Curved space-times by crystallization of liquid fiber bundles

Frédéric Hélein, Dimitri Vey

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Abstract — Motivated by the search for a Hamiltonian formulation of Einstein equations of gravity which depends in a minimal way on choices of coordinates, nor on a choice of gauge, we develop a multisymplectic formulation on the total space of the principal bundle of orthonormal frames on the 4-dimensional space-time. This leads quite naturally to a new theory which takes place on 10-dimensional manifolds. The fields are pairs of $((\alpha, \omega), \varpi)$, where (α, ω) is a 1-form with coefficients in the Lie algebra of the Poincaré group and ϖ is an 8-form with coefficients in the dual of this Lie algebra. The dynamical equations derive from a simple variational principle and imply that the 10-dimensional manifold looks locally like the total space of a fiber bundle over a 4-dimensional base manifold. Moreover this base manifold inherits a metric and a connection which are solutions of a system of Einstein–Cartan equations.

1 Introduction

General Relativity postulates among other things that the space-time is *homogeneous*, in the sense that the neighbourhoods of all of its points are endowed with the same physical laws (i.e. an extension of the Galilean inertia principle). However the space-time should also be *isotropic*, since the physical laws are independent of the point *and* of the pseudo-orthonormal or reference frame in which they are expressed. Hence we should be able to formulate physics and, in particular, General Relativity, in the total space of the bundle of pseudo-orthonormal frames on the space-time, thus replacing the local geometric model of Minkowski space by the Poincaré group.

This description may be physically relevant: as noted by M. Toller [35, 36] most local measurements in classical fields physics makes sense if a point *and* a local frame are specified. Moreover, as F. Lurçat [34] proposed in 1964, this could be a natural framework to interpret the relationship encoded in the Regge trajectories between the mass and the spin of hadron particles and resonances.

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A mathematical and geometrical analogue of these considerations appears if we start from a formulation of gravity which involves the choice of a moving frame, which is hence subject to a gauge ambiguity which, somehow, spoils the coordinate independence of the theory. A cure[1] for that consists, as in [22], in lifting the problem on the total space of the principal bundle involved, as pictured by C. Ehresmann. We then rely on ideas and points of view developed by E. Cartan [5, 6], including his theory of the equivalence problem.

However the total space of the principal bundle of pseudo-orthonormal moving frames of a space-time satisfies a priori constraints, namely the axioms of the definition of a principal bundle and of a connection, which impose a rigidity of each fiber. Since this rigidity assumption does not fit in the spirit of General Relativity (breaking a priori the symmetry of the total space of the principal bundle), we would like to release a priori the 10-dimensional total space of the principal bundle and the connection from these constraints and, instead, to recover them from the dynamical equations. It amounts to start from a 10-dimensional manifold $\mathcal{P}$ (the dimension of the Poincaré group) which can be considered as a white sheet: we don't draw on it the fibers of the principal bundle, nor a fortiori the way to quotient out this manifold to get a 4-dimensional space-time. Then we look for a variational principle which imposes dynamical equations, from which one can derive, at least locally, a fibration and the existence of a metric and connection on a 4-dimensional quotient manifold $\mathcal{X}$ which satisfy some Einstein–Cartan system of equations. (Hence the terms *crystallization of liquid fiber bundles* in our title, by analogy with nematic liquid crystals.) Such a goal was addressed and achieved by Toller in [36] (where additional matter fields are also treated). In this paper we propose an alternative approach.

In our approach and in Toller's one the gravitational field is encoded in the data of a moving frame $(e_0, \cdots, e_3, u_4, \cdots, u_9)$ defined on $\mathcal{P}$ or, equivalently, its dual coframe $(\alpha^0, \cdots, \alpha^3, \omega^4, \cdots, \omega^9)$, which can be interpreted as the expression of a section φ of the bundle $\mathfrak{p} \otimes T\mathcal{P}$, where $\mathfrak{p}$ is the Lie algebra of the Poincaré group and $T\mathcal{P}$ is the tangent bundle to $\mathcal{P}$. In both approaches we end up with a variational principle on $(\alpha^0, \cdots, \alpha^3, \omega^4, \cdots, \omega^9)$ or on φ which leads to dynamical equations which forces locally $\mathcal{P}$ to be the frame bundle over a 4-dimensional space-time manifold, satisfying an Einstein(–Cartan) system of equations. But our variational formulation differs radically from Toller's one.

Toller's variational formulation is a non standard one. Indeed the Lagrangian is built out of a 4-form λ defined on the first jet bundle $J^1(\mathcal{P}, \mathfrak{p} \otimes T\mathcal{P})$ which is everywhere pointwise proportional to a 4-form on the base manifold $\mathcal{P}$. The dynamical equations on a section φ of $\mathfrak{p} \otimes T\mathcal{P}$ over $\mathcal{P}$ follow then from the requirement that, *for any arbitrary 4-dimensional submanifold $S \subset \mathcal{P}$ with boundary, the quantity $\int_S (j^1\varphi)^*\lambda$ is stationary with respect to all first order variations of the field φ and to all first order variations of S which keep its boundary fixed* (here $j^1\varphi$ is the first jet of φ). One may figure the submanifold S as playing the role of a local section of the bundle $\mathcal{P}$ over a 4-dimensional

[1] An alternative approach would consist in building a suitable reduction of the geometry of connections on a $\mathfrak{G}$-principal bundle as for instance in [2, 3].

space-time $\mathcal{X}$, although the fibration $\mathcal{P} \longrightarrow \mathcal{X}$ is not defined a priori.

