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Bicovariograms and Euler characteristic of random fields excursions

Raphaël Lachièze-Rey *

Abstract

Let f be a C^1 bivariate function with Lipschitz derivatives, and $F = \{x \in \mathbb{R}^2 : f(x) \geq \lambda\}$ an upper level set of f , with $\lambda \in \mathbb{R}$. We give a new expression of the Euler characteristic of F in terms of the three-points indicator functions of the set, related to its bicovariograms. We also derive a bound on the number of connected components of F in terms of the values of f and its gradient, valid in higher dimensions. In dimension 2, this bound allows to pass this identity to expectations if f 's partial derivatives have Lipschitz constants with finite moments of sufficiently high order, without the requirement of a bounded conditional density. This approach provides an expression of the mean Euler characteristic in terms of the field's third order marginal. We give sufficient conditions and explicit formulas for Gaussian fields, relaxing the usual C^2 Morse hypothesis.

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1 Introduction

The geometry of random fields excursion sets has been a subject of intense research over the last two decades. Many authors are concerned with the computation of the mean [3, 4, 5, 8] or variance [12, 20] of the Euler characteristic, denoted by χ here.

As an integer-valued quantity, the Euler characteristic can be easily measured and used in many estimation and modelisation procedures. It is an important indicator of the porosity of a random media [7, 15, 24], it is used in brain imagery [17, 26], astronomy, [20, 21, 23], and many other disciplines. See also [2] for a general review of applied algebraic topology.

Most of the available works on random fields use the results gathered in the celebrated monograph [6], or similar variants. In this case, theoretical computations of the Euler characteristic emanate from Morse theory, where the focus is on the local extrema of the underlying field instead of the set itself. For the theory to be applicable, the functions must be C^2 and satisfy the Morse hypotheses, which conveys some restrictions on the set itself.

The expected Euler characteristic also turned out to be a widely used approximation of the distribution function of the maximum of a Morse random field, and attracted much interest in this direction, see [3, 8, 9, 26]. Indeed, for large $r > 0$, a well-behaved field rarely exceeds r , and if it does, it is likely to have a single highest peak, which yields that the level set of f at level r , when not empty, is most often simply connected, and has Euler characteristic 1. In this

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fashion, $\mathbf{E}\chi(\{f \geq r\}) \approx \mathbf{P}(\sup f \geq r)$, which provides an additional motivation to compute the mean Euler characteristic of random fields.

Even though [4] provides an asymptotic expression for some classes of infinitely divisible fields, most of the tractable formulae concern Gaussian fields. One of the ambition of this paper is to provide a formula that is tractable in a rather general setting, and also works in the Gaussian realm. There seems to be no particular obstacle to extend these ideas to higher dimensions in a further work.

Approach and main results

Given a set $A \subset \mathbb{R}^2$, let $\Gamma(A)$ be the class of its bounded connected components. We say that a set A is *admissible* if $\Gamma(A)$ and $\Gamma(A^c)$ are finite, and in this case its Euler characteristic is defined by

$$\chi(A) = \#\Gamma(A) - \#\Gamma(A^c),$$

where in all the paper $\#$ denotes the cardinality of a set. The theoretical results of Adler and Taylor [6] regarding the Euler characteristic of random excursions require second order differentiability of the underlying field f , but the expression of the mean Euler characteristic only involves the first-order derivatives, suggesting that second order derivatives do not matter in the computation of the Euler characteristic. In the words of Adler and Taylor (Section 11.7), regarding their Formula (11.7.6), it is a *rather surprising fact that the [mean Euler characteristic of a Gaussian field] depends on the covariance of f only through some of its derivatives at zero*, the latter referring to first-order partial derivatives. We present here a new method for which the second order differentiability is not needed. The results are valid for \mathcal{C}^1 fields with locally Lipschitz derivatives, also called $\mathcal{C}^{1,1}$ fields, relaxing slightly the classical \mathcal{C}^2 Morse hypothesis.

Our results exploit the findings of [18] connecting smooth sets Euler characteristic and variographic tools. For some $\lambda \in \mathbb{R}$ and a bi-variate function f , define

$$\delta^\eta(x, f, \lambda) = \mathbf{1}_{\{f(x) \geq \lambda, f(x+\eta\mathbf{u}_1) < \lambda, f(x+\eta\mathbf{u}_2) < \lambda\}}, \eta \in \mathbb{R}.$$

where $(\mathbf{u}_1, \mathbf{u}_2)$ denotes the canonical basis of \mathbb{R}^2 , assuming f is defined in these points. For $W \subset \mathbb{R}^2$. Let us write a corollary of our main result here. A more general statement can be found in section 3.

Corollary 1. *Let $W = [0, a] \times [0, b]$ for some $a, b > 0$, f be a \mathcal{C}^1 real random field on \mathbb{R}^2 with locally Lipschitz partial derivatives $\partial_1 f, \partial_2 f$, $\lambda \in \mathbb{R}$, and note $F = \{x \in W : f(x) \geq \lambda\}$. Assume furthermore that the following conditions are satisfied:*

- (i) *For some $\kappa > 0$, for $x \in \mathbb{R}^2$, the density of the random vector $(f(x), \partial_1 f(x), \partial_2 f(x))$ is bounded by κ on \mathbb{R}^3 .*
- (ii) *There is $p > 6$ such that*

$$\mathbf{E}[\text{Lip}(f, W)^p] < \infty, \mathbf{E}[\text{Lip}(\partial_i f, W)^p] < \infty, i = 1, 2,$$

where $\text{Lip}(\cdot, W)$ denotes the Lipschitz constant on W .

Then $\chi(F)$ is a well-defined integrable random variable and

$$\mathbf{E}\chi(F) = \lim_{\varepsilon \rightarrow 0} \sum_{x \in \varepsilon\mathbb{Z}^2} [\mathbf{E}\delta^\varepsilon(x, f\mathbf{1}_W, \lambda) - \mathbf{E}\delta^{-\varepsilon}(x, -f\mathbf{1}_W, -\lambda)] \quad (1)$$

$$= \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} \int_{\mathbb{R}^2} [\mathbf{E}\delta^\varepsilon(x, f\mathbf{1}_W, \lambda) - \mathbf{E}\delta^{-\varepsilon}(x, -f\mathbf{1}_W, -\lambda)] dx. \quad (2)$$

This quantity is related to the *bicovariogram* of the set F , defined by

$$\delta_0^{x,y}(F) = \ell(F \cap (F+x)^c \cap (F+y)^c), x, y \in \mathbb{R}^2, \quad (3)$$

in that (2) can be reformulated as

$$\mathbf{E}\chi(F) = \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} (\mathbf{E}\delta_0^{-\varepsilon \mathbf{u}_1, -\varepsilon \mathbf{u}_2}(F) - \mathbf{E}\delta_0^{\varepsilon \mathbf{u}_1, \varepsilon \mathbf{u}_2}(F^c)).$$

This approach seems new in the literature. We see that, under suitable conditions, one can directly compute the mean Euler characteristic of random level sets as a linear integral of the field's third order marginal in triples of arbitrarily close arguments. We also give in Theorem 3 a bound on the number of connected components of f 's excursion, valid in any dimension, which is finer than just bounding by the number of critical points; we could not locate an equivalent result in the literature. This topological estimate is interesting in its own and also applies uniformly to the number of components of pixel approximations of f 's excursions. We therefore use it here as a domination in the application of Lebesgue's theorem to obtain (1)-(2).

