we obtain that the estimator (\hat{a}_T, \hat{b}_T) coincides with the MLE $(\tilde{a}_T, \tilde{b}_T)$.

The paper is organised as follows. The second section contains our main results: the strong consistency of this new couple of estimators as well as its asymptotic normality. The third section deals with a comparison between the two couples of estimators, while the remaining of the paper is devoted to the proofs of main results, as well as their illustration by some numerical simulations.

2. MAIN RESULTS

Theorem 2.1. *Assume that $a > 0$ and $b < 0$. Then, the couple of estimators (\hat{a}_T, \hat{b}_T) is strongly consistent: for T tending to infinity,*

$$\begin{pmatrix} \hat{a}_T \\ \hat{b}_T \end{pmatrix} \xrightarrow{a.s.} \begin{pmatrix} a \\ b \end{pmatrix}.$$

We denote X and C the limiting distributions of X_T and $X_T + c$ respectively, as T goes to infinity.

Theorem 2.2. *Assume that $a > 0$ and $b < 0$. Then, for T going to infinity, the couple of estimators (\hat{a}_T, \hat{b}_T) satisfies the following CLT*

$$\sqrt{T} \begin{pmatrix} \hat{a}_T - a \\ \hat{b}_T - b \end{pmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}(0, 4ALA^\top),$$

where

$$(2.1) \quad A = (\mathbb{E}[C] \mathbb{E}[1/C] - 1)^{-1} \begin{pmatrix} \mathbb{E}[X^2/C] & -\mathbb{E}[X/C] \\ -\mathbb{E}[X/C] & \mathbb{E}[1/C] \end{pmatrix}$$

and

$$(2.2) \quad L = \begin{pmatrix} \mathbb{E}[X/C^2] & \mathbb{E}[X^2/C^2] \\ \mathbb{E}[X^2/C^2] & \mathbb{E}[X^3/C^2] \end{pmatrix}.$$

Remark 2.1. For the remaining of this paper, we name

$$(2.3) \quad \psi_c = \left(-\frac{bc}{2}\right)^{a/2} e^{-bc/2} \Gamma(1 - a/2, -bc/2),$$

where Γ is the upper incomplete gamma function defined for all $y \in \mathbb{R}$ and $\alpha \in \mathbb{R}_*^+$ by

$$\Gamma(\alpha, y) = \int_y^{+\infty} e^{-t} t^{\alpha-1} dt,$$

and extended, for $y \neq 0$, to any α by holomorphicity. To simplify the following expressions we also define

$$(2.4) \quad \varphi_c = \psi_c \left(1 - \frac{a}{bc}\right) - 1.$$

In the proof of Theorem 2.2, we evaluate the two matrices A and L involving c and we obtain that

$$(2.5) \quad A = \varphi_c^{-1} \begin{pmatrix} c(\psi_c - 1) - \frac{a}{b} & \psi_c - 1 \\ \psi_c - 1 & \frac{\psi_c}{c} \end{pmatrix}$$

and

$$(2.6) \quad L = \frac{1}{2} \begin{pmatrix} \frac{a}{c} \psi_c + b(1 - \psi_c) & (a + 2 - bc)(1 - \psi_c) - a \\ (a + 2 - bc)(1 - \psi_c) - a & \psi_c c(a + 4 - b) - 4c - bc^2 - \frac{2a}{b} \end{pmatrix}.$$

By a straightforward computation, we deduce that $ALA^\top = (\varphi_c)^{-2} \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$ with the variances σ_{11} and σ_{22} respectively given by

$$\sigma_{11} = (\psi_c - 1)^2 \left(\frac{a}{b} - bc^2 + \psi_c \frac{bc}{2} (c - 1) \right) - \frac{a^2}{2b} \varphi_c$$

and

$$\sigma_{22} = \left((\psi_c^2 - 1) \frac{b}{2} + \frac{\psi_c}{c} \right) \varphi_c + \frac{\psi_c}{2c} (\psi_c^2 (a - b) + \psi_c (2 - bc) - 2)$$

and the covariance σ_{12} given by $\sigma_{12} = (\psi_c - 1)^2 - \frac{a}{2} \varphi_c$.