In our approach the Lagrangian is defined by using a 10-form θ on a bundle over $\mathcal{P}$ (see below) and the dynamical equations follow by requiring that a section φ of this bundle is a critical point of the action functional $\int_{\mathcal{P}} \varphi^* \theta$ in the usual sense. But in comparison to Toller's principle, this raises difficulties in order to achieve a local fibration from such a principle. We first need to add extra fields which play the role of Lagrange multipliers for the equivariance constraints which are at the origin of a local fibration and of well-defined metrics and connections on the local quotient. One then faces the difficulty of finding the right definition of such Lagrange multipliers fields. A second difficulty is that Lagrange multipliers fields create in general sources for the gravitational fields, i.e. in the r.h.s. of the Einstein–Cartan system of equations that we shall find.

We answer to the first difficulty as follows: our theory is not based on some ad hoc construction, but on a study of the Hamiltonian structure of Einstein equations, starting from the variational Weyl–Einstein–Cartan formulation (called WEC in this paper and erroneously known as the Palatini one, see [13]). In this study we systematically privilege formulations which are as covariant as possible, which means mathematically that we look for a formulation which depends in a minimal way on choices of coordinates. (This was already one of the reasons for replacing the space-time by the total space of its frame bundle.) While several alternative theories exist for describing the Hamiltonian structure, we favour here the multisymplectic approach, since it simultaneously respects in a natural way the locality of physical theories. In a few words (see also below) the basic idea of the multisymplectic formalism, which goes back to V. Volterra, is to consider all first order derivatives of the fields as analogues of the velocity in Mechanics and to perform the Legendre transform with respect to all these first order derivatives. Here we use this theory on the total space of the frame bundle. The result is that this method produces naturally dual *multimomenta* fields, among which we find the Lagrange multipliers.

Concerning the second difficulty we observe that, under some compactness hypotheses, a miracle occurs and the right hand side of the Einstein–Cartan equation on the quotient 4-manifold simply vanishes. This holds e.g. either if we replace the local structure group $SO(1,3)$ (the Lorentz group) by the rotation group $SO(4)$, or if we assume that the dual multimomenta fields have decay properties at infinity. An interpretation of these phenomena raises subtle and challenging questions.

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1.1 Overview of the paper

The origin of the multisymplectic formalism goes back to the discovery by Volterra at the end of the ninetieth century [38, 39] of generalizations of the Hamilton equations for variational problems with several variables. These ideas were first developped in particular by C. Carathéodory [4], T. De Donder [12], H. Weyl [40], T. Lepage [32], and later by

P. Dedecker [10, 11]. In the seventies of the past century, this theory was geometrized in a way analogous to the construction of symplectic geometry by several mathematical physicists. In particular, the Polish school formulated important ideas and developed the *multiphase-space* formalism in the work of W.M. Tulczyjew [37], J. Kijowski [28], Kijowski and Tulczyjew [31], Kijowski and W. Szczyrba [29, 30]. Parallel to this development, the paper by H. Goldschmidt and S. Sternberg [19] gave a formulation of the Hamilton equations in terms of the Poincaré-Cartan form and the underlying jet bundles geometry, and a related approach was also developed by the Spanish school in P.L. García [17], García and A. Pérez-Rendón [18]. This theory has many recent developments which we cannot report here (see e.g. [7, 8, 14, 15, 20, 21, 25, 26, 27, 33]). Today the Hamilton–Volterra equations are often called the De Donder–Weyl equations for reference to [12, 40], which is inaccurate [24]. In this paper we name them the HVDW equations for Hamilton–Volterra–De Donder–Weyl.

A multisymplectic manifold is a smooth manifold $\mathcal{N}$ endowed with a *multisymplectic* $(m+1)$-form ω, i.e. ω is closed and one often assumes that it is non degenerate, i.e. that the only vector field ξ on the manifold such that $\xi \lrcorner \omega = 0$ is zero. Here m refers to the number of independent variables of the associated variational problem. An extra ingredient is a Hamiltonian function $H : \mathcal{N} \longrightarrow \mathbb{R}$. One can then describe the solutions of the HVDW equations by oriented m-dimensional submanifolds Γ of $\mathcal{N}$ which satisfy the condition that, at any point $\mathrm{M} \in \mathcal{N}$, there exists a basis $(X_1, \cdots, X_m)$ of $T_{\mathrm{M}}\Gamma$ such that $X_1 \wedge \cdots \wedge X_m \lrcorner \omega = (-1)^m dH$. Equivalently one can replace ω by its restriction to the level set $H^{-1}(0)$ and describe the solutions as the submanifolds Γ of $H^{-1}(0)$ such that $X_1 \wedge \cdots \wedge X_m \lrcorner \omega = 0$ everywhere (plus some independence conditions, see e.g. [24]).

