Bilinear Strichartz estimates for the ZK equation and applications

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BILINEAR STRICHARTZ ESTIMATES FOR THE ZAKHAROV-KUZNETSOV EQUATION AND APPLICATIONS

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Abstract. This article is concerned with the Zakharov-Kuznetsov equation

ZKO (0.1) $$\partial_t u + \partial_x \Delta u + u\partial_x u = 0.$$

We prove that the associated initial value problem is locally well-posed in $H^s(\mathbb{R}^2)$ for $s > \frac{1}{2}$ and globally well-posed in $H^1(\mathbb{R}\times\mathbb{T})$ and in $H^s(\mathbb{R}^3)$ for $s > 1$. Our main new ingredient is a bilinear Strichartz estimate in the context of Bourgain's spaces which allows to control the high-low frequency interactions appearing in the nonlinearity of (0.1). In the $\mathbb{R}^2$ case, we also need to use a recent result by Carbery, Kenig and Ziesler on sharp Strichartz estimates for homogeneous dispersive operators. Finally, to prove the global well-posedness result in $\mathbb{R}^3$, we need to use the atomic spaces introduced by Koch and Tataru.

1. Introduction

The Zakharov-Kuznetsov equation (ZK)

ZK (1.1) $$\partial_t u + \partial_x \Delta u + u\partial_x u = 0,$$

where $u = u(x, y, t)$ is a real-valued function, $t \in \mathbb{R}$, $x \in \mathbb{R}$, $y \in \mathbb{R}$, $\mathbb{T}$ or $\mathbb{R}^2$ and Δ is the laplacian, was introduced by Zakharov and Kuznetsov in [8] to describe the propagation of ionic-acoustic waves in magnetized plasma. The derivation of ZK from the Euler-Poisson system with magnetic field was performed by Lannes, Linares and Saut [10] (see also [13] for a formal derivation). Moreover, the following quantities are conserved by the flow of ZK,

M (1.2) $$M(u) = \int u(x, y, t)^2 dxdy,$$

and

H (1.3) $$H(u) = \frac{1}{2}\int \big(|\nabla u(x, y, t)|^2 - \frac{1}{3}u(x, y, t)^3\big)dxdy.$$

Therefore L^2 and H^1 are two natural spaces to study the well-posedness for the ZK equation.

In the 2D case, Faminskii proved in [3] that the Cauchy problem associated to (1.1) was well-posed in the energy space $H^1(\mathbb{R}^2)$. This result was recently improved

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by Linares and Pastor who proved well-posedness in $H^s(\mathbb{R}^2)$, for $s > 3/4$. Both results were proved by using a fixed point argument taking advantage of the dispersive smoothing effects associated to the linear part of ZK, following the ideas of Kenig, Ponce and Vega [7] for the KdV equation.

The case of the cylinder $\mathbb{R} \times \mathbb{T}$ was treated by Linares, Pastor and Saut in [12]. They obtained well-posedness in $H^s(\mathbb{R} \times \mathbb{T})$ for $s > \frac{3}{2}$. Note that the best results in the 3D case were obtained last year by Ribaud and Vento [15] (see also Linares and Saut [13] for former results). They proved local well-posedness in $H^s(\mathbb{R}^3)$ for $s > 1$ and in $B_2^{1,1}(\mathbb{R}^3)$. However that it is still an open problem to obtain global solutions in $\mathbb{R} \times \mathbb{T}$ and $\mathbb{R}^3$.

The objective of this article is to improve the local well-posedness results for the ZK equation in $\mathbb{R}^2$ and $\mathbb{R} \times \mathbb{T}$, and to prove new global well-posedness results. In this direction, we obtain the global well-posedness in $H^1(\mathbb{R} \times \mathbb{T})$ and in $H^s(\mathbb{R}^3)$ for $s > 1$. Next are our main results.

theoR2 **Theorem 1.1.** *Assume that $s > \frac{1}{2}$. For any $u_0 \in H^s(\mathbb{R}^2)$, there exists $T = T(\|u_0\|_{H^s}) > 0$ and a unique solution of (1.1) such that $u(\cdot, 0) = u_0$ and*

theoR2.1
$$u \in C([0,T] : H^s(\mathbb{R}^2)) \cap X_T^{s,\frac{1}{2}+} \ . \tag{1.4}$$

Moreover, for any $T' \in (0,T)$, there exists a neighborhood $\mathcal{U}$ of u_0 in $H^s(\mathbb{R}^2)$, such that the flow map data-solution

theoR2.2
$$S : v_0 \in \mathcal{U} \mapsto v \in C([0,T'] : H^s(\mathbb{R}^2)) \cap X_{T'}^{s,\frac{1}{2}+} \tag{1.5}$$

is smooth.

theoRT **Theorem 1.2.** *Assume that $s \geq 1$. For any $u_0 \in H^s(\mathbb{R} \times \mathbb{T})$, there exists $T = T(\|u_0\|_{H^s}) > 0$ and a unique solution of (1.1) such that $u(\cdot, 0) = u_0$ and*

theoRT.1
$$u \in C([0,T] : H^s(\mathbb{R} \times \mathbb{T})) \cap X_T^{s,\frac{1}{2}+} \ . \tag{1.6}$$

Moreover, for any $T' \in (0,T)$, there exists a neighborhood $\widetilde{\mathcal{U}}$ of u_0 in $H^s(\mathbb{R} \times \mathbb{T})$, such that the flow map data-solution

theoRT.2
$$S : v_0 \in \widetilde{\mathcal{U}} \mapsto v \in C([0,T'] : H^s(\mathbb{R} \times \mathbb{T})) \cap X_{T'}^{s,\frac{1}{2}+} \tag{1.7}$$

is smooth.

Remark 1.1. The spaces $X_T^{s,b}$ are defined in Section 2

As a consequence of Theorem 1.2, we deduce the following result by using the conserved quantities M and H defined in (1.2) and (1.3).

theoRTglobal **Theorem 1.3.** *The initial value problem associated to the Zakharov-Kuznetsov equation is globally well-posed in $H^1(\mathbb{R} \times \mathbb{T})$.*

Remark 1.2. Theorem 1.3 provides a good setting to apply the techniques of Rousset and Tzvetkov [16], [17] and prove the transverse instability of the KdV soliton for the ZK equation.

Finally, we combine the conserved quantities M and H with a well-posedness result in the Besov space $B_2^{1,1}$ and interpolation arguments to prove :

theo3 **Theorem 1.4.** *The initial value problem associated to the Zakharov-Kuznetsov equation is globally well-posed in $H^s(\mathbb{R}^3)$ for any $s > 1$.*

Remark 1.3. Note that the global well-posedness for the ZK equation in the energy space $H^1(\mathbb{R}^3)$ is still an open problem.

The main new ingredient in the proofs of Theorems 1.1, 1.2 and 1.4 is a bilinear estimate in the context of Bourgain's spaces (see for instance the work of Molinet, Saut and Tzvetkov for the the KPII equation [14] for similar estimates), which allows to control the interactions between high and low frequencies appearing in the nonlinearity of (1.1). In the $\mathbb{R}^2$ case, we also need to use a recent result by Carbery, Kenig and Ziesler on sharp Strichartz estimates for homogeneous dispersive operators. This allows us to treat the case of high-high to high frequency interactions. With those estimates in hand, we are able to derive the crucial bilinear estimates (see Propositions 4.1 and 5.1 below) and conclude the proof of Theorems 1.1 and 1.2 by using a fixed point argument in Bourgain's spaces. To prove the global well-posedness in $\mathbb{R}^3$ we follows ideas in [1] and need to get a suitable lower bound on the time before the norm of solution doubles. To get this bound we will have to work in the framework of the atomic spaces U^2_S and V^2_S introduced by Koch and Tataru in [9].

We saw very recently on the arXiv that Grünrock and Herr obtained a similar result [5] in the $\mathbb{R}^2$ case by using the same kind of techniques. Note however that they do not need to use the Strichartz estimate derived by Carbery, Kenig and Ziesler. On the other hand, they use a linear transformation on the equation to obtain a symmetric symbol $\xi^3+\eta^3$ in order to apply their arguments. Since we derive our bilinear estimate directly on the original equation, our method of proof also worked in the $\mathbb{R}\times\mathbb{T}$ setting (see the results in Theorems 1.2 and 1.3).

This paper is organized as follows: in the next section we introduce the notations and define the function spaces. In Section 3, we recall the linear Strichartz estimates for ZK and derive our crucial bilinear estimate. Those estimates are used in Section 4 and 5 to prove the bilinear estimates in $\mathbb{R}^2$ and $\mathbb{R}\times\mathbb{T}$. Finally, Section 6 is devoted to the $\mathbb{R}^3$ case.

notation

2. Notation, function spaces and linear estimates

2.1. **Notation.** For any positive numbers a and b, the notation $a\lesssim b$ means that there exists a positive constant c such that $a\leq cb$. We also write $a\sim b$ when $a\lesssim b$ and $b\lesssim a$. If $\alpha\in\mathbb{R}$, then α_+, respectively α_-, will denote a number slightly greater, respectively lesser, than α. If A and B are two positive numbers, we use the notation $A\wedge B=\min(A,B)$ and $A\vee B=\max(A,B)$. Finally, $\operatorname{mes} S$ or $|S|$ denotes the Lebesgue measure of a measurable set S of $\mathbb{R}^n$, whereas $\#F$ or $|S|$ denotes the cardinal of a finite set F.

We use the notation $|(x,y)|=\sqrt{3x^2+y^2}$ for $(x,y)\in\mathbb{R}^2$. For $u=u(x,y,t)\in\mathcal{S}(\mathbb{R}^3)$, $\mathcal{F}(u)$, or $\widehat{u}$, will denote its space-time Fourier transform, whereas $\mathcal{F}_{xy}(u)$, or $(u)^{\wedge_{xy}}$, respectively $\mathcal{F}_t(u)=(u)^{\wedge_t}$, will denote its Fourier transform in space, respectively in time. For $s\in\mathbb{R}$, we define the Bessel and Riesz potentials of order $-s$, J^s and D^s, by

$$J^s u=\mathcal{F}^{-1}_{xy}\big((1+|(\xi,\mu)|^2)^{\frac{s}{2}}\mathcal{F}_{xy}(u)\big)\quad\text{and}\quad D^s u=\mathcal{F}^{-1}_{xy}\big(|(\xi,\mu)|^s\mathcal{F}_{xy}(u)\big).$$

Throughout the paper, we fix a smooth cutoff function η such that

$$\eta\in C^\infty_0(\mathbb{R}),\quad 0\leq\eta\leq 1,\quad \eta_{|_{[-5/4,5/4]}}=1\quad\text{and}\quad \operatorname{supp}(\eta)\subset[-8/5,8/5].$$

For $k \in \mathbb{N}^\star = \mathbb{Z} \cap [1, +\infty)$, we define

$$\phi(\xi) = \eta(\xi) - \eta(2\xi), \quad \phi_{2^k}(\xi,\mu) := \phi(2^{-k}|(\xi,\mu)|).$$

and

$$\psi_{2^k}(\xi,\mu,\tau) = \phi(2^{-k}(\tau - (\xi^3 + \xi\mu^2))).$$

By convention, we also denote

$$\phi_1(\xi,\mu) = \eta(|(\xi,\mu)|), \quad \text{and} \quad \psi_1(\xi,\mu,\tau) = \eta(\tau - (\xi^3 + \xi\mu^2)).$$

Any summations over capitalized variables such as N, L, K or M are presumed to be dyadic with N, L, K or $M \geq 1$, *i.e.*, these variables range over numbers of the form $\{2^k : k \in \mathbb{N}\}$. Then, we have that

$$\sum_N \phi_N(\xi,\mu) = 1, \quad \text{supp}\,(\phi_N) \subset \{\frac{5}{8}N \leq |(\xi,\mu)| \leq \frac{8}{5}N\} =: I_N \ , \ N \geq 2,$$

and

$$\text{supp}\,(\phi_1) \subset \{|(\xi,\mu)| \leq \frac{8}{5}\} =: I_1.$$

Let us define the Littlewood-Paley multipliers by

proj (2.1) $$P_N u = \mathcal{F}^{-1}_{xy}\big(\phi_N \mathcal{F}_{xy}(u)\big), \quad Q_L u = \mathcal{F}^{-1}\big(\psi_L \mathcal{F}(u)\big).$$

Finally, we denote by $e^{-t\partial_x\Delta}$ the free group associated with the linearized part of equation (1.1), which is to say,

V (2.2) $$\mathcal{F}_{xy}\big(e^{-t\partial_x\Delta}\varphi\big)(\xi,\mu) = e^{itw(\xi,\mu)}\mathcal{F}_{xy}(\varphi)(\xi,\mu),$$

where $w(\xi,\mu) = \xi^3 + \xi\mu^2$. We also define the resonance function H by

Resonance (2.3) $$H(\xi_1,\mu_1,\xi_2,\mu_2) = w(\xi_1+\xi_2,\mu_1+\mu_2) - w(\xi_1,\mu_1) - w(\xi_2,\mu_2).$$

Straightforward computations give that

Resonance2 (2.4) $$H(\xi_1,\mu_1,\xi_2,\mu_2) = 3\xi_1\xi_2(\xi_1+\xi_2) + \xi_2\mu_1^2 + \xi_1\mu_2^2 + 2(\xi_1+\xi_2)\mu_1\mu_2.$$

We make the obvious modifications when working with $u = u(x,y)$ for $(x,y) \in \mathbb{R}\times\mathbb{T}$ and denote by q the Fourier variable corresponding to y.