It is likely that results presented in Theorem 7 hold in higher dimensions. See for instance [25], that paves the way for an extension of the results of [18] to random fields on spaces with arbitrary dimension. Also, the uniform bounded density hypothesis is relaxed and allows for the density of the $(d+1)$ -tuple $(f(x), \partial_1 f(x), \dots, \partial_d f(x))$ to be arbitrarily large in the neighbourhood of $(\lambda, 0, \dots, 0)$. Theorem 7 actually features a result where f is defined on the whole plane and the level sets are observed through a bounded window W , as is typically the case for level sets of non-trivial stationary fields, but the intersection with ∂W requires additional notation and care. See Theorem 9 for a result tailored to deal with stationary fields excursions.

Theorem 10 is specialised to the case where f is a Gaussian field. Under the additional hypotheses that f is stationary and isotropic, we retrieve in Theorem 11 the standard results of [6], assuming only $\mathcal{C}^{1,1}$ regularity (classical literature require \mathcal{C}^2 Morse fields in dimension $d \geq 2$, or \mathcal{C}^1 fields in dimension 1).

Let us explore other consequences of our results. Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be a \mathcal{C}^1 test function with compact support. Using the results of the current paper, it is shown in the follow-up article [19] that for any deterministic \mathcal{C}^2 Morse function f on \mathbb{R}^2 with compact level sets,

$$\int_{\mathbb{R}^2} \chi(\{f \geq \lambda\}) h(\lambda) d\lambda = - \sum_{i=1}^2 \mathbf{1}_{\{\nabla f(x) \in Q_i\}} [h'(f(x)) \partial_i f(x)^2 + h(f(x)) \partial_{ii} f(x)] \quad (4)$$

where

$$Q_1 = \{(x, y) \in \mathbb{R}^2 : y < x < 0\}, Q_2 = \{(x, y) \in \mathbb{R}^2 : x < y < 0\},$$

yielding applications for instance to shot-noise processes. In the context of random functions, one can pass this formula to expectations, breaking free from any density hypothesis on f 's marginals, at the contrary of analogous results, including those from the current paper. Bierme & Desolneux [10, Section 4.1] later gave another interpretation of (4), showing that if it is extended to a random isotropic stationary field, it can be rewritten as the simpler expression, after appropriate integration by parts,

$$\mathbf{E} \int_U \chi(\{f \geq \lambda; U\}) h(\lambda) d\lambda = -\ell(U) \mathbf{E} \partial_{ii} f(0) h(f(0)),$$

where U is an appropriate open set, and $\chi(\{f \geq \lambda; U\})$ is the total curvature of the level set $\{f \geq \lambda\}$ within U , generalising the Euler characteristic. They obtained this result by totally different means, via an approach involving Gauss-Bonnet theorem, and giving in general simpler formulations and proofs, without any requirement on f apart from being \mathcal{C}^2 .

2 Topological approximation

Let f be a function of class \mathcal{C}^1 over some domain $W \subset \mathbb{R}^d$, and $\lambda \in \mathbb{R}$. Define

$$F := F_\lambda(f) = \{x \in W : f(x) \geq \lambda\}, \quad F_{\lambda^+}(f) = \{x \in W : f(x) > \lambda\}.$$

Remark that $F_{\lambda^+}(f) = (F_{-\lambda}(-f))^c$. If we assume that ∇f does not vanish on $\partial F_\lambda(f)$, then $\partial F_\lambda(f) = \partial F_{\lambda^+}(f) = f^{-1}(\{\lambda\})$, and this set is furthermore Lebesgue-negligible, as a $(d-1)$ -dimensional manifold.

According to [13, 4.20], $\partial F_\lambda(f)$ is regular, in the sense that its boundary is \mathcal{C}^1 with Lipschitz normal, if ∇f is locally Lipschitz and does not vanish on $\partial F_\lambda(f)$. This condition is necessary to prevent F from having locally infinitely many connected components, which would make Euler characteristic not properly defined in dimension 2, see [18, Remark 1.15]. We call $\mathcal{C}^{1,1}$ function a differentiable function which gradient is a locally Lipschitz mapping $\partial F \rightarrow \mathbb{R}^2$. Those functions have been mainly used in optimisation problems, and as solutions of some PDEs, see for instance [16]. They can also be characterised as the functions which are locally *semiconvex* and *semiconcave*, see [11].

The results of [18] also yield that the Lipschitzness of ∇f is sufficient for the digital approximation of $\chi(\{f \geq \lambda\})$ to be valid. It seems therefore that the $\mathcal{C}^{1,1}$ assumption is the minimal ones ensuring the Euler characteristic to be computable.

Observation window

An aim of the present paper is to advocate the power of variographic tools for computing the Euler characteristic of random fields excursions. Since many applications are concerned with stationary random fields on the whole plane, we have to study the intersection of excursions with bounded windows, and assess the quality of the approximation.

To this end, call admissible *rectangle* of \mathbb{R}^d any set $W = I_1 \times \cdots \times I_d$ where the I_k are possibly infinite nonempty open intervals of \mathbb{R} . Note $\partial_k W$ its k -skeleton, as defined for instance in [6, Section 6.2], with $\partial_d W = W$. Note $\text{corners}(W) = \partial_0 W$, which number is between 0 and 2^d . Then call *polyrectangle* a finite union $W = \cup_i W_i$ where each W_i is an admissible rectangle, and for $i \neq j$, $\text{corners}(W_i) \cap \text{corners}(W_j) = \emptyset$. Call \mathscr{W}_d the class of admissible polyrectangles. For $W \in \mathscr{W}_d$, define the k -dimensional skeleton by $\partial_k W = (\cup_i \partial_k W_i \cap \partial W) \setminus (\cup_{j < k} \partial_j W)$, $1 \leq k < d$, $\partial_d W = W$. Note that the closure of W is the disjoint union of the $\partial_k W$, $0 \leq k \leq d$.

For $0 \leq k \leq d$, note \mathcal{I}_k the class of subsets of $\{1, \dots, d\}$ with k elements. For $W \in \mathscr{W}_d$, $0 \leq k \leq d$, $x \in \partial_k W$, let $I_x(W) \in \mathcal{I}_k$ the set of indexes such that the tangent space to W in x is spanned by the \mathbf{u}_i , $i \in I_x(W)$. For $d = 2$, for $x \in \partial_1 W$, denote by $\mathbf{n}_W(x)$ the outwards normal unit vector of W in x . We also call *edge of W* a segment $[x, y] \subset \partial W$ that is not strictly contained in another such segment of ∂W (in this case, $x, y \in \text{corners}(W)$).

Definition 2. Let $W \in \mathscr{W}_d$, and $f : W \rightarrow \mathbb{R}$ of class \mathcal{C}^1 . Say that f 's excursion at some level $\lambda \in \mathbb{R}$ is *regular within W at level λ* if for $0 \leq k \leq d$, $\{x \in \partial_k W : f(x) = \lambda, \partial_i f(x) = 0, i \in I_x(W)\} = \emptyset$.

For such a function f in dimension 2, it is shown in [18] that the Euler characteristic of its excursion set $F = F_\lambda(f) \cap W$ can be expressed by means of its bicovariograms, defined in (3). For ε sufficiently small

$$\chi(F) = \varepsilon^{-2} [\delta_0^{-\varepsilon \mathbf{u}_1, -\varepsilon \mathbf{u}_2}(F) - \delta_0^{\varepsilon \mathbf{u}_1, \varepsilon \mathbf{u}_2}(F^c)]. \quad (5)$$

The proof is based on the Gauss approximation of F :

$$F^\varepsilon = \bigcup_{x \in \varepsilon \mathbb{Z}^2 \cap F} (x + \varepsilon[-1/2, 1/2]^2).$$

According to [18, Theorem 1.7], for ε sufficiently small,

$$\begin{aligned}\chi(F) &= \chi(F^\varepsilon) \\ &= \sum_{x \in \varepsilon\mathbb{Z}^2} (\delta^\varepsilon(x, f\mathbf{1}_W, \lambda) - \delta^{-\varepsilon}(x, -f\mathbf{1}_W, -\lambda)), \\ &= \varepsilon^{-2} \int_{\mathbb{R}^2} (\delta^\varepsilon(x, f\mathbf{1}_W, \lambda) - \delta^{-\varepsilon}(x, -f\mathbf{1}_W, -\lambda)) dx,\end{aligned}$$

for $x \in \mathbb{R}^2$. If f is a random field, the difficulty to pass the result to expectations is to dominate the right hand side uniformly in ε by an integrable quantity, and this goes through bounding the number of connected components of F and its approximation F^ε . This is the object of the next section.

