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ALMOST GROUP THEORY

NADJA HEMPEL

Abstract. The notion of almost centralizer and almost commutator are introduced and basic properties are established. They are used to study $\widetilde{\mathfrak{M}}_c$-groups, i. e. groups for which every descending chain of centralizers each having infinite index in its predecessor stabilizes after finitely many steps in any definable section. The Fitting subgroup of such groups is shown to be nilpotent and the nilpotency criteria of Hall is generalized to almost nilpotent $\widetilde{\mathfrak{M}}_c$-groups.

1. Introduction

Groups in which every descending chain of centralizers stabilizes after finitely many steps, so called $\mathfrak{M}_c$-groups, have been of great interest to both group and model theorist. They have been studied by Altinel and Baginski [1], Bryant [4], Bryant and Hardley [3], Derakhshan and Wagner [5], Poizat and Wagner [17]. In the field of model theory they appear naturally as definable groups in stable and o-minimal theories. Passing to groups definable in simple theories or even more general rosy theories, we obtain a weaker chain condition, namely any chain of centralizers, each having infinite index in its predecessor, stabilizes after finitely many steps. We want to study group for which any definable section satisfies this chain condition which we call $\widetilde{\mathfrak{M}}_c$-groups. Examples are (group theoretically) simple pseudo-finite groups, groups definable in the theory of perfect bounded PAC-fields, and in general groups definable in any rosy theory. A useful tool in this context is the FC-centralizer of a subgroup: For a subgroup H of a group G, the FC-centralizer contains all elements whose centralizer has finite index in H. These subgroups were introduced by Haimo in [8]. Defining a suitable notion of these objects regarding A-invariant subgroups of G and establishing their basic properties is the main part of Section 2.

From a model theoretic point of view, one particular problem we are interested in is given an abelian, nilpotent or solvable subgroup H, can one find a *definable envelope* of H, that is a definable subgroup of G containing H with the same algebraic properties. Finding definable sets around non-definable once admitting similar properties brings objects outside of the scope of model theory into the category of definable. But it is not only interesting from a purely model theoretic point of view but also an important tool for applications. In the past decades there has been remarkable progress on groups fulfilling model theoretic properties (stable, simple, dependent, NTP_2) as well as in $\mathfrak{M}_c$-groups which will ensure the existence of definable envelopes of subgroups. In Section 3 we prove the existence of definable envelopes "up to finite index" for any abelian, nilpotent or solvable subgroup of an $\widetilde{\mathfrak{M}}_c$-group.

Another object of interest is the Fitting subgroup, i. e. the group generated by all normal nilpotent subgroups. It is always normal in the ambient group. Moreover, as the product of any two normal nilpotent subgroups is again nilpotent by Fitting's Theorem, we can conclude that the Fitting subgroup is locally nilpotent. It is even

nilpotent if G is finite. On the other hand, the Fitting subgroup of an infinite group might not be nilpotent. In the case of $\mathfrak{M}_c$-groups, Bryant first showed that the Fitting subgroup of any periodic $\mathfrak{M}_c$-group is nilpotent [4]. Using model theoretic techniques, Wagner proved in [19] nilpotency of the Fitting subgroup of any group whose theory is stable and later Wagner together with Derakshan obtained nilpotency of the Fitting subgroup for arbitrary $\mathfrak{M}_c$-groups in [5]. Furthermore, it has been recently generalized by Palacín and Wagner [16] to groups type-definable in simple theories. One of the main ingredient other than the chain condition on centralizers, is that the existence of a definable envelope up to finite index for any nilpotent subgroup. As mentioned before, we establish this result for $\widetilde{\mathfrak{M}}_c$-groups in Section 3 which enables us to prove nilpotency of the Fitting subgroup for $\widetilde{\mathfrak{M}}_c$-groups.

In the last section, we study subgroups of $\widetilde{\mathfrak{M}}_c$-groups which are almost nilpotent or also called FC-nilpotent, a notion which was also introduced by Haimo in [8]: a subgroup H of G is FC-nilpotent, if there is a sequence $\{1\} = H_0 < H_1 < \cdots < H_n = H$ of normal nilpotent subgroups of G such that H_{i+1}/H_i is in the FC-center of G/H_i. We first introduce the notion of the "almost commutator" of two subgroups and establish its properties. Using definability of the almost centralizer for $\widetilde{\mathfrak{M}}_c$-groups we can express being nilpotent for $\widetilde{\mathfrak{M}}_c$-groups via the almost commutator. This enables us to generalize the nilpotency criteria of Hall (G is nilpotent if one can find a nilpotent subgroup N such that G modulo the derived subgroup of N is nilpotent as well) to almost nilpotent $\widetilde{\mathfrak{M}}_c$-groups.

2. Almost centralizers

Let us first give the original definition of an FC-centralizer and related objects given by Haimo.

Definition 2.1. Let G be a group and H, K and N be three subgroups of G such that N is normalized by H. We define:

- The *FC-centralizer* of H in K modulo N:
$$\mathrm{FC}_K(H/N) = \{k \in N_K(N) : [H : C_H(k/N)] \text{ is finite}\}$$
- Suppose that $N \leq H \leq K$. Then, the nth *FC-centralizer* of H in K modulo N is defined inductively on n as the following:
$$\begin{aligned} \mathrm{FC}_K^0(H/N) &= N \\ \mathrm{FC}_K^{n+1}(H/N) &= \mathrm{FC}_H(H/\mathrm{FC}_K^n(H/N)) \cap \bigcap_{i=0}^{n} N_K(\mathrm{FC}_K^i(H/N)) \end{aligned}$$
- The nth*FC-center* of H:
$$\mathrm{FC}_n(H) = \mathrm{FC}_H^n(H)$$

Definition 2.2. Let H and K be two arbitrary subgroups of G. We say that H is *virtually contained* in K, denoted by $H \leq_v K$ if the index of $H \cap K$ is finite in H. We say that H and K are *commensurable*, denoted by $H =_v K$, if H is virtually contained in K and K is virtually contained in H.

We want to generalize these notions to suitable versions of these objects and relations regarding A-invariant subgroups of G. For two such groups $H \leq K$, we

have two options regarding the index of H in K: it is either *bounded*, i. e. it does not grow bigger than a certain cardinal while enlarging the ambient model, or for any given cardinal κ we can find an ambient model such that the index is larger than κ. Then we say that the index is *unbounded*. Note that if the index is bounded it is indeed bounded by $\left(2^{|T(A)|}\right)^{+}$. This leads to the definition below.

Definition 2.3. Let H and K be two A-invariant subgroups of G. We say that H is *almost contained* in K, denoted by $H \lesssim K$, if the index of $H \cap K$ is bounded in H. We say that H and K are *commensurate*, denoted by $H \sim K$, if H is almost contained in K and K is almost contained in H.

Let H and K be two A-invariant subgroups. Observe that $H \lesssim K$ does not depend on the model we choose. Thus $H \lesssim K$ remains true in any elementary extension. Moreover, if H and K are definable, bounded can be replaced by finite and hence being virtually contained and being almost contained coincide. Note also that being almost contained is a transitive relation and being commensurate is an equivalence relation among A-invariant subgroups of G. Furthermore, we have the following property:

Lemma 2.4. *Let G be a group and let H, K, and L be three A-invariant subgroups of G such that H normalizes K. If $H \lesssim L$ and $K \lesssim L$ then $HK \lesssim L$.*

Proof. We assume that G is sufficiently saturated. By assumption, we have that the index of $L \cap H$ in H as well as the index of $L \cap K$ in K are bounded by some cardinal κ_H and κ_K respectively which are smaller than $(2^{|T(A)|})^{+}$. Take $I_H = \{h_i : i < \kappa_H\}$ and $I_K = \{k_i : i < \kappa_K\}$ representatives of the cosets of $L \cap H$ in H and of $L \cap K$ in K respectively. Then the set $I_H \cdot I_K$ has at most size $2^{|T(A)|}$ and as H normalizes K, it contains a set of representatives of the cosets of $L \cap (HK)$ in HK. Hence the index of $L \cap (HK)$ in HK is bounded in G and whence $HK \lesssim L$. □

Definition 2.5. Let H, K and N be three A-invariant subgroups of G such that N is normalized by H. We define:

- The *almost centralizer* of H in K modulo N:
$$\widetilde{C}_K(H/N) = \{g \in N_K(N) : H \sim C_H(g/N)\}$$
- The *almost center* of H:
$$\widetilde{Z}(H) = \widetilde{C}_H(H)$$

Let us point out some important properties of the almost centralizer.

Properties 2.6. Let H, H', K, L and L' be A-invariant subgroups of G such that H and H' normalize L and L'.

(1) $\widetilde{C}_K(H)$ and $\widetilde{Z}(H)$ are A-invariant subgroup.
(2) $C_G(H) \leq \widetilde{C}_G(H)$ and $Z(G) \leq \widetilde{Z}(G)$.
(3) If H is definable, bounded can be replaced by finite and these definition coincide with the definition of the FC-centralizer and FC-center of H.
(4) $\widetilde{C}_{H'}(H/L) = \widetilde{C}_G(H/L) \cap H'$.
(5) $\widetilde{C}_G(H)$ is fixed by all automorphism of G which fix H and thus it is normalized by the normalizer of H and in particular by H. Furthermore, $\widetilde{Z}(H)$ is a characteristic subgroup of H.

(6) If $H \lesssim H'$ as well as $L \lesssim L'$ and $N_G(L) \leq N_G(L')$, we have that
$$\widetilde{C}_G(H'/L) \leq \widetilde{C}_G(H/L')$$
(7) Moreover, if $H \sim H'$ as well as $L \sim L'$ and $N_G(L) = N_G(L')$, we have that
$$\widetilde{C}_G(H'/L) = \widetilde{C}_G(H/L')$$
(8) If H is the union of A-type-definable subgroups H_α with $\alpha \in \Omega$. Then
$$\widetilde{C}_G(H) = \bigcap_{\alpha\in\Omega} \widetilde{C}_G(H_\alpha).$$
(9) If L is the intersection of A-definable subgroups L_α of G with $\alpha \in \Omega$, we have that
$$\widetilde{C}_G(H/L) \cap \bigcap_{\alpha\in\Omega} N_G(L_\alpha) = \bigcap_{\alpha\in\Omega} \widetilde{C}_G(H/L_\alpha)$$
(10) If L is the intersection of A-definable subgroups L_α of G with $\alpha \in \Omega$ all normalized by K and H,
$$H \lesssim \widetilde{C}_G(K/L) \quad \text{if and only if} \quad H \lesssim \widetilde{C}_G(K/L_\alpha) \quad \text{for all } \alpha \in \Omega.$$

As for any normal subgroup N of H, we have that $\widetilde{C}_G(H/N)$ is normalized by H, the following definition of the iterated almost centralizers is well defined.

Definition 2.7. Let H and K be two A-invariant subgroup of G such that $H \leq K$ and N be a normal A-invariant subgroup of H, then

- The nth *almost centralizer* of H in K modulo N is defined inductively on n by:
$$\begin{aligned} \widetilde{C}^0_K(H/N) &= N \\ \widetilde{C}^{n+1}_K(H/N) &= \widetilde{C}_K(H/\widetilde{C}^n_K(H/N)) \cap \bigcap_{i=0}^{n} N_K(\widetilde{C}^i_K(H/N)) \end{aligned}$$
- The nth *almost center* of H is defined as $\widetilde{Z}_n(H) = \widetilde{C}^n_H(H)$.

Note that if H and N are normal subgroups of K, the definition of the nth almost centralizer of H in K modulo N simplifies to:
$$\widetilde{C}^0_K(H/N) = N \text{ and } \widetilde{C}^{n+1}_K(H/N) = \widetilde{C}_K(H/\widetilde{C}^n_K(H/N))$$

Properties 2.8. Let G be a group, $H \trianglelefteq K$ be two A-invariant subgroup of G and let $n \in \mathbb{N}$. Then we have that
$$\widetilde{C}^n_K(H) = \widetilde{C}^n_G(H) \cap K.$$

In the rest of the section, we show properties of the almost centralizer of *ind-definable* subgroups of G. It is a model theoretic notion which generalizes type-definable subgroups and which falls into the class of invariant subgroups.

Definition 2.9. Let G be a group and A be a parameter set. An *A-ind-definable* subgroup H of G is the union of a directed system of A-type-definable subgroups of G, i. e. there is a family $\{H_\alpha : \alpha \in \Omega\}$ of type-definable subgroups of G such that for all α and β in Ω there is γ in Ω such that $H_\alpha \cup H_\beta \leq H_\gamma$ and H is equal to $\bigcup_{\alpha\in\Omega} H_\alpha$.

2.1. **Symmetry.** Observe that for two subgroups H and K of a group G, we have trivially that $H \leq C_G(K)$ if and only if $K \leq C_G(H)$. In the case of FC-centralizers and virtually containment, we will see that this is not true for arbitrary subgroups in non-saturated models. However, we obtain the same symmetry condition replacing the centralizer by the almost centralizer and containment by almost containment for ind-definable subgroups. In case, the ambient theory is simple, this was proven by Palacín and Wagner in [16].

Theorem 2.10 (Symmetry)**.** *Let G be a group, H and K be two A-ind-definable subgroups of G and let N be a subgroup of G which is a union of A-definable sets. Suppose N is normalized by H and by K. Then*

$$H \lesssim \widetilde{C}_G(K/N) \quad \textit{if and only if} \quad K \lesssim \widetilde{C}_G(H/N).$$

Proof. Let κ be equal to $(2^{|T(A)|})^+$. Assume that G is $(2^\kappa)^+$-saturated. We suppose that K is not almost contained in $\widetilde{C}_G(H/N)$. Thus, there is a set of representatives $\{k_i : i \in (2^\kappa)^+\}$ in K of different cosets of $\widetilde{C}_K(H/N)$ in K as G is sufficiently saturated. Since H is the union of type-definable subgroups H_α with α in an index set Ω of cardinality at most κ, for every i different than j in $(2^\kappa)^+$ there is $\alpha_{(i,j)}$ in Ω such that the centralizer of the element $k_i^{-1}k_j/N$ has unbounded index in $H_{\alpha_{(i,j)}}$. By Erdős-Rado, we can find a subset I_0 of $(2^\kappa)^+$ of cardinality κ^+ and α in Ω such that for all distinct i and $j$in I, we have that $\alpha_{(i,j)}$ is equal to α and thus the centralizer $C_{H_\alpha}(k_i^{-1}k_j/N)$ has infinite index in H_α. Hence, H_α can not be covered by finitely many cosets of these centralizers by a well known result of B. Neumann [15, Lemma 4.1]. As additionally the complement of N is type-definable the following partial type is consistent:

$$\pi(x_n : n \in \kappa^+) = \big\{[x_n^{-1}x_m, k_i^{-1}k_j] \notin N : n \neq m \in \kappa, i \neq j \in I_0\big\} \cup \{x_n \in H_\alpha : n \in \kappa\}$$

As G is sufficiently saturated, one can find a tuple $\bar{h}$ in G which satisfies $\pi(\bar{x})$. Fix two different elements n and m in κ^+. Then, we have that $k_i^{-1}k_j \notin C_K(h_n^{-1}h_m/N)$ for all $i \neq j$ in I_0. Hence, the subgroup $C_K(h_n^{-1}h_m/N)$ has unbounded index in K witnessed by $(k_j : j \in I_0)$, and whence the element $h_n^{-1}h_m$ does not belong to $\widetilde{C}_H(K/N)$. So $\widetilde{C}_H(K/N)$ has unboudedly many H_α-translates and therefore unbounded index in H. Thus, the group H is not almost contained in $\widetilde{C}_G(K/N)$ which finishes the proof. □

We obtain the following useful corollary.

Corollary 2.11. *Let G be an $\aleph_0$-saturated group and H and K be two definable subgroups of G. Then*

$$H \leq_v \widetilde{C}_H(K) \textit{ if and only if } K \leq_v \widetilde{C}_K(H)$$

Proof. Since almost containment and the almost centralizer satisfies symmetry, it is enough to show that for definable subgroups H and K of an $\aleph_0$-saturated group, we have that

$$H \leq_v \widetilde{C}_H(K) \text{ if and only if } H \lesssim \widetilde{C}_H(K).$$

So suppose first that $H \leq_v \widetilde{C}_H(K)$ and fix representatives $h_1, \ldots, h_n$ of the distinct classes of $\widetilde{C}_H(K)$ in H. Let H_d be the definable set $\{h \in H : [K : C_K(h)] < d\}$. As

K is definable, we have that $\widetilde{C}_H(K) = \bigcup_{d\in\mathbb{N}} H_d$. Thus

$$H = \bigcup_{i=1}^{n} h_i \cdot \bigcup_{d\in\mathbb{N}} H_d.$$

By $\aleph_0$-saturation, this remains true in any elementary extension of G and so $H \lesssim \widetilde{C}_H(K)$.

On the other hand, if $H \not\leq_v \widetilde{C}_H(K)$, then for any cardinal κ the type

$$\pi(x_i : i \in \kappa) = \{x_i \in H\} \cup \{x_i^{-1}x_j \notin H_d : i \neq j, d \in \mathbb{N}\}$$

is consistent. Hence, $H \not\lesssim \widetilde{C}_H(K)$. □

In the general context, we may ask if symmetry holds for FC-centralizers. We give a positive answer in the case that the ambient group H is an $\mathfrak{M}_c$-group. Afterwards, we give a counter-example which shows that it does not hold in general.

