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ABSTRACT. We consider a 2D quantum system of N bosons in a trapping potential $|x|^s$, interacting via a pair potential of the form $N^{2\beta-1}w(N^\beta x)$. We show that for all $0<\beta<(s+1)/(s+2)$, the leading order behavior of ground states of the many-body system is described in the large N limit by the corresponding cubic nonlinear Schrödinger energy functional. Our result covers the focusing case ($w<0$) where even the stability of the many-body system is not obvious. This answers an open question mentioned by X. Chen and J. Holmer for harmonic traps ($s=2$). Together with the BBGKY hierarchy approach used by these authors, our result implies the convergence of the many-body quantum dynamics to the focusing NLS equation with harmonic trap for all $0<\beta<3/4$.

CONTENTS

1. INTRODUCTION

Since the experimental realization of Bose-Einstein condensation (BEC) in dilute trapped Bose gases in 1995 [1, 6], it has been an ongoing challenge in mathematical physics to derive the phenomenon from the first principles of quantum mechanics (see [3, 16, 21] and references therein). The nature of the interaction between particles plays an essential role. In particular, singular and/or attractive potentials complicate the analysis dramatically.

In the present paper, we are interested in the derivation of the minimization problem for the 2D nonlinear Schödinger (NLS) energy functional

$$\mathcal{E}_{\rm NLS}(u)=\int_{\mathbb{R}^2}\Big(|(i\nabla+A(x))u|^2+V(x)|u(x)|^2+\frac{a}{2}|u(x)|^4\Big)dx \tag{1}$$

subject to the mass constraint

$$\int_{\mathbb{R}^2}|u|^2=1. \tag{2}$$

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We will show that this NLS functional arises as an effective model for large dilute 2D bosonic systems, as a consequence of the occurence of BEC in the ground states. We shall be more specifically concerned with the focusing (or attractive) case, $a \leqslant 0$.

Here V is an external potential which serves to trap the system and A is a vector potential corresponding to a magnetic field (or the effective influence of a rotation). We assume that

$$V \in L^1_{\rm loc}(\mathbb{R}^2,\mathbb{R}), \quad A \in L^2_{\rm loc}(\mathbb{R}^2,\mathbb{R}^2) \quad \text{and} \quad V(x) \geqslant C^{-1}(|A(x)|^2+|x|^s)-C \tag{3}$$

for a fixed parameter $s>0$ (we always denote by C a generic positive constant whose value alters from line to line). The case $s=2$ corresponds to the harmonic trap which is most often used in laboratory experiments.

We will assume that $a>-a^*$ where $a^*>0$ is the critical interaction strength for the existence of a ground state for the focusing NLS functional [23, 24, 11, 18]. In fact, a^* is the optimal constant of the Gagliardo-Nirenberg inequality:

$$\left(\int_{\mathbb{R}^2}|\nabla u|^2\right)\left(\int_{\mathbb{R}^2}|u|^2\right) \geqslant \frac{a^*}{2}\int_{\mathbb{R}^2}|u|^4. \tag{4}$$

Equivalently,

$$a^* = \|Q\|^2_{L^2(\mathbb{R}^2)},$$

where $Q \in H^1(\mathbb{R}^2)$ is the unique (up to translations) positive radial solution of

$$-\Delta Q + Q - Q^3 = 0 \text{ in } \mathbb{R}^2. \tag{5}$$

The linear many-body model for N identical bosons we start from is described by the Hamiltonian

$$H_N = \sum_{j=1}^N \Big((i\nabla_j + A(x_j))^2 + V(x_j)\Big) + \frac{1}{N-1}\sum_{1\leqslant i<j\leqslant N} w_N(x_i-x_j) \tag{6}$$

acting on $\mathfrak{H}^N = \bigotimes_{\rm sym}^N L^2(\mathbb{R}^2)$, the Hilbert space of square-integrable symmetric functions. The two-body interaction is chosen of the form

$$w_N(x) = N^{2\beta} w(N^\beta x) \tag{7}$$

for a fixed parameter $\beta>0$ and a fixed function w satisfying

$$w, \widehat{w} \in L^1(\mathbb{R}^2,\mathbb{R}), \quad w(x)=w(-x) \quad \text{and} \quad \int_{\mathbb{R}^2} w = a. \tag{8}$$

The coupling constant $1/(N-1)$ ensures that the total kinetic and interaction energies are comparable, so that we can expect a nontrivial effective theory in the limit $N\to\infty$.

Roughly speaking, BEC occurs when almost all particles live in a common quantum state, that is, in terms of wave functions,

$$\Psi(x_1,...,x_N) \approx u^{\otimes N}(x_1,...,x_N) := u(x_1)u(x_2)...u(x_N)$$

in an appropriate sense. By simply taking the trial wave functions $u^{\otimes N}$, we obtain the Hartree energy functional

$$\begin{aligned}\mathcal{E}_{{\rm H},N}(u) &= \frac{\langle u^{\otimes N}, H_N u^{\otimes N}\rangle}{N}\\ &= \int_{\mathbb{R}^2}\Big(|(i\nabla u(x)+A(x)u(x)|^2 + V(x)|u(x)|^2 + \frac{1}{2}|u(x)|^2(w_N * |u|^2)(x)\Big)dx. \end{aligned}\tag{9}$$

The infimum of the latter, under the mass constraint $\int |u|^2 = 1$, is thus an upper bound to the many-body ground state energy per particle. When $N \to \infty$, since

$$w_N \rightharpoonup \left(\int_{\mathbb{R}^2} w \right) \delta_0 = a\delta_0, \tag{10}$$

the Hartree functional (9) *formally* boils down to the NLS functional (1). On the other hand, the Hartree functional is stable in the limit $N \to \infty$ only if

$$\inf_{u \in H^1(\mathbb{R}^2)} \left(\frac{\displaystyle\iint_{\mathbb{R}^2 \times \mathbb{R}^2} |u(x)|^2 |u(y)|^2 w(x-y)\, dx\, dy}{2 \|u\|^2_{L^2(\mathbb{R}^2)} \|\nabla u\|^2_{L^2(\mathbb{R}^2)}} \right) \geqslant -1. \tag{11}$$