Remark 2.2. For c going to zero (for which we need a to be greater than 2), we obtain the same covariance matrix than for the MLE. Indeed, using well-known asymptotic results about the incomplete Gamma function Γ , which could be found in [9], we have that, as soon as $a > 2$,

$$(2.7) \quad \Gamma(1 - a/2, -bc/2) \xrightarrow{c \rightarrow 0} 0$$

and

$$(2.8) \quad \Gamma(1 - a/2, -bc/2) \left(-\frac{bc}{2}\right)^{a/2-1} \xrightarrow{c \rightarrow 0} \frac{1}{1 - a/2} = \frac{2}{a - 2}.$$

Thus ψ_c goes to zero for c tending to zero and $\frac{\psi_c}{c}$ converges to $\frac{-b}{a-2}$. Hence, we easily obtain that ALA^\top converges to $\begin{pmatrix} -\frac{a(a-2)}{2b} & -1 \\ -1 & -\frac{b}{2} \end{pmatrix}$ for c going to zero.

3. ASYMPTOTIC VARIANCE

Even though we considered the weighted least squares estimators in order to investigate the case $0 < a < 2$ for which the MLE is not consistent, it is interesting to compare the asymptotic variances in the CLT of this new estimators and of the MLE, in the case where $a > 2$. This comparison requires a lot of technical calculation as the asymptotic variances depends on the value of a , b and c . However, it is quite easy to compare variances in the case where we suppose one of the parameter known and we estimate the other one, as it simplifies substantially the expression of the estimators. Suppose we know a , the MLE for b is given by

$$(3.1) \quad \tilde{b}_T = \frac{X_T - aT}{\int_0^T X_t dt}$$

and satisfies the following CLT

$$\sqrt{T} (\tilde{b}_T - b) \xrightarrow{\mathcal{L}} \mathcal{N}(0, 4/\mathbb{E}[X])$$

where $\mathbb{E}[X] = -a/b$, see for instance [10]. Now, if we suppose b known, the MLE of a is given by

$$(3.2) \quad \tilde{a}_T = \frac{\int_0^T 1/X_t dX_t - bT}{\int_0^T 1/X_t dt}$$

and satisfies the following CLT

$$\sqrt{T} (\tilde{a}_T - a) \xrightarrow{\mathcal{L}} \mathcal{N}(0, 4/\mathbb{E}[X^{-1}])$$

with $\mathbb{E}[X^{-1}] = -b/(a-2)$. Whereas, the weighted least squares estimators are respectively given by

$$(3.3) \quad \hat{b}_T = \frac{\int_0^T \frac{X_t}{C_t} dX_t - a \int_0^T \frac{X_t}{C_t} dt}{\int_0^T \frac{X_t^2}{C_t} dt}$$

and

$$(3.4) \quad \hat{a}_T = \frac{\int_0^T \frac{1}{C_t} dX_t - b \int_0^T \frac{X_t}{C_t} dt}{\int_0^T \frac{1}{C_t} dt}.$$

Proposition 3.1. *Assume that $a > 0$ and $b < 0$. For T going to infinity, \hat{b}_T satisfies the following CLT:*

$$\sqrt{T} (\hat{b}_T - b) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, 4\mathbb{E}[X^3/C^2] (\mathbb{E}[X^2/C])^{-2}\right).$$

Proof. Replacing dX_t by its expression (1.1), we easily get that

$$\sqrt{T} (\hat{b}_T - b) = 2 \left(\frac{1}{T} \int_0^T \frac{X_t^2}{C_t} dt \right)^{-1} \frac{n_T}{\sqrt{T}}$$

where n_T is a martingale term given by

$$(3.5) \quad n_T = \int_0^T \frac{X_t \sqrt{X_t}}{C_t} dB_t,$$

and its bracket process equals

$$(3.6) \quad \langle n \rangle_T = \int_0^T \frac{X_t^3}{C_t^2} dt.$$

Using the ergodicity of the process, we get for T tending to infinity

$$(3.7) \quad \frac{\langle n \rangle_T}{T} \xrightarrow{a.s.} \mathbb{E} [X^3/C^2].$$

Thus, by the CLT for martingales, we obtain the following convergence in distribution

$$\frac{n_T}{\sqrt{T}} \xrightarrow{\mathcal{L}} \mathcal{N} (0, \mathbb{E} [X^3/C^2]).$$

This gives the announced result, using Slutsky's lemma and the fact that, by the ergodicity of the process, $\frac{1}{T} \int_0^T \frac{X_t^2}{C_t} dt$ converges a.s. to $\mathbb{E} [X^2/C]$ for T going to infinity. \square