2.2. **Function spaces.** For $1 \leq p \leq \infty$, $L^p(\mathbb{R}^2)$ is the usual Lebesgue space with the norm $\|\cdot\|_{L^p}$, and for $s \in \mathbb{R}$, the real-valued Sobolev space $H^s(\mathbb{R}^2)$ denotes the space of all real-valued functions with the usual norm $\|u\|_{H^s} = \|J^s u\|_{L^2}$. If $u = u(x,y,t)$ is a function defined for $(x,y) \in \mathbb{R}^2$ and t in the time interval $[0,T]$, with $T > 0$, if B is one of the spaces defined above, $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$, we will define the mixed space-time spaces $L^p_T B_{xy}$, $L^p_t B_{xy}$, $L^q_{xy} L^p_T$ by the norms

$$\|u\|_{L^p_T B_{xy}} = \Big(\int_0^T \|u(\cdot,\cdot,t)\|_B^p dt\Big)^{\frac{1}{p}} \quad , \quad \|u\|_{L^p_t B_{xy}} = \Big(\int_{\mathbb{R}} \|u(\cdot,\cdot,t)\|_B^p dt\Big)^{\frac{1}{p}},$$

and

$$\|u\|_{L^q_{xy} L^p_T} = \left(\int_{\mathbb{R}^2} \Big(\int_0^T |u(x,y,t)|^p dt\Big)^{\frac{q}{p}} dx\right)^{\frac{1}{q}},$$

if $1 \leq p,\ q < \infty$ with the obvious modifications in the case $p = +\infty$ or $q = +\infty$.

For s, $b \in \mathbb{R}$, we introduce the Bourgain spaces $X^{s,b}$ related to the linear part of (1.1) as the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^3)$ under the norm

Bourgain (2.5) $$\|u\|_{X^{s,b}} = \left(\int_{\mathbb{R}^3} \langle \tau - w(\xi,\mu)\rangle^{2b} \langle |(\xi,\mu)|\rangle^{2s} |\widehat{u}(\xi,\mu,\tau)|^2 d\xi d\mu d\tau\right)^{\frac{1}{2}},$$

where $\langle x\rangle := 1+|x|$. Moreover, we define a localized (in time) version of these spaces. Let $T>0$ be a positive time. Then, if $u:\mathbb{R}^2\times[0,T]\to\mathbb{C}$, we have that

$$\|u\|_{X_T^{s,b}} = \inf\{\|\tilde{u}\|_{X^{s,b}} \ : \ \tilde{u}:\mathbb{R}^2\times\mathbb{R}\to\mathbb{C},\ \tilde{u}|_{\mathbb{R}^2\times[0,T]}=u\}.$$

We make the obvious modifications for functions defined on $(x,y,t)\in\mathbb{R}\times\mathbb{Z}\times\mathbb{R}$. In particular, the integration over $\mu\in\mathbb{R}$ in (2.5) is replaced by a summation over $q\in\mathbb{Z}$, which is to say

Bourgainper
$$\|u\|_{X^{s,b}} = \left(\sum_{q\in\mathbb{Z}}\int_{\mathbb{R}^2}\langle\tau-w(\xi,q)\rangle^{2b}\langle|(\xi,q)|\rangle^{2s}|\widehat{u}(\xi,q,\tau)|^2d\xi d\tau\right)^{\frac{1}{2}}, \tag{2.6}$$

where $w(\xi,q)=\xi^3+\xi q^2$.

2.3. **Linear estimates in the $X^{s,b}$ spaces.** In this subsection, we recall some well-known estimates for Bourgain's spaces (see [4] for instance).

prop1.1 **Lemma 2.1** (Homogeneous linear estimate)**.** *Let $s\in\mathbb{R}$ and $b>\frac{1}{2}$. Then*

prop1.1.2
$$\|\eta(t)e^{-t\partial_x\Delta}f\|_{X^{s,b}}\lesssim\|f\|_{H^s}\ . \tag{2.7}$$

prop1.2 **Lemma 2.2** (Non-homogeneous linear estimate)**.** *Let $s\in\mathbb{R}$. Then for any $0<\delta<\frac{1}{2}$,*

prop1.2.1
$$\Big\|\eta(t)\int_0^t e^{-(t-t')\partial_x\Delta}g(t')dt'\Big\|_{X^{s,\frac{1}{2}+\delta}}\lesssim\|g\|_{X^{s,-\frac{1}{2}+\delta}}\ . \tag{2.8}$$

prop1.3b **Lemma 2.3.** *For any $T>0$, $s\in\mathbb{R}$ and for all $-\frac{1}{2}<b'\le b<\frac{1}{2}$, it holds*

prop1.3b.1
$$\|u\|_{X_T^{s,b'}}\lesssim T^{b-b'}\|u\|_{X_T^{s,b}}. \tag{2.9}$$

3. Linear and bilinear Strichartz estimates

3.1. **Linear strichartz estimates on $\mathbb{R}^2$.** First, we state a Strichartz estimate for the unitary group $\{e^{-t\partial_x\Delta}\}$ proved by Linares and Pastor (*c.f.* Proposition 2.3 in [11]).

Strichartz **Proposition 3.1.** *Let $0\le\epsilon<\frac{1}{2}$ and $0\le\theta\le1$. Assume that (q,p) satisfy $p=\frac{2}{1-\theta}$ and $q=\frac{6}{\theta(2+\epsilon)}$. Then, we have that*

Strichartz1
$$\|D_x^{\frac{\theta\epsilon}{2}}e^{-t\partial_x\Delta}\varphi\|_{L_t^qL_{xy}^p}\lesssim\|\varphi\|_{L^2} \tag{3.1}$$

for all $\varphi\in L^2(\mathbb{R}^2)$.

Then, we obtain the following corollary in the context of Bourgain' spaces.

Strichartzcoro **Corollary 3.2.** *We have that*

Strichartzcoro1
$$\|u\|_{L^4_{xyt}}\lesssim\|u\|_{X^{0,\frac{5}{6}+}}, \tag{3.2}$$

for all $u\in X^{0,\frac{5}{6}+}$.

Proof. Estimate (3.1) in the case $\epsilon=0$ and $\theta=\frac{3}{5}$ writes

Strichartzcoro2
$$\|e^{-t\partial_x\Delta}\varphi\|_{L^5_{xyt}}\lesssim\|\varphi\|_{L^2} \tag{3.3}$$

for all $\varphi\in L^2(\mathbb{R}^2)$. A classical argument (see for example [4]) yields

$$\|u\|_{L^5_{xyt}}\lesssim\|u\|_{X^{0,\frac{1}{2}+}},$$

which implies estimate (3.2) after interpolation with Plancherel's identity $\|u\|_{L^2_{xyt}} = \|u\|_{X^{0,0}}$. □

In [2], Carbery, Kenig and Ziesler proved an optimal L^4-restriction theorem for homogeneous polynomial hypersurfaces in $\mathbb{R}^3$.

CKZ **Theorem 3.3.** *Let $\Gamma(\xi,\mu) = (\xi,\mu,\Omega(\xi,\mu))$, where $\Omega(\xi,\mu)$ is a polynomial, homogeneous of degree $d \geq 2$. Then there exists a positive constant C (depending on ϕ) such that*

CKZ1
$$\Big(\int_{\mathbb{R}^2} |\widehat{f}(\Gamma(\xi,\mu))|^2 |K_\Omega(\xi,\mu)|^{\frac14} d\xi d\mu\Big)^{\frac12} \leq C\|f\|_{L^{4/3}}, \tag{3.4}$$

for all $f \in L^{4/3}(\mathbb{R}^3)$ and where

CKZ2
$$|K_\Omega(\xi,\mu)| = \big|\det Hess\,\Omega(\xi,\mu)\big|. \tag{3.5}$$

As a consequence, we have the following corollary.

CKZcoro **Corollary 3.4.** *Let $|K_\Omega(D)|^{\frac18}$ and $e^{it\Omega(D)}$ be the Fourier multipliers associated to $|K_\Omega(\xi,\mu)|^{\frac18}$ and $e^{it\Omega(\xi,\mu)}$, i.e.*

CKZcoro1
$$\mathcal{F}_{xy}\Big(|K_\Omega(D)|^{\frac18}\varphi\Big)(\xi,\mu) = |K_\Omega(\xi,\mu)|^{\frac18}\mathcal{F}_{xy}(\varphi)(\xi,\mu) \tag{3.6}$$

where $\big|K_\Omega(\xi,\mu)\big|$ is defined in (3.5)*, and*

CKZcoro2
$$\mathcal{F}_{xy}\big(e^{it\Omega(D)}\varphi\big)(\xi,\mu) = e^{it\Omega(\xi,\mu)}\mathcal{F}_{xy}(\varphi)(\xi,\mu). \tag{3.7}$$

Then,

CKZcoro3
$$\big\||K_\Omega(D)|^{\frac18}e^{it\Omega(D)}\varphi\big\|_{L^4_{xyt}} \lesssim \|\varphi\|_{L^2}, \tag{3.8}$$

for all $\varphi \in L^2(\mathbb{R}^2)$.

Proof. By duality, it suffices to prove that

CKZcoro4
$$\int_{\mathbb{R}^3} |K_\Omega(D)|^{\frac18}e^{it\Omega(D)}\varphi(x,y)f(x,y,t)dxdydt \lesssim \|\varphi\|_{L^2_{xy}}\|f\|_{L^{4/3}_{xyt}}. \tag{3.9}$$

The Cauchy-Schwarz inequality implies that it is enough to prove that

CKZcoro5
$$\Big\|\int_{\mathbb{R}} |K_\Omega(D)|^{\frac18}e^{-it\Omega(D)}f(x,y,t)dt\Big\|_{L^2_{xy}} \lesssim \|f\|_{L^{4/3}_{xyt}} \tag{3.10}$$

in order to prove estimate (3.9). But straightforward computations give

$$\mathcal{F}_{x,y}\Big(\int_{\mathbb{R}} \big|K_\Omega(D)\big|^{\frac18}e^{-it\Omega(D)}f dt\Big)(\xi,\mu) = c|K_\Omega(\xi,\mu)\big|^{\frac18}\mathcal{F}_{x,y,t}(f)(\xi,\mu,\Omega(\xi,\mu)),$$

so that estimate (3.10) follows directly from Plancherel's identity and estimate (3.4). □

Now, we apply Corollary 3.4 in the case of the unitary group $e^{-t\partial_x\Delta}$.

Strichartzlin **Proposition 3.5.** *Let $|K(D)|^{\frac18}$ be the Fourier multiplier associated to $|K(\xi,\mu)|^{\frac18}$ where*

Strichartzlin1
$$|K(\xi,\mu)| = |3\xi^2 - \mu^2| \tag{3.11}$$

Then, we have that

Strichartzlin2
$$\big\||K(D)|^{\frac18}e^{-t\partial_x\Delta}\varphi\big\|_{L^4_{xyt}} \lesssim \|\varphi\|_{L^2} \tag{3.12}$$

for all $\varphi \in L^2(\mathbb{R}^2)$, and

Strichartzlin3
$$\big\||K(D)|^{\frac{1}{8}}u\big\|_{L^4_{xyt}} \lesssim \|u\|_{X^{0,\frac{1}{2}+}} \tag{3.13}$$

for all $u \in X^{0,\frac{1}{2}+}$.

Proof. The symbol associated to $e^{-t\partial_x\Delta}$ is given by $w(\xi,\mu) = \xi^3 + \xi\mu^2$. After an easy computation, we get that

$$\det \mathrm{Hess}\, w(\xi,\mu) = 4(3\xi^2 - \mu^2).$$

Estimate (3.12) follows then as a direct application of Corollary 3.4.

□

Remark 3.1. It follows by applying estimate (3.1) with $\epsilon = 1/2-$ and $\theta = 2/3+$ that

$$\|D_x^{\frac{1}{6}} e^{-t\partial_x\Delta}\varphi\|_{L^{6-}_{xyt}} \lesssim \|\varphi\|_{L^2},$$

for all $\varphi \in L^2(\mathbb{R}^2)$, which implies in the context of Bourgain's spaces (after interpolating with the trivial estimate $\|u\|_{L^2_{xyt}} = \|u\|_{X^{0,0}}$) that

Strichartzlinremark
$$\|D_x^{\frac{1}{8}} u\|_{L^4_{xyt}} \lesssim \|u\|_{X^{0,\frac{3}{8}+}}, \tag{3.14}$$

for all $u \in X^{0,\frac{3}{8}+}$.

Estimate (3.13) can be viewed as an improvement of estimate (3.14), since outside of the lines $|\xi| = \frac{1}{\sqrt{3}}|\mu|$, it allows to recover $1/4$ of derivatives instead of $1/8$ of derivatives in L^4.

Remark 3.2. it is interesting to observe that the resonance function H defined in (2.4) cancels out on the planes $(\xi_1 = -\frac{\mu_1}{\sqrt{3}}, \xi_2 = \frac{\mu_2}{\sqrt{3}})$ and $(\xi_1 = \frac{\mu_1}{\sqrt{3}}, \xi_2 = -\frac{\mu_2}{\sqrt{3}})$.

3.2. **Bilinear Strichartz estimates.** In this subsection, we prove the following crucial bilinear estimates related to the ZK dispersion relation for functions defined on $\mathbb{R}^3$ and $\mathbb{R}\times\mathbb{T}\times\mathbb{R}$.