In fact, if (11) fails to hold, then the ground state energy of the Hartree functional converges to $-\infty$ as $N \to \infty$, see [14, Prop. 2.3]. Hence, Condition (11) is *necessary* for the many-body Hamiltonian to satisfy stability of the second kind:

$$H_N \geqslant -CN. \tag{12}$$

That the one-body stability condition (11) is also *sufficient* to ensure the many-body stability (12) is highly nontrivial and it is one of the main concerns of the present paper. As in [14], we will actually assume the strict stability

$$\inf_{u \in H^1(\mathbb{R}^2)} \left(\frac{\displaystyle\iint_{\mathbb{R}^2 \times \mathbb{R}^2} |u(x)|^2 |u(y)|^2 w(x-y)\, dx\, dy}{2 \|u\|^2_{L^2(\mathbb{R}^2)} \|\nabla u\|^2_{L^2(\mathbb{R}^2)}} \right) > -1 \tag{13}$$

which plays the same role as the assumption $a > -a^*$ in the NLS case. Note that (11) implies that $\int w \geqslant -a^*$, and (11) holds if $\int_{\mathbb{R}^2} |w_-| < a^*$.

The goal of the present paper is to improve on the results of [13] where we showed in particular that the many-body ground states converge (in terms of reduced density matrices) to those of the NLS functional (1) when $N \to \infty$, provided

$$0 < \beta < \beta_0(s) := \frac{s}{4(s+1)}.$$

Here we extend this range to

$$\boxed{0 < \beta < \beta_1(s) := \frac{s+1}{s+2}.} \tag{14}$$

Note the qualitative improvement: while $\beta_0(s) < 1/2$, we have $\beta_1(s) > 1/2$. This means that we now allow the range of the interaction to be much smaller than the typical distance between particles, of order $N^{-1/2}$. We can thus treat a dilute limit where interactions are rare but strong, as opposed to the previous result which was limited to the mean-field case with many weak interactions.

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2. Main results

2.1. **Statements.** We will prove the convergence of the ground state energy per particle of H_N to that of the NLS functional (1). These are denoted respectively by

$$e_N := N^{-1} \inf_{\Psi \in \mathfrak{H}^N, \|\Psi\|=1} \langle \Psi_N, H_N \Psi_N \rangle \quad \text{and} \quad e_{\rm NLS} := \inf_{\|u\|_{L^2}=1} \mathcal{E}_{\rm NLS}(u). \tag{15}$$

The convergence of ground states is formulated using k-particles reduced density matrices, defined for any $\Psi \in \mathfrak{H}^N$ by a partial trace

$$\gamma_\Psi^{(k)} := {\rm Tr}_{k+1\to N} |\Psi\rangle\langle\Psi|.$$

Equivalently, $\gamma_\Psi^{(k)}$ is the trace class operator on $\mathfrak{H}^k$ with kernel

$$\gamma_\Psi^{(k)}(x_1,...,x_k; y_1,...,y_k) = \int_{\mathbb{R}^{2(N-k)}} \overline{\Psi(x_1,...,x_k,Z)}\Psi(y_1,...,y_k,Z)dZ.$$

Our main result is the following

Theorem 1 (**Convergence to NLS theory**).
Assume that V, A, w satisfy (3), (8) *and* (13). *Then, for every* $0<\beta<(s+1)/(s+2)$,

$$\boxed{\lim_{N\to\infty} e_N = e_{\rm NLS} > -\infty.} \tag{16}$$

Moreover, for any ground state Ψ_N of H_N, there exists a Borel probability measure μ supported on the ground states of $\mathcal{E}_{\rm NLS}(u)$ such that, along a subsequence,

$$\boxed{\lim_{N\to\infty} {\rm Tr}\left|\gamma_{\Psi_N}^{(k)} - \int |u^{\otimes k}\rangle\langle u^{\otimes k}| d\mu(u)\right| = 0, \quad \forall k\in\mathbb{N}.} \tag{17}$$

If $\mathcal{E}_{\rm NLS}(u)$ has a unique minimizer u_0 (up to a phase), then for the whole sequence

$$\lim_{N\to\infty} {\rm Tr}\left|\gamma_{\Psi_N}^{(k)} - |u_0^{\otimes k}\rangle\langle u_0^{\otimes k}|\right| = 0, \quad \forall k\in\mathbb{N}. \tag{18}$$

Note that if $A=0$ and V is radial, one can prove the uniqueness for the NLS ground state by well-known arguments, reviewed for instance in [10]. Uniqueness can certainly fail when $A\neq 0$ (due to the occurence of quantized vortices [22]), or when $a<0$ and V has several isolated minima [2, 11].

2.2. **Focusing quantum dynamics.** Most recently, Chen and Holmer [5] considered the derivation of the time-dependent 2D focusing NLS in a harmonic trap $V(x)=|x|^2$ from many-body quantum dynamics. They proved that for all $0<\beta<1/6$, if the initial state $\Psi_N(0)$ condensates on $u(0)$ (in the sense of density matrices as in (18)), then for every time $t>0$, the evolved state $\Psi_N(t) = e^{-it\widetilde{H}_N}\Psi_N(0)$ with

$$\widetilde{H}_N = \sum_{j=1}^N \left(-\Delta_{x_j} + |x|^2\right) + \frac{1}{N-1}\sum_{1\leqslant i<j\leqslant N} N^{2\beta} w(N^\beta(x_i-x_j))$$

condensates on the solution $u(t)$ to the time-dependent NLS equation

$$i\partial_t u(t) = (-\Delta + |x|^2 + a|u(t)|^2)u(t), \quad u_{|t=0} = u(0).$$

Their approach is based on the BBGKY hierarchy method and the stability of the second kind (12), which has been established in [14] for $0<\beta<1/6$. As discussed in [5, Section 2.3], their method actually allows to treat any $0<\beta<3/4$, provided that the stability holds for this larger range of β, which they left as an open question. Theorem 1 thus provides the needed stability estimate to extend the main result in [5] to any $0<\beta<3/4$.