Proposition 2.12. *Let G be an $\mathfrak{M}_c$-group and H and K be subgroups of G. Then*

$$H \leq_v \mathrm{FC}_G(K) \quad \textit{if and only if} \quad K \leq_v \mathrm{FC}_G(H).$$

Proof. Suppose that $H \leq_v \mathrm{FC}_G(K)$. So the group $\mathrm{FC}_H(K)$ has finite index in H and is obviously contained in $\mathrm{FC}_G(K)$. Note that by the former the FC-centralizer of $\mathrm{FC}_H(K)$ in K is equal to the one of H in K. Since G is an $\mathfrak{M}_c$-group, we can find elements $h_0, \ldots, h_n$ in $\mathrm{FC}_H(K)$ such that $C_G(\mathrm{FC}_H(K))$ is equal to the intersection of the centralizers of the h_i's. As each h_i is contained in the FC-centralizer of K in H, this intersection and hence $C_K(\mathrm{FC}_H(K))$ has finite index in K. In other words, K is virtually contained in $C_K(\mathrm{FC}_H(K))$ which, on the other hand, is trivially contained in $\mathrm{FC}_K(\mathrm{FC}_H(K))$. As $\mathrm{FC}_K(\mathrm{FC}_H(K))$ coincides with $\mathrm{FC}_K(H)$ as mentioned before we can conclude. □

The next example was suggested by F. Wagner.

Example 1. Let G be a finite non-commutative group, K be $\prod_{\mathbb{N}} G$ and H be the subgroup $\bigoplus_{\mathbb{N}} G$ of K. The support of an element $(k_i)_{i\in\mathbb{N}}$ in K, denoted by $\mathrm{supp}((k_i)_{i\in\mathbb{N}})$, is the set of indices $i \in \mathbb{N}$ such that k_i is non trivial. As any element $\bar{h}$ of H has finite support and G is finite, any element of H has finitely many conjugates in K, namely at most $|G|^{|\mathrm{supp}(\bar{h})|}$ many. Thus its centralizer has finite index in K. Hence H is contained in the FC-centralizer of K. On the other hand, fix an element g of G which is not contained in the center of G. Let $\bar{k}_0$ be the neutral element of K and for $n \geq 1$ we define:

$$\bar{k}_n = (k_i)_{i\in\mathbb{N}} \text{ such that } \begin{cases} k_i = g & \text{if } i \equiv 0 \pmod{n} \\ k_i = 1 & \text{else} \end{cases}$$

Now fix some distinct natural numbers n and m. We have that the element $\bar{k}_n^{-1}\bar{k}_m$ is a sequence of the neutral element of G and infinitely many g's or g^{-1}'s. Now, we can choose an element h in G which does not commute with g and for any j in the support of $\bar{k}_n^{-1}\bar{k}_m$ we define the following elements of H:

$$\bar{l}_j = (l_i)_{i\in\mathbb{N}} \text{ such that } \begin{cases} l_i = h & \text{if } i = j \\ l_i = 1 & \text{else} \end{cases}$$

These elements witness that the set of conjugates $(\bar{k}_n^{-1}\bar{k}_m)^H$ is infinite and, as the n and m were chosen arbitrary, the $\bar{k}_n$'s are representatives of different cosets of

$FC_K(H)$ in K. Thus K is not virtually contained in the FC-centralizer of H in K which contradicts symmetry.

The previous example demonstrates that symmetry does not hold for the FC-centralizer of arbitrary subgroups in non-saturated models but the following question still remains open:

Question 1. Let H and K be two A-invariant subgroups of a group G. Then, do we have that

$$H \lesssim \widetilde{C}_G(K) \quad \text{if and only if} \quad K \lesssim \widetilde{C}_G(H)\ ?$$

2.2. **The almost three subgroups lemma.** For subgroups H, K and L of some group G we have that

$$[H, K, L] = 1 \text{ and } [K, L, H] = 1 \text{ imply } [L, H, K] = 1,$$

which is known as the three subgroups lemma. We want to generalize this result to our framework. As we have not yet introduced an "almost" version of the commutator, observe that, if H, K, and L normalize each other, we have that $[H, K, L] = 1$ if and only if $H \leq C_G(K/C_G(L))$. Thus we may state the three subgroups lemma as follows:

$$H \leq C_G(K/C_G(L)) \text{ and } K \leq C_G(L/C_G(H)) \text{ imply } L \leq C_G(H/C_G(K)).$$

We show this statement, replacing all centralizers and containment by almost centralizers and almost containment, for ind-definable subgroups which normalize each other in the following sense:

Definition 2.13. Let H, K and L be three A-ind-definable subgroups of G. We say that

- *H strongly normalizes L* if there is a set of A-type-definable subgroups $\{L_\alpha : \alpha \in \Omega\}$ of G each normalized by H such that L is equal to $\bigcup_{\alpha\in\Omega} L_\alpha$.
- *H and K similtaneously strongly normalize L* if there is a set of A-type-definable subgroups $\{L_\alpha : \alpha \in \Omega\}$ of G each normalized by H and K such that L is equal to $\bigcup_{\alpha\in\Omega} L_\alpha$.
- L is a *strongly normal* subgroup of G if G strongly normalizes L.

Note that if L is a type-definable group, it is strongly normalized by H (or respectively simultaneously strongly normalized by H and K) if and only if H normalizes L (respectively H and K normalize L).

The almost three subgroups lemma can be deduced from the lemma below:

Lemma 2.14. *Let H, K and L be three A-ind-definable subgroups of G. If H and K simultaneously strongly normalize L, then the following is equivalent:*

- $H \not\lesssim \widetilde{C}_G\big(K/\widetilde{C}_G(L)\big)$.
- *For any cardinal κ, there exists an elementary extension $\mathcal{G}$ of G and elements $(h_i : i \in \kappa)$ in $H(\mathcal{G})$, $(k_n : n \in \kappa)$ in $K(\mathcal{G})$ and $(l_s : s \in \kappa)$ in $L(\mathcal{G})$ such that*
$$[[h_i^{-1}h_j, k_n^{-1}k_m], l_s^{-1}l_t] \neq 1 \quad \forall i, j, n, m, s, t \in \kappa,\ i \neq j, n \neq m, s \neq t.$$

Proof. Let $\{L_\alpha : \alpha \in \Omega_L\}$ be a set of A-type-definable subgroups of G each normalized by H and K such that L is equal to $\bigcup_{\alpha\in\Omega_L} L_\alpha$ and $\{K_\beta : \beta \in \Omega_K\}$ be a set of A-type-definable subgroups of G such that K is equal to $\bigcup_{\beta\in\Omega_K} K_\beta$. Assume first

that $H \not\lesssim \widetilde{C}_G(K/\widetilde{C}_G(L))$. Note that as K and H normalize L, they normalize as well $\widetilde{C}_G(L)$. So $\widetilde{C}_G(K/\widetilde{C}_G(L))$ is well defined and for any $h \notin \widetilde{C}_G(K/\widetilde{C}_G(L))$, we have that $[K : C_K(h/\widetilde{C}_G(L))]$ is infinite.

Let κ be a given cardinal greater than $(2^{|T(A)|})^+$. Assume that G is $(2^{(2^\kappa)})^+$-saturated. The goal is to find elements $(h_i : i \in \kappa)$ in H, $(k_n : n \in \kappa)$ in K and $(l_s : s \in \kappa)$ in L which satisfy the second condition of the Lemma.

By saturation of G, one can find a sequence $(h_i : i \in (2^{(2^\kappa)})^+)$ of elements in H such that for non equal ordinals i and j, the element $h_i^{-1}h_j$ does not belong to $\widetilde{C}_G(K/\widetilde{C}_G(L))$ or equivalently

$$K \not\lesssim C_K(h_i^{-1}h_j/\widetilde{C}_G(L)). \quad (*)$$

Claim. *There is a subset I of $(2^{(2^\kappa)})^+$ of size κ^+, $\beta \in \Omega_K$ and $\alpha \in \Omega_L$ such that for all distinct elements i and j in I, we have that $K_\beta \not\lesssim C_{K_\beta}(h_i^{-1}h_j/\widetilde{C}_G(L_\alpha))$.*

Proof of the claim. Let i and j be two different arbitrary ordinal numbers less than $(2^{(2^\kappa)})^+$. By $(*)$ there exists a sequence $(k_n^{(i,j)} : n \in (2^\kappa)^+)$ of elements in K such that for non identical ordinals n and m less than $(2^\kappa)^+$, we have

$$\left[h_i^{-1}h_j, (k_n^{(i,j)})^{-1}k_m^{(i,j)}\right] \notin \widetilde{C}_G(L).$$

As K is the bounded union of A-type-definable subgroups K_β, by the pigeon hole principle we can find subset J of $(2^\kappa)^+$ of the same size and $\beta_{i,j}$ in Ω_K such that for all n in J, the element $k_n^{(i,j)}$ is an element of $K_{\beta_{i,j}}$. To simplify notation we may assume that J is equal to $(2^\kappa)^+$. Now, by Erdős-Rado, we can find a subset I of $(2^{(2^\kappa)})^+$ of size $(2^\kappa)^+$ and $\beta \in \Omega_K$ such that for non equal i and j, we have that $\beta_{i,j}$ is equal β. Again for convenience we assume that I equals $(2^\kappa)^+$.

To summarize, we have now found β in Ω_K, a sequence of elements $(h_i : i \in (2^\kappa)^+)$ in H and for any i different than j in $(2^\kappa)^+$ a sequence $(k_n^{(i,j)} : n \in (2^\kappa)^+)$ in K_β such that

$$\left[h_i^{-1}h_j, (k_n^{(i,j)})^{-1}k_m^{(i,j)}\right] \notin \widetilde{C}_G(L).$$

Fix again two distinct ordinal numbers i and j in $(2^\kappa)^+$. By Properties 2.6 (8), we have that the almost centralizer of L in G is the intersection of the almost centralizers of the L_α's in G. So for any non equal n and m in $(2^\kappa)^+$ one can find $\alpha_{(n,m)}^{(i,j)}$ in Ω_L such that

$$\left[h_i^{-1}h_j, (k_n^{(i,j)})^{-1}k_m^{(i,j)}\right] \notin \widetilde{C}_G\left(L_{\alpha_{(n,m)}^{(i,j)}}\right).$$

Now, we apply Erdős-Rado to the sequences of the $k_n^{(i,j)}$'s. Doing so, we obtain a subset $I_{(i,j)}$ of $(2^\kappa)^+$ of cardinality at least κ^+ and $\alpha_{(i,j)}$ in Ω_L such that for all non identical n and m in $I_{(i,j)}$, we have

$$\left[h_i^{-1}h_j, (k_n^{(i,j)})^{-1}k_m^{(i,j)}\right] \notin \widetilde{C}_G\left(L_{\alpha_{(i,j)}}\right).$$

Next, we apply Erdős-Rado to the h_i's. So, there exists a subset I of $(2^\kappa)^+$ of cardinality at least κ^+ and α in Ω_L such that $\alpha_{(i,j)}$ is equal to α for i different than j in I and thus for any such tuples we have

$$\left[h_i^{-1}h_j, (k_n^{(i,j)})^{-1}k_m^{(i,j)}\right] \notin \widetilde{C}_G\left(L_\alpha\right).$$

Thus, as for all non equal i and j in I, the index set $I_{(i,j)}$ is of cardinality $\kappa^+ > (2^{|T(A)|})^+$, we conclude that the centralizer of the element $h_i^{-1}h_j/\widetilde{C}_G(L_\alpha)$ has infinite index in K_β (witnessed by the $k_n^{(i,j)}$'s). Hence, for all distinct i and j in the index set I of cardinality κ^+, we have that $K_\beta \not\lesssim C_{K_\beta}(h_i^{-1}h_j/\widetilde{C}_G(L_\alpha))$ and the claim is established. $\square_{\text{claim}}$

The claim together with a well known result of B. Neumann [15, Lemma 4.1] yield that the group $K_\beta/\widetilde{C}_G(L_\alpha)$ can not be covered by finitely many translates of these centralizers.

Now, observe that since L_α is a type-definable group, any relatively definable subgroup of L_α has either finite or unbounded index, whence the group $\widetilde{C}_G(L_\alpha)$ is equal to the union of the following definable sets

$$S_{\phi,d} = \left\{ g \in G : \forall l_0, \ldots, l_d \bigwedge_{i=0}^{d} \phi(l_i) \to \bigvee_{i \neq j} l_i^{-1} l_j \in C_G(g) \right\},$$

where $\phi(x)$ ranges over the formulas in the type $\pi_{L_\alpha}(x)$ which defines L_α and d over all natural numbers.

By the two previous arguments, we conclude that the partial type below is consistent.

$$\begin{aligned} \pi(x_n : n \in \kappa) \quad = \quad & \{[h_i^{-1}h_j, x_n^{-1}x_m] \notin S_{\phi,d} : n \neq m \in \kappa,\, i \neq j \in I,\, d \in \mathbb{N}, \phi \in \pi_{L_\alpha}\} \\ & \cup \{x_n \in K_\beta : n \in \kappa\} \end{aligned}$$

Take $\bar{k}$ which satisfies $\pi(\bar{x})$. By construction we have that $[h_i^{-1}h_j, k_n^{-1}k_m] \notin \widetilde{C}_G(L_\alpha)$. Hence, $L_\alpha \not\lesssim C_{L_\alpha}([h_i^{-1}h_j, k_n^{-1}k_m])$. So L_α cannot be covered by finitely many translates of these centralizers. So the partial type below is again consistent.

$$\begin{aligned} \pi'(x_s : s \in \kappa) \quad = \quad & \{[[h_i^{-1}h_j, k_i^{-1}k_j], x_s^{-1}x_t] \neq 1 : s \neq t \in \kappa,\, n \neq m \in \kappa,\, i \neq j \in I, \} \\ & \cup \{x_s \in L_\alpha : s \in \kappa\} \end{aligned}$$

As L_α is a subgroup of L, a realization of this type together with the $(h_i : i \in I)$ and $(k_n : n \in \kappa)$ satisfies the required properties.

On the other hand, suppose that for any cardinal κ, there exists an extension $\mathcal{G}$ of G and elements $(h_i : i \in \kappa)$ in $H(\mathcal{G})$, $(k_n : n \in \kappa)$ in $K(\mathcal{G})$, and $(l_s : s \in \kappa)$ in $L(\mathcal{G})$ such that

$$\left[[h_i^{-1}h_j, k_n^{-1}k_m], l_s^{-1}l_t\right] \neq 1 \quad \forall i, j, n, m, s, t \in \kappa,\ i \neq j, n \neq m, s \neq t.$$

So let κ be greater than $2^{|T(A)|}$. If $H \lesssim \widetilde{C}_G(K/\widetilde{C}_G(L))$ then one can find $i \neq j$ such that $h_i^{-1}h_j$ is an element of $\widetilde{C}_G(K/\widetilde{C}_G(L))$. So the index of $C_K(h_i^{-1}h_j/\widetilde{C}_G(L))$ in K is bounded. Once more this implies that one can find $n \neq m$ such that $k_n^{-1}k_m \in C_G(h_i^{-1}h_j/\widetilde{C}_G(L))$. Thus $[h_i^{-1}h_j, k_n^{-1}k_m]$ is an element of $\widetilde{C}_G(L)$ or equivalently the index of $C_L([h_i^{-1}h_j, k_n^{-1}k_m])$ has bounded index in L. Thus there exists $s \neq t$ such that $[[h_i^{-1}h_j, k_n^{-1}k_m], l_s^{-1}l_t] = 1$ which contradicts our assumption and the Lemma is established. $\square$

Now we are ready to prove the almost three subgroups lemma.

Theorem 2.15 (almost three subgroup lemma)**.** *Let G be a group and H, K and L be three ind-definable subgroups of G which simultaneously strongly normalize each*

other. If

$$H \lesssim \widetilde{C}_G\big(K/\widetilde{C}_G(L)\big) \text{ and } K \lesssim \widetilde{C}_G\big(L/\widetilde{C}_G(H)\big) \text{ then } L \lesssim \widetilde{C}_G\big(H/\widetilde{C}_G(K)\big).$$

Proof. Assume towards a contradiction that $L \not\lesssim \widetilde{C}_G(H/\widetilde{C}_G(K))$ and let κ be equal to $(2^{|T(A)|})^+$. By the previous lemma we can find $(l_s : s \in \exp_5(\kappa)^+)$ in L, $(k_n : n \in (\exp_5(\kappa)^+)$ in K and $(h_i : i \in \exp_5(\kappa)^+)$ in H in a sufficiently saturated extension of G such that

$$\big[[l_s^{-1}l_t, h_i^{-1}h_j], k_n^{-1}k_m\big] \neq 1 \quad \forall i,j,n,m,s,t \in (2^\kappa)^+,\ i \neq j, n \neq m, s \neq t.$$

By the Witt's identity [12, Satz 1.4], for every tuple $i < j < n < m < s < t < \exp_5(\kappa)^+$ either

$$\big[[h_j^{-1}h_i, k_m^{-1}k_n], l_s^{-1}l_t\big] \neq 1 \quad \text{or} \quad \big[[k_n^{-1}k_m, l_t^{-1}l_s], h_j^{-1}h_i\big] \neq 1.$$

By Erdős-Rado we can find a subset I of cardinality κ^+ such that for all $i < j < n < m < s < t$ in I the same inequality of the two holds, say $[[h_j^{-1}h_i, k_m^{-1}k_n], l_s^{-1}l_t] \neq 1$. Now let λ be the order-type of I and note that it is greater or equal to κ^+. Identify I with λ. Thus

(1) $\big[[h_j^{-1}h_i, k_m^{-1}k_n], l_s^{-1}l_t\big] \neq 1$ for $0 \leq i < j \leq \kappa < n < m \leq 2\kappa < s < t \leq 3\kappa$

Arguing as in the proof of Theorem 2.10, we can find i, j, n, m, s and t such that $0 \leq i < j \leq \kappa < n < m \leq 2\kappa < s < t \leq 3\kappa$ and $[[h_j^{-1}h_i, k_m^{-1}k_n], l_s^{-1}l_t] = 1$. This contradicts (1) and the theorem is established. □

2.3. **Generalized Neumann Theorem.** We want to generalize a classical group theoretical result due to B. H. Neumann [15, Theorem 3.1] (Any bounded almost abelian group is finite-by-abelian). To do so, let us first introduce the following notion.