BilinStrichartzI
Proposition 3.6. *Let N_1, N_2, L_1, L_2 be dyadic numbers in $\{2^k : k \in \mathbb{N}^\star\} \cup \{1\}$. Assume that u_1 and u_2 are two functions in $L^2(\mathbb{R}^3)$ or $L^2(\mathbb{R}\times\mathbb{T}\times\mathbb{R})$. Then,*

BilinStrichartzI0
$$\begin{aligned}\|(P_{N_1}Q_{L_1}u_1)(P_{N_2}Q_{L_2}u_2)\|_{L^2} \\ \lesssim (L_1\wedge L_2)^{\frac{1}{2}}(N_1\wedge N_2)\|P_{N_1}Q_{L_1}u_1\|_{L^2}\|P_{N_2}Q_{L_2}u_2\|_{L^2}\end{aligned} \tag{3.15}$$

Assume moreover that $N_2 \geq 4N_1$ or $N_1 \geq 4N_2$. Then,

BilinStrichartzI1
$$\begin{aligned}\|(P_{N_1}Q_{L_1}u_1)(P_{N_2}Q_{L_2}u_2)\|_{L^2} \\ \lesssim \frac{(N_1\wedge N_2)^{\frac{1}{2}}}{N_1\vee N_2}(L_1\vee L_2)^{\frac{1}{2}}(L_1\wedge L_2)^{\frac{1}{2}}\|P_{N_1}Q_{L_1}u_1\|_{L^2}\|P_{N_2}Q_{L_2}u_2\|_{L^2}.\end{aligned} \tag{3.16}$$

Remark 3.3. Estimate (3.16) will be very useful to control the high-low frequency interactions in the nonlinear term of (1.1).

In the proof of Proposition 3.6 we will need some basic Lemmas stated in [14].

basicI
Lemma 3.7. *Consider a set $\Lambda \subset \mathbb{R}\times X$, where $X = \mathbb{R}$ or $\mathbb{T}$. Let the projection on the μ axis be contained in a set $I \subset \mathbb{R}$. Assume in addition that there exists $C > 0$ such that for any fixed $\mu_0 \in I \cap X$, $|\Lambda \cap \{(\xi,\mu_0) \ : \ \mu_0 \in X\}| \leq C$. Then, we get that $|\Lambda| \leq C|I|$ in the case where $X = \mathbb{R}$ and $|\Lambda| \leq C(|I|+1)$ in the case where $X = \mathbb{T}$.*

The second one is a direct consequence of the mean value theorem.

basicII **Lemma 3.8.** *Let I and J be two intervals on the real line and $f : J \to \mathbb{R}$ be a smooth function. Then,*

basicII1
$$mes\{x \in J \ : \ f(x) \in I\} \leq \frac{|I|}{\inf_{\xi\in J}|f'(\xi)|}. \tag{3.17}$$

In the case where f is a polynomial of degree 3, we also have the following result.

basicIII **Lemma 3.9.** *Let $a \neq 0$, b, c be real numbers and I be an interval on the real line. Then,*

basicIII1
$$mes\{x \in J \ : \ ax^2+bx+c \in I\} \lesssim \frac{|I|^{\frac12}}{|a|^{\frac12}}. \tag{3.18}$$

and

basicIII2
$$\#\{q \in \mathbb{Z} \ : \ aq^2+bq+c \in I\} \leq \frac{|I|^{\frac12}}{|a|^{\frac12}}+1. \tag{3.19}$$

Proof of Proposition 3.6. We prove estimates (3.15)–(3.16) in the case where $(x,y,t) \in \mathbb{R}^3$. The case $(x,y,t) \in \mathbb{R}\times\mathbb{T}\times\mathbb{R}$ follows in a similar way. The Cauchy-Schwarz inequality and Plancherel's identity yield

BilinStrichartzI3
$$\begin{aligned}\|(P_{N_1}Q_{L_1}u_1)(P_{N_2}Q_{L_2}u_2)\|_{L^2} &\\ = \|(P_{N_1}Q_{L_1}u_1)^\wedge \star (P_{N_2}Q_{L_2}u_2)^\wedge\|_{L^2} &\\ \lesssim \sup_{(\xi,\mu,\tau)\in\mathbb{R}^3}|A_{\xi\,\mu,\tau}|^{\frac12}\|P_{N_1}Q_{L_1}u_1\|_{L^2}\|P_{N_2}Q_{L_2}u_2\|_{L^2},&\end{aligned} \tag{3.20}$$

where

$$\begin{aligned}A_{\xi,\mu,\tau} =\Big\{&(\xi_1,\mu_1,\tau_1)\in\mathbb{R}^3 \ : \ |(\xi_1,\mu_1)|\in I_{N_1},\ |(\xi-\xi_1,\mu-\mu_1)|\in I_{N_2}\\ &|\tau_1-w(\xi_1,\mu_1)|\in I_{L_1},\ |\tau-\tau_1-w(\xi-\xi_1,\mu-\mu_1)|\in I_{L_2}\Big\}.\end{aligned}$$

it remains then to estimate the measure of the set $A_{\xi,\mu,\tau}$ uniformly in $(\xi,\mu,\tau)\in\mathbb{R}^3$.

To obtain (3.15), we use the trivial estimate

$$|A_{\xi\,\mu,\tau}| \lesssim (L_1\wedge L_2)(N_1\wedge N_2)^2,$$

for all $(\xi,\mu,\tau)\in\mathbb{R}^3$.

Now we turn to the proof of estimate (3.16). First, we get easily from the triangle inequality that

BilinStrichartzI4
$$|A_{\xi\,\mu,\tau}| \lesssim (L_1\wedge L_2)|B_{\xi\,\mu,\tau}|, \tag{3.21}$$

where

BilinStrichartzI40
$$\begin{aligned}B_{\xi,\mu,\tau} =\Big\{&(\xi_1,\mu_1)\in\mathbb{R}^2 \ : \ |(\xi_1,\mu_1)|\in I_{N_1},\ |(\xi-\xi_1,\mu-\mu_1)|\in I_{N_2}\\ &|\tau-w(\xi,\mu)-H(\xi_1,\xi-\xi_1,\mu_1,\mu-\mu_1)|\lesssim L_1\vee L_2\Big\}\end{aligned} \tag{3.22}$$

and $H(\xi_1,\xi_2,\mu_1,\mu_2)$ is the resonance function defined in (2.4). Next, we observe from the hypotheses on the daydic numbers N_1 and N_2 that

$$\Big|\frac{\partial H}{\partial\xi_1}(\xi_1,\xi-\xi_1,\mu_1,\mu-\mu_1)\Big| = \big|3\xi_1^2+\mu_1^2-(3(\xi-\xi_1)^2+(\mu-\mu_1)^2)\big| \gtrsim (N_1\vee N_2)^2\ .$$

Then, if we define $B_{\xi,\mu,\tau}(\mu_1) = \{\xi_1 \in \mathbb{R} \ : \ (\xi_1, \mu_1) \in B_{\xi,\mu,\tau}\}$, we deduce applying estimate (3.17) that

$$|B_{\xi,\mu,\tau}(\mu_1)| \lesssim \frac{L_1 \vee L_2}{(N_1 \vee N_2)^2} \ ,$$

for all $\mu_1 \in \mathbb{R}$. Thus, it follows from Lemma 3.7 that

BilinStrichartzI5 (3.23) $$|B_{\xi,\mu,\tau}| \lesssim \frac{N_1 \wedge N_2}{(N_1 \vee N_2)^2}(L_1 \wedge L_2) \ .$$

Finally, we conclude the proof of estimate (3.16) gathering estimates (3.20)–(3.23). □

4. Bilinear estimate in $\mathbb{R} \times \mathbb{R}$

The main result of this section is stated below.

BilinR2 **Proposition 4.1.** *Let $s > \frac{1}{2}$. Then, there exists $\delta > 0$ such that*

BilinR2.1 (4.1) $$\|\partial_x(uv)\|_{X^{s,-\frac{1}{2}+2\delta}} \lesssim \|u\|_{X^{s,\frac{1}{2}+\delta}} \|v\|_{X^{s,\frac{1}{2}+\delta}},$$

for all $u, \ v : \mathbb{R}^3 \to \mathbb{R}$ such that $u, \ v \in X^{s,\frac{1}{2}+\delta}$.

Before proving Proposition 4.1, we give a technical lemma.

technicalR2 **Lemma 4.2.** *Assume that $0 < \alpha < 1$. Then, we have that*

technicalR2.1 (4.2)
$$\begin{aligned} &|(\xi_1 + \xi_2, \mu_1 + \mu_2)|^2 \\ &\qquad \leq \big||(\xi_1,\mu_1)|^2 - |(\xi_2,\mu_2)|^2\big| + f(\alpha)\max\big\{|(\xi_1,\mu_1)|^2, |(\xi_2,\mu_2)|^2\big\}, \end{aligned}$$

for all $(\xi_1, \mu_1), \ (\xi_2, \mu_2) \in \mathbb{R}^2$ satisfying

technicalR2.2 (4.3) $$(1-\alpha)^{\frac{1}{2}}\sqrt{3}|\xi_i| \leq |\mu_i| \leq (1-\alpha)^{-\frac{1}{2}}\sqrt{3}|\xi_i|, \quad \text{for } i = 1, 2,$$

and

technicalR2.3 (4.4) $$\xi_1\xi_2 < 0 \quad \text{and} \quad \mu_1\mu_2 < 0,$$

and where f is a continuous function on $[0, 1]$ satisfying $\lim_{\alpha\to 0} f(\alpha) = 0$. We also recall te notation $|(\xi, \mu)| = \sqrt{3\xi^2 + \mu^2}$.

Proof. If we denote by $\vec{u}_1 = (\xi_1, \mu_1)$, $\vec{u}_2 = (\xi_2, \mu_2)$ and $(\vec{u}_1, \vec{u}_2)_e = 3\xi_1\xi_2 + \mu_1\mu_2$ the scalar product associated to $|\cdot|$, then (4.2) is equivalent to

technicalR2.4 (4.5) $$|\vec{u}_1 + \vec{u}_2|^2 \leq \big||\vec{u}_1|^2 - |\vec{u}_2|^2\big| + f(\alpha)\max\big\{|\vec{u}_1|^2, |\vec{u}_2|^2\big\}.$$

Moreover, without loss of generality, we can always assume that

technicalR2.5 (4.6) $$\xi_1 > 0, \ \mu_1 > 0, \ \xi_2 < 0, \ \mu_2 < 0 \text{ and } |\vec{u}_1| \geq |\vec{u}_2|.$$

Thus, it suffices to prove that

technicalR2.6 (4.7) $$(\vec{u}_1 + \vec{u}_2, \vec{u}_2)_e \leq \frac{f(\alpha)}{2}|\vec{u}_1|^2.$$

By using (4.3) and (4.4), we have that

technicalR2.7 (4.8)
$$\begin{aligned} (\vec{u}_1 + \vec{u}_2, \vec{u}_2)_e &= 3(\xi_1 + \xi_2)\xi_2 + (\mu_1 + \mu_2)\mu_2 \\ &\leq 6(\xi_1 + \xi_2)\xi_2 - 3\alpha\xi_1\xi_2 + 3\big((1-\alpha)^{-1} - 1\big)\xi_2^2 \end{aligned}$$

On the other hand, the assumptions $\xi_1 > 0$, $\xi_2 < 0$, $|\vec{u}_1| \geq |\vec{u}_2|$ and (4.3) imply that

technicalR2.8 (4.9) $$\xi_1 = |\xi_1| \geq (1 - g(\alpha))|\xi_2| = -(1 - g(\alpha))\xi_2$$

with

$$g(\alpha)=1-\Big(\frac{2-\alpha}{2+3\big((1-\alpha)^{-1}-1\big)}\Big)^{\frac12}\underset{\alpha\to0}{\longrightarrow}0.$$

Thus, it follows gathering (4.8) and (4.9) that

$$(\vec{u}_1+\vec{u}_2,\vec{u}_2)_e\leq 6g(\alpha)\xi_2^2-3\alpha\xi_1\xi_2+3\big((1-\alpha)^{-1}-1\big)\xi_2^2,$$

which implies (4.7) by choosing

$$f(\alpha)=12g(\alpha)+6\alpha+6\big((1-\alpha)^{-1}-1\big)\underset{\alpha\to0}{\longrightarrow}0.$$

□

Proof of Proposition 4.1. By duality, it suffices to prove that

BilinR2.2 (4.10) $$I\lesssim \|u\|_{L^2_{x,y,t}}\|v\|_{L^2_{x,y,t}}\|w\|_{L^2_{x,y,t}},$$

where

$$I=\int_{\mathbb{R}^6}\Gamma^{\xi_1,\mu_1,\tau_1}_{\xi,\mu,\tau}\widehat{w}(\xi,\mu,\tau)\widehat{u}(\xi_1,\mu_1,\tau_1)\widehat{v}(\xi_2,\mu_2,\tau_2)d\nu,$$

$\widehat{u}$, $\widehat{v}$ and $\widehat{w}$ are nonnegative functions, and we used the following notations

BilinR2.20 (4.11)
$$\Gamma^{\xi_1,\mu_1,\tau_1}_{\xi,\mu,\tau}=|\xi|\langle|(\xi,\mu)|\rangle^s\langle\sigma\rangle^{-\frac12+2\delta}\langle|(\xi_1,\mu_1)|\rangle^{-s}\langle\sigma_1\rangle^{-\frac12-\delta}\langle|(\xi_2,\mu_2)|\rangle^{-s}\langle\sigma_2\rangle^{-\frac12-\delta},$$
$$d\nu=d\xi d\xi_1 d\mu d\mu_1 d\tau d\tau_1,\quad \xi_2=\xi-\xi_1,\ \mu_2=\mu-\mu_1,\ \tau_2=\tau-\tau_1,$$
$$\sigma=\tau-w(\xi,\mu)\quad\text{and}\quad\sigma_i=\tau_i-w(\xi_i,\mu_i),\ i=1,2.$$