Note that if $\beta < 1/2$, the next order correction to the 2D focusing quantum dynamics can be obtained using the Bogoliubov approach [15, 19] (see [4] for the defocusing case).

2.3. **Strategy of proof.** We shall compare the many-body ground state energy per particle e_N to that of the Hartree functional (9)

$$e_{\mathrm{H},N} := \inf_{\|u\|_{L^2}=1} \mathcal{E}_{\mathrm{H},N}(u)$$

and then use that (see Appendix A)

$$\lim_{N\to\infty} e_{\mathrm{H},N} = e_{\mathrm{NLS}}.$$

The upper bound $e_N \leqslant e_{\mathrm{H},N}$ can be obtained using trial states $u^{\otimes N}$, and the difficult part is the matching lower bound.

The first ingredient of our proof of Theorem 1 is the following:

Lemma 2 (**First lower bound on the ground state energy**).
For any $\beta \geqslant 0$ we have, in the limit $N \to \infty$,

$$\begin{aligned} e_N &\geqslant \inf_{\|u\|_{L^2}=1} \int_{\mathbb{R}^2} \Big(|\nabla u(x)|^2 + V|u(x)|^2 + \frac{1}{2}|u|^2(w_N * |u|^2) \Big) dx - CN^{2\beta-1} \\ &\geqslant e^0_{\mathrm{NLS}} - o(1) - CN^{2\beta-1}. \end{aligned} \tag{19}$$

Here e^0_{NLS} denotes the NLS energy with $A \equiv 0$.

Proof. The first inequality is proved in [12, Section 3]. The second follows from the analysis of the Hartree functional in Appendix A. □

When $\beta < 1/2$ and $A \equiv 0$ (no magnetic field), Lemma 2 implies immediately the convergence of the ground state energy (16). When either $\beta \geqslant 1/2$ or $A \not\equiv 0$, the proof of the convergence (16) is more involved. In particular, when $\beta > 1/2$ and $w < 0$, the stability of the second kind (12) is not provided by Lemma 2.

The main novelty of the present paper is to obtain (12) by a bootstrap procedure, taking Lemma 2 as a starting point. As in [14], a major ingredient in our proof is a quantitative version of the quantum de Finetti theorem.

Lemma 3 (**Quantitative quantum de Finetti**).
Let $\Psi \in \mathfrak{H}^N = \bigotimes_{\mathrm{sym}}^N L^2(\mathbb{R}^2)$ and let P be a finite-rank orthogonal projector with

$$\dim(P\mathfrak{H}) = d < \infty.$$

There exists a positive Borel measure $d\mu_\Psi$ on the unit sphere $SP\mathfrak{H}$ such that

$$\mathrm{Tr}\left| \int_{SP\mathfrak{H}} |u^{\otimes 2}\rangle\langle u^{\otimes 2}| d\mu_\Psi(u) - P^{\otimes 2}\gamma_\Psi^{(2)} P^{\otimes 2} \right| \leqslant \frac{8d}{N} \tag{20}$$

and

$$\int_{SP\mathfrak{H}} d\mu_\Psi(u) \geqslant \big(\mathrm{Tr}(P\gamma_\Psi^{(1)}) \big)^2. \tag{21}$$

Proof. The first inequality (20) is contained in [14, Lemma 3.4]. The second inequality (21) is established in the course of the proof of [14, Lemma 3.8]. □

We will apply the above lemma with P a spectral projector below an energy cut-off L for the one-particle operator:

$$P := \mathbb{1}(h \leqslant L) \quad \text{with} \quad h := (i\nabla + A(x))^2 + V(x). \tag{22}$$

Note that Assumptions (3) ensure that

$$h \geqslant C^{-1}(1 - \Delta + |A(x)|^2 + V(x)) - C \geqslant C^{-1}(-\Delta + |x|^s) - C. \tag{23}$$

Therefore, we have a Cwikel-Lieb-Rosenblum type estimate (see [14, Lemma 3.3])

$$d := \dim(P\mathfrak{H}) \leqslant C L^{1+2/s}. \tag{24}$$

The first main improvement over our previous work [14] is a better way to control the error induced by using the finite-rank cut-off P. Using several Sobolev-type estimates on the interaction operator w_N (Lemma 6 below), we obtain

Lemma 4 (**Second lower bound on the ground state energy**).
Let $\beta > 0$. For every $\delta \in (0, 1/2)$, there exists a constant $C_\delta > 0$ such that for all $N \geqslant 2$, $L \geqslant 1$ and for all wave functions $\Psi_N \in \mathfrak{H}^N$:

$$\begin{aligned}\frac{\langle \Psi_N, H_N \Psi_N\rangle}{N} &\geqslant e_{\mathrm{H},N} - C_\delta L^{1+\delta}\frac{d}{N} \\ &\qquad - \frac{C_\delta}{L^{1/4-\delta/2}} \operatorname{Tr}\left(h\gamma^{(1)}_{\Psi_N}\right)^{1/4-\delta/2} \operatorname{Tr}\left(h \otimes h\gamma^{(2)}_{\Psi_N}\right)^{1/2+\delta}.\end{aligned} \tag{25}$$

Lemma 4 provides a sharp lower bound to the ground state energy if we have a strong enough *a-priori* control of the error terms in the second line of (25). This is the other important improvement of the present paper.