Before applying the multisymplectic formalism and in order to describe the theory in a way which does not depend on any choice of gauge, in Paragraph 2.2 we first translate and lift the 4-dimensional WEC (Weyl–Einstein–Cartan) variational principle to the total space $\mathcal{P}$ of the principal bundle. Note that this kind of approach shares some similarities with the use of Cartan geometries as e.g. in [41]. Denoting by $\mathfrak{p}$ the Lie algebra of the Poincaré group, the connection and the vierbein are both represented by a $\mathfrak{p}$-valued 1-form η on $\mathcal{P}$ which satisfies *normalization* (2), (4) and *equivariance* (3), (5) hypotheses. We note that the equivariance condition has the drawback of being a *non holonomic* constraint, i.e. on the first order derivatives of the field. Another preliminary step is, in Paragraph 2.3, to forget the normalization condition and to express the equivariance condition in a way which is independent on it, but relies on Cartan's theory of the equivalence problem. The subsequent computations will confirm that the normalization condition, as its name suggests it, is not essential and can be recovered by a suitable choice of coordinates. Similar considerations were done in [36], although in a slightly different setting.

Then, in Section 3, we apply the multisymplectic machinery for $m = 10$ and compute the Legendre transform by treating connections as equivariant $\mathfrak{p}$-valued 1-forms on $\mathcal{P}$. We find that the natural multisymplectic manifold can be built from the vector bundles $\mathfrak{p} \otimes T^*\mathcal{P}$ and $\mathfrak{p}^* \otimes \Lambda^8 T^*\mathcal{P}$ over $\mathcal{P}$, where $\mathfrak{p}^*$ its dual vector space of $\mathfrak{p}$. These vector bundles are endowed with a canonical $\mathfrak{p}$-valued 1-form $\eta = (\eta^0, \cdots, \eta^9)$ and a canonical $\mathfrak{p}^*$-valued 8-form $\psi = (\psi_0, \cdots, \psi_9)$ respectively. Then the multisymplectic manifold is the submanifold

$\mathcal{M}$ of the total space of the vector bundle $(\mathfrak{p} \otimes T^*\mathcal{P}) \oplus_{\mathcal{P}} (\mathfrak{p}^* \otimes \Lambda^8 T^*\mathcal{P})$, defined by the equations $\eta^a \wedge \eta^b \wedge \psi_A = \kappa_A^{ab} \eta^0 \wedge \cdots \wedge \eta^9$, $\forall a, b, A$ s.t. $0 \leq a, b \leq 3$ and $0 \leq A \leq 9$, where the coefficients κ_A^{ab} are some fixed structure constants. The manifold $\mathcal{M}$ is equipped with the 10-form $\theta = \psi \wedge (d\eta + \eta \wedge \eta)$, where the duality pairing between $\mathfrak{p}^*$ and $\mathfrak{p}$ is implicitly assumed. The solutions of the Hamilton equations are sections φ of $\mathcal{M}$ over $\mathcal{P}$ which are critical points of the action $\mathcal{A}[\varphi] = \int_{\mathcal{P}} \varphi^* \theta$. At this stage we will decide to *remove* the unnatural equivariance constraints (on $\varphi^*\eta$) and we derive the corresponding generalized Hamilton equations in Section 4. We note that the resulting theory is manifestly a gauge theory with gauge group the Poincaré group, whose importance for gravity theories is stressed in [1].

Then several interesting phenomena occur. The first one is that the dynamical equations force the manifold $\mathcal{P}$ to be locally fibered over a 4-dimensional manifold, with 6-dimensional fibers. This is the content of Lemma 5.1 in Paragraph 5.1 (which follows from similar mechanisms as in [23], see Lemma 2.1): a metric and a connection emerge spontaneously from the solution on the 4-dimensional quotient space. Moreover we can recover the normalization conditions by a suitable choice of coordinates adapted to this local fibration and, as in [22] for the Yang–Mills fields, the dynamical equations force the fields to satisfy the equivariance conditions along these fibers. The second phenomenon appears after a long computation in Paragraph 5.2, done in order to write the equations in coordinates adapted to these fibration. The metric and the connection on the 4-dimensional quotient space satisfy an Einstein–Cartan system of equations (87)

$$\left\{ \begin{array}{rcl} \mathrm{E}^b{}_a & = & \frac{1}{2} \rho_j \cdot p_a{}^{bj} \\ \mathrm{T}^a{}_{cd} & = & -\left(\mathsf{h}_{de} \delta^a_{a'} \delta^{c'}_c + \frac{1}{2} \delta^{c'}_{a'} (\delta^a_d \mathsf{h}_{ce} - \delta^a_c \mathsf{h}_{de}) \right) \rho_j \cdot p_{c'}{}^{ea'j} \end{array} \right. , \tag{1}$$

where $\mathrm{E}^b{}_a$ is the Einstein tensor, $\mathrm{T}^a{}_{cd}$ is the torsion tensor, $(\rho_j)_{4 \leq j \leq 9}$ is a left invariant moving frame on the 6-dimensional fiber and $\rho_j \cdot f$ is the derivative of f with respect to ρ_j. The right hand sides of (1) are covariant divergences involving derivatives with respect to coordinates on the fibers of the tensors $p_a{}^{bj}$ and $p_c{}^{eaj}$, which are components of $\varphi^*\psi$. They play here the role of a stress-energy tensor and an angular momentum tensor, respectively. The tensors $p_a{}^{bj}$ and $p_c{}^{eaj}$ satisfy also non homogeneous Maxwell type equations (88) which involve space-time partial derivatives and are defined up to some gauge transformations (see Section 7).