By using dyadic decompositions on the spatial frequencies of u, v and w, we rewrite I as

BilinR2.3 (4.12) $$I=\sum_{N_1,N_2,N}I_{N,N_1,N_2},$$

where

$$I_{N,N_1,N_2}=\int_{\mathbb{R}^6}\Gamma^{\xi_1,\mu_1,\tau_1}_{\xi,\mu,\tau}\widehat{P_Nw}(\xi,\mu,\tau)\widehat{P_{N_1}u}(\xi_1,\mu_1,\tau_1)\widehat{P_{N_2}v}(\xi_2,\mu_2,\tau_2)d\nu.$$

Since $(\xi,\mu)=(\xi_1,\mu_1)+(\xi_2,\mu_2)$, we can split the sum into the following cases:

(1) *Low* × *Low* → *Low* interactions: $N_1\leq 2, N_2\leq 2, N\leq 2$. In this case, we denote
$$I_{LL\to L}=\sum_{N\leq4,N_1\leq4,N_2\leq4}I_{N,N_1,N_2}.$$

(2) *Low* × *High* → *High* interactions: $4\leq N_2, N_1\leq N_2/4$ ($\Rightarrow N_2/2\leq N\leq 2N_2$). In this case, we denote
$$I_{LH\to H}=\sum_{4\leq N_2,N_1\leq N_2/4,N_2/2\leq N\leq2N_2}I_{N,N_1,N_2}.$$

(3) *High* × *Low* → *High* interactions: $4\leq N_1, N_2\leq N_1/4$ ($\Rightarrow N_1/2\leq N\leq 2N_1$). In this case, we denote
$$I_{HL\to H}=\sum_{4\leq N_1,N_2\leq N_1/4,N_1/2\leq N\leq2N_1}I_{N,N_1,N_2}.$$

(4) *High* $\times$ *High* $\to$ *Low* interactions: $4 \leq N_1, N \leq N_1/4$ ($\Rightarrow N_1/2 \leq N_2 \leq 2N_1$) or $4 \leq N_2, N \leq N_2/4$ ($\Rightarrow N_2/2 \leq N_1 \leq 2N_2$) . In this case, we denote

$$I_{HH\to L} = \sum_{4\leq N_1, N\leq N_1/4, N_2/2\leq N_1\leq 2N_2} I_{N,N_1,N_2}.$$

(5) *High* $\times$ *High* $\to$ *High* interactions: $N_2 \geq 4$, $N_1 \geq 4$, $N_2/2 \leq N_1 \leq 2N_2$, $N_1/2 \leq N \leq 2N_1$ and $N_2/2 \leq N \leq 2N_2$. In this case, we denote

$$I_{HH\to H} = \sum_{N_2/2\leq N_1\leq 2N_2, N_1/2\leq N\leq 2N_1, N_2/2\leq N\leq 2N_2} I_{N,N_1,N_2}.$$

Then, we have

BilinR2.4 (4.13) $$I = I_{LL\to L} + I_{LH\to H} + I_{HL\to H} + I_{HH\to L} + I_{HH\to H}.$$

1. Estimate for $I_{LL\to L}$. We observe from Plancherel's identity, Hölder's inequality and estimate (3.2) that

BilinR2.40 (4.14)
$$\begin{aligned} I_{N,N_1,N_2} &\lesssim \Big\| \Big(\frac{\widehat{P_{N_1}u}}{\langle\sigma_1\rangle^{\frac12+\delta}} \Big)^\vee \Big\|_{L^4} \Big\| \Big(\frac{\widehat{P_{N_2}v}}{\langle\sigma_2\rangle^{\frac12+\delta}} \Big)^\vee \Big\|_{L^4} \|P_N w\|_{L^2} \\ &\lesssim \|P_{N_1}u\|_{L^2}\|P_{N_2}v\|_{L^2}\|P_N w\|_{L^2}, \end{aligned}$$

which yields

BilinR2.400 (4.15) $$I_{LL\to L} \lesssim \|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}.$$

2. Estimate for $I_{LH\to H}$. In this case, we also use dyadic decompositions on the modulations variables σ, σ_1 and σ_2, so that

BilinR2.5 (4.16) $$I_{N,N_1,N_2} = \sum_{L,L_1,L_2} I^{L,L_1,L_2}_{N,N_1,N_2},$$

where

$$I^{L,L_1,L_2}_{N,N_1,N_2} = \int_{\mathbb{R}^6} \Gamma^{\xi_1,\mu_1,\tau_1}_{\xi,\mu,\tau} \widehat{P_N Q_L} w(\xi,\mu,\tau) \widehat{P_{N_1} Q_{L_1}} u(\xi_1,\mu_1,\tau_1) \widehat{P_{N_2} Q_{L_2}} v(\xi_2,\mu_2,\tau_2) d\nu.$$

Hence, by using the Cauchy-Schwarz inequality in (ξ,μ,τ), we can bound $I^{L,L_1,L_2}_{N,N_1,N_2}$ by

$$N_2 N_1^{-s} L^{-\frac12+2\delta} L_1^{-\frac12-\delta} L_2^{-\frac12-\delta} \|(P_{N_1}Q_{L_1}u)(P_{N_2}Q_{L_2}v)\|_{L^2} \|P_N Q_L w\|_{L^2}.$$

Now, estimate (3.16) provides the following bound for $I_{LH\to H}$,

$$\sum_{L,L_1,L_2} L^{-\frac12+2\delta} L_1^{-\delta} L_2^{-\delta} \sum_{N\sim N_2, N_1\leq N_2/4} N_1^{-(s-\frac12)} \|P_{N_1}Q_{L_1}u\|_{L^2} \|P_{N_2}Q_{L_2}v\|_{L^2} \|P_N Q_L w\|_{L^2}.$$

Therefore, we deduce after summing over L, L_1, L_2, N_1 and applying the Cauchy-Schwarz inequality in $N \sim N_2$ that

BilinR2.7 (4.17)
$$\begin{aligned} I_{LH\to H} &\lesssim \|u\|_{L^2} \sum_{N\sim N_2} \|P_{N_2}v\|_{L^2}\|P_N w\|_{L^2} \\ &\lesssim \|u\|_{L^2} \Big(\sum_{N_2} \|P_{N_2}v\|_{L^2}^2 \Big)^{\frac12} \Big(\sum_{N} \|P_N w\|_{L^2}^2 \Big)^{\frac12} \\ &\lesssim \|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}. \end{aligned}$$

3. Estimate for $I_{HL\to H}$. Arguing similarly, we get that

BilinR2.8 (4.18) $$I_{HL\to H} \lesssim \|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}.$$

4. Estimate for $I_{HH\to L}$. We use the same decomposition as in (4.16). By using the Cauchy-Schwarz inequality, we can bound $I^{L,L_1,L_2}_{N,N_1,N_2}$ by

BilinR2.9
$$L^{-\frac12+2\delta}L_1^{-\frac12-\delta}L_2^{-\frac12-\delta}\frac{N^{s+1}}{N_1^sN_2^s}\big\|\widetilde{(P_{N_1}Q_{L_1}u)}(P_NQ_Lw)\big\|_{L^2}\|P_{N_2}Q_{L_2}v\|_{L^2}, \tag{4.19}$$

where $\tilde f(\xi,\mu,\tau)=f(-\xi,-\mu,-\tau)$. Moreover, observe interpolating (3.15) and (3.16) that

BilinR2.10
$$\begin{aligned}&\|\widetilde{(P_{N_1}Q_{L_1}u)}(P_NQ_Lw)\|_{L^2}\\ &\lesssim \frac{(N_1\wedge N)^{\frac12(1+\theta)}}{(N_1\vee N)^{1-\theta}}(L_1\vee L)^{\frac12(1-\theta)}(L_1\wedge L)^{\frac12}\|P_{N_1}Q_{L_1}u\|_{L^2}\|P_NQ_Lw\|_{L^2},\end{aligned} \tag{4.20}$$

for all $0\le\theta\le1$. Without loss of generality, we can assume that $L=L\vee L_1$ (the case $L_1=L\vee L_1$ is actually easier). Hence, we deduce from (4.19) and (4.20) that

BilinR2.11
$$I^{L,L_1,L_2}_{N,N_1,N_2}\lesssim L_1^{-\delta}L_2^{-\frac12-\delta}L^{2\delta-\frac{\theta}{2}}N^{\frac12+\theta}N_1^{-(s-\theta)}\|P_{N_1}Q_{L_1}u\|_{L^2}\|P_NQ_Lw\|_{L^2}\|P_{N_2}Q_{L_2}v\|_{L^2} \tag{4.21}$$

Now, we choose $0<\theta<1$ and $\delta>0$ satisfying $0<2\theta<s-\frac12$ and $0<\delta<\frac{\theta}{4}$. It follows after summing (4.21) over $L,\ L_1,\ L_2$ and performing the Cauchy-Schwarz inequality in N and N_1 that

BilinR2.12
$$\begin{aligned}I_{HH\to L}&\lesssim\sum_{N_1}N_1^{-(s-\frac12-2\theta)}\|P_{N_1}u\|_{L^2}\big(\sum_N\|P_Nw\|_{L^2}^2\big)^{\frac12}\|v\|_{L^2}\\ &\lesssim\|u\|_{L^2}\|w\|_{L^2}\|v\|_{L^2}.\end{aligned} \tag{4.22}$$

5. Estimate for $I_{HH\to H}$. Let $0<\alpha<1$ be a small positive number such that $f(\alpha)=\frac{1}{1000}$, where f is defined in Lemma 4.2. In order to simplify the notations, we will denote $(\xi,\mu,\tau)=(\xi_0,\mu_0,\tau_0)$. We split the integration domain in the following subsets:

$$\mathcal{D}_1=\big\{(\xi_1,\mu_1,\tau_1,\mu,\xi,\tau)\in\mathbb{R}^6\ :\ (1-\alpha)^{\frac12}\sqrt{3}|\xi_i|\le|\mu_i|\le(1-\alpha)^{-\frac12}\sqrt{3}|\xi_i|,\ i=1,2\big\},$$
$$\mathcal{D}_2=\big\{(\xi_1,\mu_1,\tau_1,\mu,\xi,\tau)\in\mathbb{R}^6\ :\ (1-\alpha)^{\frac12}\sqrt{3}|\xi_i|\le|\mu_i|\le(1-\alpha)^{-\frac12}\sqrt{3}|\xi_i|,\ i=0,1\big\},$$
$$\mathcal{D}_3=\big\{(\xi_1,\mu_1,\tau_1,\mu,\xi,\tau)\in\mathbb{R}^6\ :\ (1-\alpha)^{\frac12}\sqrt{3}|\xi_i|\le|\mu_i|\le(1-\alpha)^{-\frac12}\sqrt{3}|\xi_i|,\ i=0,2\big\},$$
$$\mathcal{D}_4=\mathbb{R}^6\setminus\bigcup_{j=1}^3\mathcal{D}_j\ .$$

Then, if we denote by $I^j_{HH\to H}$ the restriction of $I_{HH\to H}$ to the domain $\mathcal{D}_j$, we have that

BilinR2.13
$$I_{HH\to H}=\sum_{j=1}^4I^j_{HH\to H}. \tag{4.23}$$

5.1. Estimate for $I^1_{HH\to H}$. We consider the following subcases.

(i) *Case $\{\xi_1\xi_2>0$ and $\mu_1\mu_2>0\}$.* We define

$$\mathcal{D}_{1,1}=\big\{(\xi_1,\mu_1,\tau_1,\mu,\xi,\tau)\in\mathcal{D}_1\ :\ \xi_1\xi_2>0\text{ and }\mu_1\mu_2>0\big\}$$

and denote by $I^{1,1}_{HH\to H}$ the restriction of $I^1_{HH\to H}$ to the domain $\mathcal{D}_{1,1}$. We observe from (2.4) and the frequency localization that

BilinR2.i.1
$$\max\{|\sigma|,|\sigma_1|,|\sigma_2|\}\gtrsim|H(\xi_1,\mu_1,\xi_2,\mu_2)|\gtrsim N_1^3 \tag{4.24}$$

in the region $\mathcal{D}_{1,1}$. Therefore, it follows arguing exactly as in (4.14) that

BilinR2.i.2
$$I^{1,1}_{HH\to H} \lesssim \|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}. \tag{4.25}$$

(ii) *Case* $\{\xi_1\xi_2 > 0$ *and* $\mu_1\mu_2 < 0\}$ *or* $\{\xi_1\xi_2 < 0$ *and* $\mu_1\mu_2 > 0\}$. We define

$$\mathcal{D}_{1,2} = \big\{(\xi_1,\mu_1,\tau_1,\mu,\xi,\tau) \in \mathcal{D}_1 \ : \ \xi_1\xi_2 > 0,\ \mu_1\mu_2 < 0 \text{ or } \xi_1\xi_2 < 0,\ \mu_1\mu_2 > 0\big\}$$

and denote by $I^{1,2}_{HH\to H}$ the restriction of $I^1_{HH\to H}$ to the domain $\mathcal{D}_{1,2}$. Moreover, we use dyadic decompositions on the variables σ, σ_1 and σ_2 as in (4.16). Plancherel's identity and the Cauchy-Schwarz inequality yield

BilinR2.i.3
$$I^{L,L_1,L_2}_{N,N_1,N_2} \lesssim N^{1-s}L^{-\frac{1}{2}+2\delta}L_1^{-\frac{1}{2}-\delta}L_2^{-\frac{1}{2}-\delta}\|(P_{N_1}Q_{L_1}u)(P_{N_2}Q_{L_2}v)\|_{L^2}\|w\|_{L^2}. \tag{4.26}$$