Lemma 5 (**Moments estimates**).
Let $0 < \beta < 1$ and $\Psi_N \in \mathfrak{H}^N$ be a ground state of H_N. For all $\varepsilon \in (0, 1)$ we have

$$\operatorname{Tr}\left(h\gamma^{(1)}_{\Psi_N}\right) \leqslant C\frac{1 + |e_{N,\varepsilon}|}{\varepsilon} \quad \textit{and} \quad \operatorname{Tr}\left(h \otimes h\gamma^{(2)}_{\Psi_N}\right) \leqslant C\left(\frac{1 + |e_{N,\varepsilon}|}{\varepsilon}\right)^2 \tag{26}$$

where

$$e_{N,\varepsilon} := N^{-1} \inf_{\Psi \in \mathfrak{H}^N, \|\Psi\|=1} \left\langle \Psi, \left(H_N - \varepsilon \sum_{j=1}^N h_j\right)\Psi \right\rangle. \tag{27}$$

This is reminiscent of similar estimates used by Erdös, Schlein and Yau for the time-dependent problem [9, 7, 8]. Recently, related ideas were also adapted to the ground state problem in [20]. Note, however, that these previous applications were limited to the defocusing case $w \geqslant 0$. Then $|e_{N,\varepsilon}|$ is clearly bounded independently of N and the moments estimates above allow to derive the NLS theory for any $\beta < 1$ (an even larger range of β can be dealt with when $A = 0$, using the methods of [17]).

In the focusing case, we do not obtain actual a priori bounds by Lemma 5, since the estimates depend on $|e_{N,\varepsilon}|$, which is essentially of the same order of magnitude as $|e_N|$. The uniform bound $|e_{N,\varepsilon}| \leqslant C$ will be obtained by a bootstrap argument: Lemma 2 provides the starting point, and then the bounds in Lemmas 4 and 5 can be improved step by step, provided (14) holds. Once stability of the second kind is proved, the convergence of the ground state energy (16) follows immediately from Lemma 4. The convergence of density matrices (17) is a consequence of the proof of (16) and the quantum de Finetti Theorem, just as in [14].

Organization of the paper. We will prove Lemma 5 in Section 3, then Lemma 4 in Section 4. The proof of the main Theorem 1 is concluded in Section 5. Appendix A contains the needed estimate to pass to the limit in the Hartree functional.

3. Moments estimates: Proof of Lemma 5

Since we can always add a constant to V if necessary, from now on we will assume that $V \geqslant 1$, and hence $h := (i\nabla + A(x))^2 + V(x) \geqslant 1$. We will need the following

Lemma 6 (**Operator bounds for two-body interactions**). *For every $W \in L^1 \cap L^2(\mathbb{R}^2, \mathbb{R})$, the multiplication operator $W(x-y)$ on $L^2((\mathbb{R}^2)^2)$ satisfies*

$$|W(x-y)| \leqslant C\|W\|_{L^2} h_x, \tag{28}$$

$$|W(x-y)| \leqslant C_\delta \|W\|_{L^1} (h_x h_y)^{1/2+\delta}, \quad \forall \delta > 0, \tag{29}$$

$$\pm \Big(h_x W(x-y) + W(x-y) h_x \Big) \leqslant C\|W\|_{L^2} h_x h_y. \tag{30}$$

Lemma 6 is the 2D analogue of [20, Lemma 3.2]. The proof is similar and we omit it for shortness. Now we come to the

Proof of Lemma 5. Note that $C \geqslant e_{\mathrm{H},N} \geqslant e_N \geqslant e_{N,\varepsilon}$, and hence $|e_N| \leqslant C(1+|e_{N,\varepsilon}|)$. Clearly

$$H_{N,\varepsilon} := H_N - \varepsilon \sum_{j=1}^N h_j \geqslant N e_{N,\varepsilon}. \tag{31}$$

Taking the expectation against Ψ_N and using the definition of the one-body density matrix we obtain the first inequality in (26) immediately. To obtain the second inequality in (26), we use the ground state equation

$$H_N \Psi_N = N e_N \Psi_N$$

to write

$$\begin{aligned} \frac{1}{N^2} \left\langle \Psi_N, \Big(\Big(\sum_{j=1}^N h_j \Big) H_N + H_N \Big(\sum_{j=1}^N h_j \Big) \Big) \Psi_N \right\rangle \\ = \frac{2e_N}{N} \left\langle \Psi_N, \sum_{j=1}^N h_j \Psi_N \right\rangle \leqslant \frac{C(1+|e_{N,\varepsilon}|)^2}{\varepsilon}. \end{aligned} \tag{32}$$

Now we are after an operator lower bound on

$$\begin{aligned} \frac{1}{N^2} \Big(\sum_{j=1}^N h_j \Big) H_N + \frac{1}{N^2} H_N \Big(\sum_{j=1}^N h_j \Big) = \frac{2}{N^2} \Big(\sum_{j=1}^N h_j \Big)^2 \\ + \frac{1}{N^2(N-1)} \sum_{i=1}^N \sum_{j<k} \big(h_i w_N(x_j - x_k) + w_N(x_j - x_k) h_i \big). \end{aligned} \tag{33}$$

For every $i = 1, 2, ..., N$, we have

$$\begin{aligned} \frac{1}{N-1} \sum_{i \neq j < k \neq i} w_N(x_j - x_k) &= H_{N,\varepsilon} - (1-\varepsilon) \sum_{j=1}^N h_j - \frac{1}{N-1} \sum_{j \neq i} w_N(x_i - x_j) \\ &\geqslant N e_{N,\varepsilon} - \left(1 - \varepsilon + \frac{N^\beta}{N} \right) \sum_{j=1}^N h_j \end{aligned} \tag{34}$$

where we have used $H_{N,\varepsilon} \geqslant Ne_{N,\varepsilon}$ and applied (28) to obtain $w_N(x_i - x_j) \leqslant CN^\beta h_j$. Note that both sides of (34) commute with h_i. Therefore, we can multiply (34) with h_i and then take the sum over i to obtain

$$\frac{1}{N^2(N-1)} \sum_{i=1}^N \sum_{i\neq j<k\neq i} (h_i w_N(x_j - x_k) + w_N(x_j - x_k)h_i)$$
$$\geqslant \frac{2e_{N,\varepsilon}}{N} \sum_{j=1}^N h_j - \frac{2}{N^2}\Big(1 - \varepsilon + \frac{CN^\beta}{N}\Big)\Big(\sum_{j=1}^N h_j\Big)^2. \tag{35}$$