Proposition 3.2. *Assume that $a > 0$ and $b < 0$. For T going to infinity, \hat{a}_T satisfies the following CLT:*

$$\sqrt{T} (\hat{a}_T - a) \xrightarrow{\mathcal{L}} \mathcal{N} (0, 4 \mathbb{E} [X/C^2] (\mathbb{E} [1/C])^{-2}).$$

Proof. It works as in the previous proof. We first notice that

$$\sqrt{T} (\hat{a}_T - a) = 2 \left(\frac{1}{T} \int_0^T \frac{1}{C_t} dt \right)^{-1} \frac{m_T}{\sqrt{T}}$$

where m_T is a martingale term given by

$$(3.8) \quad m_T = \int_0^T \frac{\sqrt{X_t}}{C_t} dB_t.$$

Thus, for T going to infinity,

$$(3.9) \quad \frac{\langle m \rangle_T}{T} = \frac{1}{T} \int_0^T \frac{X_t}{C_t^2} dt \xrightarrow{a.s.} \mathbb{E} [X/C^2]$$

and we conclude using the CLT for martingales, the ergodicity of the process and Slutsky's lemma. \square

Proposition 3.3. *Assume that $a > 2$ is known and $b < 0$. Then, the MLE of b satisfies a CLT with a smaller asymptotic variance than the weighted least squares estimator.*

Proof. Using Cauchy-Schwarz Inequality, we notice that

$$(\mathbb{E} [X^2/C])^2 = \left(\mathbb{E} \left[\sqrt{X} \times X^{3/2}/C \right] \right)^2 \leq \mathbb{E} [X] \mathbb{E} [X^3/C^2]$$

which immediately leads to the result. \square

Proposition 3.4. *Assume that $a > 2$ and $b < 0$ is known. Then, the MLE of a satisfies a CLT with a smaller asymptotic variance than the weighted least squares estimator.*

Proof. Using Cauchy-Schwarz Inequality, we notice that

$$(\mathbb{E} [1/C])^2 = \left(\mathbb{E} [X^{-1/2} \times X^{1/2}/C] \right)^2 \leq \mathbb{E} [1/X] \mathbb{E} [X/C^2]$$

which immediately leads to the result. \square

Remark 3.1. *Thus, the weighted least squares estimator is less efficient than the MLE in the case where this later is easily manageable. This could seem to be contradictory to Remark 4.4 of [11] which deals with the discrete-time counterpart of the process. In fact, they compare the weighted least squares with the conditional least squares estimator which does not coincide with the MLE.*

4. KEYSTONE LEMMAS

In order to prove Theorem 2.1, we need to investigate the almost sure convergence of all the integrals involved in the definition of the estimators. Overbeck has shown in Lemma 3(i) of [10] that, for T going to infinity, X_T converges in distribution to X with Gamma $\Gamma(a/2, -b/2)$ distribution, whose density is given by

$$(4.1) \quad f(y) = (\Gamma(a/2))^{-1} (-b/2)^{a/2} y^{a/2-1} e^{yb/2} \mathbf{1}_{y>0}.$$

Thus, by Lemma 3(ii) of [10], for T going to infinity,

$$\frac{1}{T} \int_0^T g(X_t) dt \xrightarrow{a.s.} \mathbb{E}[g(X)] = \int_0^{+\infty} g(y) f(y) dy.$$

for any function g such that the right-hand side exists.

By an integration by part, we easily show the two following properties of the incomplete gamma function, which will be very useful in the rest of the proofs:

$$(4.2) \quad \Gamma(\alpha + 1, y) = y^\alpha e^{-y} + \alpha \Gamma(\alpha, y)$$

and

$$(4.3) \quad \Gamma(\alpha + 2, y) = y^\alpha e^{-y} (y + \alpha + 1) + \alpha(\alpha + 1) \Gamma(\alpha, y).$$

We are now able to prove the following lemmas.

Lemma 4.1. $\mathbb{E}[1/C] = \frac{\psi_c}{c}$.

Proof. We have

$$(4.4) \quad \mathbb{E}[1/C] = \int_0^{+\infty} \frac{1}{y+c} f(y) dy,$$

where f is given by (4.1). Formula 3.38(10) of [7] gives that

$$\int_0^{+\infty} \frac{1}{y+c} y^{a/2-1} e^{yb/2} dy = c^{a/2-1} e^{-bc/2} \Gamma(a/2) \Gamma(1-a/2, -bc/2),$$

which leads to

$$\mathbb{E}[1/C] = \frac{1}{c} \left(-\frac{bc}{2}\right)^{a/2} e^{-bc/2} \Gamma(1-a/2, -bc/2)$$

and ensures the announced result. \square

Lemma 4.2. $\mathbb{E}[X/C] = 1 - \psi_c$.