At this point come some difficulties but also some challenging questions, discussed in Section 6. A natural question is to know in which circumstance the r.h.s. of (1) vanish, in order to recover the standard vacuum Einstein equations of gravity. This is actually the case if we replace the Lorentz group by $SO(4)$ (or its universal cover $Spin(4)$): then, as shown in Theorem 6.1, under reasonable hypotheses, one can show that the r.h.s. of (1) vanish and hence we recover exactly all the orthonormal frame bundles of Einstein manifolds. The main reason here is that $SO(4)$ or $Spin(4)$ are *compact*, as in [22] for Yang–Mills. This is also the case if the structure group is $SO(1,3)$ or $Spin(1,3)$ and if we have some control on the dual fields $p_a{}^{bj}$ and $p_a{}^{bcj}$ at infinity. However, without such an hypothesis, since $SO(1,3)$ is not compact we cannot conclude that the r.h.s. of (1) vanish in general. Thus we are led to consider a larger class of solutions than the classical

Einstein metrics in vacuum. One needs for that purpose to understand Equations (88) and to know whether one could assume physically relevant hypotheses on ${p_a}^{bj}$ and ${p_a}^{bcj}$ which would imply that the r.h.s. of (1) vanish or, at least, satisfy some equations (besides the usual conservation law satisfied by the stress-energy tensor and the angular momentum tensor). It would be also interesting to see whether the r.h.s. of (1) could be interpreted as a dark matter and/or a dark energy source. In a broader framework, it would interesting to study similar models coupled with matter fields and to understand the possible role of the extra fields $\varphi^*\psi$ (or their generalizations) in the interaction between gravity and the other fields.

1.2 Summary of notations

- $\mathbb{M}$ is a 4-dimensional real affine space and $\vec{\mathbb{M}}$ is the associate vector space endowed with a non degenerate symmetric bilinear form h (either the Minkowski metric or the standard Euclidean one); (E_0, E_1, E_2, E_3) is an orthonormal basis of $(\vec{\mathbb{M}}, \mathsf{h})$.

- $\mathfrak{G}$ is the group of linear isometries of $(\vec{\mathbb{M}}, \mathsf{h})$ or its universal cover (either $SO(1,3)$ or $SL(2,\mathbb{C})$ if h is the Minkowski metric or $SO(4)$ or $Spin(4)$ if h is the Euclidean metric); $\mathfrak{g}$ is the Lie algebra of $\mathfrak{G}$.

- $(u_4, u_5, u_6, u_7, u_8, u_9)$ is a basis of $\mathfrak{g}$ and c_{ij}^k $(4 \leq i, j, k \cdots \leq 9)$ are the structure coefficients of $\mathfrak{g}$ in this basis, so that $[u_i, u_j] = c_{ij}^k u_k$.

- If $\mathcal{R} : \mathfrak{G} \longrightarrow GL(\vec{\mathbb{M}})$ is the standard linear representation, then $\forall g \in \mathfrak{G}$, $({g^a}_b)_{0\leq a,b\leq 3}$ are the coefficients of the matrix of $\mathcal{R}(g)$ in the basis (E_0, E_1, E_2, E_3), i.e. $\mathcal{R}(g)(E_b) = E_a {g^a}_b$.

- Similarly, if $\mathcal{R} : \mathfrak{g} \longrightarrow gl(\vec{\mathbb{M}})$ is the standard linear representation, $\forall \xi \in \mathfrak{g}$, $({\xi^a}_b)_{0\leq a,b\leq 3}$ are the coefficients of the matrix of $\mathcal{R}(\xi)$ in the basis (E_0, E_1, E_2, E_3). We then have $\xi^{ab} + \xi^{ba} = 0$, where $\xi^{ab} = {\xi^a}_{b'}\mathsf{h}^{b'b}$, for $0 \leq a, b, c, d, \cdots \leq 3$.

- In particular, for $4 \leq i \leq 9$, $(u_{ib}^a)_{0\leq a,b\leq 3}$ are the coefficients of the matrix of $\mathcal{R}(u_i)$; we set $u_i^{ab} := u_{ib'}^a \mathsf{h}^{b'b}$ (see Paragraph 8.1). Then, for $0 \leq a, b \leq 3$ and $0 \leq A \leq 9$, κ_A^{ab} is defined by: $\kappa_c^{ab} = 0$ for $0 \leq c \leq 3$ and $\kappa_i^{ab} = 2u_i^{ab}$ for $4 \leq i \leq 9$.