On the other hand, for every $j \neq k$, by (30) we have

$$h_j w_N(x_j - x_k) + w_N(x_j - x_k)h_j \geqslant -CN^\beta h_j h_k.$$

Therefore,

$$\frac{1}{N^2(N-1)} \sum_{j\neq k} \Big(h_j w_N(x_j - x_k) + w_N(x_j - x_k)h_j\Big) \geqslant -CN^{\beta-3}\Big(\sum_{j=1}^N h_j\Big)^2. \tag{36}$$

Inserting (35) and (36) into (33), we find the operator bound

$$\frac{1}{N^2}\Big(\sum_{j=1}^N h_j\Big)H_N + \frac{1}{N^2}H_N\Big(\sum_{j=1}^N h_j\Big)$$
$$\geqslant \frac{2}{N^2}\Big(\varepsilon - \frac{CN^\beta}{N}\Big)\Big(\sum_{j=1}^N h_j\Big)^2 - \frac{C(1+|e_{N,\varepsilon}|)}{N}\sum_{j=1}^N h_j. \tag{37}$$

Taking the expectation against Ψ_N and using the first inequality in (26), we get

$$\frac{1}{N^2}\left\langle \Psi_N, \Big(\Big(\sum_{j=1}^N h_j\Big)H_N + H_N\Big(\sum_{j=1}^N h_j\Big)\Big)\Psi_N \right\rangle$$
$$\geqslant \frac{2}{N^2}\Big(\varepsilon - \frac{N^\beta}{N}\Big)\left\langle \Psi_N, \Big(\sum_{j=1}^N h_j\Big)^2 \Psi_N\right\rangle - \frac{C(1+|e_{N,\varepsilon}|)^2}{\varepsilon}. \tag{38}$$

Putting (32) and (38) together, we deduce that

$$\frac{2}{N^2}\Big(\varepsilon - \frac{N^\beta}{N}\Big)\left\langle \Psi_N, \Big(\sum_{j=1}^N h_j\Big)^2 \Psi_N\right\rangle \leqslant \frac{C(1+|e_{N,\varepsilon}|)^2}{\varepsilon}.$$

If $\beta < 1$, then $\varepsilon - CN^{\beta-1} \geqslant \varepsilon/2 > 0$ for large N. Therefore, we conclude

$$\frac{1}{N^2}\left\langle \Psi_N, \Big(\sum_{j=1}^N h_j\Big)^2 \Psi_N\right\rangle \leqslant \frac{C(1+|e_{N,\varepsilon}|)^2}{\varepsilon^2} \tag{39}$$

and the second inequality in (26) follows by definition of the two-body density matrix. □

4. Lower bound via de Finetti: Proof of Lemma 4

Again we can assume without loss of generality that $V \geqslant 1$, and hence $h \geqslant 1$. Take an arbitrary wave function $\Psi_N \in \mathfrak{H}^N$. We have

$$\frac{\langle \Psi_N, H_N \Psi_N\rangle}{N} = \mathrm{Tr}\Big(K_2 \gamma^{(2)}_{\Psi_N}\Big) \quad \text{where} \quad K_2 = \frac{1}{2}\Big(h_x + h_y + w_N(x-y)\Big).$$

Let Ψ_N be a many-body wave function and $d\mu_{\Psi_N}$ the associated de Finetti measure defined in Lemma 3 with the projector P as in (22). We write

$$\begin{aligned}\operatorname{Tr}\left(K_2\gamma^{(2)}_{\Psi_N}\right) &= \int \langle u^{\otimes 2}, K_2 u^{\otimes 2}\rangle d\mu_{\Psi_N}(u) + \operatorname{Tr}(K_2(\gamma^{(2)}_{\Psi_N} - P^{\otimes 2}\gamma^{(2)}_{\Psi_N}P^{\otimes 2}))\\ &\quad + \operatorname{Tr}\left(K_2\left(P^{\otimes 2}\gamma^{(2)}_{\Psi_N}P^{\otimes 2} - \int |u^{\otimes 2}\rangle\langle u^{\otimes 2}| d\mu_{\Psi_N}(u)\right)\right)\end{aligned} \tag{40}$$

and bound the right side from below term by term.

Main term. By the variational principle we have

$$\int \langle u^{\otimes 2}, K_2 u^{\otimes 2}\rangle d\mu_{\Psi_N}(u) = \int \mathcal{E}_{\mathrm{H},N}(u) d\mu_{\Psi_N}(u) \geqslant e_{\mathrm{H},N}\int d\mu_{\Psi_N}. \tag{41}$$

On the other hand, using (21) and $Q \leqslant L^{-1}h$ with $Q := \mathbb{1} - P$, we have

$$\begin{aligned}\int d\mu_{\Psi_N} &\geqslant \left(\operatorname{Tr}\left(P\gamma^{(1)}_{\Psi_N}\right)\right)^2 = \left(1 - \operatorname{Tr}\left(Q\gamma^{(1)}_{\Psi_N}\right)\right)^2\\ &\geqslant 1 - 2\operatorname{Tr}\left(Q\gamma^{(1)}_{\Psi_N}\right) \geqslant 1 - 2L^{-1}\operatorname{Tr}\left(h\gamma^{(1)}_{\Psi_N}\right).\end{aligned}$$