Proof. As in the previous proof, we have

$$(4.5) \quad \mathbb{E}[X/C] = \int_0^{+\infty} \frac{y}{y+c} f(y) dy.$$

By formula 3.38(10) of [7], we know that

$$(4.6) \quad \int_0^{+\infty} \frac{1}{y+c} y^{a/2} e^{yb/2} dy = c^{a/2} e^{-bc/2} \Gamma(a/2+1) \Gamma(-a/2, -bc/2).$$

With formula (4.2), we easily obtain that

$$(4.7) \quad \Gamma(-a/2, -bc/2) = \left(-\frac{2}{a}\right) \left(\Gamma(1-a/2, -bc/2) - \left(-\frac{bc}{2}\right)^{-a/2} e^{bc/2} \right).$$

Combining (4.5), (4.6), (4.7) and the fact that $\Gamma(a/2+1) = a/2 \times \Gamma(a/2)$, we deduce the announced result. \square

Lemma 4.3. $\mathbb{E}[X^2/C] = c(\psi_c - 1) - \frac{a}{b}$.

Proof. We have

$$\mathbb{E} \left[\frac{X^2}{C} \right] = \mathbb{E} \left[\frac{(X+c-c)^2}{X+c} \right] = \mathbb{E}[X] - c + c^2 \mathbb{E} \left[\frac{1}{C} \right],$$

and we conclude using Lemma 4.1 and the fact that $\mathbb{E}[X] = -a/b$. \square

Thus, with the three previous lemmas, we are able to establish the almost sure convergence of TA_T^{-1} as T goes to infinity. Only the right-hand side vector of (1.6) remains to be studied. It is the aim of the following lemmas.

Lemma 4.4. $\mathbb{E}[X/C^2] = \frac{a}{2c}\psi_c + \frac{b}{2}(1 - \psi_c)$.

Proof. By the very definition of f given by (4.1), we have

$$(4.8) \quad \mathbb{E}[X/C^2] = \int_0^{+\infty} \frac{y}{(y+c)^2} f(y) dy = \frac{(-b/2)^{a/2}}{\Gamma(a/2)} \int_0^{+\infty} \frac{y^{a/2}}{(y+c)^2} e^{yb/2} dy.$$

Integrating the right-hand side of (4.8) by part, we obtain that

$$(4.9) \quad \mathbb{E}[X/C^2] = \frac{(-b/2)^{a/2}}{\Gamma(a/2)} \left[\frac{a}{2} \int_0^{+\infty} \frac{y^{a/2-1}}{y+c} e^{yb/2} dy + \frac{b}{2} \int_0^{+\infty} \frac{y^{a/2}}{y+c} e^{yb/2} dy \right].$$

We have already computed both integrals in the proofs of respectively Lemma 4.1 and Lemma 4.2, which leads to

$$(4.10) \quad \mathbb{E}[X/C^2] = \frac{a}{2} \mathbb{E}[1/C] + \frac{b}{2} \mathbb{E}[X/C] = \frac{a}{2c} \psi_c + \frac{b}{2} (1 - \psi_c).$$

\square

Lemma 4.5. $\mathbb{E}[X^2/C^2] = \frac{1}{2} ((a+2-bc)(1 - \psi_c) - a)$.

Proof. Integrating by part and using Lemmas 4.3 and 4.4,

$$\begin{aligned} \mathbb{E}[X^2/C^2] &= \frac{(-b/2)^{a/2}}{\Gamma(a/2)} \int_0^{+\infty} \frac{y^{a/2+1}}{(y+c)^2} e^{yb/2} dy \\ &= \frac{(-b/2)^{a/2}}{\Gamma(a/2)} \left[\frac{a+2}{2} \int_0^{+\infty} \frac{y^{a/2}}{y+c} e^{yb/2} dy + \frac{b}{2} \int_0^{+\infty} \frac{y^{a/2+1}}{y+c} e^{yb/2} dy \right] \\ &= \frac{1}{2} \left((a+2) \mathbb{E}[X/C] + \frac{b}{2} \mathbb{E}[X^2/C] \right) \\ &= \frac{1}{2} \left((a+2)(1 - \psi_c) + b \left(c(\psi_c - 1) - \frac{a}{b} \right) \right). \end{aligned}$$

\square

Lemma 4.6. $\mathbb{E}[X^3/C^2] = \frac{c}{2} (a+4-b) \psi_c - 2c - \frac{bc^2}{2} - \frac{a}{b}$.