Next, we argue as in (3.20) to estimate $\|(P_{N_1}Q_{L_1}u)(P_{N_2}Q_{L_2}v)\|_{L^2}$. Moreover, we observe that

$$\Big|\frac{\partial H}{\partial\mu_1}(\xi_1,\xi-\xi_1,\mu_1,\mu-\mu_1)\Big| = 2\big|\mu_1\xi_1 - \mu_2\xi_2\big| \gtrsim N^2$$

in the region $\mathcal{D}_{1,2}$. Thus, we deduce from Lemma 3.7, estimates (3.17) and (3.20) and (3.21) that

BilinR2.i.4
$$\begin{aligned}&\|(P_{N_1}Q_{L_1}u)(P_{N_2}Q_{L_2}v)\|_{L^2}\\ &\qquad\lesssim N^{-\frac{1}{2}}(L_1\vee L_2)^{\frac{1}{2}}(L_1\wedge L_2)^{\frac{1}{2}}\|P_{N_1}Q_{L_1}u\|_{L^2}\|P_{N_2}Q_{L_2}v\|_{L^2}.\end{aligned} \tag{4.27}$$

Therefore, we deduce combining estimates (4.26) and (4.27) and summing over $L,\ L_1,\ L_2$ and $N\sim N_1\sim N_2$ that

BilinR2.i.5
$$I^{1,2}_{HH\to H} \lesssim \|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}. \tag{4.28}$$

(iii) *Case* $\{\xi_1\xi_2 < 0$ *and* $\mu_1\mu_2 < 0\}$. We define

$$\mathcal{D}_{1,3} = \big\{(\xi_1,\mu_1,\tau_1,\mu,\xi,\tau) \in \mathcal{D}_1 \ : \ \xi_1\xi_2 < 0 \text{ and } \mu_1\mu_2 < 0\big\}$$

and denote by $I^{1,3}_{HH\to H}$ the restriction of $I^1_{HH\to H}$ to the domain $\mathcal{D}_{1,3}$. Moreover, we observe due to the frequency localization that there exists some $0<\gamma\ll 1$ such that

BilinR2.i.6
$$\Big||(\xi_2,\mu_2)|^2 - |(\xi_1,\mu_1)|^2\Big| \geq \gamma\max\big\{|(\xi_1,\mu_1)|^2, |(\xi_2,\mu_2)|^2\big\} \tag{4.29}$$

in $\mathcal{D}_{1,3}$. Indeed, if estimate (4.29) does not hold for all $0<\gamma\leq\frac{1}{1000}$, then estimate (4.2) with $f(\alpha)=\frac{1}{1000}$ would imply that

$$|(\xi,\mu)|^2 \leq \frac{1}{500}\max\big\{|(\xi_1,\mu_1)|^2, |(\xi_2,\mu_2)|^2\big\}$$

which would be a contradiction since we are in the $High\times High\to High$ interactions case. Thus, we deduce from (4.29) that

$$\Big|\frac{\partial H}{\partial\xi_1}(\xi_1,\xi-\xi_1,\mu_1,\mu-\mu_1)\Big| = \Big||(\xi_2,\mu_2)|^2 - |(\xi_1,\mu_1)|^2\Big| \gtrsim N^2.$$

We can then reapply the arguments in the proof of Proposition 3.6 to show that estimate (4.27) still holds true in this case. Therefore, we conclude arguing as above that

BilinR2.i.7
$$I^{1,3}_{HH\to H} \lesssim \|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}. \tag{4.30}$$

Finally, estimates (4.25), (4.28) and (4.30) imply that

BilinR2.i.8 $$I^1_{HH\to H} \lesssim \|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}. \tag{4.31}$$

5.2. Estimate for $I^2_{HH\to H}$ and $I^3_{HH\to H}$. Arguing as for $I^1_{HH\to H}$, we get that

BilinR2.ii.1 $$I^2_{HH\to H} + I^3_{HH\to H} \lesssim \|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}. \tag{4.32}$$

We explain for example how to deal with $I^2_{HH\to H}$. It suffices to rewrite I_{N,N_1,N_2} as

$$I_{N,N_1,N_2} = \int_{\mathcal{D}_2} \Gamma^{\tilde\xi_1,\tilde\mu_1,\tilde\tau_1}_{\xi,\mu,\tau} \widehat{P_N w}(\xi,\mu,\tau)\widehat{\widetilde{P_{N_1} u}}(\tilde\xi_1,\tilde\mu_1,\tilde\tau_1)\widehat{P_{N_2} v}(\xi_2,\mu_2,\tau_2)d\tilde\nu,$$

where

$$d\tilde\nu = d\xi d\xi_2 d\mu d\mu_2 d\tau d\tau_2, \quad \tilde\xi_1 = \xi_2 - \xi,\ \tilde\mu_1 = \mu_2 - \mu,\ \tilde\tau_1 = \tau_2 - \tau,$$

and $\Gamma^{\tilde\xi_1,\tilde\mu_1,\tilde\tau_1}_{\xi,\mu,\tau}$ is defined as in (4.11). Moreover, we observe that

$$\tilde H = H(\xi,\xi_2-\xi,\mu,\mu_2-\mu) = w(\xi_2,\mu_2) - w(\xi,\mu) - w(\xi_2-\xi,\mu_2-\mu)$$

satisfies

$$\Big|\frac{\partial \tilde H}{\partial \xi}\Big| = \big|3\xi^2 + \mu^2 - (3\tilde\xi_1^2 + \tilde\mu_1^2)\big| \quad \text{and} \quad \Big|\frac{\partial \tilde H}{\partial \mu}\Big| = 2\big|\xi\mu - \tilde\xi_1\tilde\mu_1\big|.$$

Therefore, we divide in the subregions $\{\xi\tilde\xi_1 > 0,\ \mu\tilde\mu_1 > 0\}$, $\{\xi\tilde\xi_1 < 0,\ \mu\tilde\mu_1 > 0\}$, $\{\xi\tilde\xi_1 > 0,\ \mu\tilde\mu_1 < 0\}$ and $\{\xi\tilde\xi_1 < 0,\ \mu\tilde\mu_1 < 0\}$ and use the same arguments as above.

5.3. Estimate for $I^4_{HH\to H}$. Observe that in the region $\mathcal{D}_4$, we have

BilinR2.iv.1 $$|\mu_i^2 - 3\xi_i^2| > \frac{\alpha}{2}|(\xi_i,\mu_i)|^2 \quad \text{and} \quad |\mu_j^2 - 3\xi_j^2| > \frac{\alpha}{2}|(\xi_j,\mu_j)|^2, \tag{4.33}$$

for at least a combination (i,j) in $\{0,1,2\}$. Without loss of generality[1], we can assume that $i=1$ and $j=2$ in (4.33). Then, we deduce from Plancherel's identity and Hölder's inequality that

$$I^4_{HH\to H} \lesssim \sum_{N_2\sim N_1} N_1^{-(s-\frac12)} \|K(D)^{\frac18}\Big(\frac{\widehat{P_{N_1}u}}{\langle\sigma_1\rangle^{\frac12+\delta}}\Big)^\vee\|_{L^4}\|K(D)^{\frac18}\Big(\frac{\widehat{P_{N_2}v}}{\langle\sigma_2\rangle^{\frac12+\delta}}\Big)^\vee\|_{L^4}\|w\|_{L^2},$$

where the operator $K(D)^{\frac18}$ is defined in Proposition 3.5. Therefore, estimate (3.13) implies that

BilinR2.iv.2 $$I^4_{HH\to H} \lesssim \|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}. \tag{4.34}$$

Finally, we conclude the proof of estimate (4.1) gathering estimates (4.13), (4.15), (4.17), (4.18), (4.22), (4.23), (4.31), (4.32) and (4.34). □

At this point, we observe that the proof of Theorem 1.1 follows from Proposition 4.1 and the linear estimates (2.7), (2.8) and (2.9) by using a fixed point argument in a closed ball of $X_T^{s,\frac12+\delta}$ (see for example [14] for more details).

[1] in the other cases, we cannot use estimate (3.13) directly, but need to interpolate it with estimate (3.2) as previously.

5. Bilinear estimate in $\mathbb{R}\times\mathbb{T}$

The main result of this section is stated below.

BilinRT **Proposition 5.1.** *Let $s\geq 1$. Then, there exists $\delta>0$ such that*

BilinRT.1
$$\|\partial_x(uv)\|_{X^{s,-\frac{1}{2}+2\delta}}\lesssim \|u\|_{X^{s,\frac{1}{2}+\delta}}\|v\|_{X^{s,\frac{1}{2}+\delta}}, \tag{5.1}$$

for all $u,\ v:\mathbb{R}\times\mathbb{T}\times\mathbb{R}\to\mathbb{R}$ such that $u,\ v\in X^{s,\frac{1}{2}+\delta}$.

Proof. By duality, it suffices to prove that

BilinRT.2
$$J\lesssim \|u\|_{L^2_{x,y,t}}\|v\|_{L^2_{x,y,t}}\|w\|_{L^2_{x,y,t}}, \tag{5.2}$$

where

$$J=\sum_{q,q_1\in\mathbb{Z}^2}\int_{\mathbb{R}^4}\Gamma^{\xi_1,q_1,\tau_1}_{\xi,q,\tau}\widehat{w}(\xi,q,\tau)\widehat{u}(\xi_1,q_1,\tau_1)\widehat{v}(\xi_2,q_2,\tau_2)d\nu,$$

$\widehat{u}$, $\widehat{v}$ and $\widehat{w}$ are nonnegative functions, and we used the following notations

BilinRT.3 (5.3)

$$\Gamma^{\xi_1,q_1,\tau_1}_{\xi,q,\tau}=|\xi|\langle|(\xi,q)|\rangle^s\langle\sigma\rangle^{-\frac{1}{2}+2\delta}\langle|(\xi_1,q_1)|\rangle^{-s}\langle\sigma_1\rangle^{-\frac{1}{2}-\delta}\langle|(\xi_2,q_2)|\rangle^{-s}\langle\sigma_2\rangle^{-\frac{1}{2}-\delta},$$
$$d\nu=d\xi d\xi_1 d\tau d\tau_1,\quad \xi_2=\xi-\xi_1,\ q_2=q-q_1,\ \tau_2=\tau-\tau_1,$$
$$\sigma=\tau-w(\xi,q)\quad\text{and}\quad \sigma_i=\tau_i-w(\xi_i,q_i),\ i=1,2.$$

By using dyadic decompositions on the spatial frequencies of u, v and w, we rewrite J as

BilinRT.4
$$J=\sum_{N_1,N_2,N}J_{N,N_1,N_2}, \tag{5.4}$$

where

$$J_{N,N_1,N_2}=\sum_{q,q_1\in\mathbb{Z}^2}\int_{\mathbb{R}^4}\Gamma^{\xi_1,q_1,\tau_1}_{\xi,q,\tau}\widehat{P_Nw}(\xi,q,\tau)\widehat{P_{N_1}u}(\xi_1,q_1,\tau_1)\widehat{P_{N_2}v}(\xi_2,q_2,\tau_2)d\nu.$$

Now, we use the decomposition

BilinRT.5
$$J=J_{LL\to L}+J_{LH\to H}+J_{HL\to H}+J_{HH\to L}+J_{HH\to H}, \tag{5.5}$$

where $J_{LL\to L}$, $J_{LH\to H}$, $J_{HL\to H}$, $J_{HH\to L}$, respectively $J_{HH\to H}$, denote the $Low\times Low\to Low$, $Low\times High\to High$, $High\times Low\to High$, $High\times High\to Low$, respectively $High\times High\to High$ contributions for J as defined in the proof of Proposition 4.1.

1. Estimate for $J_{LH\to H}+J_{HL\to H}+J_{HH\to L}$. Since Proposition 3.6 also holds in the $\mathbb{R}\times\mathbb{T}$ case, we deduce arguing as in (4.17), (4.18) and (4.22) that

BilinRT.6
$$J_{LH\to H}+J_{HL\to H}+J_{HH\to L}\lesssim \|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}. \tag{5.6}$$

2. Estimate for $J_{HH\to H}$. We recall that $N\sim N_1\sim N_2$ in this case. We divide the integration domain in several regions.