- $\mathfrak{T}$ is the Abelian Lie group of translations on the Minkowski space, and $\mathfrak{t}$ is its trivial Lie algebra, with basis (t_0, t_1, t_2, t_3).

- $\mathfrak{P} = \mathfrak{G} \ltimes \mathfrak{T}$ is the group of affine isometries of $\mathbb{M}$ (or its universal cover), with Lie algebra $\mathfrak{p} = \mathfrak{g} \oplus \mathfrak{t}$. We denote by $(\mathfrak{l}_A)_{0\leq A\leq 9} = (t_0, \cdots, t_3, u_4 \cdots, u_9)$, a basis of $\mathfrak{p}$. If $\mathbb{M}$ is the Minkowski space, $\mathfrak{P}$ is the Poincaré Lie group.

- $\mathfrak{g}^*$, $\mathfrak{t}^*$ and $\mathfrak{p}^*$ are the dual vector spaces of respectively $\mathfrak{g}$, $\mathfrak{t}$ and $\mathfrak{p}$.

- If (e^0, e^1, e^2, e^3) is a coframe on a 4-dimensional manifold $\mathcal{X}$ (i.e. a collection of four 1-forms e^0, e^1, e^2, e^3 defined on an open subset of $\mathcal{X}$ which is everywhere of rank 4) and if we denote by $(\frac{\partial}{\partial e^0}, \frac{\partial}{\partial e^1}, \frac{\partial}{\partial e^2}, \frac{\partial}{\partial e^3})$ the dual frame, we set $e^{(4)} := e^0 \wedge e^1 \wedge e^2 \wedge e^3$ and
$$e^{(3)}_a := \frac{\partial}{\partial e^a} \lrcorner e^{(4)}, \quad e^{(2)}_{ab} := \frac{\partial}{\partial e^b} \lrcorner e^{(4)}_a, \quad e^{(1)}_{abc} := \frac{\partial}{\partial e^c} \lrcorner e^{(3)}_{ab}$$
(note that $e^{(1)}_{abc} = \epsilon_{abcd} e^d$).

- if $(e^0, \cdots, e^3, \gamma^4, \cdots, \gamma^9)$ is a coframe on a 10-dimensional manifold $\mathcal{P}$ and if $(\frac{\partial}{\partial e^0}, \cdots, \frac{\partial}{\partial e^3}, \frac{\partial}{\partial \gamma^4}, \cdots, \frac{\partial}{\partial \gamma^9})$ is its dual frame, we set:
$$\begin{array}{rclcrcl} e^{(4)} & := & e^0 \wedge \cdots \wedge e^3, & & \gamma^{(6)} & := & \gamma^4 \wedge \cdots \wedge \gamma^9 \\ e^{(3)}_a & := & \frac{\partial}{\partial e^a} \lrcorner e^{(4)}, & & \gamma^{(5)}_i & := & \frac{\partial}{\partial \gamma^i} \lrcorner \gamma^{(6)} \\ e^{(2)}_{ab} & := & \frac{\partial}{\partial e^b} \lrcorner e^{(3)}_a, & & \gamma^{(4)}_{ij} & := & \frac{\partial}{\partial \gamma^j} \lrcorner \gamma^{(5)}_i \\ e^{(1)}_{abc} & := & \frac{\partial}{\partial e^c} \lrcorner e^{(2)}_{ab}, & & \gamma^{(3)}_{ijk} & := & \frac{\partial}{\partial \gamma^k} \lrcorner \gamma^{(4)}_{ij} \end{array}$$

- Similarly, if $(\alpha^0, \cdots, \alpha^3, \omega^4, \cdots, \omega^9)$ is another coframe on $\mathcal{P}$, if $\alpha^{(4)} := \alpha^0 \wedge \alpha^1 \wedge \alpha^2 \wedge \alpha^3$, $\omega^{(6)} := \omega^4 \wedge \cdots \wedge \omega^9$ and if $(\frac{\partial}{\partial \alpha^0}, \cdots, \frac{\partial}{\partial \alpha^3}, \frac{\partial}{\partial \omega^4}, \cdots, \frac{\partial}{\partial \omega^9})$ is its dual frame we use the same conventions: $\alpha^{(3)}_a := \frac{\partial}{\partial \alpha^a} \lrcorner \alpha^{(4)}$, $\alpha^{(2)}_{ab} := \frac{\partial}{\partial \alpha^a} \wedge \frac{\partial}{\partial \alpha^b} \lrcorner \alpha^{(4)} = \frac{\partial}{\partial \alpha^b} \lrcorner \alpha^{(3)}_a$, etc., $\omega^{(5)}_i := \frac{\partial}{\partial \omega^i} \lrcorner \omega^{(6)}$, $\omega^{(4)}_{ij} := \frac{\partial}{\partial \omega^i} \wedge \frac{\partial}{\partial \omega^j} \lrcorner \omega^{(6)} = \frac{\partial}{\partial \omega^j} \lrcorner \omega^{(5)}_i$, etc.

2 The starting point of the approach

Our first task consists in recasting the usual Weyl–Einstein–Cartan formulation of gravity on the total space of the principal bundle of lorentzian frames on space-time in an invariant way.