Since $|e_{\mathrm{H},N}| \leqslant C$, (41) reduces to

$$\int \langle u^{\otimes 2}, K_2 u^{\otimes 2}\rangle d\mu_{\Psi_N}(u) \geqslant e_{\mathrm{H},N} - CL^{-1}\operatorname{Tr}(h\gamma^{(1)}_{\Psi_N}). \tag{42}$$

First error term. Using $Ph \leqslant LP$ and Lemma 3 we find that

$$\left|\operatorname{Tr}\left((h_1+h_2)\left(P^{\otimes 2}\gamma^{(2)}_{\Psi_N}P^{\otimes 2} - \int |u^{\otimes 2}\rangle\langle u^{\otimes 2}| d\mu_{\Psi_N}(u)\right)\right)\right| \leqslant CL\frac{d}{N}.$$

On the other hand, using Equation (29), we have

$$P^{\otimes 2}|w_N(x_1 - x_2)|P^{\otimes 2} \leqslant C_\delta((Ph)_1 \otimes (Ph)_2)^{1/2+\delta} \leqslant CL^{1+2\delta}P^{\otimes 2}$$

for all $\delta > 0$. Therefore, using Lemma 3 again, we find

$$\left|\operatorname{Tr}\left(w_N\left(P^{\otimes 2}\gamma^{(2)}_{\Psi_N}P^{\otimes 2} - \int |u^{\otimes 2}\rangle\langle u^{\otimes 2}| d\mu_{\Psi_N}(u)\right)\right)\right| \leqslant C_\delta L^{1+2\delta}\frac{d}{N}.$$

Thus for all $\delta > 0$,

$$\operatorname{Tr}\left(K_2\left(P^{\otimes 2}\gamma^{(2)}_{\Psi_N}P^{\otimes 2} - \int |u^{\otimes 2}\rangle\langle u^{\otimes 2}| d\mu_{\Psi_N}(u)\right)\right) \geqslant -C_\delta L^{1+2\delta}\frac{d}{N}. \tag{43}$$

Second error term. Since h commutes with P and $h \geqslant hP$, we have

$$\operatorname{Tr}\left((h_1+h_2)(\gamma^{(2)}_{\Psi_N} - P^{\otimes 2}\gamma^{(2)}_{\Psi_N}P^{\otimes 2})\right) = \operatorname{Tr}\left(\left[(h_1+h_2) - P^{\otimes 2}(h_1+h_2)P^{\otimes 2}\right]\gamma^{(2)}_{\Psi_N}\right) \geqslant 0.$$

Using the Cauchy-Schwarz inequality for operators

$$\pm(AB + B^*A^*) \leqslant \eta^{-1}AA^* + \eta B^*B, \quad \forall \eta > 0,$$

we find that

$$\begin{aligned}&\pm 2\left(\gamma^{(2)}_{\Psi_N} - P^{\otimes 2}\gamma^{(2)}_{\Psi_N}P^{\otimes 2}\right)\\ &= \pm\Big[(1-P^{\otimes 2})\gamma^{(2)}_{\Psi_N} + \gamma^{(2)}_{\Psi_N}(1-P^{\otimes 2}) + P^{\otimes 2}\gamma^{(2)}_{\Psi_N}(1-P^{\otimes 2}) + (1-P^{\otimes 2})\gamma^{(2)}_{\Psi_N}P^{\otimes 2}\Big]\\ &\leqslant 2\eta^{-1}(1-P^{\otimes 2})\gamma^{(2)}_{\Psi_N}(1-P^{\otimes 2}) + \eta(\gamma^{(2)}_{\Psi_N} + P^{\otimes 2}\gamma^{(2)}_{\Psi_N}P^{\otimes 2})\end{aligned}$$

for all $\eta > 0$. Taking the trace against $(w_N)_\pm$ and optimizing over $\eta > 0$ we find that

$$\operatorname{Tr}(w_N(\gamma^{(2)}_{\Psi_N} - P^{\otimes 2}\gamma^{(2)}_{\Psi_N}P^{\otimes 2})) \geqslant -\sqrt{2}\Big(\operatorname{Tr}\left(|w_N|(\gamma^{(2)}_{\Psi_N} + P^{\otimes 2}\gamma^{(2)}_{\Psi_N}P^{\otimes 2})\right)\Big)^{1/2} \times \Big(\operatorname{Tr}\left(|w_N|(1-P^{\otimes 2})\gamma^{(2)}_{\Psi_N}(1-P^{\otimes 2})\right)\Big)^{1/2}. \tag{44}$$

Using again Equation (29) and the elementary fact

$$t^r = \inf_{\eta>0}\left(r\eta^{-1}t + (1-r)\eta^{r/(1-r)}\right) \quad \text{for all } t \geqslant 0, r \in (0,1) \tag{45}$$

we get

$$|w_N(x-y)| \leqslant C_\delta (h_x h_y)^{1/2+\delta} \leqslant C_\delta(\eta^{-1}h_x h_y + \eta^{\frac{1+2\delta}{1-2\delta}}) \quad \text{for all } \delta \in (0,1/2), \eta > 0.$$