Definition 2.16. A group G is *almost abelian* if the centralizer of any of its element has finite index in G. If there is a natural number d such that the index of the centralizer of any element of G in G is smaller than d, we say that G is a *bounded almost abelian group.*

Remark 2.17. If we consider a definable almost abelian subgroup of an $\aleph_0$-saturated group, we can always bound the index of the centralizers by some natural number d by compactness. Hence, any definable almost abelian subgroups of any $\aleph_0$-saturated group is a *bounded almost abelian group.*

Analyzing and adapting the proof of B. Neumann result led to the following theorem:

Theorem 2.18. *Let G be a group and let H and K be two subgroups of G. Suppose that*

- *H normalizes K;*
- *$H \leq \widetilde{C}_G(K)$;*
- *$K \leq \widetilde{C}_G(H)$, moreover there is $d \in \mathbb{N}$ such that for all k in K the set of conjugates k^H has size at most d.*

Then the group $[K, H]$ is finite.

In the proof, we use the following fact:

Fact 2.19. [2] *Let G be a group and let K and H be two subgroups of G such that H normalizes K. If the set of commutators*

$$\{[k,h] : k \in K, h \in H\}$$

is finite, then the group $[K,H]$ is finite.

Proof of Theorem 2.18. Let d be the minimal bound for the size of conjugacy classes of elements of K by H. Fix some element k of K for which the conjugacy class of k in H has size d and let $1, h_2, \ldots, h_d$ be a set of right coset representatives of H modulo $C_H(k)$. Thus

$$k_1 = k, \quad k_2 = k^{h_2}, \quad \ldots, \quad k_d = k^{h_d}$$

are the d distinct conjugates of k by H. We let C be equal to the centralizer $C_K(h_2, \ldots, h_d)$. As H is contained in $\widetilde{C}_G(K)$, we have that the group C has finite index in K. Choose some representatives $a_1, \ldots, a_n$ of right cosets of K modulo C. Note that their conjugacy classes by H are finite by assumption. Let F be the finite set $k^H \cup a_1^H \cup \cdots \cup a_n^H$ and let E be the set $\{x_0 \cdot x_1 \cdot x_2 \cdot x_3 : x_i \in F \cup F^{-1}, i < 4\}$ which is finite as well. Note that K is equal to CF.

Now, we want to prove that E contains the set

$$D := \{[g,h] : g \in K, h \in H\}.$$

So let $g \in K$ and $h \in H$ be a arbitrary elements. Choose c in C, f in F, such that $g = cf$. We have that

$$[g,h] = [cf,h] = [c,h]^f[f,h] = f^{-1}[c,h] \cdot f^h$$

As f^{-1} belongs to F^{-1} and f^h belong to F, it remains to show that $[c,h]$ can be written as a product of two elements in $F \cup F^{-1}$.

Let $w = ck$. As c commutes with $h_2, \ldots, h_d$ the conjugates

$$w = ck, \quad w^{h_2} = ck_2, \quad \ldots, \quad w^{h_d} = ck_d$$

are all different. As d was chosen to be maximal, these have to be all conjugates of w by H. So there are i and j less or equal than d, such that

$$h^{-1}wh = ck_i \quad \text{and} \quad h^{-1}kh = k_j$$

and we have that

$$[c,h] = c^{-1}h^{-1}ch = c^{-1}(h^{-1}ckh)(h^{-1}k^{-1}h) = c^{-1}ck_ik_j^{-1} = k_ik_j^{-1}.$$

As all k_i's belong to F, we can conclude that D is a subset of E and therefore finite. Hence $[K,H]$ is finite by Fact 2.19. □

Corollary 2.20. *Let G be an $\aleph_0$-saturated group and let H and K be two definable subgroups of G such that H normalizes K. Suppose that*

$$K \leq \widetilde{C}_G(H) \quad \textit{and} \quad H \leq \widetilde{C}_L(K).$$

Then the group $[K,H]$ is finite.

Proof. As G is $\aleph_0$-saturated, the fact that $K \leq \widetilde{C}_G(H)$ implies that there is $d \in \mathbb{N}$ such that for all k in K the set of conjugates k^H has size at most d. So all hypotheses of Theorem 2.18 are satisfied and we can conclude. □

2.4. $\widetilde{\mathfrak{M}}_c$-groups.

Definition 2.21. A group G is called *$\widetilde{\mathfrak{M}}_c$-group* if for any two definable subgroups H and N, such that N is normalized by H, there exists natural numbers n_{HN} and d_{HN} such that any chain of centralizers

$$C_{H/N}(h_0N) \geq \ldots \geq C_{H/N}(h_0N, \ldots, h_mN) \geq \ldots \quad (h_i \in H)$$

each having index at least d_{HN} in its predecessor has length at most n_{HN}.

Remark 2.22. Note that any definable subgroup, any definable quotient and any elementary extension of G is again an $\widetilde{\mathfrak{M}}_c$-group.

One of the crucial property of subgroups of an $\widetilde{\mathfrak{M}}_c$-group G is that the iterated almost centralizers are definable which we prove below.

Proposition 2.23. *Let G be an $\widetilde{\mathfrak{M}}_c$-group, let H be a subgroup of G and let N be a definable subgroup of G which is contained in and normalized by H.*

(1) Then all iterated FC-centralizers $FC^n_G(H/N)$ are definable.
(2) If H is an A-invariant group, then all iterated almost centralizers $\widetilde{C}^n_G(H/N)$ are definable.

Proof. The proofs for the two cases are identical just replacing the iterated almost centralizers by the iterated FC-centralizers and bounded by finite. We give the proof using the notion $\widetilde{C}^n_G(H/N)$.

For n equals to 0 there is nothing to show as N is definable by assumption.

Now, let $n \in \mathbb{N}$ and assume that $\widetilde{C}^i_G(H/N)$ is definable for all $i \leq n$. This yields that $\bigcap_{i=0}^n N_K(\widetilde{C}^i_K(H/N))$ is a definable subgroup of G and thus an $\widetilde{\mathfrak{M}}_c$-group as well. Moreover, as $\widetilde{C}^{n+1}_G(H/N)$ only contains elements which belong to this intersection we may replace G by this intersection and assume that $\widetilde{C}^n_G(H/N)$ is a normal subgroup. Since G is an $\widetilde{\mathfrak{M}}_c$-group, there are $g_0, \ldots, g_m \in \widetilde{C}^{n+1}_G(H/N)$ and $d \in \mathbb{N}$ such that for all $h \in \widetilde{C}^{n+1}_G(H/N)$:

$$\left[\bigcap_{i=0}^{i=m} C_G\big(g_i/\widetilde{C}^n_G(H/N)\big) : \bigcap_{i=0}^{i=m} C_G\big(g_i/\widetilde{C}^n_G(H/N)\big) \cap C_G\big(h/\widetilde{C}^n_G(H/N)\big)\right] < d$$

Let D be equal to the definable group $\bigcap_{i=0}^{i=m} C_G(g_i/\widetilde{C}^n_G(H/N))$. Then the following set is definable.

$$S := \left\{g \in G : \left[D : C_D(g/\widetilde{C}^n_G(H/N))\right] < d\right\}$$

We show that $S = \widetilde{C}^{n+1}_G(H/N)$. The inclusion $\widetilde{C}^{n+1}_G(H/N) \subset S$ is obvious by choice of the g_i's and d. So let $g \in S$. To prove the inverse inclusion, we may compute:

$$\begin{aligned}
[H : C_H(g/\widetilde{C}^n_G(H/N))] &\leq [H : H \cap D] \cdot \left[H \cap D : C_{H\cap D}(g/\widetilde{C}^n_G(H/N))\right] \\
&\leq [H : H \cap D] \cdot \left[D : C_D(g/\widetilde{C}^n_G(H/N))\right] \\
&< \infty \quad \text{(i. e. finite for 1. and bounded for 2.)}
\end{aligned}$$

Thus g belongs to $\widetilde{C}^{n+1}_G(H/N)$. Hence $\widetilde{C}^{n+1}_G(H/N)$ is equal to S, and whence definable. $\square$

Remark 2.24. Note that all iterated centralizers of H in G are stabilized by any automorphism which fixes H set wise. So, if H is an A-invariant group, all its iterated almost centralizers are indeed definable over A. Moreover, for any (type-, ind-) definable (resp. A-invariant) subgroup H, the iterated almost centers of H are (type-, ind-) definable (resp. A-invariant).

3. Definable envelopes in $\widetilde{\mathfrak{M}}_c$-groups

In this section, we analyze arbitrary abelian, nilpotent and (normal) solvable subgroups of $\widetilde{\mathfrak{M}}_c$-groups. We prove the existence of definable envelopes up to finite index, which is inspired by the result in simple theories.

As the following remark shows, it is impossible to obtain the same result on definable envelopes as for stable groups in groups with a simple theory and thus $\widetilde{\mathfrak{M}}_c$-groups:

Remark 3.1. Let T be the theory of an infinite vector space over $\mathbb{F}_p$ with $p > 2$ together with a non-degenerate skew symmetric bilinear form. Then T is supersimple of SU-rank 1 and in any model of T one can define an "extraspecial p-group" G, i. e. G is infinite, every non-trivial element of G has order p, the center of G is cyclic of order p and is equal to the derived group of G. This group has SU-rank 1 and as any centralizer has finite index, one can find an infinite abelian subgroup. On the other hand, there are no abelian subgroups of finite index in G. However, any definable infinite subgroup of G has SU-rank 1 and thus finite index in G. Thus an infinite abelian subgroup of G cannot have a definable abelian envelope.

A model theoretic study of extra special p-groups can be found in [6].

So one has to find the modify the notion of definable envelopes which is adapted to the new context. These are results due to Milliet.

Fact 3.2. *Let G be a group definable in a simple theory and H be a subgroup of G.*

- [14, Proposition 5.6.] *If H is abelian, then there exists a definable finite-by-abelian subgroup of G which contains H.*
- [13] *If H is nilpotent (resp. solvable), then one can find a definable nilpotent (resp. solvable) subgroup of class at most $2n$ which virtually contains H.*

To find envelopes for $\widetilde{\mathfrak{M}}_c$-groups, we make use of a definable version of Schlichting's Theorem [18], which can be found in [20, Theorem 4.2.4].

In the rest of the section we prove the following theorem:

Theorem 3.3. *Let G be an $\widetilde{\mathfrak{M}}_c$-group and H be a subgroup of G. Then the following hold:*

(1) If H is almost abelian, then there exists a definable finite-by-abelian subgroup of G which contains H and which is normalized by $N_G(H)$.
(2) If H is almost nilpotent of class n, then there is a definable nilpotent subgroup N of G of class at most $2n$ which is normalized by $N_G(H)$ and virtually contains H.
In particular, if H is normal in G, we have that HN is a definable normal nilpotent subgroup of G of class at most $3n$ which contains H.

(3) *If H is almost solvable of class n, then there exists a definable solvable subgroup S of G of class at most $2n$ which is normalized by $N_G(H)$ and virtually contains H.*
In particular, if H is normal in G, the group HS is a definable normal solvable subgroup of G of class at most $3n$ which contains H.

3.1. **Abelian groups.** We first investigate the abelian case. The proof is inspired by the one of the corresponding theorem for simple theories in [14].

Proof of 3.3(1). Let H be an almost abelian subgroup of the $\widetilde{\mathfrak{M}}_c$-group G. As G is an $\widetilde{\mathfrak{M}}_c$-group there are elements $h_0, \dots, h_{n-1}$ in H and a natural number d such that for every element h in H, the index $[C : C \cap C_G(h)]$ is smaller than d for $C := \bigcap_{i=0}^{n-1} C_G(h_i)$. Observe additionally that H is virtually contained in C. Moreover, the following set

$$\mathcal{F} = \{C^h : h \in N_G(H)\}$$

is a family of uniformly commensurable definable subgroups of G. Thus applying Schlichting's Theorem to this family of subgroups, we obtain a definable subgroup D which is normalized by $N_G(H)$ and commensurable with C. So D virtually contains H and thus DH is a finite extension of D and thus definable. Note that:

- $\widetilde{Z}(DH)$ is a definable almost abelian group as DH is a definable subgroup of an $\widetilde{\mathfrak{M}}_c$-group.
- $H \leq \widetilde{Z}(DH)$ as DH is commensurable with C and thus the centralizer of any element of H has finite index in DH.
- $\widetilde{Z}(DH)$ is normalized by $N_G(H)$ as both D and H are.

So the definable almost abelian (thus finite-by-abelian) group $\widetilde{Z}(DH)$ contains H and is normalized by $N_G(H)$. □

3.2. **Solvable groups.** To prove the solvable case we introduce the following notations:

Definition 3.4. A group G is *almost solvable* if there exists a normal *almost abelian series* of finite length, i. e. a finite sequence

$$\{1\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$$

of normal subgroups of G such that G_{i+1}/G_i is an almost abelian group for all $i \in n$. The least such natural number $n \in \mathbb{N}$ is called the *almost solvable class* of G.

Definition 3.5. Let G be a group and S be a definable almost solvable subgroup. We say that S *admits a definable almost abelian series of length n* if there exists a family of definable normal subgroups $\{S_i : i \leq n\}$ of S such that S_0 is the trivial group, S_n is equal to S and S_{i+1}/S_i is almost abelian and normalized by S.

In an arbitrary group, a priori not every almost solvable group admits a definable almost abelian series.

By the following Lemma we only need to concentrate on building a definable almost series.

Lemma 3.6. *Any definable almost solvable subgroup H of an $\aleph_0$-saturated group G which admits a definable almost series of length n*

$$H = H_0 \trianglerighteq H_1 \trianglerighteq \ldots \trianglerighteq H_n = \{1\}.$$

Then H has a definable subgroup of finite index which is solvable of class at most $2n$ and which is normalized by $\bigcap_i N_G(H_i)$.

The proof is analogous to the one of Corollary 4.12 in [13]. Although is done there in the context of a simple theory, the proof is almost identically in our context. The only difference is that in the beginning of the proof, one has to replace G by the definable subgroup $\bigcap_i N_G(H_i)$ and thus we may suppose that all H_i are normal in G. Moreover, in the end, instead of taking the connected component, one has to take the intersection of all normal subgroups of G of finite index and so we have that the definable solvable subgroup H we obtain is additionally normalized by $\bigcap_i N_G(H_i)$.

Proposition 3.7. *Let H be an almost solvable subgroup of class n of an $\widetilde{\mathfrak{M}}_c$-group G. Then there exists a definable almost solvable subgroup of class n which is normalized by $N_G(H)$ and admits a definable almost abelian series containing H.*

Proof. Let $\{1\} = H_0 \leq \cdots \leq H_n = H$ be an almost abelian series for H. We construct recursively a definable almost abelian series

$$\{1\} = S_0 \leq \cdots \leq S_n$$

such that for all $i \leq n$, $H_i \leq S_i$ and S_i is normalized by $N_G(H)$.

As S_0 is the trivial group, we may let $0 < i < n$ and suppose that S_{i-1} has been constructed. Since S_{i-1} is definable and normalized by $N_G(H)$, we can replace G by the definable section $\mathbf{G}_i = N_G(S_{i-1})/S_{i-1}$. Note that this is an $\widetilde{\mathfrak{M}}_c$-group and that H_i/S_{i-1} is an almost abelian subgroup. Thus by the almost abelian case, there exists a definable almost abelian subgroup $\mathbf{S}_i$ of $\mathbf{G}_i$ which is normalized by $N_{\mathbf{G}_i}(H_i/S_{i-1})$ containing H_i/S_{i-1}. As H_i is a characteristic subgroup of H and S_{i-1} is normalized by $N_G(H)$, the normalizer of H_i/S_{i-1} and thus of $\mathbf{S}_i$ contains $N_G(H)/S_{i-1}$. Now defining S_i to be the pullback of $\mathbf{S}_i$ in G, we conclude. □

Proof of 3.3(3). Proposition 3.7 applied to H gives us a definable almost solvable group K of class n containing H which admits a definable almost series for which each member is normalized by $N_G(H)$. By Lemma 3.6, the group K has a definable subgroup S of finite index which is solvable of class at most $2n$ and which is normalized by $N_G(H)$. □

3.3. **Nilpotent groups.**

Definition 3.8. A group H is *almost nilpotent* if there exists an *almost central series* of finite length, i. e. a sequence of normal subgroups of H

$$\{1\} \leq H_0 \leq H_1 \leq \cdots \leq H_n = H$$

such that H_{i+1}/H_i is a subgroup of $\widetilde{Z}(H/H_i)$ for every $i \in \{0, \ldots, n-1\}$. We call the least such $n \in \mathbb{N}$, the *almost nilpotency class* of H.

Remark 3.9. The iterated almost centers of any almost nilpotent group H of class n form an *almost central series* of length n.

In this section we prove that any almost nilpotent subgroup of class n is virtually contained in a definable nilpotent group of class at most $2n$. To do so, we need the following consequences of Corollary 2.20 and Theorem 2.10.