2.1 Topological estimates

The next result, valid in all dimension $d \geq 1$, does not concern directly the Euler characteristic. Its purpose is to bound the number of connected components of $F_\lambda(f) \cap W$ by an expression depending on f and its partial derivatives. It turns out that a similar bound holds for the excursion approximation $(F_\lambda(f) \cap W)^\varepsilon$ in dimension 2, uniformly in ε , enabling the application of Lebesgue's theorem to the point-wise convergence (5).

Traditionally, see for instance [12, Prop. 1.3], the number of connected components of the excursion set, or its Euler characteristic, is bounded by using the number of critical points, or by the number of points on the level set where f 's gradient is directed towards a predetermined direction. Here, we use another method based on the idea that in a small connected component, a critical point is necessarily close to the boundary, where $f - \lambda$ vanishes. That gives the relatively closed expression (6) as a bound on the number of connected components. It also allows in Section 3, devoted to random fields, to relax the usual uniform density assumption on the marginals of the $(d+1)$ -tuple $(f, \partial_i f, i = 1, \dots, d)$, leaving the possibility that the density is unbounded around $(\lambda, 0, \dots, 0)$.

Denote by $\text{Lip}(g; A) \in \mathbb{R}_+ \cup \{\infty\}$, or just $\text{Lip}(g)$, the Lipschitz constant of a function g going from a metric space A to another. Let $W \in \mathcal{W}_d, g : W \rightarrow \mathbb{R}, \mathcal{C}^1$ with Lipschitz derivatives. Note \mathcal{H}_d^k the k -dimensional Hausdorff measure in \mathbb{R}^d . Define the possibly infinite quantity, for $1 \leq k \leq d$,

$$I_k(g; W) := \max(\text{Lip}(g), \text{Lip}(\partial_i g), 1 \leq i \leq d) \int_{\partial_k W} \frac{\mathcal{H}_d^k(dx)}{\max(|g(x)|, |\partial_i g(x)|, i \in I_x(W))^k},$$

and $I_0(g) = \#\text{corners}(W)$. For undetermined cases, put $I_k(g; W) = 0$ if $\text{Lip}(g) = 0, 1 \leq k \leq d$.

Theorem 3. *Let $W \in \mathcal{W}_d$, and $f : W \rightarrow \mathbb{R}$ be a $\mathcal{C}^{1,1}$ function. Let $F = F_\lambda(f)$ or $F = F_{\lambda+}(f)$ for some $\lambda \in \mathbb{R}$. Assume that ∇f does not vanish on ∂F . We have*

(i)

$$\#\Gamma(F \cap W) \leq \sum_{k=0}^d 2^k \kappa_k^{-1} I_k(f - \lambda; W), \quad (6)$$

where κ_k is the volume of the k -dimensional unit sphere.

(ii) If $d = 2$,

$$\#\Gamma((F \cap W)^\varepsilon) \leq C \sum_{k=0}^2 I_k(f - \lambda; W) \quad (7)$$

for some $C > 0$ not depending on f, λ , or ε .

The proof is given in Section 4.

Remark 4. Nothing ensures that right hand sides are finite in the bounds above, as it depends on how fast f drifts from the value λ . Theorem 7 gives conditions on the marginal densities of a bivariate random field so that the right hand member has finite expectation.

Remark 5. Similar results hold if f 's partial derivatives are only assumed to be Holder, i.e. if there is $\delta > 0$ and $H_i > 0, i = 1, \dots, d$, such that $\|\partial_i f(x) - \partial_i f(y)\| \leq H_i \|x - y\|^\delta$ for x, y such that $[x, y] \subset W$. Namely, we have to change constants and replace the exponent k on the max by an exponent $k\delta$. We don't treat such cases here because, as noted at the beginning of Section 2, if the partial derivatives are not Lipschitz, the upper level set is not regular enough to compute the Euler characteristic from the bicovariogram, but the proof is exactly the same as for the $\mathcal{C}^{1,1}$ case.

Remark 6. Calling B the right hand term of (7) and noticing that $F_{\lambda+}(f)^c$ is an upper level set of $-f$, an easy reasoning yields (see [18, Remark 1.17])

$$|\chi((F_\lambda(f) \cap W)^\varepsilon)| \leq 2B.$$

3 Mean Euler characteristic of random excursions

To avoid measurability issues, we call here \mathcal{C}^1 random field over a set $\Omega \subseteq \mathbb{R}^d$ the data of a family of random variables $\{f(x); x \in \Omega\}$, such that in each point $x \in \Omega$, the variables

$$\partial_i f(x) := \lim_{s \rightarrow 0} \frac{f(x + s\mathbf{u}_i) - f(x)}{s}, i = 1, 2,$$

exist a.s., and the fields $(\partial_i f(x), x \in \Omega), i = 1, \dots, d$, are a.s. continuous. See [1, 6] for a discussion on the regularity properties of random fields. Say that the random field is $\mathcal{C}^{1,1}$ if the partial derivatives are a.s. locally Lipschitz.

Many sets of conditions allowing to take the expectation in (5) can be derived from Theorem 3, we give below a compromise between optimality and compactness. Given a random closed set F , call $\text{supp}(F)$ the smallest compact set K satisfying $F \subset K$ a.s. See [22] for a formal introduction to the theory of random closed sets.

Theorem 7. Let $W \in \mathcal{W}_d$, and f be a $\mathcal{C}^{1,1}$ random field on W , $\lambda \in \mathbb{R}$ such that, with $F = F_\lambda(f)$, $\text{supp}(F) \cap W$ is bounded. Assume that the following conditions are satisfied:

(i) For some $\kappa > 0, \alpha > 1$, for $1 \leq k \leq d, x \in \partial_k W, I \subset \mathcal{I}_k$, the density of the random $(k+1)$ -tuple $(f(x) - \lambda, \partial_i f(x), i \in I)$ satisfies

$$\mathbf{P}(|f(x) - \lambda| \leq \varepsilon, |\partial_i f(x)| \leq \varepsilon, i \in I) \leq \kappa \varepsilon^{\alpha k}, \varepsilon > 0,$$

(ii) for some $p > d\alpha(\alpha - 1)^{-1}$,

$$\mathbf{E}[\text{Lip}(f)^p] < \infty, \mathbf{E}[\text{Lip}(\partial_i f)^p] < \infty, i = 1, \dots, d.$$

Then $\mathbf{E}\#\Gamma(F \cap W) + \mathbf{E}\#\Gamma((F \cap W)^c) < \infty$ and f is a.s. regular within W at level λ . In the context $d = 2$, F is a.s. admissible, and (1)-(2) hold with the current W .

Remark 8. In the case where the $\text{Lip}(f), \text{Lip}(\partial_i f), i = 1, \dots, d$ have a finite moment of order $> d(d+1)$, the hypotheses are satisfied if for instance $(f(x) - \lambda, \partial_i f(x), 1 \leq i \leq d)$ has a uniformly bounded density, in which case $\alpha = (d+1)/d$ is suitable. If $\alpha < (d+1)/d$, i.e. if the density is unbounded around 0, it requires higher moments for the Lipschitz constants.