2.1 The Weyl–Einstein–Cartan action

Consider a 4-dimensional manifold $\mathcal{X}$, the space-time. Dynamical fields in the Weyl–Einstein–Cartan formulation can be defined locally as being pairs (e, A), where $e = (e^0, e^1, e^2, e^3)$ is a moving coframe on $\mathcal{X}$ (defining the metric $\mathsf{h}_{ab} e^a \otimes e^b$ on the tangent bundle $T\mathcal{X}$) and A is a $\mathfrak{g}$-valued connection 1-form on $\mathcal{X}$. The WEC (Weyl–Einstein–Cartan) action then reads
$$\mathcal{A}_{EWC}[e, A] = \int_{\mathcal{X}} \frac{1}{2} \epsilon_{abc}{}^d e^a \wedge e^b \wedge (dA + A \wedge A)^c{}_d = \int_{\mathcal{X}} \frac{1}{2} \epsilon_{abcd} e^a \wedge e^b \wedge F^{cd},$$
where $F := dA + A \wedge A$ and $F^{cd} := F^c{}_{d'} \mathsf{h}^{dd'}$. Alternatively, by Lemma 8.3,
$$\mathcal{A}_{EWC}[e, A] = \int_{\mathcal{X}} e^{(2)}_{ab} \wedge F^{ab} = \int_{\mathcal{X}} u^{ab}_i e^{(2)}_{ab} \wedge F^i.$$

It is possible to understand pairs (e, A) in a more global and geometric way by assuming that a rank 4 vector bundle $V\mathcal{X}$ has been chosen over $\mathcal{X}$, equipped with a pseudo-metric h. Then A represents a connection of $V\mathcal{X}$ which respects the pseudo-metric h and e represents a *solder form*, i.e. a rank 4 section of the vector bundle over $\mathcal{X}$ whose fiber over $\mathsf{x} \in \mathcal{X}$ is the set of linear maps from $T_{\mathsf{x}}\mathcal{X}$ to $V_{\mathsf{x}}\mathcal{X}$. By choosing a family of four local sections of $V\mathcal{X}$ that forms an orthonormal basis of $V\mathcal{X}$, we may decompose locally e and A in terms of real valued 1-forms e^a and $A^c{}_d$ and recover the previous description. Note that this description still has the drawback that it rests on the *a priori* choice of a vector bundle $V\mathcal{X}$ over $\mathcal{X}$. This drawback will be removed in the model proposed in the following.

2.2 Lifting to the principal bundle

It is well-known that the previous action is invariant by gauge transformations of the form

$$(e, A) \longmapsto (g^{-1}e, g^{-1}dg + g^{-1}Ag),$$

or, in indices,

$$e^a \longmapsto (g^{-1})^a{}_{a'}e^{a'}, \quad A^a{}_b \longmapsto (g^{-1})^a{}_{a'}dg^{a'}{}_b + (g^{-1})^a{}_{a'}A^{a'}{}_{b'}g^{b'}{}_b$$

where $g : \mathcal{X} \longrightarrow \mathfrak{G}$. One way to picture geometrically this ambiguity is to lift the variational problem on the *total space* $\mathcal{P}$ of the principal bundle of orthonormal frames on $V\mathcal{X}$ (with the right action of $\mathfrak{G}$ denoted by $\mathcal{P} \times \mathfrak{G} \ni (\mathsf{z}, g) \longmapsto \mathsf{z} \cdot g \in \mathcal{P}$). This amounts roughly speaking to consider all possible gauge transformations of a given field (e, A) simultaneously. We then represent each pair (e, A) by a pair of 1-forms (α, ω) on $\mathcal{P}$ with values in the Poincaré Lie algebra $\mathfrak{p}$, i.e. α takes values $\mathfrak{t}$ and ω takes values in $\mathfrak{g}$.

The price to pay however is that we need to assume that the $\mathfrak{p}$-valued 1-form (α, ω) satisfies normalization and equivariance constraints. To write them, use the basis $(u_4, \cdots, u_9)$ of $\mathfrak{g}$ and, for any $i = 4, \cdots, 9$, let ρ_i be the tangent vector field on $\mathcal{P}$ induced by the right action of u_i on $\mathcal{P}$. Indeed we assume that the lift ω of A satisfies the following *normalization* and *equivariance* properties respectively (see [22])

$$\rho_i \lrcorner \omega = u_i, \tag{2}$$

$$L_{\rho_i}\omega + [u_i, \omega] = 0, \tag{3}$$

where L_ρ denotes the Lie derivative with respect to a vector field ρ. Similarly α satisfies respectively the *normalization* and *equivariance* properties

$$\rho_i \lrcorner \alpha = 0, \tag{4}$$

$$L_{\rho_i}\alpha + u_i\alpha = 0. \tag{5}$$

The relationship with the previous description is as follows: for any $\mathfrak{p}$-valued 1-form (α, ω) on $\mathcal{P}$ which satisfies (2), (3), (4) and (5) and for any local section $\sigma : \mathcal{X} \longrightarrow \mathcal{P}$, we obtain a pair (e, A) on $\mathcal{X}$ simply by setting $e = \sigma^*\alpha$ and $A = \sigma^*\omega$.