Proposition 3.10. *Let G be an $\widetilde{\mathfrak{M}}_c$-group. Then the commutator $[\widetilde{Z}(G), \widetilde{C}_G(\widetilde{Z}(G))]$ is finite.*

Proof. We may assume that G is $\aleph_0$-saturated. As G is an $\widetilde{\mathfrak{M}}_c$-group, the normal subgroups $\widetilde{Z}(G)$ and $\widetilde{C}_G(\widetilde{Z}(G))$ are definable. As trivially $\widetilde{C}_G(\widetilde{Z}(G))$ is contained in itself and

$$\widetilde{Z}(G) = \widetilde{C}_G(G) \leq \widetilde{C}_G(\widetilde{C}_G(\widetilde{Z}(G))),$$

we may apply Corollary 2.20 to these two subgroups and obtain the result. □

Corollary 3.11. *Let G be an $\widetilde{\mathfrak{M}}_c$-group and H be an A-ind-definable subgroup of G. Then*

$$H \lesssim \widetilde{C}_G(\widetilde{C}_G(H))$$

Proof. Trivially, we have that $\widetilde{C}_G(H) \leq \widetilde{C}_G(H)$. Since G is an $\widetilde{\mathfrak{M}}_c$-group, the almost centralizer $\widetilde{C}_G(H)$ is definable and thus by symmetry, we obtain the result. □

Proof of 3.3(2). We construct inductively on $i \leq n$ the following subgroups of G: In the ith step we find a definable subgroup G_i of G and two definable normal subgroups N_{2i-1} and N_{2i} of G_i all normalized by $N_G(H)$ such that:

- $H \leq_v G_i$;
- $\mathrm{FC}_i(H) \cap G_i \leq N_{2i}$;
- $[N_{2i-1}, G_i] \leq N_{2(i-1)}$;
- $[N_{2i}, G_i] \leq N_{2i-1}$;
- $G_i \leq G_{i-1}$.

Once the construction is done, letting N be equal to N_{2n} gives a definable nilpotent subgroup normalized by $N_G(H)$ and of class at most $2n$ which is witnessed by the sequence

$$\{1\} = N_0 \cap G_n \leq N_1 \cap G_n \leq \cdots \leq N_{2n} \cap G_n.$$

So, let N_0 be the trivial group and G_0 be equal to G.

Now, assume that $i > 0$ and that for $j < i$ and $k < 2i - 1$ the groups N_k and G_j have been constructed. We work in the quotient $\mathbf{G} = G_{i-1}/N_{2(i-1)}$ which is an $\widetilde{\mathfrak{M}}_c$-group and we let $\mathbf{H} = (H \cap G_{i-1})/N_{2(i-1)}$ which is obviously normalized by $N_G(H)$. The first step is to replace $\mathbf{G}$ by a definable subgroup $\mathbf{C}$ which virtually contains $\mathbf{H}$ and such that $\mathrm{FC}_{\mathbf{G}}(\mathbf{H}) = \widetilde{Z}(\mathbf{C})$. Observe that the preimage of $\mathrm{FC}_{\mathbf{G}}(\mathbf{H})$ in G_{i-1} contains $\mathrm{FC}_i(H) \cap G_{i-1}$ as $\mathrm{FC}_{i-1}(H) \cap G_{i-1}$ is contained in $N_{2(i-1)}$.

If there is $g_0/N_{2(i-1)} \in \mathrm{FC}_{\mathbf{G}}(\mathbf{H}) \setminus \widetilde{Z}(\mathbf{G})$, we consider the family

$$\mathcal{H} = \{C_{\mathbf{G}}(g_0^h/N_{2(i-1)}) : h \in N_G(H)\}$$

Note that as $\mathbf{H}$ is normalized by $N_G(H)$ all members of $\mathcal{H}$ virtually contain $\mathbf{H}$. Moreover, as $\mathbf{G}$ is an $\widetilde{\mathfrak{M}}_c$-group there exists a finite intersection $\mathbf{F}$ of groups in $\mathcal{H}$ such that any $\mathbf{K}$ in $\mathcal{H}$ we have that the index $[\mathbf{F} : \mathbf{F} \cap \mathbf{K}]$ is at most d. Thus the family

$$\{\mathbf{F}^h : h \in N_G(H)\}$$

is uniformly commensurable. So, by Schlichting's Theorem there is a definable subgroup $\mathbf{C}_0$ of $\mathbf{G}$ which is invariant under all automorphisms which stabilizes the family setwise, thus normalized by $N_G(H)$, and commensurable with $\mathbf{F}$. Moreover $\mathbf{F} \cap \mathbf{H}$ is commensurable with $C_{\mathbf{H}}(g_0/N_{2(i-1)})$ as $g_0/N_{2(i-1)}$ belongs to $\mathrm{FC}_{\mathbf{G}}(\mathbf{H})$. Over all we obtain that

$$\mathbf{C}_0 \cap \mathbf{H} =_v \mathbf{H} \quad \text{and} \quad \mathbf{C}_0 \leq_v C_{\mathbf{G}}(g_0/N_{2(i-1)}). \quad (*)$$

If now, there is $g_1/N_{2(i-1)} \in \widetilde{C}_{\mathbf{C}_0}(\mathbf{H} \cap \mathbf{C}_0) \setminus \widetilde{Z}(\mathbf{C}_0)$, we can redo the same construction and obtain a $\mathbf{C}_1$. By $(*)$ and g_1 not belonging to $\widetilde{Z}(\mathbf{C}_0)$, we have that $C_{\mathbf{G}}(g_0/N_{2(i-1)}, g_1/N_{2(i-1)})$ has infinite index in $C_{\mathbf{G}}(g_0/N_{2(i-1)})$. Then we can iterated this process. It has to stop after finitely many steps, as for every j the index of $C_{\mathbf{G}}(g_0/N_{2(i-1)}, \ldots, g_{j+1}/N_{2(i-1)})$ in $C_{\mathbf{G}}(g_0/N_{2(i-1)}, \ldots, g_j/N_{2(i-1)})$ is infinite by construction, contradicting the fact that $\mathbf{G}$ is an $\widetilde{\mathfrak{M}}_c$-group. Letting $\mathbf{C}$ be equal to $\bigcap_i \mathbf{C}_i$, we found a definable subgroup of $\mathbf{G}$ (thus an $\widetilde{\mathfrak{M}}_c$-group), such that $\mathrm{FC}_{\mathbf{C}}(\mathbf{H}) = \widetilde{Z}(\mathbf{C})$, which is normalized by $N_G(H)$ and whose intersection with $\mathbf{H}$ has finite index in $\mathbf{H}$.

The next step is to define G_i, N_{2i-1} and N_{2i}. As $\mathbf{C}$ is an $\widetilde{\mathfrak{M}}_c$-group, Proposition 3.10 yields that the commutator $\mathbf{Z} = [\widetilde{Z}(\mathbf{C}), \widetilde{C}_{\mathbf{C}}(\widetilde{Z}(\mathbf{C}))]$ is finite. Since $\widetilde{Z}(\mathbf{C})$ and $\widetilde{C}_{\mathbf{C}}(\widetilde{Z}(\mathbf{C}))$ are characteristic subgroups of $\mathbf{C}$, we have that $\mathbf{Z}$ is normalized by $N_G(H)$ and contained in $\widetilde{Z}(\mathbf{C})$. Note additionally that the group $\widetilde{C}_{\mathbf{C}}(\widetilde{Z}(\mathbf{C}))$ has finite index in $\mathbf{C}$ by Corollary 3.11. Thus $\mathbf{G}_i = \widetilde{C}_{\mathbf{C}}(\widetilde{Z}(\mathbf{C})) \cap C_{\mathbf{C}}(\mathbf{Z})$ has finite index in $\mathbf{C}$. We let $\mathbf{N}_1 = \mathbf{Z} \cap \mathbf{G}_i$, a finite subgroup of the center of $\mathbf{G}_i$, and $\mathbf{N}_2 = \widetilde{Z}(\mathbf{C}) \cap \mathbf{G}_i = \widetilde{Z}(\mathbf{G}_i)$, which is contained in $Z(\mathbf{G}_i/\mathbf{N}_1)$. Note that all groups used to define $\mathbf{G}_i$, $\mathbf{N}_1$ and $\mathbf{N}_2$ are all characteristic subgroups of $\mathbf{C}$ and thus $\mathbf{G}_i$, $\mathbf{N}_1$ and $\mathbf{N}_2$ are normalized $N_G(H)$. Moreover, $\mathbf{N}_1$ and $\mathbf{N}_2$ are normal subgroups of $\mathbf{G}_i$. Let G_i, N_{2i-1} and N_{2i} be the preimages of $\mathbf{G}_i$, $\mathbf{N}_1$ and $\mathbf{N}_2$ in G respectively. They satisfy all requirements, finishing the construction and therefore the proof. □

4. Fitting subgroup of $\widetilde{\mathfrak{M}}_c$-groups

In this section we analyze the Fitting subgroup $F(G)$ and the almost Fitting subgroup of an $\widetilde{\mathfrak{M}}_c$-group as well as locally nilpotent $\widetilde{\mathfrak{M}}_c$-groups.

The first step is to show that any locally nilpotent subgroup of an $\widetilde{\mathfrak{M}}_c$-group, thus in particular the Fitting subgroup, is solvable.

Proposition 4.1. *Any locally nilpotent subgroup of an $\widetilde{\mathfrak{M}}_c$-group is solvable. In particular, the Fitting subgroup of an $\widetilde{\mathfrak{M}}_c$-group is solvable.*

The proof is analog to the corresponding result for type-definable groups in simple theories [16, Lemma 3.6].

In the next lemma we deal with a definable section of some $\widetilde{\mathfrak{M}}_c$-group acting via conjugation on another definable section. We recall and introduce some facts and notations:
Let G be a group that acts on an abelian group A by automorphisms. If B is a subgroup of A and g an element of G we denote by $C_B(g)$ the group of elements b in B on which g acts trivially, i. e. $gb = b$. Furthermore, if H is a subgroup of G and a an element of A, we denote by $C_H(a)$ all elements h in H which act trivially on

a. This yields the natural definition of an almost centralizer via this group action, namely for any subgroup B of A and H of G, we have that

$$\widetilde{C}_B(H) = \{b \in B : [H : C_H(b)] \text{ is finite}\}$$

$$\widetilde{C}_H(B) = \{h \in H : [B : C_B(h)] \text{ is finite}\}$$

Note that this group action defines a semidirect product $A \rtimes G$. Within this group, the above defined almost centralizer $\widetilde{C}_B(H)$ (respectively $\widetilde{C}_H(B)$) corresponds to the projection of $\widetilde{C}_{B\rtimes 1}(1 \rtimes H)$ to its first coordinate (respectively $\widetilde{C}_{1\rtimes H}(B \rtimes 1)$ to its second coordinate). So one obtains immediately the following symmetry for the above almost centralizer using Theorem 2.10 for $A \rtimes G$.

Lemma 4.2. *Let G be a group that acts on an abelian group A by automorphisms. Let H be a definable subgroup of G and B be a definable subgroup of A, then we have that*

$$H \lesssim \widetilde{C}_G(B) \text{ if and only if } B \lesssim \widetilde{C}_A(H).$$

Remark 4.3. Let G be a group and K, A, N and M be subgroups of G such that:

$$M \trianglelefteq K \quad \text{and} \quad N \trianglelefteq A.$$

We say that the quotient K/M *acts by conjugation on* A/N if the action by K/M on A/N via conjugation is well-defined, i. e.

- $K \leq N_G(A) \cap N_G(N)$;
- $M \leq C_G(A/N)$.

Lemma 4.4. *Let $\boldsymbol{K}$ and $\boldsymbol{A}$ be quotients of definable subgroups of an $\widetilde{\mathfrak{M}}_c$-group G such that $\boldsymbol{K}$ acts by conjugation on $\boldsymbol{A}$. Then the $\widetilde{C}_{\boldsymbol{K}}(\boldsymbol{A})$ and $\widetilde{C}_{\boldsymbol{A}}(\boldsymbol{K})$ are definable.*

Proof. The lemma is an immediate consequence of the following fact which is an easy consequence of the descending chain condition in $\widetilde{\mathfrak{M}}_c$-groups:
There are natural numbers n and d (resp. n' and d') such that any descending chain of centralizers

$$C_{\mathbf{A}}(\mathbf{k}_0) \geq C_{\mathbf{A}}(\mathbf{k}_0, \mathbf{k}_1) \geq \cdots \geq C_{\mathbf{A}}(\mathbf{k}_0, \ldots, \mathbf{k}_m) \geq \ldots \quad (\mathbf{k}_i \in \mathbf{K})$$

$$\big(\text{resp. } C_{\mathbf{K}}(\mathbf{a}_0) \geq C_{\mathbf{K}}(\mathbf{a}_0, \mathbf{a}_1) \geq \cdots \geq C_{\mathbf{K}}(\mathbf{a}_0, \ldots, \mathbf{a}_m) \geq \ldots \quad (\mathbf{a}_i \in \mathbf{A})\big)$$

each of index greater than d (resp. d') in its predecessor has length at most n (resp. n'). □

The proof of [16, Lemma 3.8] which is stated for groups type-definable in a simple theory uses only symmetry of the almost centralizer and that they are definable. Hence it remains true for $\widetilde{\mathfrak{M}}_c$-groups.

Lemma 4.5. *Let $\boldsymbol{K}$ and $\boldsymbol{A}$ be definable sections of an $\widetilde{\mathfrak{M}}_c$-group G such that $\boldsymbol{A}$ is abelian and $\boldsymbol{K}$ acts by conjugation on $\boldsymbol{A}$. Suppose that $\boldsymbol{H}$ is an arbitrary abelian subgroup of $\boldsymbol{K}$ and that there are a tuple $\bar{\boldsymbol{h}} = (\boldsymbol{h}_i : i < \ell)$ in $\boldsymbol{H}$ and natural numbers $(m_i : i < \ell)$ such that*

- *$(\boldsymbol{h}_i - 1)^{m_i}\boldsymbol{A}$ is finite $\forall i < \ell$;*
- *for any $\boldsymbol{h}$ in $\boldsymbol{H}$ the index of $C_{\boldsymbol{A}}(\bar{\boldsymbol{h}}, \boldsymbol{h})$ in $C_{\boldsymbol{A}}(\bar{\boldsymbol{h}})$ is finite.*

Then there is a definable subgroup $\boldsymbol{L}$ of $\boldsymbol{K}$ which contains $\boldsymbol{H}$ and a natural number m such that $\widetilde{C}^m_{\boldsymbol{A}}(\boldsymbol{L})$ has finite index in $\boldsymbol{A}$.

Proof. Let

$$\mathbf{L} = \widetilde{C}_{C_{\mathbf{K}}(\bar{\mathbf{h}})}(C_{\mathbf{A}}(\bar{\mathbf{h}})) = \{\mathbf{k} \in C_{\mathbf{K}}(\bar{\mathbf{h}}) : [C_{\mathbf{A}}(\bar{\mathbf{h}}) : C_{\mathbf{A}}(\bar{\mathbf{h}}, \mathbf{k})] < \infty\}$$

with $\bar{\mathbf{h}}$ given by the statement (note that $C_{\mathbf{K}}(\bar{\mathbf{h}})$ denotes the centralizer within the group $\mathbf{K}$ and $C_{\mathbf{A}}(\bar{\mathbf{h}})$ denotes the centralizer given by the group action of $\mathbf{K}$ on $\mathbf{A}$). Observe that L contains $\mathbf{H}$ by assumption and that it is definable by Lemma 4.4.

Let m be equal to $1+\sum_{i=0}^{\ell-1}(m_i-1)$ and fix an arbitrary tuple $\bar{n} = (n_0, \dots, n_{m-1})$ in $\ell^{\times m}$. By the pigeonhole principle and the choice of m there is at least one i less than ℓ such that at least m_i many coordinates of $\bar{n}$ are equal to i. As the group ring $\mathbb{Z}(\mathbf{H})$ is commutative and $(\mathbf{h}_i - 1)^{m_i} A$ is finite for all i less than ℓ by assumption, we have that

$$(\mathbf{h}_{n_0} - 1)(\mathbf{h}_{n_1} - 1) \dots (\mathbf{h}_{n_{m-1}} - 1)\mathbf{A}$$

is finite.