The proof is deferred to Section 4. We give an explicit expression in the case where f is stationary. Boundary terms involve the perimeter of F , so we introduce the related notation below. Note \mathcal{C}_c^1 the class of compactly supported functions on \mathbb{R}^2 . For a measurable set A , and $\mathbf{u} \in \mathcal{S}^1$, the unit circle in \mathbb{R}^2 , define the variational perimeter of A in direction \mathbf{u} by

$$\text{Per}_{\mathbf{u}}(A) = \sup_{\varphi \in \mathcal{C}_c^1: \|\varphi(x)\| \leq 1} \int_A \langle \nabla \varphi(x), \mathbf{u} \rangle dx,$$

and the $\|\cdot\|_{\infty}$ -perimeter

$$\text{Per}_{\infty}(A) = \text{Per}_{\mathbf{u}_1}(A) + \text{Per}_{\mathbf{u}_2}(A),$$

named so because it is the analogue of the classical perimeter when the Euclidean norm is replaced by the $\|\cdot\|_{\infty}$ -norm, see [14].

Theorem 9. Let f be a $\mathcal{C}^{1,1}$ stationary random field, $\lambda \in \mathbb{R}$, and $W \in \mathcal{W}_2$ bounded. Assume that $(f(0), \partial_1 f(0), \partial_2 f(0))$ has a bounded density, and that there is $p > 6$ such that

$$\mathbf{E}[\text{Lip}(f; W)^p] < \infty, \mathbf{E}[\text{Lip}(\partial_i f; W)^p] < \infty, i = 1, 2.$$

Then the following limits exist:

$$\begin{aligned} \overline{\chi}(f) &:= \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} [\mathbf{E}\delta^{\varepsilon}(0, f, \lambda) - \mathbf{E}\delta^{-\varepsilon}(0, -f, -\lambda)] \\ \overline{\text{Per}}_{\mathbf{u}_i}(f) &:= \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \mathbf{P}(f(0) \geq \lambda, f(\varepsilon \mathbf{u}_i) < \lambda), i = 1, 2, \\ \overline{\text{Vol}}(f) &:= \mathbf{P}(f(0) \geq \lambda), \end{aligned}$$

and we have, with $\overline{\text{Per}}_{\infty} = \overline{\text{Per}}_{\mathbf{u}_1} + \overline{\text{Per}}_{\mathbf{u}_2}$,

$$\begin{aligned} \mathbf{E}\chi(F \cap W) &= \text{Vol}(W)\overline{\chi}(F) + \frac{1}{4}(\text{Per}_{\mathbf{u}_2}(W)\overline{\text{Per}}_{\mathbf{u}_1}(F) + \text{Per}_{\mathbf{u}_1}(W)\overline{\text{Per}}_{\mathbf{u}_2}(F)) \\ &\quad + \chi(W)\overline{\text{Vol}}(F) \end{aligned} \tag{8}$$

$$\mathbf{E}\text{Per}_{\infty}(F \cap W) = \text{Vol}(W)\overline{\text{Per}}_{\infty}(F) + \text{Per}_{\infty}(W)\overline{\text{Vol}}(F) \tag{9}$$

$$\mathbf{E}\text{Vol}(F \cap W) = \text{Vol}(W)\overline{\text{Vol}}(F). \tag{10}$$

The proof of Theorem 7 establishes that the expectations contained in [18, (12)] are finite. Therefore the result above is a consequence of that proof and [18, Proposition 2.1].

3.1 Gaussian level sets

Let $(f(x), x \in W)$ be a Gaussian field on some $W \in \mathcal{W}_d$. We assume throughout the section for simplification that each variable $f(x), x \in W$, is centred and has variance 1. Let the covariance function be defined by

$$\sigma(x, y) = \mathbf{E}f(x)f(y), \quad x, y \in W.$$

The book [1] gives some background on Gaussian fields and their regularity. Theorem 2.2.2 states that if the derivative $\partial^2\sigma(x, y)/\partial x_{[i]}\partial y_{[i]}$ exists and is finite at each point $(x, x), x \in W$, the limits

$$\partial_i f(x) := \lim_{\varepsilon \rightarrow 0} \frac{f(x + \varepsilon \mathbf{u}_i) - f(x)}{\varepsilon}, \quad 1 \leq i \leq d, x \in W,$$

exist in the L^2 sense, and they form a Gaussian field. Also, the covariance function of $\partial_i f$ is $(x, y) \mapsto \partial^2\sigma(x, y)/\partial x_{[i]}\partial y_{[i]}, i = 1, \dots, d$.

We are interested here in the case where f is of class $\mathcal{C}^{1,1}$. For W bounded, since $\|\partial_i f\| := \sup_{x \in W} |\partial_i f(x)|$ is a.s. finite, $\|\partial_i f\|$ has finite moments of every order, see (2.1.4) in [6].

Theorem 10. *Let f be a $\mathcal{C}^{1,1}$ Gaussian field on a bounded closed set $W \in \mathcal{W}_d$. Assume that for $x \in W$, $(f(x), \partial_i f(x), i = 1, \dots, d)$ is non-degenerate, and that for some $p > d(d+1)$, for $i = 1, 2$,*

$$\mathbf{E} \text{Lip}(\partial_i f)^p < \infty.$$

Then for any $\lambda \in \mathbb{R}$, $F = F_\lambda(f)$ satisfies the conclusions of Theorem 7.

Proof. Put for notational convenience $f^{(0)} := f, f^{(i)} = \partial_i f, i = 1, \dots, d$. We have for $i, j \in \{0, \dots, d\}$,

$$\begin{aligned} & |\mathbf{E} f^{(i)}(x) f^{(j)}(x) - f^{(i)}(y) f^{(j)}(y)| \\ & \leq \left| \mathbf{E} \left[\left(f^{(i)}(x) - f^{(i)}(y) \right) f^{(j)}(x) \right] \right| + \left| \mathbf{E} \left[f^{(i)}(y) \left(f^{(j)}(x) - f^{(j)}(y) \right) \right] \right| \\ & \leq \mathbf{E} \sup_W |f^{(j)}| \text{Lip}(f^{(i)}) \|x - y\| + \mathbf{E} \sup_W |f^{(i)}| \text{Lip}(f^{(j)}) \|x - y\|, \end{aligned}$$

which yields that the covariance function with values in the space of $(d+1) \times (d+1)$ matrices,

$$x \mapsto \Sigma(x) := \text{cov}(f(x), \partial_i f(x), 1 \leq i \leq d)$$

is Lipschitz on W . In particular, since $\det(\Sigma(x))$ does not vanish on W , it is bounded from below by some $c > 0$, whence the density of $(f(x), \partial_1 f(x), \partial_2 f(x)), x \in W$, is uniformly bounded by $(2\pi)^{-d/2} c^{-1/2}$, and assumption (i) from Theorem 7 is satisfied with $\alpha = (d+1)/d$. Since $\text{Lip}(f) = \sup_W \|\nabla f\|$ and $\mathbf{E} \|\nabla f\|^q < \infty$ for any $q \geq 1$,

$$\mathbf{E} \text{Lip}(f)^q < \infty.$$

Hence indeed Theorem 7-(ii) is satisfied. \square

Let us give the mean Euler characteristic in dimension 2 under the simplifying assumptions that the law of f is invariant under translations and rotations of \mathbb{R}^2 . In combination with the constant variance assumption, it eases certain computations. This implies for instance that in every $x \in \mathbb{R}^2$, $f(x), \partial_1 f(x)$ and $\partial_2 f(x)$ are independent, see for instance [6] Section 5.6 and (5.7.3). A nice feature of the following result is that the hypotheses on f match the result, in the sense that the mean Euler characteristic only depends on the properties of ∇f , and that the number of times f should be continuously differentiable is only 1.