2.1 Estimate for $J_{HH\to H}$ in the region $|\xi|\leq 100$. We denote by $J^1_{HH\to H}$ the restriction of $J_{HH\to H}$ to the region $|\xi|\leq 100$ and use dyadic decompositions on the variables σ, σ_1, σ_2 and ξ, so that

BilinRT.7
$$J_{N,N_1,N_2}=\sum_{k\geq 0}\sum_{L,L_1,L_2}J^{L,L_1,L_2}_{N,N_1,N_2,k}, \tag{5.7}$$

where $J^{L,L_1,L_2}_{N,N_1,N_2,k}$ is given by the expression

$$\sum_{q,q_1\in\mathbb{Z}^2}\int_{\mathcal{E}_k}\Gamma^{\xi_1,q_1,\tau_1}_{\xi,q,\tau}\widehat{P_NQ_Lw}(\xi,q,\tau)\widehat{P_{N_1}Q_{L_1}u}(\xi_1,q_1,\tau_1)\widehat{P_{N_2}Q_{L_2}v}(\xi_2,q_2,\tau_2)d\nu,$$

with $\mathcal{E}_k=\{(\xi,\xi_1,\tau,\tau_1)\in\mathbb{R}^4\ :\ 2^{-(k+1)}100\le|\xi|\le 2^{-k}100\}$. Thus, by using the Cauchy-Schwarz inequality, we get that

BilinRT.8 (5.8)
$$J^{L,L_1,L_2}_{N,N_1,N_2,k}\lesssim 2^{-k}N_1^{-s}L^{-\frac12+2\delta}L_1^{-\frac12-\delta}L_2^{-\frac12-\delta}\|(P_{N_1}Q_{L_1}u)(P_{N_2}Q_{L_2}v)\|_{L^2}\|w\|_{L^2}.$$

Next, we argue as in (3.20) to estimate $\|(P_{N_1}Q_{L_1}u)(P_{N_2}Q_{L_2}v)\|_{L^2}$. Moreover, we observe that

$$\Big|\frac{\partial^2 H}{\partial\xi_1^2}(\xi,\xi_1,q,q_1)\Big|=6|\xi|\sim 2^{-k}.$$

Thus, it follows from Lemma 3.7, estimates (3.18), (3.20) and (3.21) that

BilinRT.9 (5.9)
$$\begin{aligned}&\|(P_{N_1}Q_{L_1}u)(P_{N_2}Q_{L_2}v)\|_{L^2}\\&\quad\lesssim 2^{\frac{k}{4}}N_1^{\frac12}(L_1\wedge L_2)^{\frac12}(L_1\vee L_2)^{\frac14}\|P_{N_1}Q_{L_1}u\|_{L^2}\|P_{N_2}Q_{L_2}v\|_{L^2}.\end{aligned}$$

Therefore, we deduce combining (5.8) and (5.9) and summing over L, L_1, L_2, $N\sim N_1\sim N_2$ and $k\in\mathbb{N}$ that

BilinRT.10 (5.10)
$$J^1_{HH\to H}\lesssim\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}.$$

2.2 Estimate for $J_{HH\to H}$ in the region $|\xi|\ge 100$, and $|\xi_1|\wedge|\xi_2|\le 100$. We denote by $J^2_{HH\to H}$ the restriction of $J_{HH\to H}$ to this region and use dyadic decompositions on the variables σ, σ_1, σ_2, so that

BilinRT.11 (5.11)
$$J_{N,N_1,N_2}=\sum_{L,L_1,L_2}J^{L,L_1,L_2}_{N,N_1,N_2},$$

where $J^{L,L_1,L_2}_{N,N_1,N_2}$ is given by the expression

BilinRT.110 (5.12)
$$\sum_{q,q_1\in\mathbb{Z}^2}\int_{\mathbb{R}^4}\Gamma^{\xi_1,q_1,\tau_1}_{\xi,q,\tau}\widehat{P_NQ_Lw}(\xi,q,\tau)\widehat{P_{N_1}Q_{L_1}u}(\xi_1,q_1,\tau_1)\widehat{P_{N_2}Q_{L_2}v}(\xi_2,q_2,\tau_2)d\nu.$$

Thus, the Caucy-Schwarz inequality implies that

BilinRT.12 (5.13)
$$J^{L,L_1,L_2}_{N,N_1,N_2}\lesssim L^{-\frac12+2\delta}L_1^{-\frac12-\delta}L_2^{-\frac12-\delta}\|(P_{N_1}Q_{L_1}u)(P_{N_2}Q_{L_2}v)\|_{L^2}\|w\|_{L^2},$$

where we used the bound $|\xi|\le N\sim N_1\sim N_2$ and $s\ge 1$. This time, we observe that

$$\Big|\frac{\partial^2 H}{\partial q_1^2}(\xi,\xi_1,q,q_1)\Big|=2|\xi|\gtrsim 1.$$

in order to estimate $\|(P_{N_1}Q_{L_1}u)(P_{N_2}Q_{L_2}v)\|_{L^2}$. Then, since $|\xi_1|\wedge|\xi_2|\le 1$, it follows from Lemma 3.7, estimates (3.19), (3.20) and (3.21) that

BilinRT.13 (5.14)
$$\begin{aligned}&\|(P_{N_1}Q_{L_1}u)(P_{N_2}Q_{L_2}v)\|_{L^2}\\&\quad\lesssim(L_1\wedge L_2)^{\frac12}\big(1+(L_1\vee L_2)^{\frac14}\big)\|P_{N_1}Q_{L_1}u\|_{L^2}\|P_{N_2}Q_{L_2}v\|_{L^2}.\end{aligned}$$

Therefore, we deduce combining (5.13) and (5.14) and summing over L, L_1, L_2 and $N\sim N_1\sim N_2$ (here we use the Cauchy-Schwarz inequality in N_1) that

BilinRT.14 (5.15)
$$J^2_{HH\to H}\lesssim\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}.$$

2.3 Estimate for $J_{HH\to H}$ in the region $|\xi_i|\geq 100$ for $i=1,2,3$. We denote by $J^3_{HH\to H}$ the restriction of $J_{HH\to H}$ to this region. Once again, we use dyadic decompositions on the variables σ, σ_1 and σ_2 as in (5.11). In order to simplify the notations, we will denote $(\xi,q,\tau)=(\xi_0,q_0,\tau_0)$. Next, for $0<\delta\ll 1$, we split the integration domain in the following subregions

$$\begin{aligned}\mathcal{F}_{3.1}=\big\{&(\xi,\xi_1,\tau,\tau_1,q,q_1)\in\mathbb{R}^4\times\mathbb{Z}^2\ :|\xi_i|\geq 100,\ \forall i\in\{0,1,2\}\\ &\text{and }\exists\,(i,j)\in\{0,1,2\}\text{ with }\big||(\xi_i,q_i)|^2-|(\xi_j,q_j)|^2\big|\geq NL^{6\delta}\ \big\},\\ \mathcal{F}_{3.2}=\big\{&(\xi,\xi_1,\tau,\tau_1,q,q_1)\in\mathbb{R}^4\times\mathbb{Z}^2\ :|\xi_i|\geq 100,\ \forall i\in\{0,1,2\}\\ &\text{and }\big||(\xi_i,q_i)|^2-|(\xi_j,q_j)|^2\big|\leq NL^{6\delta},\ \forall\,(i,j)\in\{0,1,2\}\ \big\}.\end{aligned}$$

and denote by $J^{3,1}_{HH\to H}$, respectively $J^{3,2}_{HH\to H}$, the restriction of $J_{HH\to H}$ to $\mathcal{F}_{3.1}$, respectively $\mathcal{F}_{3.2}$.

2.3.1 Estimate for $J^{3,1}_{HH\to H}$. Without loss of generality, we can assume that

BilinRT.15
$$\big||(\xi,q)|^2-|(\xi_1,q_1)|^2\big|\geq NL^{6\delta}. \tag{5.16}$$

By using the Cauchy-Schwarz inequality and the fact that $|\xi|\lesssim N\sim N_1\sim N_2$ and $s\geq 1$, we obtain that

BilinRT.16
$$J^{L,L_1,L_2}_{N,N_1,N_2}\lesssim L^{-\frac12+2\delta}L_1^{-\frac12-\delta}L_2^{-\frac12-\delta}\big\|\widetilde{(P_{N_1}Q_{L_1}u)}(P_NQ_Lw)\big\|_{L^2}\|P_{N_2}Q_{L_2}v\|_{L^2}, \tag{5.17}$$

where $\tilde f(\xi,q,\tau)=f(-\xi,-q,-\tau)$. Moreover, we observe arguing exactly as in the proof of Proposition 3.6 and by using (5.16) that

BilinRT.17
$$\begin{aligned}&\|\widetilde{(P_{N_1}Q_{L_1}u)}(P_NQ_Lw)\|_{L^2}\\ &\lesssim\frac{(N_1\wedge N_2)^{\frac12}}{N^{\frac12}L^{3\delta}}(L_1\vee L)^{\frac12}(L_1\wedge L)^{\frac12}\|P_{N_1}Q_{L_1}u\|_{L^2}\|P_NQ_Lw\|_{L^2}.\end{aligned} \tag{5.18}$$

Therefore, we deduce combining (5.17) and (5.18) and summing over L, L_1, L_2 and $N\sim N_1\sim N_2$ (by using the Cauchy-Schwarz inequality in N) that

BilinRT.18
$$J^{3,1}_{HH\to H}\lesssim\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}. \tag{5.19}$$

2.3.2 Estimate for $J^{3,2}_{HH\to H}$. In the region $\mathcal{F}_{3,2}$, it holds that

BilinRT.19
$$\big||(\xi_i,q_i)|^2-|(\xi_j,q_j)|^2\big|\leq NL^{6\delta},\ \forall\,(i,j)\in\{0,1,2\}. \tag{5.20}$$

Then, we deduce from the definition of H in (2.3), the definition $|(\xi,q)|=\sqrt{3\xi^2+q^2}$ and the assumptions (5.20) that

BilinRT.20
$$\begin{aligned}H(\xi,\xi_1,q,q_1)&=(\xi-\xi_1-\xi_2)|(\xi_{i_0},q_{i_0})|^2-6\xi\xi_1\xi_2+\Theta(\xi,\xi_1,q,q_1)\\ &=-6\xi\xi_1\xi_2+\Theta(\xi,\xi_1,q,q_1),\end{aligned} \tag{5.21}$$

for $i_0\in\{1,2,3\}$ such that $|\xi_{i_0}|=\max\{|\xi_j|\ :\ j=1,2,3\}$ and $\Theta(\xi,\xi_1,q,q_1)$ satisfies

BilinRT.21
$$\big|\Theta(\xi,\xi_1,q,q_1)\big|\leq\sum_{i\neq i_0}|\xi_i|\big||(\xi_i,q_i)|^2-|(\xi_j,q_j)|^2\big|\leq|\xi_{med}|NL^{6\delta}. \tag{5.22}$$

It follows combining (5.21) and (5.22) that

BilinRT.22
$$\Big|H(\xi,\xi_1,q,q_1)\Big|\geq|\xi_{med}|\Big(6|\xi_{max}||\xi_{min}|-NL^{6\delta}\Big). \tag{5.23}$$

Then, we subdivide the region $\mathcal{F}_{1.2}$ in the following subregions

$$\mathcal{F}_{3.2.1}=\big\{(\xi,\xi_1,\tau,\tau_1,q,q_1)\in\mathcal{F}_{1.2}\ :|\xi_{max}||\xi_{min}|\geq NL^{6\delta}\big\},$$

$$\mathcal{F}_{3.2.2}=\big\{(\xi,\xi_1,\tau,\tau_1,q,q_1)\in\mathcal{F}_{1.2}\ :|\xi_{max}||\xi_{min}|\leq NL^{6\delta}\big\},$$

and denote by $J^{3,2,1}_{HH\to H}$, respectively $J^{3,2,2}_{HH\to H}$, the restriction of $J^{3,2}_{HH\to H}$ to $\mathcal{F}_{3.2.1}$, respectively $\mathcal{F}_{3.2.2}$.

2.3.2.1 Estimate for $J^{3,2,1}_{HH\to H}$. Due to (5.23), we have that

BilinRT.23 (5.24) $$\max\{|\sigma|,|\sigma_1|,|\sigma_2|\}\gtrsim|\xi_{min}||\xi_{max}|^2,$$

in $\mathcal{F}_{3.2.1}$. Without loss of generality[2], we assume that $\max\{|\sigma|,|\sigma_1|,|\sigma_2|\}=|\sigma|$. Then, by using the Cauchy-Schwarz inequality, we deduce that

BilinRT.24 (5.25) $$J^{L,L_1,L_2}_{N,N_1,N_2}\lesssim N_1^{-\frac12}L^{-\delta}L_1^{-\frac12-\delta}L_2^{-\frac12-\delta}\|(P_{N_1}Q_{L_1}u)(P_{N_2}Q_{L_2}v)\|_{L^2}\|w\|_{L^2}.$$

where we used that $\big||\xi|N_1^{\frac12-s}\big(|\xi_{max}|^2|\xi_{min}|\big)^{-\frac12+3\delta}\big|\lesssim 1$ for $s\geq 1$ and $0<\delta\ll 1$. Moreover, we use that

$$\Big|\frac{\partial^2 H}{\partial\xi_1^2}(\xi,\xi_1,q,q_1)\Big|=6|\xi|\gtrsim 1,$$

Lemma 3.7, estimates (3.18), (3.20) and (3.21) lead to

BilinRT.25 (5.26) $$\begin{aligned}&\|(P_{N_1}Q_{L_1}u)(P_{N_2}Q_{L_2}v)\|_{L^2}\\&\qquad\lesssim N_1^{\frac12}(L_1\wedge L_2)^{\frac12}(L_1\vee L_2)^{\frac14}\|P_{N_1}Q_{L_1}u\|_{L^2}\|P_{N_2}Q_{L_2}v\|_{L^2}.\end{aligned}$$

We deduce combining (5.25) and (5.26) and summing over L, L_1, L_2 and using the Cauchy-Schwarz inequality in $N_1\sim N_2$ that

BilinRT.26 (5.27) $$J^{3,2,1}_{HH\to H}\lesssim\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}.$$

2.3.2.2 Estimate for $J^{3,2,2}_{HH\to H}$. This time, we perform also dyadic decompositions in the ξ_1, ξ_2 and ξ variables. We denote by R_K the Littlewood-Paley projectors , *i.e.* R_K is defined by $R_Ku=\mathcal{F}_x^{-1}\big(\phi(K^{-1}\xi)\mathcal{F}_x(u)\big)$, for any dyadic number $K\geq 1$. Then, we have that

BilinRT.27 (5.28) $$J^{L,L_1,L_2}_{N,N_1,N_2}=\sum_{100\leq K_1,K_2,K_3\lesssim N}J^{L,L_1,L_2}_{N,N_1,N_2}(K_1,K_2,K_3),$$

where $J^{L,L_1,L_2}_{N,N_1,N_2}(K_1,K_2,K_3)$ is defined by the expression

$$\begin{aligned}J^{L,L_1,L_2}_{N,N_1,N_2}(K_1,K_2,K_3)=&\sum_{q,q_1\in\mathbb{Z}^2}\int_{\mathbb{R}^4}\Gamma^{\xi_1,q_1,\tau_1}_{\xi,q,\tau}\big(P_NQ_LR_Kw\big)^\wedge(\xi,q,\tau)\\&\times\big(P_{N_1}Q_{L_1}R_{K_1}u\big)^\wedge(\xi_1,q_1,\tau_1)\big(P_{N_2}Q_{L_2}R_{K_2}v\big)^\wedge(\xi_2,q_2,\tau_2)d\nu.\end{aligned}$$

By using the Cauchy-Schwarz inequality, we can bound $J^{L,L_1,L_2}_{N,N_1,N_2}(K_1,K_2,K_3)$ by

BilinRT.28 (5.29) $$KK_{min}^{-1}K_{max}^{-1}N^{1-s}L^{-\frac12+8\delta}L_1^{-\frac12-\delta}L_2^{-\frac12-\delta}\|(P_{N_1}Q_{L_1}u)(P_{N_2}Q_{L_2}v)\|_{L^2}\|w\|_{L^2}.$$

since $K_{min}K_{max}\lesssim NL^{6\delta}$ in the region $\mathcal{F}_{3,2,2}$. Moreover, noticing that

$$\Big|\frac{\partial^2 H}{\partial q_1^2}(\xi,\xi_1,q,q_1)\Big|=6|\xi|\gtrsim K,$$

[2]In the other cases we need to interpolate (5.26) with (3.15) as previously.