Conversely, given a pair (e, A) on $\mathcal{X}$ and a local section $\sigma : \mathcal{X} \longrightarrow \mathcal{P}$, this provides us with a local trivialization

$$\begin{array}{rccc} \mathcal{T}: & \mathcal{P} & \longrightarrow & \mathcal{X} \times \mathfrak{G} \\ & \mathsf{z} & \longmapsto & (\mathsf{x}, g) \end{array}$$

where (x, g) is s.t. $\mathsf{z} = \sigma(\mathsf{x}) \cdot g$. We can then associate to (e, A) a $\mathfrak{p}$-valued 1-form (α, ω) on $\mathcal{P}$ which satisfies (2), (3), (4) and (5) given by $\alpha = \mathcal{T}^*(g^{-1}e)$ and $\omega = \mathcal{T}^*(g^{-1}Ag + g^{-1}dg)$ (see [22]).

Lastly let us define $\gamma := \mathcal{T}^*(g^{-1}dg)$ and denote by $\gamma^4, \cdots, \gamma^9$ the components of γ in the basis $(u_4, \cdots, u_9)$, i.e. s.t. $\gamma = u_i\gamma^i$. We can lift the action $\mathcal{A}_{EWC}$ to a functional on the space of $\mathfrak{p}$-valued 1-forms (α, ω) by setting:

$$\widehat{\mathcal{A}}_{EWC}[\alpha, \omega] = \int_{\mathcal{P}} \frac{1}{2}\epsilon_{abc}{}^d \alpha^a \wedge \alpha^b \wedge (d\omega + \omega \wedge \omega)^c{}_d \wedge \gamma^1 \wedge \cdots \wedge \gamma^6.$$

Alternatively, by setting $\alpha^{(2)}_{ab} := \frac{1}{2}\epsilon_{abcd}\alpha^c \wedge \alpha^d$, $\Omega := d\omega + \omega \wedge \omega$, $\Omega^{ab} := \Omega^a{}_{b'}\mathsf{h}^{bb'}$ and $\gamma^{(6)} := \gamma^4 \wedge \cdots \wedge \gamma^9$, we can write

$$\widehat{\mathcal{A}}_{EWC}[\alpha, \omega] = \int_{\mathcal{P}} \alpha^{(2)}_{ab} \wedge \Omega^{ab} \wedge \gamma^{(6)} = \int_{\mathcal{P}} u_i^{ab}\alpha^{(2)}_{ab} \wedge \Omega^i \wedge \gamma^{(6)}. \tag{6}$$

Then critical points of $\mathcal{A}_{EWC}$ correspond to critical points of $\widehat{\mathcal{A}}_{EWC}$ under the constraints (2), (3), (4) and (5).

2.3 Forgetting the fibration

A key step for our purpose is to translate the previous conditions on (α, ω) in a situation where the fibration $\mathcal{P} \longrightarrow \mathcal{X}$ is not given a priori. For that we claim that the normalization conditions (2) and (4) are not essential (this will be confirmed by the following). We hence translate the equivariance conditions (3) and (5) without reference to the normalization conditions.

We first observe that, if (2) holds, then $L_{\rho_i}\omega = \rho_i \lrcorner d\omega + d(\rho_i \lrcorner \omega) = \rho_i \lrcorner d\omega + du_i = \rho_i \lrcorner d\omega$ and $\rho_i \lrcorner \omega \wedge \omega = [\omega(\rho_i), \omega] = [u_i, \omega]$; hence the l.h.s. of (3) is equal to $L_{\rho_i}\omega + [u_i, \omega] = \rho_i \lrcorner d\omega + \rho_i \lrcorner \omega \wedge \omega$. Thus, assuming (2), (3) is equivalent to

$$\rho_i \lrcorner (d\omega + \omega \wedge \omega) = 0, \quad \forall i = 1, \cdots, 6. \tag{7}$$

Similarly, if (2) and (4) hold, $L_{\rho_i}\alpha = \rho_i \lrcorner d\alpha + d(\rho_i \lrcorner \alpha) = \rho_i \lrcorner d\alpha + d0 = \rho_i \lrcorner d\alpha$ and $\rho_i \lrcorner \omega \wedge \alpha = [u_i, \alpha]$. Hence, if we assume (2) and (4), (5) is equivalent to

$$\rho_i \lrcorner (d\alpha + \omega \wedge \alpha) = 0, \quad \forall i = 1, \cdots, 6. \tag{8}$$

Now both equations (7) and (8) are linear in ρ_i and so are also valid if we replace ρ_i by any tangent vector field ρ on $\mathcal{P}$ which is a linear combination of $\rho_4, \cdots, \rho_9$. Such vector fields are tangent to the fibers of $\mathcal{P} \longrightarrow \mathcal{X}$ or, equivalentely, are characterized by the property $\rho \lrcorner \alpha^a = 0$, $\forall a = 0, \cdots, 3$. Hence (7) and (8) are equivalent to the implication