Theorem 11. *Let $f = (f(x); x \in \mathbb{R}^2)$ be a $\mathcal{C}^{1,1}$ stationary isotropic centred Gaussian field on \mathbb{R}^2 with constant variance equal to 1, $\lambda \in \mathbb{R}$, $F = \{x : f(x) \geq \lambda\}$, and let $W \in \mathcal{W}_2$ bounded. Assume that*

$$\mathbf{E} \text{Lip}(\partial_1 f, W)^p < \infty$$

for some $p > 6$. Define $\mu = \mathbf{E}\partial_1 f(0)^2$, and $\Phi(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\lambda}^{\infty} \exp(-t^2/2) dt$. Then

$$\mathbf{E}\text{Vol}(F \cap W) = \text{Vol}(W)\Phi(\lambda), \quad (11)$$

$$\mathbf{E}\text{Per}_{\infty}(F \cap W) = \text{Vol}(W)2\frac{\sqrt{\mu}}{\pi} \exp(-\lambda^2/2) + \text{Per}_{\infty}(W)\Phi(\lambda), \quad (12)$$

$$\mathbf{E}\chi(F \cap W) = \left(\text{Vol}(W) \frac{\mu\lambda}{(2\pi)^{3/2}} + \text{Per}_{\infty}(W) \frac{\sqrt{\mu}}{4\pi} \right) e^{-\lambda^2/2} + \frac{1}{\sqrt{2\pi}} \Phi(\lambda)\chi(W). \quad (13)$$

Remark 12. If W is a square, the relation (13) coincides with [6, (11.7.14)], obtained under stronger requirements for f .

Proof. (10) immediately yields (11). To prove (13), first remark that the stationarity of the field and the fact that it is not constant a.s. entail that $(f(0), \partial_1 f(0), \partial_2 f(0)) \stackrel{(d)}{=} (f(x), \partial_1 f(x), \partial_2 f(x)), x \in \mathbb{R}^2$ is non-degenerate. Let us show

$$\bar{\chi}(F) = \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} \mathbf{E} [\delta^{\varepsilon}(0, f, \lambda) - \delta^{-\varepsilon}(0, -f, -\lambda)] = \frac{\mu\lambda \exp(-\lambda^2/2)}{(2\pi)^{3/2}}. \quad (14)$$

Fix $\varepsilon > 0$. Let M_{ε} be the 3×3 covariance matrix of $(f(0), f(\varepsilon \mathbf{u}_1), f(\varepsilon \mathbf{u}_2))$. Straightforward computations show that $\det(M_{\varepsilon}) = \varepsilon^4 \mu^2 + o(\varepsilon^4)$, and

$$M_{\varepsilon}^{-1} = \frac{1}{\det(M_{\varepsilon})} (\varepsilon^2 W_{\varepsilon} + \varepsilon^4 D_{\varepsilon}), \quad (15)$$

where the sum of each line and each column of W_{ε} is 0, for $\varepsilon > 0$, and

$$W_{\varepsilon} \rightarrow W := \mu \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, D_{\varepsilon} \rightarrow D := \frac{\mu^2}{4} \begin{pmatrix} -4 & 2 & 2 \\ 2 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix},$$

as $\varepsilon \rightarrow 0$. Denote by $\mathbf{1}$ the vector $(1, 1, 1)$, and let $\Lambda = \lambda \mathbf{1}$, $Q = \{(t, s, z) : t \geq 0, s < 0, z < 0\}$. Denote by A' the transpose of a matrix (or a vector) A . We have

$$\mathbf{E}\delta^{\varepsilon}(0, f, \lambda) = \frac{1}{\sqrt{(2\pi)^3 \det(M_{\varepsilon})}} \int_{Q+\Lambda} \exp\left(-\frac{1}{2}(t, s, z)' M_{\varepsilon}^{-1}(t, s, z)\right) dt ds dz$$

and by isotropy and symmetry, for $\lambda \in \mathbb{R}$,

$$\mathbf{E}\delta^{-\varepsilon}(0, -f, -\lambda) = \mathbf{E}\delta^{\varepsilon}(0, -f, -\lambda) = \mathbf{E}\delta^{\varepsilon}(0, f, -\lambda).$$

Therefore, (8) yields that $\bar{\chi}(F) = \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} (\mathbf{E}\delta^{\varepsilon}(0, f, \lambda) - \mathbf{E}\delta^{\varepsilon}(0, f, -\lambda))$. Let $X = (t, s, z) \in Q$, $Y = \frac{\varepsilon}{\sqrt{\det(M_{\varepsilon})}} X$. Since ΛW_{ε} and $W_{\varepsilon} \Lambda$ are 0, we have

$$\begin{aligned} & (X + \Lambda)' M_{\varepsilon}^{-1} (X + \Lambda) \\ &= \underbrace{Y'(W_{\varepsilon} + \varepsilon^2 D_{\varepsilon})Y}_{=: \gamma_{\varepsilon}(Y)} + \frac{2\varepsilon^3}{\sqrt{\det(M_{\varepsilon})}} Y' D_{\varepsilon} \Lambda + \frac{\varepsilon^4}{\det(M_{\varepsilon})} \Lambda' D_{\varepsilon} \Lambda \\ \mathbf{E}\delta^{\varepsilon}(0, f, \lambda) &= \frac{\left(\frac{\sqrt{\det(M_{\varepsilon})}}{\varepsilon}\right)^3 \exp\left(-\lambda^2 \frac{\varepsilon^4}{2 \det(M_{\varepsilon})} \mathbf{1}' D_{\varepsilon} \mathbf{1}\right)}{\sqrt{(2\pi)^3 \det(M_{\varepsilon})}} \int_Q \exp\left(-\frac{1}{2} \gamma_{\varepsilon}(Y) - \varepsilon^3 \frac{Y' D_{\varepsilon} \Lambda}{\sqrt{\det(M_{\varepsilon})}}\right) dY \end{aligned}$$

and, for some $\theta = \theta(\varepsilon, Y, \lambda) \in [-\varepsilon^3 \det(M_\varepsilon)^{-1/2}, \varepsilon^3 \det(M_\varepsilon)^{-1/2}]$,

$$\exp\left(-\frac{\varepsilon^3 Y' D_\varepsilon \Lambda}{\sqrt{\det(M_\varepsilon)}}\right) - \exp\left(\frac{\varepsilon^3 Y' D_\varepsilon \Lambda}{\sqrt{\det(M_\varepsilon)}}\right) = -2 \frac{\varepsilon^3 Y' D_\varepsilon \Lambda}{\sqrt{\det(M_\varepsilon)}} \exp(\theta \Lambda' D_\varepsilon Y)$$

Therefore, as $\varepsilon \rightarrow 0$, $\varepsilon^{-2}(\mathbf{E}\delta^\varepsilon(0, f, \lambda) - \mathbf{E}\delta^\varepsilon(0, f, -\lambda))$ is equivalent to

$$\varepsilon^{-2} \frac{\exp(-\lambda^2/2) \det(M_\varepsilon)}{\varepsilon^3 \sqrt{(2\pi)^3}} \int_Q \exp(-\gamma_\varepsilon(Y)/2) \frac{-2\varepsilon^3 Y' D_\varepsilon \Lambda}{\sqrt{\det(M_\varepsilon)}} \exp(\theta Y' D_\varepsilon \Lambda) dY \quad (16)$$

$$\sim \frac{-\exp(-\lambda^2/2)\mu}{\sqrt{2\pi^3}} \int_Q \exp(-\gamma_\varepsilon(Y)/2) Y' D_\varepsilon \Lambda \exp(\theta Y' D_\varepsilon \Lambda) dY. \quad (17)$$