Proof. Noticing that $X^3 = X(X+c)^2 - 2cX^2 - c^2X$, we obtain that

$$\mathbb{E}[X^3/C^2] = \mathbb{E}[X] - 2c \mathbb{E}[X^2/C^2] - c^2 \mathbb{E}[X/C^2]$$

and we conclude using Lemmas 4.4 and 4.5. \square

5. PROOF OF THE CONSISTENCY

We are now in position to prove Theorem 2.1. We first rewrite (1.6) using (1.1):

$$(5.1) \quad \begin{aligned} \begin{pmatrix} \widehat{a}_T \\ \widehat{b}_T \end{pmatrix} &= A_T^{-1} \begin{pmatrix} \int_0^T \frac{1}{C_t} \left((a + bX_t)dt + 2\sqrt{X_t} dB_t \right) \\ \int_0^T \frac{X_t}{C_t} \left((a + bX_t)dt + 2\sqrt{X_t} dB_t \right) \end{pmatrix} \\ &= A_T^{-1} A_T \begin{pmatrix} a \\ b \end{pmatrix} + A_T^{-1} M_T, \end{aligned}$$

where M_T is a martingale given by

$$(5.2) \quad M_T = \begin{pmatrix} \int_0^T \frac{2\sqrt{X_t}}{C_t} dB_t \\ \int_0^T \frac{2\sqrt{X_t}X_t}{C_t} dB_t \end{pmatrix} = 2 \begin{pmatrix} m_T \\ n_T \end{pmatrix}.$$

Proof of Theorem 2.1. First of all, we have

$$\frac{1}{T^2} \det A_T = \frac{1}{T} \int_0^T \frac{1}{C_t} dt \times \frac{1}{T} \int_0^T \frac{X_t^2}{C_t} dt - \left(\frac{1}{T} \int_0^T \frac{X_t}{C_t} dt \right)^2.$$

Thus, as the process is ergodic, we get, for T going to infinity,

$$(5.3) \quad \frac{1}{T^2} \det A_T \xrightarrow{a.s.} \mathbb{E}[1/C] \mathbb{E}[X^2/C] - (\mathbb{E}[X/C])^2$$

and

$$(5.4) \quad T A_T^{-1} \xrightarrow{a.s.} A$$

where A is given by

$$A = (\mathbb{E}[C] \mathbb{E}[1/C] - 1)^{-1} \begin{pmatrix} \mathbb{E}[X^2/C] & -\mathbb{E}[X/C] \\ -\mathbb{E}[X/C] & \mathbb{E}[1/C] \end{pmatrix}.$$

A straightforward application of Lemmas 4.1 to 4.3 gives that

$$(5.5) \quad A = \frac{1}{\psi_c \left(1 - \frac{a}{bc}\right) - 1} \begin{pmatrix} c(\psi_c - 1) - \frac{a}{b} & \psi_c - 1 \\ \psi_c - 1 & \frac{\psi_c}{c} \end{pmatrix}.$$

Besides, the martingale M_T satisfies for T going to infinity

$$(5.6) \quad \frac{M_T}{T} \xrightarrow{a.s.} 0.$$

Indeed, by equations (3.7) and (3.9), we know that $\langle n \rangle_T = \mathcal{O}(T)$ and $\langle m \rangle_T = \mathcal{O}(T)$. Thus, as both $\frac{n_T}{\langle n \rangle_T}$ and $\frac{m_T}{\langle m \rangle_T}$ go to zero a.s. for T going to infinity,

$$\frac{n_T}{T} \xrightarrow{a.s.} 0 \quad \text{and} \quad \frac{m_T}{T} \xrightarrow{a.s.} 0.$$

Combining (5.4) and (5.6) with (5.1), we obtain the announced almost sure convergence. \square

6. PROOF OF THE ASYMPTOTIC NORMALITY

Proof of Theorem 2.2. First of all, we deduce from (5.1) that

$$(6.1) \quad \sqrt{T} \begin{pmatrix} \widehat{a}_T - a \\ \widehat{b}_T - b \end{pmatrix} = TA_T^{-1} \frac{M_T}{\sqrt{T}}.$$