Note the following:

(†) Let $\mathbf{k}$ be an element of $\mathbf{K}$ and $\mathbf{B}$ be a subgroup of $\mathbf{A}$. Then we have that the set $(\mathbf{k} - 1)\mathbf{B}$ is finite if and only if $\mathbf{B} \lesssim C_{\mathbf{A}}(\mathbf{k})$

So, applying (†) to $\mathbf{B} = (\mathbf{h}_{n_1} - 1) \dots (\mathbf{h}_{n_{m-1}} - 1)\mathbf{A}$, for all $i \leq n$ we obtain that

$$(\mathbf{h}_{n_1} - 1) \dots (\mathbf{h}_{n_{m-1}} - 1)\mathbf{A} \lesssim C_{\mathbf{A}}(\mathbf{h}_i).$$

Thus

$$(\mathbf{h}_{n_1} - 1) \dots (\mathbf{h}_{n_{m-1}} - 1)\mathbf{A} \lesssim C_{\mathbf{A}}(\bar{\mathbf{h}}).$$

Since for all $\mathbf{k}_0$ in $\mathbf{L}$, we have that $C_{\mathbf{A}}(\bar{\mathbf{h}}) \lesssim C_{\mathbf{A}}(\mathbf{k}_0)$, we have as well that

$$(\mathbf{h}_{n_1} - 1) \dots (\mathbf{h}_{n_{m-1}} - 1)\mathbf{A} \lesssim C_{\mathbf{A}}(\mathbf{k}_0)$$

and again by (†) we deduce that

$$(\mathbf{k}_0 - 1)(\mathbf{h}_{n_1} - 1) \dots (\mathbf{h}_{n_{m-1}} - 1)\mathbf{A}$$

is finite. As $\mathbf{L}$ is contained in the centralizer of $\bar{\mathbf{h}}$, the previous line is equal to

$$(\mathbf{h}_{n_1} - 1) \dots (\mathbf{h}_{n_{m-1}} - 1)(\mathbf{k}_0 - 1)\mathbf{A}.$$

We repeat the previous process m times and obtain that for any m-tuple $(\mathbf{k}_0, \dots \mathbf{k}_{m-1})$ in $\mathbf{L}$ we have that the set

$$(\mathbf{k}_{m-1} - 1) \dots (\mathbf{k}_1 - 1)(\mathbf{k}_0 - 1)\mathbf{A}$$

is finite. As the tuple is arbitrary, we have that for any $\mathbf{k}$ in $\mathbf{L}$ the group

$$(\mathbf{k}_{m-2} - 1) \dots (\mathbf{k}_1 - 1)(\mathbf{k}_0 - 1)\mathbf{A}$$

is almost contained in the centralizer $C_{\mathbf{A}}(\mathbf{k})$, i. e.

$$\mathbf{L} \leq \widetilde{C}_{\mathbf{K}}((\mathbf{k}_{m-2} - 1) \dots (\mathbf{k}_1 - 1)(\mathbf{k}_0 - 1)\mathbf{A})$$

By symmetry we have that

$$(\mathbf{k}_{m-2} - 1) \dots (\mathbf{k}_1 - 1)(\mathbf{k}_0 - 1)\mathbf{A} \lesssim \widetilde{C}_{\mathbf{A}}(\mathbf{L})$$

By Lemma 4.4, we have that $\widetilde{C}_{\mathbf{A}}(\mathbf{L})$ is definable. Thus we may work modulo this group as A is abelian and obtain that

$$(\mathbf{k}_{m-2} - 1) \dots (\mathbf{k}_1 - 1)(\mathbf{k}_0 - 1)\mathbf{A}/\widetilde{C}_{\mathbf{A}}(\mathbf{L})$$

is finite for all choices of an $(m-1)$-tuple $(\mathbf{k}_0, \dots, \mathbf{k}_{m-2})$ in $\mathbf{L}$. Thus as before we obtain by (†) and symmetry that

$$(\mathbf{k}_{m-3}-1)\dots(\mathbf{k}_1-1)(\mathbf{k}_0-1)\mathbf{A} \lesssim \widetilde{C}_{\mathbf{A}}(\mathbf{L}/\widetilde{C}_{\mathbf{A}}(\mathbf{L})) = \widetilde{C}^2_{\mathbf{A}}(\mathbf{L})$$

Repeating this process m times yields that $\mathbf{A} \lesssim \widetilde{C}^m_{\mathbf{A}}(\mathbf{L})$. □

Theorem 4.6. *The Fitting subgroup of an $\widetilde{\mathfrak{M}}_c$-group is nilpotent and definable.*

Proof. Note first, that the Fitting subgroup $F(G)$ of G is solvable by Proposition 4.1. So there exists a natural number r such that the rth derived subgroup $F(G)^{(r)}$ of $F(G)$ is trivial, hence nilpotent. Now we will show that if $F(G)^{(n+1)}$ is nilpotent, then so is $F(G)^{(n)}$.

So, suppose that $F(G)^{(n+1)}$ is nilpotent. As it is additionally normal in G, using Theorem 3.3 (2) we can find a definable normal nilpotent subgroup N of G containing $F(G)^{(n+1)}$. Note additionally that the central series

$$\{1\} = N_0 < N_1 < \dots < N_k = N$$

with $N_i = Z_i(N)$ consists of definable normal subgroups of G such that $[N, N_{i+1}] \leq N_i$.

Observe that it is enough to show that $F(G)^{(n)}$ is almost nilpotent: If $F(G)^{(n)}$ is almost nilpotent it has a normal nilpotent subgroup F of finite index by Theorem 3.3(2). As $F(G)^{(n)}$ is a subgroup of the Fitting subgroup, any finite subset is contained in a normal nilpotent subgroup of G. Thus, there is a normal nilpotent subgroup that contains a set of representatives of cosets of F in $F(G)^{(n)}$. Hence the group $F(G)^{(n)}$ is a product of two normal nilpotent subgroups, whence nilpotent by Fitting's Theorem [7].

As $F(G)^{(n)}/N$ is abelian and G/N is an $\widetilde{\mathfrak{M}}_c$-group, by Proposition 3.3(1) one can find a definable subgroup A' of G which contains $F(G)^{(n)}$ such that A'/N is an FC-group, i. e. $A' \leq \widetilde{C}_G(A'/N)$. Moreover, the group A'/N is normalized by the normalizer of $F(G)^{(n)}/N$ and thus A' is normal in G. The next step is to find a definable subgroup A of A' which still contains $F(G)^{(n)}$ and a natural number m for which $N \leq \widetilde{C}^m_G(A)$. This will imply that $A \leq \widetilde{C}_G(A/N) \leq \widetilde{C}_G(A/\widetilde{C}^m_G(A)) = \widetilde{C}^{m+1}_G(A)$. As A contains $F(G)^{(n)}$, the group $F(G)^{(n)}$ would be nilpotent by the above.

Fix now some $i > 0$. For any g in $F(G)^{(n)}$ there is some normal nilpotent subgroup H_g which contains g. So $N_i H_g$ is nilpotent by Fitting's Theorem [7]. Therefore, we can find a natural number m_g such that $[N_i \,,_{m_g} g] \leq \{1\}$ or seen with the group action as in Lemma 4.5

$$(g-1)^{m_g} N_i = \{1\}.$$

Additionally, as G is an $\widetilde{\mathfrak{M}}_c$-group, we can find a finite tuple $\bar{g}$ in $F(G)^{(n)}$ such that for any $g \in F(G)^{(n)}$ the index $[C_{N_i}(\bar{g}/N_{i-1}) : C_{N_i}(\bar{g}/N_{i-1}, g/N_{i-1})]$ is finite. So we may apply Lemma 4.5 to G/N acting on N_i/N_{i-1} and the abelian subgroup $F(G)^{(n)}/N$. Thus, there is a natural number m_i and a definable group K_i that contains $F(G)^{(n)}$ such that $N_i \lesssim \widetilde{C}^{m_i}_G(K_i/N_{i-1})$. Then the finite intersection $A = A' \cap \bigcap_i K_i$ is a definable subgroup of G which still contains $F(G)^{(n)}$. As for A', we have that

$A \leq \widetilde{C}_G(A/N)$. Additionally:

$$N_i \lesssim \widetilde{C}_G^{m_i}(K_i/N_{i-1}) \leq \widetilde{C}_G^{m_i}(A/N_{i-1})$$

and inductively

$$\begin{aligned} N &\lesssim \widetilde{C}_G^{m_k}(A/N_{k-1}) \\ &\leq \widetilde{C}_G^{m_k}(A/(C_G^{m_{k-1}}(A/N_{k-2}))) = C_G^{m_k+m_{k-1}}(A/N_{k-2}) \\ &\leq \ldots \leq C_G^{m_k+\cdots+m_1}(A) \end{aligned}$$

Using that $A \leq \widetilde{C}_G(A/N)$, we obtain that $A \leq \widetilde{C}_G^m(A)$ for $m = m_k + \cdots + m_1 + 1$.

Overall, we get that $F(G)^{(n)}$ is nilpotent for all n. In particular, the Fitting subgroup $F(G)$ of G is nilpotent. And finally by a result of Ould Houcine [11] we deduce that it is definable as well. □

Now, we want to study the almost Fitting subgroup:

Definition 4.7. The *almost Fitting subgroup* of a group G is the group generated by all its normal almost nilpotent subgroups. We denote this subgroup by $\widetilde{F}(G)$.

Hickin and Wenzel show in [10] that the product of two normal almost nilpotent subgroups is again normal almost nilpotent. Hence the almost Fitting subgroup of any group G is locally almost nilpotent but it might not be almost nilpotent. For $\widetilde{\mathfrak{M}}_c$-groups we show the following:

Proposition 4.8. *The almost Fitting subgroup of an $\widetilde{\mathfrak{M}}_c$-group is almost solvable.*

Proof. Let G be an $\widetilde{\mathfrak{M}}_c$-group and g be an element of its almost Fitting subgroup. Then there is a normal almost nilpotent subgroup H of G which contains g. By Theorem 3.3(2), we deduce that H has a nilpotent subgroup of finite index which is normal in G. Thus, the quotient $H/F(G)$ is finite. Since additionally H is a normal subgroup of G, we deduce that any element of H has finitely many conjugates modulo $F(G)$. Hence the group H and therefore $\widetilde{F}(G)$ are contained in $\widetilde{C}_G(G/F(G))$. As $F(G)$ is nilpotent by Theorem 4.6 and $\widetilde{C}_G(G/F(G))/F(G)$ is almost abelian, we deduce that $\widetilde{C}_G(G/F(G))$ is almost solvable. As any subgroup of an almost solvable group is almost solvable, we conclude that $\widetilde{F}(G)$ is almost solvable which finishes the proof. □

We finish this section with two proposition about locally nilpotent $\widetilde{\mathfrak{M}}_c$-group.

Proposition 4.9. *Let G be a locally nilpotent $\aleph_0$-saturated $\widetilde{\mathfrak{M}}_c$-group. Then G is nilpotent-by-finite.*

Proof. Note first of all, that it is enough to show that G is almost nilpotent as any almost nilpotent subgroup of an $\widetilde{\mathfrak{M}}_c$-group is nilpotent-by-finite by Theorem 3.3(2).

As G is locally nilpotent, it is solvable by Proposition 4.1. So, we may inductively assume that G' is almost nilpotent. Thus G' is virtually contained in a definable normal nilpotent subgroup N of G by Theorem 3.3(2). We claim that it is enough to show that for some natural number n, the group N is contained in $\widetilde{Z}_n(G)$: If so, we have that $G/\widetilde{Z}_n(G)$ is an almost abelian group as G/N is an almost abelian group and thus G is contained in $\widetilde{Z}_{n+1}(G)$.

Now, we prove inductively that for every natural number $i \leq m$, we can find a natural number j such that $Z_i(N)$ is contained in $\widetilde{Z}_j(G)$.

For i equals 0 this is trivially true. Thus, suppose that for $Z_i(N)$ we have found j such that $Z_i(N)$ is contained in $\widetilde{Z}_j(G)$. We work in $\mathbf{G} = G/\widetilde{Z}_j(G)$ which is again an $\widetilde{\mathfrak{M}}_c$-group. We set

$$\mathbf{N} := N\widetilde{Z}_j(G)/\widetilde{Z}_j(G) \quad \text{and} \quad \mathbf{N}_{i+1} := Z_{i+1}(N)\widetilde{Z}_j(G)/\widetilde{Z}_j(G).$$

As

$$[Z_{i+1}(N), N] \leq Z_i(N) \leq \widetilde{Z}_j(G),$$

we have that $[\mathbf{N}_{i+1}, \mathbf{N}] = 1$. Moreover, since G/N is an almost abelian group, so is $\mathbf{G}/\mathbf{N}$. We fix additionally the following notation:
For any subgroup $\mathbf{H}$ of $\mathbf{G}$, by $\mathbf{H}^*$ we denote $\mathbf{H}/\mathbf{N}$ and for any element h of $\mathbf{H}$ we write h^* for its class modulo $\mathbf{N}$. So, the group $\mathbf{G}^*$ acts on $\mathbf{N}_{i+1}$ by conjugation and we may regard $\mathbf{N}_{i+1}$ as an $\mathbf{G}^*$-module as $[N_{i+1}, N] = 1$.

Since $\mathbf{G}$ is an $\widetilde{\mathfrak{M}}_c$-group, we can find a finite tuple $\bar{g} = (g_0, \ldots, g_m)$ of elements in $\mathbf{G}$ such that for any g in $\mathbf{G}$ the index $[C_{\mathbf{G}}(\bar{g}) : C_{\mathbf{G}}(\bar{g}, g)]$ is finite. Let $\mathbf{K}$ be equal to $C_{\mathbf{G}}(\bar{g}/\mathbf{N})$ which has finite index in $\mathbf{G}$ as $\mathbf{G}/\mathbf{N}$ is almost abelian. For any $a \in \mathbf{N}_{i+1}$, we have that the group generated by a and $\bar{g}$ is a finitely generated subgroup of a locally nilpotent group and must be nilpotent. Thus for a given a in $\mathbf{N}_{i+1}$ there is a choice $h_0, \ldots, h_{d_a}$ of elements all belonging to the tuple $\bar{g}$ such that in the right module notation

$$(h_0^* - 1)(h_1^* - 1) \ldots (h_{d_a}^* - 1)a = 0.$$

As $\mathbf{N}_{i+1}$ is definable and G is $\aleph_0$-saturated, there is an upper bound for the choice of d_a which we denote by d.

Thus, for any choice of $h_0, \ldots, h_d$ each being an element of the tuple $\bar{g}$ and any element a of $\mathbf{N}_{i+1}$ we have in the right module notation

$$(h_0^* - 1)(h_1^* - 1) \ldots (h_d^* - 1)a = 0.$$

As a was arbitrary in $\mathbf{N}_{i+1}$, we obtain that

$$(h_0^* - 1)(h_1^* - 1) \ldots (h_d^* - 1)\mathbf{N}_{i+1} = 0.$$

Moreover, since h_0 is an arbitrary element of $\bar{g}$, the previous equation yields that

$$(h_1^* - 1) \ldots (h_d^* - 1)\mathbf{N}_{i+1} \leq C_{\mathbf{G}}(\bar{g}).$$

Let k_0 be any element of $\mathbf{K}$, by the choice of $\bar{g}$, we obtain that

$$(h_1^* - 1) \ldots (h_d^* - 1)\mathbf{N}_{i+1} \lesssim C_{\mathbf{G}}(k_0)$$

or in other words

$$(k_0^* - 1)(h_1^* - 1) \ldots (h_d^* - 1)\mathbf{N}_{i+1} \quad \text{is finite.}$$

As k_0 is an element of $C_{\mathbf{G}}(\bar{g}/\mathbf{N})$ and $\mathbf{N}_{i+1}$ is commutative, this finite set equals

$$(h_1^* - 1) \ldots (h_d^* - 1)(k_0^* - 1)\mathbf{N}_{i+1}$$

Iterating this process, we obtain that for any tuple of elements $(k_0, \ldots, k_d)$ in $\mathbf{K}$ we have that

$$(k_d^* - 1) \ldots (k_1^* - 1)(k_0^* - 1)\mathbf{N}_{i+1} \quad \text{is finite.}$$

Since the tuple was taken arbitrary, we have that for any k in $\mathbf{K}$ the group

$$(k_{d-1}^* - 1) \ldots (k_1^* - 1)(k_0^* - 1)\mathbf{N}_{i+1}$$

is almost contained in the centralizer $C_{\mathbf{N}_{i+1}}(k)$, i. e.

$$\mathbf{K} \leq \widetilde{C}_G((k^*_{d-1}-1)\dots(k^*_1-1)(k^*_0-1)\mathbf{N}_{i+1})$$

By symmetry we have that

$$(k^*_{d-1}-1)\dots(k^*_1-1)(k^*_0-1)\mathbf{N}_{i+1} \lesssim \widetilde{C}_{\mathbf{N}_{i+1}}(\mathbf{K})$$

As $\mathbf{N}_{i+1}$ is an $\widetilde{\mathfrak{M}}_c$-group, the group $\widetilde{C}_{\mathbf{N}_{i+1}}(\mathbf{K})$ is definable, thus we may work modulo $\widetilde{C}_{\mathbf{N}_{i+1}}(\mathbf{K})$ and obtain that

$$(k^*_{d-1}-1)\dots(k^*_1-1)(k^*_0-1)\mathbf{N}_{i+1}/\widetilde{C}_{\mathbf{N}_{i+1}}(\mathbf{K})$$

is finite for all choices of an $d-1$ tuple $(k_0,\dots,k_{m-2})$ in $\mathbf{K}$. Thus as before we obtain by symmetry that

$$(k^*_{d-2}-1)\dots(k^*_1-1)(k^*_0-1)\mathbf{N}_{i+1} \lesssim \widetilde{C}_{\mathbf{N}_{i+1}}(\mathbf{K}/\widetilde{C}_{\mathbf{N}_{i+1}}(\mathbf{K})) = \widetilde{C}^2_{\mathbf{N}_{i+1}}(\mathbf{K}).$$

Repeating this process m many times yields that $\mathbf{N}_{i+1} \lesssim \widetilde{C}^d_{\mathbf{N}_{i+1}}(\mathbf{K}) = \widetilde{C}^d_{\mathbf{N}_{i+1}}(\mathbf{G}) \leq \widetilde{Z}_d(\mathbf{G})$. Thus $Z_{i+1}(N) \lesssim \widetilde{C}^d_G(G/\widetilde{Z}_j(G)) = \widetilde{Z}_{d+j}(G)$. As N and thus $Z_{i+1}(N)$ are normal in G, this yields immediately that $Z_{i+1}(N) \leq \widetilde{Z}_{d+j+1}(G)$ which finishes the proof. □

Proposition 4.10. *Let G be a locally nilpotent $\widetilde{\mathfrak{M}}_c$-group such that $G/\widetilde{Z}_k(G)$ has finite exponent for some natural number k. Then G is nilpotent-by-finite.*

Proof. First of all note, that it is enough to show that $G/\widetilde{Z}_k(G)$ is almost nilpotent, as this implies that G is almost nilpotent and any almost nilpotent subgroup of an $\widetilde{\mathfrak{M}}_c$-group is nilpotent-by-finite by Theorem 3.3(2). So let us replace G by $G/\widetilde{Z}_k(G)$ which is as well an $\widetilde{\mathfrak{M}}_c$-group by definition, locally nilpotent and of finite exponent.