Lemma 3.7, estimates (3.19), (3.20) and (3.21) yield

BilinRT.29 (5.30)
$$\begin{aligned}&\|(P_{N_1}Q_{L_1}u)(P_{N_2}Q_{L_2}v)\|_{L^2}\\&\lesssim (K_1\wedge K_2)^{\frac12}(L_1\wedge L_2)^{\frac12}(1+K^{-\frac14}(L_1\vee L_2)^{\frac14})\|P_{N_1}Q_{L_1}u\|_{L^2}\|P_{N_2}Q_{L_2}v\|_{L^2}.\end{aligned}$$

Now, we observe that

BilinRT.30 (5.31)
$$K(K_1\wedge K_2)^{\frac12}K_{min}^{-1}K_{max}^{-1}\lesssim K_{min}^{-\frac12}.$$

Assume without loss of generality that $K_{min}=K$. Therefore, it follows combining (5.28)–(5.31), summing over L, L_1, L_2 and K_{min} and applying Cauchy-Schwarz in $K_1\sim K_2$ and in $N_1\sim N_2$ that

BilinRT.31 (5.32)
$$\begin{aligned}J_{HH\to H}^{3,2,2}&\lesssim \sum_{N\sim N_1\sim N_2}\sum_{100\le K_1\sim K_2\lesssim N}\|P_{N_1}R_{K_1}u\|_{L^2}\|P_{N_2}R_{K_2}v\|_{L^2}\|P_Nw\|_{L^2}\\&\lesssim \sum_{N_1\sim N_2}\Big(\sum_{K_1\le N_1}\|P_{N_1}R_{K_1}u\|_{L^2}^2\Big)^{\frac12}\Big(\sum_{K_2\le N_2}\|P_{N_2}R_{K_2}v\|_{L^2}^2\Big)^{\frac12}\|w\|_{L^2}\\&\lesssim \Big(\sum_{N_1}\|P_{N_1}u\|_{L^2}^2\Big)^{\frac12}\Big(\sum_{N_2}\|P_{N_2}v\|_{L^2}^2\Big)^{\frac12}\|w\|_{L^2}\\&\lesssim \|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}.\end{aligned}$$

Thus, we deduce combining (5.10), (5.15), (5.19), (5.27) and (5.32) that

BilinRT.32 (5.33)
$$J_{HH\to H}\lesssim \|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}.$$

2.3 Estimate for $J_{LL\to L}$. We get arguing exactly as in the cases *2.1* and *2.2* that

BilinRT.33 (5.34)
$$J_{LL\to L}\lesssim \|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}.$$

Finally, we conclude the proof of estimate (5.1) gathering (5.5), (5.6), (5.33) and (5.34). □

We observe that the proof of Theorem 1.2 follows from Proposition 5.1 and the linear estimates (2.7), (2.8) and (2.9) by using a fixed point argument in a closed ball of $X_T^{s,\frac12+\delta}$ (see for example [14] for more details).

6. Global existence in $H^s(\mathbb{R}^3)$ for $s>1$

In this section we prove the global well-posedness in $H^s(\mathbb{R}^3)$ for $s>1$. To this aim we combine the conservation laws (1.2) and (1.3), a well-posedness result in the Besov space $B_2^{1,1}(\mathbb{R}^3)$ and follow ideas in [1] (see [18] for the same kind of arguments). One crucial tool will also be the atomic spaces U^2 and V^2 introduced by Koch-Tataru in [9]. Recall that the Besov space $B_2^{1,1}(\mathbb{R}^3)$ is the space of all functions $g\in\mathcal{S}'(\mathbb{R}^3)$ such that

defB (6.1)
$$\|g\|_{B_2^{1,1}}:=\sum_N N\|P_Ng\|_{L^2}<\infty\,,$$

where the Fourier projector P_N is the $\mathbb{R}^3$-version of the one defined in (2.1).

Before stating the local existence theorem let us give the definition of a "doubling time" that will appear in the statement of this theorem. Let be given a Cauchy

problem locally well-posed in some Banach space B with a minimum time of existence depending on the B-norm of the initial data and let $C_0 \geq 1$ be given. For any $u_0 \in B$ we call "doubling time", the infinite or finite positive real number

$$T_{C_0}(u_0) = \sup\Big\{ t > 0 \,:\, \|u(\theta)\|_B \leq 2C_0\|u_0\|_B \text{ on } [0,t] \Big\} .$$

prop2 **Theorem 6.1.** *The Cauchy problem associated to* (1.1) *is locally well-posed in* $H^s(\mathbb{R}^3)$ *for* $s > 1$. *Moreover, there exists* $C_0 \geq 1$ *and* $C > 0$ *such that for any* $u_0 \in H^s(\mathbb{R}^3)$, *the doubling time* T_{C_0} *satisfies*

Z3 (6.2) $$T_{C_0}(u_0) \geq \frac{C}{\|u_0\|^2_{B^{1,1}_2}} .$$

Remark 6.1. The local well-posedness of ZK in $H^s(\mathbb{R}^3)$ for $s > 1$ was already proven in [15]. The only new result here is the estimate from below of the doubling time.

With Theorem 6.1 in hand we will now prove the Theorem 1.4. The proof of Theorem 6.1 is postponed at the end of this section.

Proof of Theorem 1.4. Let us fix $s > 1$. For any $g \in H^s(\mathbb{R}^3)$ and any $k \geq 1$ it holds

$$\begin{aligned} \|g\|_{B^{1,1}_2} &= \sum_{j=0}^{k-1} 2^j \|P_{2^j} g\|_{L^2} + \sum_{j=k}^{\infty} 2^{j(1-s)} 2^{js} \|P_{2^j} g\|_{L^2} \\ &\lesssim \sqrt{k}\|g\|_{H^1} + 2^{k(1-s)}\|g\|_{H^s} . \end{aligned}$$

Therefore, taking $k = \frac{\ln(1+\|g\|_{H^s})}{(s-1)\ln 2}$ we get

z1 (6.3) $$\|g\|_{B^{1,1}_2} \leq C_s \left(1 + \|g\|_{H^1} \ln(1 + \|g\|_{H^s})^{1/2}\right)$$

for some $C_s > 0$.
Now, let $u_0 \in H^s(\mathbb{R}^3)$ and u be the solution of ZK emanating from u_0. Combining Theorem 6.1 and (6.3) we get

$$T_{C_0}(u_0) \geq \frac{C}{1 + \|u_0\|^2_{H^1} \ln(1 + \|u_0\|_{H^s})} .$$

If $T_{C_0}(u_0) = +\infty$ then we are done. Otherwise we set $u_1 := u(T_{C_0}(u_0))$. In the same way as above we have

$$T_{C_0}(u_1) \geq \frac{C}{1 + \|u_1\|^2_{H^1} \ln(1 + \|u_1\|_{H^s})} .$$

From the definition of the doubling time, it holds $\|u_1\|_{H^s} = 2C_0\|u_0\|_{H^s}$ and from the conservation of the quantities $M(u)$ and $H(u)$ and classical Sobolev inequalities we infer that

$$\|u_1\|^2_{H^1} \leq C' E(u_1) = C' E(u_0) ,$$

for some positive constant C' independent of u_1. Therefore, setting $E_0 := E(u_0)$, we obtain

$$T_{C_0}(u_1) \geq \frac{C}{1 + C' E_0 \ln(1 + 2C_0\|u_0\|_{H^s})} .$$

Repeating this argument n-times (assuming that all doubling times $T_{C_0}(u_k)$, $k = 1, 2, .., n-1$, are finite, since otherwise we are done), we get

z2 (6.4) $$T_{C_0}(u_n) \geq \frac{C}{1 + C' E_0 \ln(1 + (2C_0)^n \|u_0\|_{H^s})} \gtrsim \frac{1}{n} .$$

Since $\sum_{n-1} 1/n = +\infty$ this ensures that for any given $T > 0$ there exists $n \geq 1$ such that $\sum_{k=0}^{n-1} T_{C_0}(u_k) > T$ and thus the solution is global in time. □

Remark 6.2. Actually, it is not too hard to check that the lower bound (6.4) leads to a double exponential upper bound on the solution u, i.e. there exists constants K_1, K_2 and K_3 only depending on $\|u_0\|_{H^s}$ such that for all $t \geq 0$,

$$\|u(t)\|_{H^s} \leq K_1 \exp\Big(K_2 \exp(K_3 t)\Big) .$$

6.1. **Proof of Theorem 6.1.**

6.1.1. *Resolution spaces.* We start by recalling the definition of the function spaces U^2 and V^2 (see [9] and [6]).

Definition 6.2. Let $\mathcal{Z}$ be the set of finite partitions $-\infty = t_0 < t_1 < \cdot\cdot < t_K = +\infty$. For $\{t_k\}_{k=0}^K \in \mathcal{Z}$ and $\{\phi_k\}_{k=0}^{K-1} \subset L^2(\mathbb{R}^3)$ with $\sum_{k=0}^{K-1} \|\phi_k\|_{L^2}^2 = 1$ and $\phi_0 = 0$ we call the function $a : \mathbb{R} \to L^2(\mathbb{R}^3)$ given by

$$a = \sum_{k=1}^{K} \mathbb{1}_{[t_{k-1}, t_k)} \phi_{k-1}$$

a U^2-atom and we define the atomic space

$$U^2 := \Big\{ u = \sum_{j=1}^{\infty} \lambda_j a_j \ : \ a_j \ U^2\text{-atom and } \lambda_j \in \mathbb{R} \text{ with } \sum_{j=1}^{\infty} |\lambda_j| < \infty \Big\}$$

with norm

$$\|u\|_{U^2} := \inf\Big\{ \sum_{j=1}^{\infty} |\lambda_j| \ : \ u = \sum_{j=1}^{\infty} \lambda_j a_j \text{ with } \lambda_j \in \mathbb{R} \text{ and } a_j \, U^2\text{-atom} \Big\}$$

The function space V^2 is defined as the normed space of all functions $v : \mathbb{R} \to L^2(\mathbb{R}^3)$ such that $\lim_{t \to \mp\infty} v(t)$ exists and for which the norm

$$\|v\|_{V^2} := \sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}} \Big(\sum_{k=1}^{K} \|v(t_k) - v(t_{k-1})\|_{L^2}^2 \Big)^{1/2}$$

is finite, where we use the convention that $v(-\infty) = \lim_{t \to -\infty} v(t)$ and $v(+\infty) = 0$.