Taking the trace against $\gamma^{(2)}_{\Psi_N} + P^{\otimes 2}\gamma^{(2)}_{\Psi_N}P^{\otimes 2}$ and optimizing over $\eta > 0$ (cf. (45)), we get

$$\operatorname{Tr}\left(|w_N|\left(\gamma^{(2)}_{\Psi_N} + P^{\otimes 2}\gamma^{(2)}_{\Psi_N}P^{\otimes 2}\right)\right) \leqslant 2C_\delta\left(\operatorname{Tr}\left(h\otimes h\gamma^{(2)}_{\Psi_N}\right)\right)^{1/2+\delta}$$

for all $\delta \in (0,1/2)$. Similarly, from (29), (45) and $Q \leqslant L^{-1}h$, we find that

$$\begin{aligned}
(1-P^{\otimes 2})|w_N|(1-P^{\otimes 2}) &\leqslant C_\delta(1-P^{\otimes 2})(h\otimes h)^{1/2+\delta}(1-P^{\otimes 2})\\
&\leqslant C_\delta\Big[(Qh^{1/2+\delta})\otimes h^{1/2+\delta} + h^{1/2+\delta}\otimes(Qh^{1/2+\delta})\Big]\\
&\leqslant \frac{C_\delta}{L^{1/2-\delta}}\Big[h\otimes h^{1/2+\delta} + h^{1/2+\delta}\otimes h\Big]\\
&\leqslant \frac{C_\delta}{L^{1/2-\delta}}\Big[\eta^{-1}h\otimes h + \eta^{\frac{1+2\delta}{1-2\delta}}(h\otimes 1 + 1\otimes h)\Big]
\end{aligned}$$

for all $\delta \in (0,1/2)$ and $\eta > 0$. Taking the trace against $\gamma^{(2)}_{\Psi_N}$ and optimizing over $\eta > 0$ we deduce that

$$\operatorname{Tr}\left(|w_N|(1-P^{\otimes 2})\gamma^{(2)}_{\Psi_N}(1-P^{\otimes 2})\right) \leqslant \frac{C_\delta}{L^{1/2-\delta}}\left(\operatorname{Tr}\left(h\gamma^{(1)}_{\Psi_N}\right)\right)^{1/2-\delta}\left(\operatorname{Tr}\left(h\otimes h\gamma^{(2)}_{\Psi_N}\right)\right)^{1/2+\delta}$$

for all $\delta \in (0,1/2)$. Therefore, ir follows from (44) that

$$\begin{aligned}
&\operatorname{Tr}(w_N(\gamma^{(2)}_{\Psi_N} - P^{\otimes 2}\gamma^{(2)}_{\Psi_N}P^{\otimes 2}))\\
&\geqslant -\frac{C_\delta}{L^{1/4-\delta/2}}\left(\operatorname{Tr}\left(h\gamma^{(1)}_{\Psi_N}\right)\right)^{1/4-\delta/2}\left(\operatorname{Tr}\left(h\otimes h\gamma^{(2)}_{\Psi_N}\right)\right)^{1/2+\delta}.
\end{aligned} \tag{46}$$

Summary. Inserting the estimates (42), (43) and (46) in (40) we find the desired lower bound. □

5. Final energy estimate: Proof of Theorem 1

We again assume, without loss of generality, that $V \geqslant 1$. We apply Lemma 4 to a ground state Ψ_N of H_N, then insert the dimension estimate (24) and the results of Lemma 5 (recall the definition (27)). This gives

$$e_{\mathrm{H},N} \geqslant e_N \geqslant e_{\mathrm{H},N} - C_\delta\left(\frac{L^{2+2/s+2\delta}}{N} + \frac{1}{L^{1/4-\delta/2}}\left(\frac{1+|e_{N,\varepsilon}|}{\varepsilon}\right)^{5/4+3\delta/2}\right) \tag{47}$$

for all $\varepsilon > 0$, $\delta \in (0,1/2)$, $N \geqslant 2$ and $L \geqslant 1$.

Stability of the second kind. We will deduce from (47) that $|e_{N,\varepsilon}| \leqslant C$ for $\varepsilon > 0$ small, provided (14) holds. Using (47) with w replaced by $(1-\varepsilon)^{-1}w$, we have

$$e^{\varepsilon}_{\mathrm{H},N} \geqslant e_{N,\varepsilon} \geqslant e^{\varepsilon}_{\mathrm{H},N} - C_\delta \left(\frac{L^{2+2/s+2\delta}}{N} + \frac{1}{L^{1/4-\delta/2}} \left(\frac{1+|e_{N,\varepsilon'}|}{\varepsilon'-\varepsilon} \right)^{5/4+3\delta/2} \right) \tag{48}$$

for all $1 > \varepsilon' > \varepsilon > 0$ and $\delta \in (0,1/2)$, where $e^{\varepsilon}_{\mathrm{H},N}$ is the ground state energy of the Hartree functional with h replaced by $(1-\varepsilon)h$ (similarly as in (27)). Using Assumption (13), Lemma 7 and the diamagnetic inequality $\langle u, hu\rangle \geqslant \int |\nabla|u||^2$, we find that there exists some $\varepsilon_0 > 0$ (depending only on w) such that

$$e^{\varepsilon}_{\mathrm{H},N} \geqslant -C \quad \text{for all } 0 < \varepsilon < \varepsilon_0. \tag{49}$$

We make the induction hypothesis (labeled I_η)

$$\limsup_{N\to\infty} \frac{|e_{N,\varepsilon}|}{1+N^\eta} < \infty \text{ for all } 0 < \varepsilon < \varepsilon_0. \tag{50}$$

Note that I_η holds for $\eta = 2\beta - 1$ by Lemma 2, and we ultimately aim at proving I_0. From (47) and (49), by choosing $L = N^\tau$ with $\tau > 0$, we deduce that if I_η holds for some $\eta \leqslant 2\beta - 1$, then $I_{\eta'}$ also holds provided that

$$\eta' > \max\left\{ \tau(2+2/s) - 1, (5\eta - \tau)/4 \right\} \quad \text{for some } \tau > 0. \tag{51}$$

With the optimal choice $\tau = s(5\eta+4)/(9s+8)$, the requirement (51) reduces to

$$\eta' > \eta - \frac{s - \eta(s+2)}{9s+8}. \tag{52}$$

When $\beta < (s+1)/(s+2)$, we can choose a constant c such that

$$0 < c < \frac{s-(2\beta-1)(s+2)}{9s+8}$$

and it is clear that (52) holds with $\eta' = \eta - c$ because $\eta \leqslant 2\beta - 1$. Thus we have shown that I_η implies $I_{\eta - c}$ for some constant $c > 0$ independent of η. Repeating the argument sufficiently many times we finally deduce that I_0 holds, which is the desired stability bound.