The rest of the proof is analogous to the previous one. Using the same notation as before, the only difference is the way to find the bound d such that for any choice of $h_0,\dots,h_d$ each being an element of the tuple $\bar{g}$ and any element of $\mathbf{N}_{i+1}$ we have in the right module notation that

$$(h^*_0-1)(h^*_1-1)\dots(h^*_d-1)a = 0.$$

In this context, we know that $\mathbf{G}$ has finite exponent, say e. Thus, the group generated by $\bar{g}$ has finite order, say f. So for any $a \in \mathbf{N}_{i+1}$, the group generated by a and $\bar{g}$ has order at most $d = e^f \cdot f$ and as it is a finitely generated subgroup of a locally nilpotent group, it is nilpotent. Thus it is nilpotent of class at most d which gives the bound. □

5. Almost commutators and almost nilpotent subgroups of $\widetilde{\mathfrak{M}}_c$-groups

In section 2 we introduced the almost centralizer which is a centralizer "up to finite index". Thus one might ask, if there exists a corresponding notion of an "almost commutator". In this section we introduce such a notion and establish its basic properties. Even though, this notion might not have the desired properties in the general context, it has once we work in $\widetilde{\mathfrak{M}}_c$-groups. This allows us to generalize results on nilpotent group to almost nilpotent $\widetilde{\mathfrak{M}}_c$-groups.

For the rest of the section we fix a parameter set A and let G be an $|A|^+$-saturated and $|A|^+$-homogeneous group.

Definition 5.1. Let $\mathcal{G}$ be family of all A-definable subgroups of G. For two A-ind-definable subgroups H and K of G and , we define:

$$\tilde{[}H,K\tilde{]}_A := \bigcap\{L \in \mathcal{G} : \, L = L^{N_G(H)} = L^{N_G(K)}, \, H \lesssim \widetilde{C}_G(K/L)\}$$

and call it the *almost A-commutator* of H and K. If A is the empty set we omit the index and just say the almost commutator.

By Theorem 2.10 the almost commutator is symmetric, i. e. for two A-ind-definable subgroups H and K, we have $\tilde{[}H,K\tilde{]}_A = \tilde{[}K,H\tilde{]}_A$. Moreover, it is the intersection of definable subgroups of G. Note that the ordinary commutator of two A-ind-definable groups is not necessary definable nor the intersection of definable subgroups, and hence one cannot compare it with its approximate version, contrary to the almost centralizer.

As the final results on almost nilpotent subgroups of $\widetilde{\mathfrak{M}}_c$-groups we obtain only deals with normal subgroups, we **restrict our framework** from now on **to normal subgroups**. In this case, the subgroup $\tilde{[}H,K\tilde{]}$ is the intersection of normal subgroups in G which simplifies not only the definition but also many arguments and ambiguities in numerous proofs. Note anyhow that all results in this section could be generalized to arbitrary subgroups.

So let from now on $\mathcal{F}$ be the family of all A-definable normal subgroups of G. Note that this family is stable under finite intersections and finite products.

Then the definition of the almost commutator of two ind-definable normal subgroups H and K of G simplifies to:

$$\tilde{[}H,K\tilde{]}_A := \bigcap\{L \in \mathcal{F} : \, H \lesssim \widetilde{C}_G(K/L)\}.$$

As $H \lesssim \widetilde{C}_G(K/L)$ does not depend on the model we choose, the almost commutator does not depend on G. In other words, in any elementary extension of G, it will correspond to the intersection of the same A-definable groups.

To simplify notation in the rest of the section, **we add A as constants to the language** and thus for any two A-ind-definable subgroups H and K of G, **the almost commutator $\tilde{[}H,K\tilde{]}$ and the A-almost commutator $\tilde{[}H,K\tilde{]}_A$ in the new language coincide**. Therefore, we may omit A in the index in the rest of the section.

For two A-ind-definable normal subgroups H and K of G and L the intersection of A-definable subgroups of G, we obtain immediately that

$$H \lesssim \widetilde{C}_G(K/L) \quad \text{implies} \quad \tilde{[}H,K\tilde{]} \leq L.$$

The other implications is a consequence of the following result:

Lemma 5.2. *For any A-ind-definable normal subgroups H and K of G, we have that*

$$H \lesssim \widetilde{C}_G\left(K \,/\, \tilde{[}H,K\tilde{]}\right).$$

Moreover, $\tilde{[}H,K\tilde{]}$ is the smallest intersection of A-definable normal subgroups for which this holds.

Proof. We let $\mathcal{L}$ be the family of all A-definable normal subgroups L of G such that $H \lesssim \widetilde{C}_G(K/L)$. Suppose that $H \not\lesssim \widetilde{C}_G(K/\tilde{[}H,K\tilde{]})$. As $\tilde{[}H,K\tilde{]}$ is the intersection of

the normal subgroups L in $\mathcal{L}$, Properties 2.6 (10) yields that there is an L in $\mathcal{L}$ such that $H \not\lesssim \widetilde{C}_G(K/L)$. This contradicts the choice of the $\mathcal{L}$ and the first part of the lemma is established.

Now, let L be an intersection of A-definable normal subgroups such that $H \lesssim \widetilde{C}_G(K/L)$. Then, this holds for any of the definable subgroups in the intersection. Thus, those subgroups contain $\tilde{[}H, K\tilde{]}$ and therefore L contains $\tilde{[}H, K\tilde{]}$. □

Using the previous lemma we obtain immediately the following corollaries.

Corollary 5.3. *Let H and K be two A-ind-definable normal subgroups of G and L be an intersection of A-definable normal subgroups of G. Then, we have that $H \lesssim \widetilde{C}_G(K/L)$ if and only if $\tilde{[}H, K\tilde{]} \leq L$.*

Corollary 5.4. *For any almost commutator of two A-ind-definable normal subgroups H and K and any intersection L of A-definable normal subgroups, we have that $\tilde{[}H, K\tilde{]} \lesssim L$ if and only if $\tilde{[}H, K\tilde{]} \leq L$*

Proof. The implication from right to left is trivial. So suppose that $\tilde{[}H, K\tilde{]} \lesssim L$. Lemma 5.2 yields that $H \lesssim \widetilde{C}_G(K/\tilde{[}H, K\tilde{]})$. Furthermore, by assumption we have that the intersection of A-definable subgroups $\tilde{[}H, K\tilde{]} \cap L$ has bounded index in $\tilde{[}H, K\tilde{]}$, i. e. we have that $\tilde{[}H, K\tilde{]} \cap L \sim \tilde{[}H, K\tilde{]}$. So Properties 2.6 (7) yields that $H \lesssim \widetilde{C}_G(K/(\tilde{[}H, K\tilde{]} \cap L))$. As $\tilde{[}H, K\tilde{]}$ is the smallest subgroup for which this holds, we obtain the result. □

Lemma 5.5. *Let H, K, N and M be A-ind-definable normal subgroups of G.*

(1) If $N \lesssim H$ and $M \lesssim K$ then $\tilde{[}N, M\tilde{]} \leq \tilde{[}H, K\tilde{]}$.
(2) If H (resp. K) is an intersection of definable groups $\tilde{[}H, K\tilde{]}$ is contained in H (resp. K).

Proof. (1) Let L be an arbitrary A-definable normal subgroup of G such that H is almost contained in $\widetilde{C}_G(K/L)$. Since $K \cap M$ is a subgroup of K, we have that H is almost contained in $\widetilde{C}_G(K \cap M/L)$ as well. As N is almost contained in H, we may replace H by N and obtain N is almost contained in $\widetilde{C}_G(K \cap M/L)$. Additionally, the almost centralizer of two commensurate A-ind-definable subgroups such as M and $K \cap M$ coincides. Thus we conclude that N is almost contained in $\widetilde{C}_G(M/L)$ or in orther words $\tilde{[}N, M\tilde{]}$ is a subgroup of L. As L was arbitrary, the almost commutator $\tilde{[}N, M\tilde{]}$ is contained in $\tilde{[}H, K\tilde{]}$.

(2) We have trivially that $H \leq \widetilde{C}_G(K/H)$. So if H is the intersection of definable groups, we conclude that the almost commutator of H and K is contained in H.

□

An easy application of compactness yields:

Lemma 5.6. *Let H and K be two A-type-definable normal subgroups of an $|A|^+$-saturated group G. Fix $\{H_i : i \in I\}$ and $\{K_s : s \in S\}$ two projective systems of A-definable sets such that $H = \bigcap_{i \in I} H_i$ and $K = \bigcap_{s \in S} K_s$ (i. e. for any i, j in I and s, t in S there exists n in I and m in S such that $H_i \cap H_j \supseteq H_n$ and $K_s \cap K_t \supseteq K_m$).*

Then, we have that

$$HK = \bigcap_{(i,s)\in I\times S} H_i K_s.$$

Lemma 5.7. *Let H, K, and L be A-ind-definable normal subgroups of G. Then we have*

$$\tilde{[}HK, L\tilde{]} \leq \tilde{[}H, L\tilde{]} \cdot \tilde{[}K, L\tilde{]}.$$

Proof.

$$\begin{aligned} \tilde{[}H, L\tilde{]} \cdot \tilde{[}K, L\tilde{]} &= \bigcap\{M \in \mathcal{F} : H \lesssim \widetilde{C}_G(L/M)\} \cdot \bigcap\{N \in \mathcal{F} : K \lesssim \widetilde{C}_G(L/N)\} \\ &\overset{5.6}{=} \bigcap\{M \cdot N : M, N \in \mathcal{F},\, H \lesssim \widetilde{C}_G(L/M),\, K \lesssim \widetilde{C}_G(L/N)\} \end{aligned}$$

As the product of two groups in $\mathcal{F}$ is again a subgroup which belongs to $\mathcal{F}$ and since $H \lesssim \widetilde{C}_G(L/M)$ and $K \lesssim \widetilde{C}_G(L/N)$, by Properties 2.6 we have that $H \lesssim \widetilde{C}_G(L/MN)$ and $K \lesssim \widetilde{C}_G(L/MN)$. So by Lemma 2.4 we obtain $HK \lesssim \widetilde{C}_G(L/MN)$. Thus, the previous set contains the following one:

$$\begin{aligned} &\supseteq \bigcap\{P \in \mathcal{F} : HK \lesssim \widetilde{C}_G(L/P)\} \\ &= \tilde{[}HK, L\tilde{]} \end{aligned}$$

This finishes the proof. □

Another useful behavior of the almost centralizer is the following:

Lemma 5.8. *Let H and K be two A-ind-definable normal subgroups of G and L be an intersection of A-definable normal subgroups of G. If $\tilde{[}H, K\tilde{]} \leq L$ then $H \leq \widetilde{C}_G^2(K/L)$.*

Proof. Let $\tilde{[}H, K\tilde{]}$ be contained in L. By Corollary 5.3, we have that $H \lesssim \widetilde{C}_G(K/L)$. So $H/\widetilde{C}_G(K/L)$ is a bounded group and as H is normal in G, it contains $h^k \cdot \widetilde{C}_G(K/L)$ for all h in H and k in K. Hence the set $\{h^k : k \in K\}/\widetilde{C}_G(K/L)$ of conjugates of any element h in H by K modulo $\widetilde{C}_G(K/L)$ is bounded. As the size of this set corresponds to the index of $C_K(h/\widetilde{C}_K(K/L))$ in K, the group H is contained in the almost centralizer $\widetilde{C}_G(K/\widetilde{C}_K(K/L))$, i. e. the group H is contained in $\widetilde{C}_G^2(K/L)$. □

5.1. **Almost nilpotent subgroups of $\widetilde{\mathfrak{M}}_c$-groups.** A consequence of the definability of the almost centralizer in $\widetilde{\mathfrak{M}}_c$-groups (Proposition 2.23) is that the almost commutator is "well behaved". For example, we obtain the lemma below:

Lemma 5.9. *Let G be an $\widetilde{\mathfrak{M}}_c$-group and H be an A-ind-definable normal subgroup of G. For any natural number n, so we have that*

$$\tilde{[}H, \widetilde{C}_G^n(H)\tilde{]} \leq \widetilde{C}_G^{n-1}(H)$$

Proof. We have that

$$\tilde{[}H, \widetilde{C}_G^n(H)\tilde{]} = \tilde{[}H, \widetilde{C}_G\left(H \,/\, \widetilde{C}_G^{n-1}(H)\right)\tilde{]}$$

by definition of the almost centralizer. Moreover, the almost centralizer $\widetilde{C}_G^{n-1}(H)$ is an A-definable subgroup of G since G is an $\widetilde{\mathfrak{M}}_c$-group. Thus

$$\tilde{[}H, \widetilde{C}_G\left(H \,/\, \widetilde{C}_G^{n-1}(H)\right)\tilde{]} \leq \widetilde{C}_G^{n-1}(H)$$

as $\widetilde{C}_G(H/\widetilde{C}_G^{n-1}(H))$ is trivially contained in itself and we obtain the result. □

The main goal is to show a version of the Hall nilpotency criteria [9, Theorem 7] for almost nilpotent $\widetilde{\mathfrak{M}}_c$-groups. To do so, we first have to state the approximate three subgroups lemma in terms of the almost commutator.

Notation. Let H, K and L be A-ind-definable normal subgroups of a given group G. We write $\tilde{[}H, K, L\tilde{]}$ for $\tilde{[}\tilde{[}H, K\tilde{]}, L\tilde{]}$. Note that the group $\tilde{[}H, K\tilde{]}$ is an A-ind-definable normal subgroup of G and thus $\tilde{[}\tilde{[}H, K\tilde{]}, L\tilde{]}$ is well defined.

Now, given an $\widetilde{\mathfrak{M}}_c$-group G, we have that the almost centralizer of any A-ind-definable subgroup in G is definable. Thus for H, K and L such that H and K normalize L, we have that $H \lesssim \widetilde{C}_G(K/\widetilde{C}_G(L))$ if and only if $\tilde{[}H, K, L\tilde{]}$ is trivial. With this equivalence, we may phrase Theorem 2.15 for $\widetilde{\mathfrak{M}}_c$-groups as below.

Corollary 5.10. *Let H, K and L be three A-ind-definable strongly normal subgroups of an $\widetilde{\mathfrak{M}}_c$-group G. Then for any M which is an intersection of A-definable normal subgroups of G, we have that*

$$\tilde{[}H, K, L\tilde{]} \leq M \text{ and } \tilde{[}K, L, H\tilde{]} \leq M \text{ imply } \tilde{[}L, H, K\tilde{]} \leq M.$$

Proof. Let M be equal to the intersection of definable subgroups M_i with $i < \kappa$. For any i less than κ, we may work in the group G modulo M_i which is a quotient of an $\widetilde{\mathfrak{M}}_c$-group by a definable normal subgroup and so an $\widetilde{\mathfrak{M}}_c$-group as well. Hence, Theorem 2.15 (working modulo the definable group M_i) yields that

$$H \lesssim \widetilde{C}_G\left(K \,/\, \widetilde{C}_G(L/M_i)\right) \text{ and } K \lesssim \widetilde{C}_G\left(L \,/\, \widetilde{C}_G(H/M_i)\right)$$

imply

$$L \lesssim \widetilde{C}_G\left(H \,/\, \widetilde{C}_G(K/M_i)\right).$$

Which we can translate to

$$\tilde{[}H, K, L\tilde{]} \leq M_i \text{ and } \tilde{[}K, L, H\tilde{]} \leq M_i \text{ imply } \tilde{[}L, H, K\tilde{]} \leq M_i$$

So the statement is true for any M_i and hence for the intersection. □

Now, we want to define the notion of an *almost lower central series* and find a characterization of being almost nilpotent via this series. In literature the ordinary lower central series of a subgroup H of G is defined as follows:

$$\gamma_1 H = H \quad \text{and} \quad \gamma_{i+1} H = [\gamma_i H, H].$$

Analogously, we introduce a notion of the almost lower central series:

Definition 5.11. We define the *almost lower A-central series* of an A-ind-definable subgroup H of G as follows:

$$(\widetilde{\gamma}_1 H)_A = H \quad \text{and} \quad (\widetilde{\gamma}_{i+1} H)_A = \tilde{[}\widetilde{\gamma}_i H, H\tilde{]}_A.$$

We also refer to $(\widetilde{\gamma}_n H)_A$ as the *iterated n^{th} almost commutator* of H. Again, if A is the empty set we omit the index.

As we have added A as constants to the language, we may omit it again in the subscript of the iterated n^{th} almost commutator for the rest of the section.

Remark 5.12. The almost lower center series is well-defined as $\widetilde{[H,H]}$ is the intersection of A-definable groups and hence A-type-definable. Thus, by induction we see that $\widetilde{\gamma}_{i+1}H = \widetilde{[\widetilde{\gamma}_i H, H]}$ is again an A-type-definable subgroup.

To make the proofs more readable, we fix the following notation:

Notation. If $K_1, \dots, K_n$ are A-ind-definable subgroups of G, let

$$\widetilde{\gamma}_n(K_1, \dots, K_n) := \widetilde{[\dots \widetilde{[\widetilde{[K_0, K_1]}, K_2]}, \dots, K_n]}.$$

If $K_i, \dots, K_{i+j-1}$ are all equal to K we can replace the sequence by K^j, i. e. write $\widetilde{\gamma}_n(K_1, \dots, K_n)$ as $\widetilde{\gamma}_n(K_1, \dots K_{i-1}, K^j, K_{i+j}, \dots, K_n)$. Also,

$$\widetilde{\gamma}_{i+0+j}(K_1, \dots K_i, K^0, K_{i+1}, \dots, K_{i+j}) = \widetilde{\gamma}_{i+j}(K_1, \dots K_i, K_{i+1}, \dots, K_{i+j}).$$

Observe that $\widetilde{\gamma}_n(H^n)$ is another way of writing $(\widetilde{\gamma}_n H)$.