For $Y = (x, y, z) \in Q$, we have

$$\frac{Y' W Y}{\mu} = 2x^2 + y^2 + z^2 - 2xy - 2xz = 2x^2 + y^2 + z^2 + 2|xy| + 2|xz| \geq \|Y\|^2.$$

Since $W_\varepsilon + \varepsilon^2 D_\varepsilon \rightarrow W$ as $\varepsilon \rightarrow 0$, $\gamma_\varepsilon(Y) \geq \mu \|Y\|^2/2$ for ε sufficiently small, uniformly in $Y \in Q$. This yields a clear domination and Lebesgue's theorem gives $\bar{\chi}(F) = -\mu \exp(-\lambda^2/2) I (2\pi^3)^{-1/2}$ with $I = \int_Q \exp(-\frac{1}{2} Y' W Y) Y' D \Lambda dY = 2\lambda J$ where

$$J = \int_Q \exp(-(2t^2 + s^2 + z^2 - 2ts - 2tz))(s + z) dt ds dz = -1/4$$

with the change of variables $u = t - s, v = t - z, w = t$. Reporting in (16) proves (14). The computation of $\overline{\text{Per}}_\infty(F)$ is similar and simpler and is omitted here. \square

4 Proofs

We need some general notation before turning to the proofs. For a set $A \subset \mathbb{R}^d$, and $r > 0$, call $A^{\oplus r} = \{x : d(x, A) \leq r\}$, where $d(x, A)$ is the Euclidean distance between x and A in \mathbb{R}^d . We also note, for $x \in \mathbb{R}^d$, $(x_{[1]}, \dots, x_{[d]})$ its coordinates in the canonical basis. If φ is an application with values in \mathbb{R}^d , denote its coordinates by $(\varphi(\cdot)_{[1]}, \dots, \varphi(\cdot)_{[d]})$.

4.1 Proof of Theorem 3

(i) Assume wlog $\lambda = 0$ in all the proof. Recall that $\Gamma(F \cap W)$ is the collection of bounded connected components of $F \cap W$. For $0 \leq k \leq d$, note $\Gamma_k(F \cap W)$ the elements of W that hit $\partial_k W$, and define recursively $\Gamma_k^+(F; W) = \Gamma_k(F \cap W) \setminus \Gamma_{k-1}^+(F; W)$, $0 \leq k \leq d$, with the convention $\Gamma_{-1}(F; W) = \emptyset$.

Let $1 \leq k \leq d$, $C' \in \Gamma_k^+(F; W)$, C' arbitrarily chosen in $\Gamma(C' \cap \partial_k W)$. Since C' does not touch $\partial_{k-1} W$, it is included in the relative interior of $\partial_k W$ within the affine k -dimensional tangent space to W that contains C' . Let $I \in \mathcal{I}_k$ such that for $x \in C'$, $I_x(W) = I$. Let $x_C \in \text{cl}(C')$ such that $f(x_C) = \sup_{C'} f$. Since $f \geq \lambda$ on C' , and the gradient ∇f does not vanish on $\partial C'$, there is a neighbourhood of x_C that does not touch any other connected component of F , whence x_C is a local maximum of f within $\partial_k W$. The Lagrange multipliers Theorem yields that $\partial_i f(x_C) = 0$ for $i \in I$. Call r_C the maximal radius such that $B_C := (B(x_C, r_C) \cap \partial_k W) \subset C'$. Since B_C touches ∂F , f has a zero on B_C . It follows that $|f(x)| \leq 2\text{Lip}(f)r_C$ and $|\partial_i f(x)| \leq \text{Lip}(\partial_i f)r_C$

for $x \in B_C, i \in I$. Call I' the set I from which have been removed indexes $i \in I$ such that $\text{Lip}(\partial_i f) = 0$, and hence $\partial_i f = 0$ on B_C . Define

$$M(x) = \max(|f(x)|/2\text{Lip}(f), |\partial_i f(x)|/\text{Lip}(\partial_i f), i \in I') \in \mathbb{R}_+, x \in \partial_k W.$$

We have

$$\begin{aligned} 1 &= \frac{1}{\mathcal{H}_d^k(B_C)} \int_{B_C} \mathbf{1}_{\{M(x) \leq r_C\}} \mathcal{H}_d^k(dx) = \kappa_k^{-1} r_C^{-k} \int_{B_C} \mathbf{1}_{\{r_C^{-1} \leq M(x)^{-1}\}} \mathcal{H}_d^k(dx) \\ &\leq \kappa_k^{-1} \int_{B_C} M(x)^{-k} \mathcal{H}_d^k(dx). \end{aligned}$$

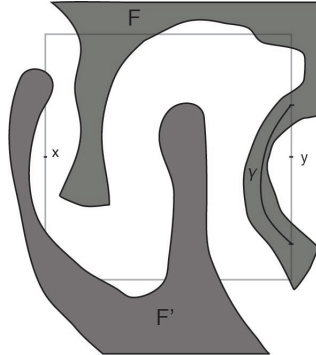
Noticing that

$$M(x) \geq \frac{\max(|f(x)|, |\partial_i f(x)|, i \in I')}{\max(\text{Lip}(f), \text{Lip}(\partial_i f), i \in I')} = \frac{\max(|f(x)|, |\partial_i f(x)|, i \in I)}{\max(\text{Lip}(f), \text{Lip}(\partial_i f), i \in I)},$$

and since the B_C are pairwise disjoint, summing over all the $C \in \Gamma_k^+(F; W)$ and $k \in \{1, \dots, d\}$ gives the result (with $\#\Gamma_0(F; W) \leq \#\text{corners}(W)$).

(ii) Theorem 1.16 in the companion paper [18], in the context $d = 2$, features a bound on $\chi((F \cap W)^\varepsilon)$ in terms of the number of occurrences of local configurations called *entanglement points of F* . Roughly, an entanglement point occurs when two close points of F are connected by a tight path in F . As a consequence, if F is sampled with an insufficiently high resolution in this region, the connecting path is not detected, and F looks locally disconnected. For formal definitions, for $x, y \in \varepsilon\mathbb{Z}^2$ at distance ε , introduce $\mathbb{P}_{x,y}$ the closed square with side-length ε such that x and y are the midpoints of two opposite sides. Denote $\mathbb{P}'_{x,y} = \partial\mathbb{P}_{x,y} \setminus \{x, y\}$, which has two connected components. Then $\{x, y\}$ is an *entanglement pair of points of F* if $x, y \notin F$ and $(\mathbb{P}'_{x,y} \cup F) \cap \mathbb{P}_{x,y}$ is connected. We call $\mathcal{N}_\varepsilon(F)$ the family of such pairs of points. See Figure 1 for an example.

Figure 1: *Entanglement point*: In this example, $\{x, y\} \in \mathcal{N}_\varepsilon(F)$ because the two connected components of $\mathbb{P}'_{x,y}$, in lighter grey, are connected through $\gamma \subseteq (F \cap \mathbb{P}_{x,y})$. We don't have $\{x, y\} \in \mathcal{N}_\varepsilon(F')$.



We introduce the notation $(x, y) = \varepsilon\mathbb{Z}^2 \cap [x, y] \setminus \{x, y\}$, for $x, y \in \varepsilon\mathbb{Z}^2$. To account for boundary effects, we also consider grid points $x, y \in \varepsilon\mathbb{Z}^2 \cap W \cap F$, on the same line or column of $\varepsilon\mathbb{Z}^2$, such that

- x, y are within distance ε from one of the edges of W (the same edge for x and y)

- $\langle x, y \rangle \neq \emptyset$
- $\langle x, y \rangle \subseteq \varepsilon \mathbb{Z}^2 \cap F^c \cap F^{\oplus \varepsilon}$.