Conclusion. Now, using $|e_{N,\varepsilon}| \leqslant C$ for $\varepsilon > 0$ small, (47) reduces to

$$e_{\mathrm{H},N} \geqslant e_N \geqslant e_{\mathrm{H},N} - C_\delta \Big(\frac{L^{2+2/s+2\delta}}{N} + \frac{1}{L^{1/4-\delta/2}} \Big) \tag{53}$$

for all $\delta \in (0,1/2)$ and $L \geqslant 1$. By choosing $L = N^{4/(9+8/s)}$ we conclude that

$$e_{\mathrm{H},N} \geqslant e_N \geqslant e_{\mathrm{H},N} - C_\alpha N^{-\alpha} \tag{54}$$

for every $0 < \alpha < s/(9s+8)$. The desired energy convergence (16) follows from (54) and $\lim_{N\to\infty} e_{\mathrm{H},N} = e_{\mathrm{NLS}}$ (see Appendix A). Once the convergence of the energy is established, the convergence of states (17) follows exactly as in [14, Section 4.3], and we omit the details.

Appendix A. From Hartree to NLS

Here we prove an elementary lemma which, together with the variational principle, implies that $\lim_{N\to\infty} e_{\mathrm{H},N} = e_{\mathrm{NLS}}$.

Lemma 7 (**Limit of the Hartree interaction energy**).
For every $w \in L^1(\mathbb{R}^2)$ with $\int w = a$,

$$\lim_{\lambda\to\infty} \sup_{\substack{u\in H^1\\ u\neq 0}} \||u\||_{H^1}^{-4} \left| \iint |u(x)|^2 \lambda^3 w(\lambda(x-y))|u(y)|^2 dxdy - a\int |u(x)|^4 dx \right| = 0.$$

Proof. It suffices to consider the case when $u \geqslant 0$. By introducing the variable $z = \lambda(x-y)$, we can write

$$\begin{aligned} &\iint |u(x)|^2\lambda^3 w(\lambda(x-y))|u(y)|^2 dxdy - a\int |u(x)|^4 dx \\ =& \iint |u(x)|^2 w(z)\Big(|u(x-\lambda^{-1}z)|^2 - |u(x)|^2\Big) dxdz. \end{aligned} \tag{55}$$

Now we pick $L > 0$ and decompose

$$w(z) = \mathbb{1}(|z| > L)w(z) + \mathbb{1}(|z| \leqslant L)w(z).$$

By the Cauchy-Schwarz inequality

$$2|u(x)|^2|u(x-\lambda^{-1}z)|^2 \leqslant |u(x)|^4 + |u(x-\lambda^{-1}z)|^4$$

and the Sobolev's embedding $\|u\|_{L^4} \leqslant C\|u\|_{H^1}$ we have

$$\begin{aligned} &\left|\iint |u(x)|^2 \mathbb{1}(|z|>L)w(z)\Big(|u(x-\lambda^{-1}z)|^2 - |u(x)|^2\Big)dxdz\right| \\ \leqslant& \iint \Big(\frac{3}{2}|u(x)|^4 + \frac{1}{2}|u(x-\lambda^{-1}z)|^4\Big)\mathbb{1}(|z|>L)|w(z)|dxdz \\ =& 2\|u\|_{L^4}^4 \int_{|z|>L} |w(z)|dz \leqslant C\|u\|_{H^1}^4 \int_{|z|>L}|w(z)|dz. \end{aligned} \tag{56}$$

On the other hand, note that

$$\begin{aligned} &\big||u(x-\lambda^{-1}z)|^2 - |u(x)|^2\big| = \left|\int_0^1 (-\lambda^{-1}z).(\nabla|u|^2)(x-t\lambda^{-1}z)dt\right| \\ \leqslant& 2\lambda^{-1}|z|\int_0^1 |\nabla u(x-t\lambda^{-1}z)|.|u(x-t\lambda^{-1}z)|dt \end{aligned}$$

where we have used $|\nabla(u^2)| \leqslant 2|\nabla u|.|u|$ in the last estimate. Combining with Fubini's theorem and Sobolev's inequality $\|u\|_{L^6} \leqslant C\|u\|_{H^1}$, we find that

$$\left|\iint |u(x)|^2\mathbb{1}(|z|\leqslant L)w(z)\Big(|u(x-\lambda^{-1}z)|^2 - |u(x)|^2\Big)dxdz\right| \tag{57}$$

$$\begin{aligned} \leqslant& 2\lambda^{-1}L\int_0^1\int |w(z)|\left(\int |u(x)|^2|\nabla u(x-\lambda^{-1}z)|.|u(x-\lambda^{-1}z)|dx\right)dzdt \\ \leqslant& 2\lambda^{-1}L\int_0^1\int |w(z)|\Big(\int |u(x)|^6dx\Big)^{1/3}\Big(\int|\nabla u(x-\lambda^{-1}z)|^2dx\Big)^{1/2}\times \\ &\qquad\qquad\times\Big(\int |u(x-\lambda^{-1}z)|^6dx\Big)^{1/6}dzdt \\ \leqslant& C\lambda^{-1}L\|u\|_{H^1}^4\|w\|_{L^1}. \end{aligned} \tag{58}$$

From (55), (56) and (58), it follows that

$$\|u\|_{H^1}^{-4}\left|\iint |u(x)|^2\lambda^3 w(\lambda(x-y))|u(y)|^2 dxdy - a\int |u(x)|^4 dx\right|$$
$$\leqslant C\left(\int_{|z|>L} |w(z)| + \lambda^{-1}L\|w\|_{L^1}\right).$$

The conclusion follows by choosing $1 \ll L \ll \lambda$ (for example $L = \sqrt{\lambda}$). $\square$

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