We want to establish a connection between the triviality of the nth iterated almost commutator of a normal subgroup H of G and the almost nilpotency class of H.

Lemma 5.13. *If H is an A-ind-definable normal subgroup of an $\widetilde{\mathfrak{M}_c}$-group G and almost nilpotent of class n, then $\widetilde{\gamma}_{n+1}H$ is trivial. Conversely, if $\widetilde{\gamma}_{n+1}H$ is trivial, then H is almost nilpotent of class at most $n+1$.*

Proof. To prove the first result, we show by induction on $i \leq n$ that the almost commutator $\widetilde{\gamma}_{i+1}H$ is contained in $\widetilde{C}_G^{n-i}(H)$. As H is almost nilpotent of class n, i. e. $H \leq \widetilde{C}_G^n(H)$, the inclusion is satisfied for i equals to zero. Now suppose it holds for all natural numbers smaller or equal to i. The induction hypothesis together with Lemma 5.5(1) implies that $\widetilde{\gamma}_{i+2}H = \widetilde{[\widetilde{\gamma}_{i+1}H, H]}$ is contained in $\widetilde{[\widetilde{C}_G^{n-i}(H), H]}$. Moreover, by Lemma 5.9 we have that $\widetilde{[\widetilde{C}_G^{n-i}(H), H]}$ is contained in $\widetilde{C}_G^{n-i-1}(H)$. Hence $\widetilde{\gamma}_{i+2}H$ is also contained in $\widetilde{C}_G^{n-i-1}(H)$ which finishes the induction. Letting i be equal to n, we obtain that $\widetilde{\gamma}_{n+1}H$ is contained in $\widetilde{C}_G^0(H)$ which is the trivial group by definition.

For the second result, we first show the following inclusion by induction that for i less or equal to $n-1$:

$$\widetilde{\gamma}_{(n+1)-i}H \leq \widetilde{C}_G^i(H).$$

For $i = 0$, the inequality holds by hypothesis. Now we assume, the inequality holds for $i < n-1$. Thus $\widetilde{\gamma}_{(n+1)-i}H \leq \widetilde{C}_G^i(H)$ or in other words $\widetilde{[\widetilde{\gamma}_{(n+1)-(i+1)}H, H]} \leq \widetilde{C}_G^i(H)$. By Corollary 5.3, we have that

$$\widetilde{\gamma}_{(n+1)-(i+1)}H \lesssim \widetilde{C}_G\left(H \,/\, \widetilde{C}_G^i(H)\right) = \widetilde{C}_G^{i+1}(H).$$

By Corollary 5.4, as $(n+1)-(i+1)$ is at least 2, finally we obtain $\widetilde{\gamma}_{(n+1)-(i+1)}H \leq \widetilde{C}_G^{i+1}(H)$ which finishes the induction.

Now, we let i be equal to $n-1$ we obtain: $\widetilde{[H,H]} \leq \widetilde{C}_G^{n-1}(H)$. Then by Lemma 5.8 we have that $H \leq \widetilde{C}_G^{n+1}(H)$ and hence H is almost nilpotent of class $n+1$. □

The next three lemmas are the preparation to finally show the approximate version of Hall's nilpotency criteria.

Lemma 5.14. *Let N be a normal subgroup of an $\widetilde{\mathfrak{M}}_c$-group G. Then for all positive natural numbers n and m, we have that*

$$\tilde{[}\widetilde{\gamma}_n N, \widetilde{\gamma}_m N\tilde{]} = \widetilde{\gamma}_{n+m} N.$$

The proof is an easy induction on m.

Lemma 5.15. *Let N be an A-ind-definable normal subgroup of an $\widetilde{\mathfrak{M}}_c$-group G. Then, for any natural numbers $n \geq 2$, i and j we have that*

$$\tilde{\big[}\widetilde{\gamma}_n N, \widetilde{\gamma}_{i+j}(N^i, G^j)\tilde{\big]} \leq \widetilde{\gamma}_{n+i} N$$

where $\tilde{\big[}\widetilde{\gamma}_n N, \widetilde{\gamma}_{i+j}(N^i, G^j)\tilde{\big]}$ for $i = j = 0$ equals $\widetilde{\gamma}_n N$.

Proof. Note first that as n is at least 2, the group $\widetilde{\gamma}_n N$ is an intersection of normal A-definable groups. Thus for i equal to 0, we have that $\tilde{[}\widetilde{\gamma}_n N, \widetilde{\gamma}_j G\tilde{]} \leq \widetilde{\gamma}_n N$ by Lemma 5.5(2).

Now, let i be equal to 1. Note first that by Lemma 5.5(1)+(2),

$$\tilde{\big[}\widetilde{\gamma}_n N, \widetilde{\gamma}_{1+j}(N, G^j)\tilde{\big]} \leq \tilde{[}\widetilde{\gamma}_n N, \tilde{[}N, G\tilde{]}\tilde{]}. \quad (*)$$

Furthermore, we have the following:

$$\tilde{\big[}\tilde{[}\widetilde{\gamma}_n N, G\tilde{]}, N\tilde{\big]} \overset{5.5(1)+(2)}{\leq} \tilde{[}\widetilde{\gamma}_n N, N\tilde{]} = \widetilde{\gamma}_{n+1} N,$$

$$\tilde{\big[}\tilde{[}\widetilde{\gamma}_n N, N\tilde{]}, G\tilde{\big]} \overset{5.5(2)}{\leq} \tilde{[}\widetilde{\gamma}_n N, N\tilde{]} = \widetilde{\gamma}_{n+1} N.$$

Hence, as $\widetilde{\gamma}_{n+1} N$ is the intersection of A-definable subgroups, the three subgroups lemma (Corollary 5.10) yields that $\tilde{[}\widetilde{\gamma}_n N, \tilde{[}N, G\tilde{]}\tilde{]}$ is contained in $\widetilde{\gamma}_{n+1} N$. Now, by $(*)$ we conclude for i equals to 1.

If i is greater than 1, we have that

$$\tilde{\big[}\widetilde{\gamma}_n N, \widetilde{\gamma}_{i+j}(N^i, G^j)\tilde{\big]} \overset{5.5(1)+(2)}{\leq} \tilde{[}\widetilde{\gamma}_n N, \widetilde{\gamma}_i N\tilde{]}.$$

By Lemma 5.14, we obtain that $\tilde{[}\widetilde{\gamma}_n N, \widetilde{\gamma}_i N\tilde{]}$ is contained in $\widetilde{\gamma}_{n+i} N$ which finishes the proof. □

The following lemma is [9, Lemma 7] generalized to our framework.

Lemma 5.16. *Let N be a A-ind-definable normal subgroup of an $\widetilde{\mathfrak{M}}_c$-group G and suppose that there exists a natural number $m > 0$ such that $\widetilde{\gamma}_{m+1}(N, G^m) \lesssim \tilde{[}N, N\tilde{]}$. Then, for all natural numbers $r > 0$ we have that*

$$\widetilde{\gamma}_{rm+1}(N^r, G^{rm-r+1}) \leq \widetilde{\gamma}_{r+1} N.$$

Proof. We start this proof with the following claim.

Claim. *Let X be an ind-definable normal subgroup of G. Then for any $n > 0$, we have that*

$$\widetilde{\gamma}_{n+2}(X, N, G^n) \leq \prod_{i=0}^{n} \tilde{\big[}\widetilde{\gamma}_{i+1}(X, G^i),\, \widetilde{\gamma}_{n-i+1}(N, G^{n-i})\tilde{\big]}. \tag{2}$$

Proof of the claim. We prove the claim by induction on $n > 0$. Let n be equal to 1. Trivially we have that

$$\tilde{[}N, G, X\tilde{]} \leq \tilde{[}X, \tilde{[}N, G\tilde{]}\tilde{]} \cdot \tilde{[}\tilde{[}X, G\tilde{]}, N\tilde{]}$$

and

$$\tilde{[}G, X, N\tilde{]} \leq \tilde{[}X, \tilde{[}N, G\tilde{]}\tilde{]} \cdot \tilde{[}\tilde{[}X, G\tilde{]}, N\tilde{]}.$$

The three subgroups lemma (Corollary 5.10) insures that

$$\tilde{[}X, N, G\tilde{]} \leq \tilde{[}X, \tilde{[}N, G\tilde{]}\tilde{]} \cdot \tilde{[}\tilde{[}X, G\tilde{]}, N\tilde{]}$$

and so the claim holds for $n = 1$.

Now, assume the claim holds for some $n > 0$. We compute:

$$\begin{aligned} \widetilde{\gamma}_{n+3}(X, N, G^{n+1}) &= \tilde{[}\widetilde{\gamma}_{n+2}(X, N, G^n), G\tilde{]} \\ &\underset{5.5(1)}{\overset{IH}{\leq}} \tilde{\Big[}\prod_{i=0}^{n} \tilde{[}\widetilde{\gamma}_{i+1}(X, G^i),\, \widetilde{\gamma}_{n-i+1}(N, G^{n-i})\tilde{]}, G\tilde{\Big]}. \end{aligned}$$

As all factors are invariant normal subgroups of G we may apply Lemma 5.7 finitely many times to the last expression and continue the computation:

$$\leq \prod_{i=0}^{n} \tilde{\Big[}\tilde{[}\widetilde{\gamma}_{i+1}(X, G^i),\, \widetilde{\gamma}_{n-i+1}(N, G^{n-i})\tilde{]}, G\tilde{\Big]}. \tag{3}$$

To simplify notation, we let $X_i = \widetilde{\gamma}_{i+1}(X, G^i)$ and $N_j = \widetilde{\gamma}_{j+1}(N, G^j)$. Now, fix some i less or equal to n. We obtain that

$$\begin{aligned} \tilde{[}\tilde{[}\widetilde{\gamma}_{i+1}(X, G^i), G\tilde{]}, \widetilde{\gamma}_{n-i+1}(N, G^{n-i})\tilde{]} &= \tilde{[}\widetilde{\gamma}_{i+2}(X, G^{i+1}), \widetilde{\gamma}_{n-i+1}(N, G^{n-i})\tilde{]} \\ &= \tilde{[}X_{i+1}, N_{n-i}\tilde{]} \end{aligned}$$

and

$$\begin{aligned} \tilde{[}\tilde{[}\widetilde{\gamma}_{n-i+1}(N, G^{n-i}), G\tilde{]}, \widetilde{\gamma}_{i+1}(X, G^i)\tilde{]} &= \tilde{[}\widetilde{\gamma}_{n-i+2}(N, G^{n+1-i}), \widetilde{\gamma}_{i+1}(X, G^i)\tilde{]} \\ &= \tilde{[}N_{n-i+1}, X_i\tilde{]} \\ &= \tilde{[}X_i, N_{n-i+1}\tilde{]}. \end{aligned}$$

As the groups on the right are intersection of definable subgroups of G, using the approximate three subgroups lemma (Corollary 5.10), we obtain the following inequation for the ith factor of (3):

$$\tilde{\Big[}\tilde{[}\widetilde{\gamma}_{i+1}(X, G^i),\, \widetilde{\gamma}_{n-i+1}(N, G^{n-i})\tilde{]}, G\tilde{\Big]} \leq \tilde{[}X_{i+1}, N_{n-i}\tilde{]} \cdot \tilde{[}X_i, N_{n-i+1}\tilde{]}.$$

Over all, we get that

$$\widetilde{\gamma}_{n+3}(X, N, G^n) \leq \prod_{i=0}^{n+1} \tilde{[}X_i, N_{n-i+1}\tilde{]} = \prod_{i=0}^{n+1} \tilde{[}\widetilde{\gamma}_{i+1}(X, G^i),\, \widetilde{\gamma}_{n+1-i+1}(N, G^{n+1-i})\tilde{]}.$$

$\square_{\text{claim}}$

Now, we prove the Lemma by induction on $r > 0$. By Corollary 5.4, the almost inequality $\widetilde{\gamma}_{m+1}(N, G^m) \lesssim \tilde{[}N, N\tilde{]}$ implies immediately $\widetilde{\gamma}_{m+1}(N, G^m) \leq \tilde{[}N, N\tilde{]}$. Thus,

for r equals to 1 the lemma holds trivially by the hypothesis. Assume that the result holds for a given r greater or equal to 1. We want to prove that

$$\widetilde{\gamma}_{(r+1)m+1}(N^{r+1}, G^{(r+1)m-r}) \leq \widetilde{\gamma}_{r+2}N.$$

Now consider equation (2) with $n = (r+1)m - r$ and X replaced by $\widetilde{\gamma}_r N^r$. This gives us:

$$\begin{aligned} (4)\ \widetilde{\gamma}_{(r+1)m+1}(N^{r+1}, G^{(r+1)m-r}) &= \widetilde{\gamma}_{((r+1)m-r)+2}(\widetilde{\gamma}_r N, N, G^{(r+1)m-r}) \\ (5) \qquad &\leq \prod_{i=0}^{(r+1)m-r} \tilde{[}\widetilde{\gamma}_{i+1}(\widetilde{\gamma}_r N, G^i),\ \widetilde{\gamma}_{n-i+1}(N, G^{n-i})\tilde{]}. \end{aligned}$$

The group on the left hand side is the one we want to analyze. The goal is to prove that all factors on the right hand side are contained in $\widetilde{\gamma}_{r+2}N$. So, we consider the factor indexed by i.

Supoose first that i is greater than $rm - r$. By induction hypothesis, we have that

$$\widetilde{\gamma}_{rm+1}(N^r, G^{rm-r+1}) \leq \widetilde{\gamma}_{r+1}N.$$

As $\widetilde{\gamma}_{rm+1}(N^r, G^{rm-r+1})$ is normal in G and an intersection of A-definable groups, using Lemma 5.5 (2) we obtain that $\widetilde{\gamma}_{r+i}(N^r, G^i) \leq \widetilde{\gamma}_{r+1}N$ and

$$\begin{aligned} \tilde{[}\widetilde{\gamma}_{i+1}(\widetilde{\gamma}_r N, G^i),\ \widetilde{\gamma}_{n-i+1}(N, G^{n-i})\tilde{]} &\overset{5.5(1)}{\leq} \tilde{[}\widetilde{\gamma}_{r+1}N,\ \widetilde{\gamma}_{n-i+1}(N, G^{n-i})\tilde{]} \\ &\overset{5.15}{\leq} \widetilde{\gamma}_{r+2}N. \end{aligned}$$

Now, assume that $i \leq rm - r$. By the case $r = 1$, we have that $\widetilde{\gamma}_{m+1}(N, G^m) \leq \tilde{[}N, N\tilde{]}$. As $n - i$ is greater than m and $\widetilde{\gamma}_{m+1}(N, G^m)$ is an intersection of normal subgroup of G, we also have that $\widetilde{\gamma}_{n-i+1}(N, G^{n-i}) \leq \tilde{[}N, N\tilde{]}$. So we may compute:

$$\begin{aligned} \tilde{[}\widetilde{\gamma}_{i+1}((\widetilde{\gamma}_r N^r), G^i),\ \widetilde{\gamma}_{n-i+1}(N, G^{n-i})\tilde{]} &\overset{5.5(1)}{\leq} \tilde{[}\widetilde{\gamma}_{i+r}(N^r, G^i),\ \tilde{[}N, N\tilde{]}\tilde{]} \\ &\overset{5.15}{\leq} \widetilde{\gamma}_{r+2}N. \end{aligned}$$

Hence all factors, and therefore $\widetilde{\gamma}_{(r+1)m+1}(N^{r+1}, G^{(r+1)m-r})$, are contained in $\widetilde{\gamma}_{r+2}N$. This finishes the proof. □

Now, we are ready to generalize Hall's nilpotency criteria to $\widetilde{\mathfrak{M}}_c$-groups.

Corollary 5.17. *Let N be an A-ind-definable normal subgroup of an $\widetilde{\mathfrak{M}}_c$-group G. If N is almost nilpotent of class m and $G/\tilde{[}N, N\tilde{]}$ is almost nilpotent of class n then G is almost nilpotent of class at most $\binom{m+1}{2}n - \binom{n}{2} + 1$.*

Proof. By hypothesis and Lemma 5.13 we have that

$$\widetilde{\gamma}_{m+1}N = 1 \quad \text{and} \quad \widetilde{\gamma}_{n+1}G \leq \tilde{[}N, N\tilde{]}. \quad (*)$$

Hence

$$\widetilde{\gamma}_{n+1}(N, G^n) \leq \tilde{[}N, N\tilde{]}$$

and whence N satisfies the hypothesis of Lemma 5.16. Thus

$$(6) \qquad \widetilde{\gamma}_{rn+1}(N^r, G^{rn-r+1}) \leq \widetilde{\gamma}_{r+1}N$$

holds for all natural numbers r. Let $f(x) = \binom{x+1}{2}\, n - \binom{x}{2}$. For every i greater than 1, we obtain that

$$\widetilde{\gamma}_{f(i)+1}G \leq \widetilde{\gamma}_{i+1}N$$

Choosing i to be m we get that

$$\widetilde{\gamma}_{f(m)+1}G \leq \widetilde{\gamma}_{m+1}N = \{1\}.$$

So Lemma 5.13 yields that G is almost nilpotent of class at most $\binom{m+1}{2}n - \binom{n}{2} + 1$. □

Corollary 5.18. *Let H and K be A-ind-definable normal subgroups of an $\widetilde{\mathfrak{M}}_c$-group G.*

(1) If $\widetilde{[H,H]} = \widetilde{[G,G]}$, then for all $r \geq 2$, we have $\widetilde{\gamma}_r H = \widetilde{\gamma}_r G$.
(2) If $\widetilde{[H,K]}$ and $\widetilde{[H,H]}$ are contained in $\widetilde{[K,K]}$, then for all $r \geq 2$, the almost commutator $\widetilde{\gamma}_r H$ is contained in $\widetilde{\gamma}_r K$.