The family of such pairs of points $\{x, y\}$ is noted $\mathcal{N}'_\varepsilon(F; W)$.

It is proved in [18, Theorem 1.16] that

$$\#\Gamma((F \cap W)^\varepsilon) \leq 2\#(\mathcal{N}_\varepsilon(F) \cap W^{\oplus \varepsilon}) + 2\#\mathcal{N}'_\varepsilon(F, W) + \#\Gamma(F \cap W) + 2\#\text{corners}(W). \quad (18)$$

It therefore only remains to bound $\#(\mathcal{N}_\varepsilon(F) \cap W^{\oplus \varepsilon})$ and $\#\mathcal{N}'_\varepsilon(F, W)$ to achieve (7). For $m \geq 1$ and a function $g : A \subseteq \mathbb{R}^m \rightarrow \mathbb{R}$, introduce the continuity modulus

$$\omega(g, A) = \sup_{x \neq y \in A} |g(x) - g(y)|.$$

The bound will follow from the following lemma.

Lemma 13. (i) For $\{x, y\} \in \mathcal{N}_\varepsilon(F)$, we have for some $i \in \{1, 2\}$, and i' such that $\{i, i'\} = \{1, 2\}$,

$$\begin{aligned} |f(x)| &\leq \omega(f, [x, y]) \leq \text{Lip}(f)\varepsilon \\ |\partial_i f(x)| &\leq \omega(\partial_i f, [x, y]) \leq \text{Lip}(\partial_i f)\varepsilon \\ |\partial_{i'} f(x)| &\leq 2\omega(\partial_i f, P_{x,y}) + \omega(\partial_{i'} f, P_{x,y}) \leq \sqrt{2}\varepsilon(2\text{Lip}(\partial_i f) + \text{Lip}(\partial_{i'} f)), \end{aligned}$$

and idem for y .

(ii) For $x, y \in \mathcal{N}'_\varepsilon(F, W)$, there is $z = z(x, y) \in \langle x, y \rangle$, $i \in \{1, 2\}$, such that

$$\begin{aligned} |f(z)| &\leq \text{Lip}(f)\varepsilon \\ |\partial_i f(z)| &\leq \text{Lip}(\partial_i f)\varepsilon. \end{aligned}$$

The lemma is proved later for convenience. To obtain the integral upper bounds from (7), note that there is $c > 0$ such that for $\varepsilon > 0$ sufficiently small, for every $x, y \in W$, neighbours in $\varepsilon \mathbb{Z}^2$, $\text{Vol}((B(x, \varepsilon) \cup B(y, \varepsilon)) \cap W) \geq \varepsilon^2/c$. Define the possibly infinite quantity, for $z \in W$,

$$M(z) = \max(|f(z)|/2\text{Lip}(f), |\partial_i f(z)|/2\text{Lip}(\partial_i f), |\partial_{i'} f(z)|/(2\sqrt{2}(\text{Lip}(\partial_1 f) + \text{Lip}(\partial_2 f))))$$

Treating undetermined cases 0/0 can be done like in the proof of (i). Lemma 13 then yields

$$\begin{aligned} \mathcal{N}'_\varepsilon(F) &\leq \sum_{x, y \in \mathcal{N}_\varepsilon(F)} \mathbf{1}_{\{\text{for } z \in B(x, \varepsilon) \cup B(y, \varepsilon) \cap W, M(z) \leq \varepsilon\}} \\ &\leq \sum_{x, y \in \mathcal{N}_\varepsilon(F)} c\varepsilon^{-2} \int_{(B(x, \varepsilon) \cup B(y, \varepsilon)) \cap W} \mathbf{1}_{\{M(z)^{-1} \geq \varepsilon^{-1}\}} dz \\ &\leq 4c \int_W M(z)^{-2} dz \leq c' I_2(f), \end{aligned}$$

for some $c' > 0$, because for every $z \in W$ there are at most 4 couples $\{x, y\} \in \mathcal{N}_\varepsilon(F)$ such that $z \in B(x, \varepsilon) \cup B(y, \varepsilon)$.

Now, given $w \in \partial W$, there can be at most 3 pairs $\{x, y\} \in \mathcal{N}'_\varepsilon(F)$ such that w is on the closest edge of W parallel to $[x, y]$ and $z = z(x, y)$ (defined in Lemma 13) is within distance 3ε from w , and in this case $|f(w)| \leq 4\text{Lip}(f)\varepsilon$ and $|\partial_i f(w)| \leq 4\text{Lip}(\partial_i f)\varepsilon$ for some $i \in \{1, 2\}$. We

have $\mathcal{H}_2^1(B(z, 3\varepsilon) \cap \partial W) \geq \varepsilon$, because z is within distance 2ε from a segment of ∂W parallel to $[x, y]$. It follows that, with $M_i(w) = \max(|f(w)|/4\text{Lip}(f), |\partial_i f(w)|/4\text{Lip}(\partial_i f))$

$$\begin{aligned} \#\mathcal{N}'_\varepsilon(F, W) &\leq \sum_{x, y \in \mathcal{N}'_\varepsilon(F, W)} \sum_{i=1}^2 \mathbf{1}_{\{\text{for } w \in B(z, 3\varepsilon) \cap \partial W, M_i(w) \leq \varepsilon\}} \\ &\leq \sum_{i=1}^2 \sum_{x, y \in \mathcal{N}'_\varepsilon(F)} \frac{1}{\varepsilon} \int_{\partial W \cap B(z, 3\varepsilon)} \mathbf{1}_{\{M_i(w)^{-1} \geq \varepsilon^{-1}\}} \mathcal{H}_2^1(dw) \\ &\leq \sum_{i=1}^2 3 \int_{\partial W} M_i(w)^{-1} \mathcal{H}_2^1(dw) = 24I_1(f; 0). \end{aligned}$$

Proof of Lemma 13. (i) Let $x, y \in \mathcal{N}_\varepsilon(F)$. The definition of $\mathcal{N}_\varepsilon(F)$ yields a connected path $\gamma \subseteq (F \cap \mathbb{P}_{x, y})$ going through some $z \in [x, y]$ and connecting the two connected components of $\mathbb{P}'_{x, y}$. Since $f(x) \geq 0$ and $f(z) \leq 0$, there is a point z' of $[x, y]$ satisfying $f(z') = 0$, hence $|f(x)| \leq \omega(f, [x, y])$. Note for later that for $t \in \mathbb{P}_{x, y}$ $|f(t)| \leq \omega(f, \mathbb{P}_{x, y}) \leq \text{Lip}(f)\sqrt{2\varepsilon}$.

We assume without loss of generality that $[x, y]$ is horizontal. Let $[z', z'']$ be the (also horizontal) connected component of $F \cap [x, y]$ containing z . After choosing a direction on $[x, y]$, z' and z'' are entry and exit points for F , and their normal vectors $\mathbf{n}_F(z')$, $\mathbf{n}_F(z'')$ point towards the outside of F . Therefore they satisfy $\mathbf{n}_F(z')_{[1]} \mathbf{n}_F(z'')_{[1]} \leq 0$, and so $\partial_1 f(z') \partial_1 f(z'') \leq 0$. This gives us by continuity the existence of a point $w \in [x, y]$ such that $0 = \partial_1 f(w)$, whence $|\partial_1 f(x)| \leq \omega(\partial_1 f, [x, y])$. Note for later that $|\partial_1 f(t)| \leq \omega(\partial_1 f, \mathbb{P}_{x, y})$ on $\mathbb{P}_{x, y}$. If $[x, y]$ is vertical, $\partial_2 f$ verifies the inequality instead. Let us keep assuming that $[x, y]$ is horizontal for the sequel of the proof.