The spaces U^2 and V^2 are Banach spaces. They will serve as substitutes of the Besov type spaces $\tilde{B}_2^{1/2,1}(L^2(\mathbb{R}^3))$ and $\tilde{B}_2^{1/2,\infty}(L^2(\mathbb{R}^3))$ that where first used in [19] in the context of Bourgain's method. Denoting by Δ_j the Fourier multiplier by[3] $\phi(2^{-j}\tau)$ for $j \geq 1$ and $\eta(\tau)$ for $j = 0$, these last spaces are respectively endowed with the norms

$$\|u\|_{\tilde{B}_2^{1/2,1}(L^2(\mathbb{R}^3))} := \sum_{j \geq 0} 2^{j/2} \|\Delta_j u\|_{L^2(\mathbb{R}^4)}$$

[3]See Section 2 for the definition of ϕ and η.

and

$$\|u\|_{\tilde{B}_2^{1/2,\infty}(L^2(\mathbb{R}^3))} := \sup_{j\geq 0} 2^{j/2}\|\Delta_j u\|_{L^2(\mathbb{R}^4)} .$$

The crucial point for us will be that, from the definition of the function space V^2, for a smooth function $\psi \in C_c^\infty(\mathbb{R})$ and any $0 < T < 1$, it holds

zzz
$$\|\psi(\cdot/T)f\|_{L^2(\mathbb{R}^4))} \lesssim T^{1/2}\| f\|_{L_t^\infty L^2(\mathbb{R}^3)} \lesssim T^{1/2}\|f\|_{V^2}, \quad \forall f \in C_c^\infty(\mathbb{R}^4), \tag{6.5}$$

whereas we only have

$$\|\psi(\cdot/T)f\|_{L^2(\mathbb{R}^4))} \lesssim T^{1/2}|\ln T|\|f\|_{\tilde{B}_2^{1/2,\infty}(L^2(\mathbb{R}^3))}, \quad \forall f \in C_c^\infty(\mathbb{R}^4) .$$

This last inequality would lead to a lower bound

$$T(u_0) \gtrsim \frac{1}{\|u_0\|^2_{B_2^{1,1}}|\ln(\|u_0\|_{B_2^{1,1}})|^2}$$

of the doubling time that will not be sufficient to get the global existence result. This is the reason why we will work with the couple of spaces U^2 and V^2 and not with the more usual couple of spaces $\tilde{B}_2^{1/2,1}(L^2(\mathbb{R}^3))$ and $\tilde{B}_2^{1/2,\infty}(L^2(\mathbb{R}^3))$.
Then denoting by $S(t) := e^{-t\partial_x\Delta}$ the linear group associated with ZK, we define the spaces

$$U_S^2 = S(\cdot)U^2 \text{ with norm } \|u\|_{U_S^2} = \|S(-\cdot)u\|_{U^2}$$

$$\text{and } V_S^2 = S(\cdot)V^2 \text{ with norm } \|u\|_{V_S^2} = \|S(-\cdot)u\|_{V^2} .$$

The properties of these spaces we need in the sequel are summarized in the following propositions (see [6]).

prop3
Proposition 6.3. *Let $\psi \in C_c^\infty(\mathbb{R})$ then*

$$\|\psi S(\cdot)u_0\|_{U_S^2} \lesssim \|u_0\|_{L^2}, \quad \forall u_0 \in L^2(\mathbb{R}^3)$$

and

$$\Big\|\psi(t)\int_0^t S(t-t')f(t',\cdot)\,dt'\Big\|_{U_S^2} \lesssim \sup_{\|v\|_{V_S^2}=1}\Big|\int_{\mathbb{R}^4} fv\Big|, \quad \forall f \in C_c^\infty(\mathbb{R}^4) .$$

prop4
Proposition 6.4. *Let $T_0 : L^2 \times \cdots \times L^2 \to L^1_{loc}(\mathbb{R}^3;\mathbb{R})$ be a n-linear operator. Assume that for some $2 \leq p, q \leq \infty$,*

$$\|T_0(S(\cdot)\phi_1, \cdots, S(\cdot)\phi_n)\|_{L_t^p(\mathbb{R};L^q(\mathbb{R}^3))} \lesssim \prod_{i=1}^n \|\phi_i\|_{L^2} .$$

Then there exists $T : U_S^2 \times \cdots \times U_S^2 \to L_t^p(\mathbb{R};L^q(\mathbb{R}^3))$ satisfying

$$\|T(u_1, \cdots, u_n)\|_{L_t^p(\mathbb{R};L^q(\mathbb{R}^3))} \lesssim \prod_{i=1}^n \|u_i\|_{U_S^2}$$

such that $T(u_1, \cdots, u_n)(t)(x, y, z) = T_0(u_1(t), \cdots, u_n(t))(x, y, z)$ almost everywhere.

We are now ready to define our resolution spaces : we denote by $Y^{1,1}$ the space of all functions $u \in \mathcal{S}'(\mathbb{R}^4)$ such that

$$\|u\|_{Y^{1,1}} := \sum_N N\|P_N u\|_{U_S^2} < \infty$$

and by $Y^{s,2}$ the space of all functions $u \in \mathcal{S}'(\mathbb{R}^4)$ such that

$$\|u\|_{Y^{s,2}} := \Big(\sum_N N^{2s}\|P_N u\|^2_{U^2_S}\Big)^{1/2} < \infty \, .$$

Here, the Fourier projector P_N is the $\mathbb{R}^3$-version of the one defined in (2.1), i.e. P_1 localized in frequencies $3\xi^2+\mu^2+\eta^2 \lesssim 1$ while for $N \geq 2$, P_N localized in frequencies $3\xi^2+\mu^2+\eta^2 \sim N$.

6.1.2. *Local existence estimate.* Note that Proposition 6.3 ensures that

65 (6.6) $$\|\psi(\cdot)S(\cdot)u_0\|_{Y^{1,1}} \lesssim \|u_0\|_{B^{1,1}_2}, \quad \forall u_0 \in B^{1,1}_2(\mathbb{R}^3),$$

and

(6.7) $$\|\psi(\cdot)S(\cdot)u_0\|_{Y^{s,2}} \lesssim \|u_0\|_{H^s}, \quad \forall u_0 \in H^s(\mathbb{R}^3) \, .$$

Moreover, Proposition 6.4 lead to the following estimates in U^2_S :

lem5 **Lemma 6.5.** *Let $\psi \in C^\infty_c(\mathbb{R})$. For any $u \in U^2_S$ it holds*

$$\|\psi u\|_{L^4} \lesssim \|u\|_{U^2_S} \, .$$

For any couple $u, v \in U^2_S$ and any couple (N_1, N_2) of dyadic number such that $N_1 \geq 4N_2$ it holds

$$\|\psi P_{N_1} u P_{N_2} v\|_{L^2} \lesssim \frac{N_2}{N_1}\|P_{N_1}u\|_{U^2_S}\|P_{N_2}v\|_{U^2_S} \, .$$

Proof. The first estimate is a direct combination of the Strichartz estimate for the ZK equation in $\mathbb{R}^3$ (see [13][4])

strichartzR3 (6.8) $$\|\psi S(\cdot)g\|_{L^4(\mathbb{R}^4)} \lesssim \|g\|_{L^2(\mathbb{R}^3)}$$

with Proposition 6.4. To prove the second estimate we notice that since

$$\begin{aligned}&\Big|\frac{\partial H}{\partial \xi_1}(\xi_1, \xi-\xi_1, \mu_1, \mu-\mu_1, \eta_1, (\eta-\eta_1))\Big| \\ &\quad = \big|3\xi_1^2+\mu_1^2+\eta_1^2-(3(\xi-\xi_1)^2+(\mu-\mu_1)^2+(\eta-\eta_1)^2)\big| \gtrsim N_1^2 \, .\end{aligned}$$

where H is the resonance function in dimension 3, the $\mathbb{R}^3$-version of the bilinear estimate (3.16) reads

$$\|(P_{N_1}Q_{L_1}u_1)(P_{N_2}Q_{L_2}u_2)\|_{L^2} \lesssim \frac{N_2}{N_1}(L_1 \vee L_2)^{\frac{1}{2}}(L_1 \wedge L_2)^{\frac{1}{2}}\|P_{N_1}Q_{L_1}u_1\|_{L^2}\|P_{N_2}Q_{L_2}u_2\|_{L^2} \, .$$

Since for $\psi \in C^\infty_c(\mathbb{R})$, $g \in L^2(\mathbb{R}^3)$ and any dyadic number $L \geq 1$ it holds

$$\|Q_L \psi S(\cdot)g\|_{L^2} \lesssim L^{-4}\|g\|_{L^2}$$

this ensures that

$$\|(\psi P_{N_1}S(\cdot)g)(\psi P_{N_2}S(\cdot)f)\|_{L^2} \lesssim \frac{N_2}{N_1}\|P_{N_1}g\|_{L^2}\|P_{N_2}f\|_{L^2} \, .$$

The desired estimate follows by applying Proposition 6.4. □

We are now in position to prove the needed estimates on the retarded Duhamel operator.

[4]Estimate (6.8) would correspond to the case $\epsilon = 0$ and $\theta = 1/2$ of Proposition 3.1 in [13], but the case $\epsilon = 0$ is not included in the hypotheses. Note however that this case follows by arguing exactly as in [13].

prop6 **Proposition 6.6.** *Let $0<T<1$. For all $u,v\in Y^{1,1}$ with compact support in time in $]-T,T[$ it holds*

po1 (6.9) $$\Big\|\psi(t/T)\int_0^t S(t-t')\partial_x(uv)(t')\,dt'\Big\|_{Y^{1,1}}\lesssim T^{1/2}\|u\|_{Y^{1,1}}\|v\|_{Y^{1,1}}\,.$$

For all $u,v\in Y^{s,2}$, $s>1$, with compact support in time in $]-T,T[$ it holds

po2 (6.10) $$\Big\|\psi(t/T)\int_0^t S(t-t')\partial_x(uv)(t')\,dt'\Big\|_{Y^{s,2}}\lesssim T^{1/2}\Big(\|u\|_{Y^{1,1}}\|v\|_{Y^{s,2}}+\|u\|_{Y^{s,2}}\|v\|_{Y^{1,1}}\Big)\,.$$

Proof. We separate the contribution of $\sum_{N_1\not\sim N_2}P_{N_1}uP_{N_2}v$ and the one of $\sum_{N_1\sim N_2}P_{N_1}uP_{N_2}v$. We use Proposition 6.3, Lemma 6.5 and (6.5). For the first one we assume without loss of generality that $N_1\geq 4N_2$ to get

$$\begin{aligned}\sum_N\sum_{N_1\geq 4N_2}N\Big\|\psi(t)\int_0^t S(t-t')\partial_xP_N(P_{N_1}uP_{N_2}v)(t')\,dt'\Big\|_{U^2_S}&\\
\lesssim\sup_{\|w\|_{V^2_S}=1}\Big(\sum_{N_1\geq 4N_2}N_1\|\partial_x(P_{N_1}uP_{N_2}v)\|_{L^2}\|\psi(\frac{\cdot}{T})P_{N_1}w\|_{L^2}\Big)&\\
\lesssim T^{1/2}\sup_{\|w\|_{V^2_S}=1}\Big(\sum_{N_1\geq 4N_2}N_1^2\frac{N_2}{N_1}\|P_{N_1}u\|_{U^2_S}\|P_{N_2}v\|_{U^2_S}\|P_{N_1}w\|_{V^2_S}\Big)&\\
\lesssim T^{1/2}\|u\|_{Y^{1,1}}\|v\|_{Y^{1,1}}\,.&\end{aligned}$$

Whereas the contribution of the second one is easily estimated by

$$\begin{aligned}\sum_N\sum_{N_1\sim N_2\gtrsim N}N\Big\|\psi(t)\int_0^t S(t-t')\partial_xP_N(P_{N_1}uP_{N_2}v)(t')\,dt'\Big\|_{U^2_S}&\\
\lesssim\sup_{\|w\|_{V^2_S}=1}\Big(\sum_N N^2\sum_{N_1\sim N_2\gtrsim N}\|P_{N_1}u\|_{U^2_S}\|P_{N_2}v\|_{U^2_S}\|\psi(\frac{\cdot}{T})P_Nw\|_{L^2}\Big)&\\
\lesssim T^{1/2}\sum_{N_1}N_1^{-2}\sum_{l\geq 0}2^{-l}N_1^2(N_1\|P_{N_1}u\|_{U^2_S})(N_1\|P_{N_1}v\|_{U^2_S})&\\
\lesssim T^{1/2}\|u\|_{Y^{1,1}}\|v\|_{Y^{1,1}}\,.&\end{aligned}$$

Finally the proof of (6.10) follows the same lines and thus will be omitted. □

Note that the definition of the function space U^2_S ensures that for any $0<T<1$ and any smooth function $\psi\in C_c^\infty(\mathbb{R})$ it holds

$$\|\psi(\cdot/T)u\|_{U^2_S}\lesssim\|u\|_{U^2_S},\quad\forall u\in U^2_S\,.$$

Therefore, combining (6.6) and Proposition 6.6, we deduce that for any $0<T<1$, the functional

$$\mathcal{G}_T(w)(t,\cdot):=\psi(t)S(t)u_0-\frac{1}{2}\int_0^t S(t-t')\partial_x(\psi(\cdot/T)w)^2(t',\cdot)\,dt'$$

maps $Y^{1,1}$ into itself and satisfies

$$\|\mathcal{G}_T(w)\|_{Y^{1,1}}\lesssim\|u_0\|_{B^{1,1}_2}+T^{1/2}\|w\|^2_{Y^{1,1}}\,.$$

This ensures that there exists $C\geq 1$ such that, for $T\lesssim\|u_0\|^{-2}_{B^{1,1}_2}$, $\mathcal{G}_T$ is strictly contractive in the ball of $Y^{1,1}$ centered at the origin of radius $2C\|u_0\|_{B^{1,1}_2}$. By the

Banach fixed point theorem, it follows that $\mathcal{G}_T$ has got a fixed point u satisfying $\|u\|_{Y^{1,1}} \lesssim 2C\,\|u_0\|_{B_2^{1,1}}$. Since $Y^{1,1} \hookrightarrow L_t^\infty B_2^{1,1}$, this proves the local existence and uniqueness in the time restriction space $Y_T^{1,1}$ of the solution $u \in C([-T,T];B_2^{1,1})$ of ZK emanating from $u_0 \in B_2^{1,1}(\mathbb{R}^3)$ with a doubling time satisfying (6.2) for some constant $C_0 \geq 1$. The result for $u_0 \in H^s(\mathbb{R}^3)$, $s>1$, follows by noticing that (6.10) implies that $\mathcal{G}_T$ maps as well $Y^{s,2}$ into itself with

$$\|\mathcal{G}_T(w)\|_{Y^{s,2}} \lesssim \|u_0\|_{H^s} + T^{1/2}\|w\|_{Y^{1,1}}\|w\|_{Y^{s,2}}\ .$$

This completes the proof of Theorem 6.1.

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