Proof. (1) As H is a subgroup of G, we have that $\widetilde{\gamma}_r H \leq \widetilde{\gamma}_r G$ holds trivial for all $r \geq 2$. We prove the inverse inclusion by induction on r. For r equals to 2, the statement holds by hypothesis. Now suppose that the statement holds for all natural numbers smaller than $r > 2$. Thus,

$$\widetilde{\gamma}_r G \leq \widetilde{\gamma}_r(H^{r-1}, G).$$

Furthermore, $\widetilde{[H,G]} \leq \widetilde{[H,H]}$, hence we may apply Lemma 5.16 with $m = 1$ and obtain that

$$\widetilde{\gamma}_r(H^{r-1}, G) \leq \widetilde{\gamma}_r H$$

which finishes the proof.

(2) Consider $L = HK$. Then we can compute that

$$\widetilde{[L,L]} = \widetilde{[HK,HK]} \overset{5.7}{\leq} \widetilde{[H,H]} \cdot \widetilde{[K,K]} \cdot \widetilde{[H,K]} = \widetilde{[K,K]}.$$

By the first part of the corollary we can conclude that $\widetilde{\gamma}_r H^r \leq \widetilde{\gamma}_r L^r = \widetilde{\gamma}_r K^r$. □

5.2. **Other applications of the almost three subgroups lemma and results on almost nilpotent groups.** Using symmetry of the almost centralizer, the three subgroups lemma and the definabilily of the almost centralizer, we may generalize a theorem due to Hall [12, Satz III.2.8] for the ordinary centralizer to our context.

Proposition 5.19. *Let G be an $\widetilde{\mathfrak{M}}_c$-group, $N_0 \geq N_1 \geq \cdots \geq N_m \geq \ldots$ be a descending sequence of A-definable normal subgroups of G, and H be an A-ind-definable normal subgroup of G. Suppose that for all $i \in \mathbb{N}$, we have $H \lesssim \widetilde{C}_G(N_i/N_{i+1})$. We define for $i > 0$,*

$$H_i := \bigcap_{k \in \mathbb{N}} \widetilde{C}_H(N_k/N_{k+i}).$$

Then we have that for all positive natural numbers i and j, the group H_i is almost contained in $\widetilde{C}_G(H_j/H_{i+j})$, the group H is almost contained in $\widetilde{C}_G^i\left(H \,/\, \widetilde{C}_G(N_{j-1}/N_{i+j})\right)$ and therefore $\widetilde{[\widetilde{\gamma}_{i+1}H, N_{j-1}]} \leq N_{i+j}$.

Remark 5.20. The non-approximate version [12, Satz III.2.8] states that for H_i defined as $\bigcap_{k \in \mathbb{N}} C_H(N_k/N_{k+i})$ we have that for all positive natural numbers i and j, $[H_i, H_j] \leq H_{i+j}$ and $[\gamma_{i+1}H, N_{j-1}] \leq N_{i+j}$.

Proof. Note that H is equal to $\bigcap_{k\in\mathbb{N}} \widetilde{C}_G(N_k/N_{k+i}) \cap H$ and thus the intersection of an ind-definable subgroup and boundedly many definable subgroups. So H_i is as well an ind-definable subgroup of G.

As $\widetilde{C}_G(N_k/N_{k+i+j})$ is definable for any natural number k, Properties 2.6 (9) yields that

$$H_i \lesssim \widetilde{C}_G(H_j/H_{i+j}) = \widetilde{C}_G\left(H_j \Big/ \bigcap_{k\in\mathbb{N}} \widetilde{C}_G(N_k/N_{k+i+j})\right)$$

if and only if for all natural number k we have that

$$H_i \lesssim \widetilde{C}_G\left(H_j \,/\, \widetilde{C}_G(N_k/N_{k+i+j})\right).$$

So it is enough to show the latter result for any natural number $k \in \mathbb{N}$. So fix some k, i and j in $\mathbb{N}$. By the definition of H_j we have that $H_j \leq \widetilde{C}_G(N_{k+i}/N_{k+i+j})$. Symmetry modulo definable subgroups for almost centralizers yields that $N_{k+i} \lesssim \widetilde{C}_G(H_j/N_{k+i+j})$. This implies that

$$H_i \;\leq\; \widetilde{C}_G\left(N_k/N_{k+i}\right) \leq \widetilde{C}_G\left(N_k \,/\, \widetilde{C}_G(H_j/N_{k+i+j})\right). \tag{7}$$

Exchanging the role of i and j we obtain as well that

$$H_j \;\leq\; \widetilde{C}_G\left(N_k \,/\, \widetilde{C}_G(H_i/N_{k+j+i})\right) = \widetilde{C}_G\left(N_k \,/\, \widetilde{C}_G(H_i/N_{k+i+j})\right). \tag{8}$$

Using again symmetry modulo definable subgroups for almost centralizers to (7), we get:

$$N_k \;\lesssim\; \widetilde{C}_G\left(H_i \,/\, \widetilde{C}_G(H_j/N_{k+i+j})\right). \tag{9}$$

Working in G/N_{k+i+j}, we can apply the three subgroups lemma (Theorem 2.15) to the equalities (8) and (9) since all N_i's and all H_i's normalize each other and obtain

$$H_i \lesssim \widetilde{C}_G\left(H_j \,/\, \widetilde{C}_G(N_k/N_{k+i+j})\right).$$

As k was arbitrary, this establishes the first part of the theorem.

In particular, we have that for any natural numbers i and j greater than 0

$$\begin{aligned} H_1 \;&\lesssim\; \widetilde{C}_G(H_1/H_2) \lesssim \widetilde{C}_G\left(H_1 \,/\, \widetilde{C}_G(H_1/H_3)\right) = \widetilde{C}_G^2(H_1/H_3) \\ &\lesssim\; \ldots \lesssim \widetilde{C}_G^i(H_1/H_{i+1}) \lesssim \widetilde{C}_G^i\left(H_1 \,/\, \widetilde{C}_G(N_{j-1}/N_{i+j})\right) \end{aligned}$$

By hypothesis we have that H_1 is a bounded intersection of groups which are commensurate with H and whence it is itself commensurate with H. As two commensurate groups have the same almost centralizer, the same almost inclusion holds for H which finishes the proof. □

Using the previous result and definability of the almost centralizers, we may find a version of [4, Lemma 2.4] in terms of the almost centralizer:

Corollary 5.21. *Let H be an A-ind-definable normal subgroup of an $\widetilde{\mathfrak{M}}_c$-group G. Then for any $0 < i < j$, we have that*

$$H \lesssim \widetilde{C}_G^i\left(H \Big/ \widetilde{C}_G\left(\widetilde{C}_G^j(H) \,/\, \widetilde{C}_G^{j-i-1}(H)\right)\right)$$

Proof. For $k < 2j-1$, we let $N_k = \widetilde{C}_G^{2j-1-k}(H)$ and for $k \geq 2j-1$, we let N_k be the trivial group. As G is an $\widetilde{\mathfrak{M}}_c$-group, all N_k are definable. Note that for any natural number n, the almost centralizer $\widetilde{C}_G^n(H)$ is definable and $\widetilde{C}_G^{n+1}(H) = \widetilde{C}_G(H/\widetilde{C}_G^n(H))$ is contained in itself. Hence, symmetry of the almost centralizer (Theorem 2.10) yield that

$$H \lesssim \widetilde{C}_G\left(\widetilde{C}_G^{n+1}(H)/\widetilde{C}_G^n(H)\right)$$

and whence

$$H \lesssim \widetilde{C}_G(N_k/N_{k+1}).$$

So we may apply Proposition 5.19 to the ind-definable subgroup H and the sequence of definable groups N_i. This gives us that

$$H \lesssim \widetilde{C}_G^i\left(H \,/\, \widetilde{C}_G(N_{j-1}/N_{i+j})\right) = \widetilde{C}_G^i\left(H \,\Big/\, \widetilde{C}_G\left(\widetilde{C}_G^j(H) \,/\, \widetilde{C}_G^{j-i-1}(H)\right)\right)$$

□

Using the new notion of almost commutator and the fact that the almost centralizer of any subgroup in an $\widetilde{\mathfrak{M}}_c$-group is definable, we may state the previous lemma in this terminology which resembles more to the ordinary result.

Corollary 5.22. *Let H be an A-ind-definable normal subgroup of the $\widetilde{\mathfrak{M}}_c$-group G. Then for any $0 < i < j$, we have that*

$$\widetilde{[}\widetilde{\gamma}_{i+1}H, \widetilde{C}_G^j(H)\widetilde{]} \leq \widetilde{C}_G^{j-i-1}(H).$$

In the next lemma, we use the almost three subgroups lemma in terms of the almost commutator to generalize [4, Lemma 2.5] to our framework.

Lemma 5.23. *Let H and K be two A-ind-definable normal subgroups of G with $K \leq H$ and $\ell > 0$. If*

$$\widetilde{C}_G(\widetilde{\gamma}_t K) \sim \widetilde{C}_G(\widetilde{\gamma}_t H) \quad t = 1, \ldots, \ell$$

then $\widetilde{C}_G^\ell(K) \sim \widetilde{C}_G^\ell(H)$.

Proof. The case ℓ equals 1 is trivial. So let's assume that the lemma holds for $\ell - 1$. We need to prove the following intermediate result:

Claim. $\widetilde{[}\widetilde{\gamma}_{\ell-t}H, \widetilde{C}_G^\ell(K)\widetilde{]} \leq \widetilde{C}_G^t(H)$ *holds for all* $t = 0, \ldots, \ell-1$.

Proof. We show the claim by induction on the tuple (ℓ, t) (ordered lexicographically) with $t < \ell$. First we treat the cases $(\ell, 0)$ for any natural number ℓ:
Replacing H by K, i by $\ell-1$, and j by ℓ in Corollary 5.22, we obtain $\widetilde{[}\widetilde{\gamma}_\ell K, \widetilde{C}_G^\ell(K)\widetilde{]} = 1$. This implies that $\widetilde{C}_G^\ell(K)$ is almost contained in $\widetilde{C}_G(\widetilde{\gamma}_\ell K)$ which is, by the hypothesis of the lemma, commensurate with $\widetilde{C}_G(\widetilde{\gamma}_\ell H)$. Thus $\widetilde{C}_G^\ell(K) \lesssim \widetilde{C}_G(\widetilde{\gamma}_\ell H)$ or in other words $\widetilde{[}\widetilde{\gamma}_\ell H, \widetilde{C}_G^\ell(K)\widetilde{]} = 1$. Hence the claim holds for $(\ell, 0)$ with $\ell > 0$.

Now, let $0 < t < \ell$ and assume additionally that the claim holds for any tuple $(k, s) < (\ell, t)$ in the lexicographical order.
Then using Lemma 5.5 (1) and the induction hypothesis for $(\ell, t-1)$ (in the equation marked as $(*)$ below) and for $(\ell-1, t-1)$ (in the equation marked as $(**)$ below) we may compute

$$\widetilde{[}\widetilde{[}\widetilde{\gamma}_{\ell-t}H, K\widetilde{]}, \widetilde{C}_G^\ell(K)\widetilde{]} \overset{5.5(1)}{\underset{K\leq H}{\leq}} \widetilde{[}\widetilde{[}\widetilde{\gamma}_{\ell-t}H, H\widetilde{]}, \widetilde{C}_G^\ell(K)\widetilde{]} = \widetilde{[}\widetilde{\gamma}_{\ell-(t-1)}H, \widetilde{C}_G^\ell(K)\widetilde{]} \overset{(*)}{\leq} \widetilde{C}_G^{t-1}(H)$$

and

$$\tilde{[}\widetilde{\gamma}_{\ell-t}H, \tilde{[}K, \widetilde{C}_G^{\ell}(K)\tilde{]}\tilde{]} \leq \tilde{[}\widetilde{\gamma}_{\ell-t}H, \widetilde{C}_G^{\ell-1}(K)\tilde{]} = \tilde{[}\widetilde{\gamma}_{(\ell-1)-(t-1)}H, \widetilde{C}_G^{\ell-1}(H)\tilde{]} \overset{(**)}{\leq} \widetilde{C}_G^{t-1}(H).$$

Thus by Corollary 5.10 we have

$$\tilde{[}\tilde{[}\widetilde{\gamma}_{l-t}H, \widetilde{C}_G^{\ell}(K)\tilde{]}, K\tilde{]} \leq \widetilde{C}_G^{t-1}(H).$$

As $t-1$ is less than ℓ, we have, by the hypothesis of the outer induction, that $\widetilde{C}_G^{t-1}(H)$ is commensurate with $\widetilde{C}_G^{t-1}(K)$ and so $\tilde{[}\tilde{[}\widetilde{\gamma}_{\ell-t}H, \widetilde{C}_G^{\ell}(K)\tilde{]}, K\tilde{]}$ is almost contained in $\widetilde{C}_G^{t-1}(K)$. As $\widetilde{C}_G^{t-1}(K)$ is A-definable, using Corollary 5.4, we obtain that

$$\tilde{[}\tilde{[}\widetilde{\gamma}_{\ell-t}H, \widetilde{C}_G^{\ell}(K)\tilde{]}, K\tilde{]} \leq \widetilde{C}_G^{t-1}(K).$$

Thus $\tilde{[}\widetilde{\gamma}_{\ell-t}H, \widetilde{C}_G^{\ell}(K)\tilde{]}$ is almost contained in $\widetilde{C}_G^{t}(K)$ which is commensurate once more with $\widetilde{C}_G^{t}(H)$ by the outer induction hypothesis. Again by Corollary 5.4 almost contained can be replaced by contained, which gives us

$$\tilde{[}\widetilde{\gamma}_{\ell-t}H, \widetilde{C}_G^{\ell}(K)\tilde{]} \leq \widetilde{C}_G^{t}(H).$$

Thus the claim holds for the tuple (ℓ, t) which finishes the induction and hence the proof of the claim. $\square_{(\text{claim})}$

Now taking t equals to $\ell-1$, we obtain $\tilde{[}H, \widetilde{C}_G^{\ell}(K)\tilde{]} \leq \widetilde{C}_G^{\ell-1}(H)$ which implies that $\widetilde{C}_G^{\ell}(K)$ is almost contained in $\widetilde{C}_G^{\ell}(H)$. On the other hand, we have that

$$\tilde{[}K, \widetilde{C}_G^{\ell}(H)\tilde{]} \underset{K\leq H}{\overset{5.5(1)}{\leq}} \tilde{[}H, \widetilde{C}_G^{\ell}(H)\tilde{]} \overset{5.9}{\leq} \widetilde{C}_G^{\ell-1}(H) \overset{\text{hyp.}}{\sim} \widetilde{C}_G^{\ell-1}(K).$$

Again by Corollary 5.3 we obtain that $\tilde{[}K, \widetilde{C}_G^{\ell}(H)\tilde{]} \leq \widetilde{C}_G^{\ell-1}(K)$ and so $\widetilde{C}_G^{\ell}(H)$ is almost contained in $\widetilde{C}_G^{\ell}(K)$. Combining these two results, we obtain that $\widetilde{C}_G^{\ell}(K)$ is commensurate with $\widetilde{C}_G^{\ell}(H)$ which finishes the proof. $\square$

We finish this section with another result on almost nilpotent $\widetilde{\mathfrak{M}}_c$-groups which do not use the almost three subgroups lemma.

Lemma 5.24. *Let G be almost nilpotent $\widetilde{\mathfrak{M}}_c$-group and N be a nontrivial intersection of A-definable normal subgroups of G. Then $\tilde{[}N, G\tilde{]}$ is properly contained in N and $N \cap \widetilde{Z}(G)$ is a nontrivial subgroup of G. In particular, any minimal A-invariant normal subgroup of G is contained in the almost center of G.*

Proof. As N is an intersection of A-definable normal subgroups of G and we have trivially that $N \lesssim \widetilde{C}_G(G/N)$, the group $\tilde{[}N, G\tilde{]}$ is contained in N. Additionally, the commutator $\tilde{[}N, G\tilde{]}$ is also contained in $\tilde{[}G, G\tilde{]}$ by Lemma 5.5. Inductively we obtain $\widetilde{\gamma}_{i+1}(N, G^i) \leq N \cap \widetilde{\gamma}_{i+1}G$. As G is almost nilpotent $\widetilde{\gamma}_m G$ is trivial for some natural number m. Hence $\tilde{[}N, G\tilde{]}$ has to be properly contained in N because if not $\widetilde{\gamma}_m(N, G^{m-1})$ would be equal to N as well. This proves the first part of the Lemma.

Moreover, again by Lemma 5.5, we have that $\widetilde{\gamma}_m(N, G^{m-1}) \leq \widetilde{\gamma}_m G$ and thus it is also trivial. Now choose n such that $\widetilde{\gamma}_{n+1}(N, G^n)$ is trivial and properly contained in $\widetilde{\gamma}_n(N, G^{n-1})$. Hence

$$\widetilde{\gamma}_n(N, G^{n-1}) \lesssim \widetilde{Z}(G).$$

Since the almost center of G is definable, Corollary 5.4 yields that $\widetilde{\gamma}_n(N, G^{n-1})$ is actually contained in $\widetilde{Z}(G)$. As additionally the group $\widetilde{\gamma}_n(N, G^{n-1})$ is nontrivial and contained in N, the subgroup $N \cap \widetilde{Z}(G)$ is nontrivial as well. $\square$

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