We claim that $|\partial_2 f(x)| \leq 2\omega(\partial_1 f, \mathbb{P}_{x, y}) + \omega(\partial_2 f, \mathbb{P}_{x, y})$, and consider two cases to prove it.

- First case $\partial_2 f(z') \partial_2 f(z'') \leq 0$, and by continuity we have $w \in [x, y]$ such that $0 = \partial_2 f(w)$, whence $|\partial_2 f(\cdot)| \leq \omega(\partial_2 f, \mathbb{P}_{x, y})$ on the whole pixel $\mathbb{P}_{x, y}$. The desired inequality follows.
- Second case $\partial_2 f(z') > 0, \partial_2 f(z'') > 0$ (equivalent treatment if they are both < 0). Assume for instance that z' is the leftmost point, and that $|\partial_2 f(x)| > 2\omega(\partial_1 f, \mathbb{P}_{x, y}) + \omega(\partial_2 f, \mathbb{P}_{x, y})$, otherwise the claim is proved. It implies in particular that $|\partial_2 f(\cdot)| > 2\omega(\partial_1 f, \mathbb{P}_{x, y})$ on the whole pixel $\mathbb{P}_{x, y}$. Since $|\partial_1 f(\cdot)| \leq \omega(\partial_1 f, \mathbb{P}_{x, y})$ on $\mathbb{P}_{x, y}$, the implicit function theorem yields a function φ (resp. ψ): $[z'_{[1]}, z''_{[1]}] \rightarrow \mathbb{R}$ such that $|\varphi'| \leq 1/2$, (resp. $|\psi'| \leq 1/2$), $\varphi([z'_{[1]}, z''_{[1]}]) \subset (z'_{[2]} + (-\varepsilon/2, \varepsilon/2))$, (resp. $\psi([z'_{[1]}, z''_{[1]}]) \subset (z'_{[2]} + (-\varepsilon/2, \varepsilon/2))$) and the graph of φ (resp. ψ) coincides with $\partial F \cap ([z'_{[1]}, z''_{[1]}] \times (z'_{[2]} + [-\varepsilon/2, \varepsilon/2]))$. In particular, $\varphi = \psi$, and its graph cannot touch the upper half of $\partial \mathbb{P}_{x, y}$. Applying this to every maximal segment $[z', z''] \subset (F \cap [x, y])$, we see that every connected component of F touching $[x, y]$, and hence γ , cannot meet the upper half of $\mathbb{P}_{x, y}$. In particular, it contradicts the definition of $\mathcal{N}_\varepsilon(F)$, whence indeed the assumption is proved by contradiction.

We indeed proved that $|\partial_2 f(x)| \leq 2\omega_1(\partial_2 f, \mathbb{P}_{x, y}) + \omega_2(\partial_2 f, \mathbb{P}_{x, y})$.

(ii) Let now $\{x, y\}$ be an element of $\mathcal{N}'_\varepsilon(f, W)$. We know that $(x, y) \cap F^c \neq \emptyset$. Let $[z', z''] \subset [x, y]$ a connected component of $F^c \cap [x, y]$. If $[z', z'']$ is, say, horizontal, since $\mathbf{n}_F(\cdot)_{[1]}$ changes sign between z' and z'' , so does $\partial_1 f$, and by continuity there is $w \in [z', z'']$ where $\partial_1 f(w) = 0$. Calling z the closest point from w in (x, y) , $\|z - w\| \leq \varepsilon$, and by definition of $\mathcal{N}'_\varepsilon(F, W)$, z is also at distance ε from $\partial F = \{f = 0\}$. It follows that $|\partial_1 f(z)| \leq \text{Lip}(\partial_1 f)\varepsilon, |f(z)| \leq \text{Lip}(f)\varepsilon$. \square

4.2 Proof of Theorem 7

Assume without loss of generality $\lambda = 0$. Let us prove that F is a.s. regular within W at level λ . For $0 \leq k \leq d, I \in \mathcal{I}_k$, note $\tilde{\partial}_k W = \partial_k W \cap \text{supp}(F)$, and define

$$\theta_{k,I} = \{x \in \tilde{\partial}_k W : f(x) = 0, \partial_i f(x) = 0, i \in I\}.$$

There is $c > 0$ such that for $\varepsilon > 0$ sufficiently small, for all $x \in \partial_k W$, $\mathcal{H}_d^k(\partial_k W \cap B(x, \varepsilon)) \geq \varepsilon^k / c$. Define $M(y) = \max(|f(y)|/\text{Lip}(f), |\partial_i f(y)|/\text{Lip}(\partial_i f), i \in I)$, with undetermined cases treated like in the proof of Theorem 3-(i). For $x \in \theta_{k,I}, y \in B(x, \varepsilon) \cap \partial_k W$, $M(y) \leq \varepsilon$. Therefore, for $\eta > 0$,

$$\begin{aligned} \#\theta_{k,I} &\leq \liminf_{\varepsilon \rightarrow 0} \int_{\tilde{\partial}_k W} \frac{1}{\mathcal{H}_d^k(B(x, \varepsilon))} \mathbf{1}_{\{M(y) \leq \varepsilon\}} \mathcal{H}_d^k(dy) \\ &\leq c \liminf_{\varepsilon \rightarrow 0} \varepsilon^\eta \int_{\tilde{\partial}_k W} M(y)^{-(k+\eta)} \mathcal{H}_d^k(dy). \end{aligned}$$

Fatou's lemma yields, with $L := \max(\text{Lip}(f), \text{Lip}(\partial_i f), i = 1, \dots, d), m_y = \max(|f(y)|, |\partial_i f(y)|, i \in I)$,

$$\mathbf{E} \#\theta_{k,I} \leq \liminf_{\varepsilon \rightarrow 0} \varepsilon^\eta \int_{\partial_k W} \mathbf{E} \left[\frac{L^{k+\eta}}{m_y^{k+\eta}} \right] \mathcal{H}_d^k(dy).$$

Since $p > \frac{\alpha d}{\alpha-1} \geq \frac{\alpha k}{\alpha-1}$, we can choose $q > 1$ such that $q < \alpha$ and $q'k < p$, with $q' = (1 - q^{-1})^{-1}$. In particular, if η is chosen sufficiently small, $\mathbf{E} L^{q'(k+\eta)} < \infty$ for $1 \leq k \leq d$. The fact that $\mathbf{E} \sum_{I \in \mathcal{I}_k} \#\theta_{k,I} = 0$ follows from a bound uniform in $y \in \partial_k W$ on

$$\begin{aligned} \mathbf{E} L^{q'(k+\eta)} \mathbf{E} m_y^{-q(k+\eta)} &\leq C \int_{\mathbb{R}_+} \mathbf{P}(m_y \leq t^{-1/(q(k+\eta))}) dt \\ &\leq C \int_{\mathbb{R}_+} 1 \vee t^{-\frac{\alpha k}{q(k+\eta)}} dt. \end{aligned}$$

Assume wlog that η is chosen so that $q(k+\eta) < \alpha k$, so that indeed for all $1 \leq k \leq d$, the previous bound is finite (uniformly in y).

Therefore, if $d = 2$, using (5), $\chi(F \cap W) = \lim_{\varepsilon \rightarrow 0} \chi((F \cap W)^\varepsilon)$ holds a.s. According to Theorem 3, to apply Lebesgue's theorem and show that this limit can be passed to expectations, yielding (1)-(2), it only remains to show $\mathbf{E} I_1(f; W) < \infty, \mathbf{E} I_2(f; W) < \infty$. This follows from the exact same computation as before, with $k = 1, 2$, and $\eta = 0$.

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