We already saw that TA_T^{-1} converges almost surely as T goes to infinity and its limit A is given by (5.5). We now have to establish the asymptotic normality of $\frac{M_T}{\sqrt{T}}$. The bracket process of M_T is given by

$$\langle M \rangle_T = 4 \begin{pmatrix} \langle m \rangle_T & \int_0^T \frac{X_t^2}{C_t^2} dt \\ \int_0^T \frac{X_t^2}{C_t^2} dt & \langle n \rangle_T \end{pmatrix},$$

where $\langle m \rangle_T$ and $\langle n \rangle_T$ are respectively given by (3.9) and (3.7). Thus, by the ergodic Theorem, we get

$$\frac{\langle M \rangle_T}{T} \xrightarrow{a.s.} 4L$$

where

$$L = \begin{pmatrix} \mathbb{E}[X/C^2] & \mathbb{E}[X^2/C^2] \\ \mathbb{E}[X^2/C^2] & \mathbb{E}[X^3/C^2] \end{pmatrix}.$$

As a straightforward consequence of Lemmas 4.4 to 4.6, we obtain that

$$L = \frac{1}{2} \begin{pmatrix} \frac{a}{c} \psi_c + b(1 - \psi_c) & (a + 2 - bc)(1 - \psi_c) - a \\ (a + 2 - bc)(1 - \psi_c) - a & \psi_c c(a + 4 - b) - 4c - bc^2 - \frac{2a}{b} \end{pmatrix}.$$

We deduce from the CLT for martingales that

$$\frac{M_T}{\sqrt{T}} \xrightarrow{\mathcal{L}} \mathcal{N}(0, 4L),$$

which leads, via Slutsky's Lemma, to the result:

$$\sqrt{T} \begin{pmatrix} \widehat{a}_T - a \\ \widehat{b}_T - b \end{pmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}(0, 4ALA^\top).$$

□

7. NUMERICAL SIMULATIONS

The efficient discretization of the CIR process is a challenging question, see for example [2] and [1]. We choose to implement the QE-algorithm based on quadratic-exponential approximations proposed in [2]. Andersen introduced this algorithm to deal with the case $a < 2$, for which common discretization schemes are not accurate. The two following figures illustrate our main results (consistency and asymptotic normality) in the case $a = 1$ and $b = -2$. The red curves in the second figure display the standard normal distribution.

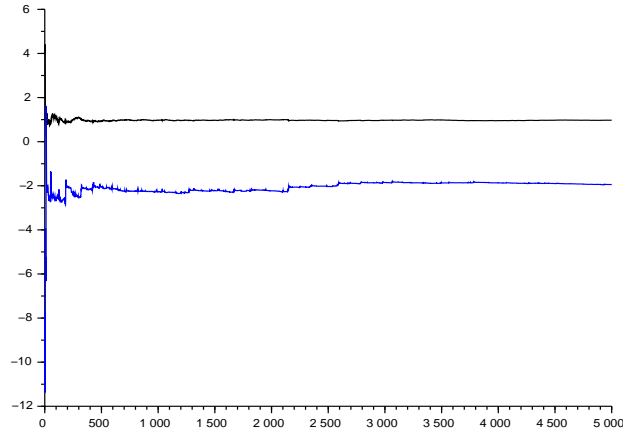


FIGURE 1. Consistency: $(\hat{a}_T)_T$ in black and $(\hat{b}_T)_T$ in blue

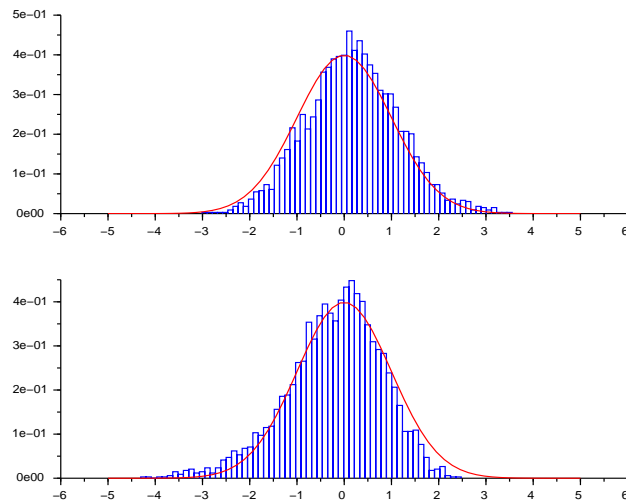


FIGURE 2. Histograms of 3000 outcomes of $\sqrt{T/4\sigma_{11}}(\hat{a}_T - a)$ and $\sqrt{T/4\sigma_{22}}(\hat{b}_T - b)$ at time $T = 70$

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