$[\rho \lrcorner \alpha = 0] \Longrightarrow [\rho \lrcorner (d\omega + \omega \wedge \omega) = \rho \lrcorner (d\alpha + \omega \wedge \alpha) = 0]$. This is also equivalent to claim that there exists functions $Q^a{}_{bcd}$ and $Q^a{}_{cd}$ on $\mathcal{P}$ s.t.

$$(d\alpha + \omega \wedge \alpha)^a = \frac{1}{2} Q^a{}_{cd} \alpha^c \wedge \alpha^d \quad \text{and} \quad (d\omega + \omega \wedge \omega)^a{}_b = \frac{1}{2} Q^a{}_{bcd} \alpha^c \wedge \alpha^d. \tag{9}$$

Note that, if we set $Q^{ab}{}_{cd} := Q^a{}_{b'cd} \mathsf{h}^{b'b}$, we have $Q^{ab}{}_{cd} + Q^{ba}{}_{cd} = 0$ and that we may assume w.l.g. that $Q^a{}_{bcd} + Q^a{}_{bdc} = Q^a{}_{cd} + Q^a{}_{dc} = 0$.

Now let us return to the action. A key observation is that, since $\omega = \gamma + \mathcal{T}^*(g^{-1}Ag)$ and since $\mathcal{T}^*(g^{-1}Ag)$ is a linear combination of $\alpha^0, \alpha^1, \alpha^2$ and α^3, we have

$$\alpha^{(4)} \wedge \gamma^4 \wedge \cdots \wedge \gamma^9 = \alpha^{(4)} \wedge \omega^4 \wedge \cdots \wedge \omega^9, \quad \forall c, d, \tag{10}$$

where the ω^i are the coefficients of the decomposition of ω in the basis $(u_4, \cdots, u_9)$.

But if we assume that (9) is satisfied we have $\Omega^{ab} = \frac{1}{2} Q^{ab}{}_{cd} \alpha^c \wedge \alpha^d$ and hence $\alpha^{(2)}_{ab} \wedge \Omega^{ab} = Q^{ab}{}_{ab} \alpha^{(4)}$ (see Lemma 8.2). Hence, by using (6) and (10), it follows that, *if* (α, ω) *satisfies (9)*,

$$\widehat{\mathcal{A}}_{EWC}[\alpha, \omega] = \int_{\mathcal{P}} \alpha^{(2)}_{ab} \wedge \Omega^{ab} \wedge \omega^4 \wedge \cdots \wedge \omega^9 = \int_{\mathcal{P}} u_i^{ab} \alpha^{(2)}_{ab} \wedge \Omega^i \wedge \omega^{(6)}, \tag{11}$$

where $\omega^{(6)} := \omega^4 \wedge \cdots \wedge \omega^9$. Thus we are led to study critical points of the action defined in (11) under the constraints (9). As in [22] such constraints are non-holonomic and thus a source of difficulties. We will follow a similar approach to the one in [22] and perform a Legendre transform of the former variational problem within the multisymplectic framework.

3 Towards a multisymplectic formulation

3.1 The canonical 1-form on $\mathfrak{p} \otimes T^*\mathcal{P}$

In order to facilitate the computation, we introduce the vector bundle $\mathfrak{p} \otimes T^*\mathcal{P}$ over $\mathcal{P}$, whose fiber at point $\mathsf{z} \in \mathcal{P}$ is the tensor product $\mathfrak{p} \otimes T^*_{\mathsf{z}}\mathcal{P}$ and can be canonically identified with the space of linear maps from $T_{\mathsf{z}}\mathcal{P}$ to the Poincaré Lie algebra $\mathfrak{p}$. A point in $\mathfrak{p} \otimes T^*\mathcal{P}$ will be denoted by (z, y), where $\mathsf{z} \in \mathcal{P}$ and $\mathsf{y} \in \mathfrak{p} \otimes T^*_{\mathsf{z}}\mathcal{P}$. This bundle is equipped with the canonical $\mathfrak{p}$-valued 1-form η (a section of $\mathfrak{p} \otimes T^*(\mathfrak{p} \otimes T^*\mathcal{P})$) defined by

$$\forall (\mathsf{z}, \mathsf{y}) \in \mathfrak{p} \otimes T^*\mathcal{P}, \forall v \in T_{(\mathsf{z},\mathsf{y})}(\mathfrak{p} \otimes T^*\mathcal{P}), \quad \eta_{(\mathsf{z},\mathsf{y})}(v) = \mathsf{y}(d\pi_{(\mathsf{z},\mathsf{y})}(v)),$$

where $\pi = \pi_{\mathfrak{p} \otimes T^*\mathcal{P}} : \mathfrak{p} \otimes T^*\mathcal{P} \longrightarrow \mathcal{P}$ is the canonical projection map. This $\mathfrak{p}$-valued 1-form can be decomposed as $\eta = \mathfrak{l}_A \eta^A$, where each η^A is a 1-form on $\mathcal{P}$.

We introduce the following coordinates on $\mathfrak{p} \otimes T^*\mathcal{P}$:

- $(z^I)_{1 \leq I \leq 10}$ are local coordinates on $\mathcal{P}$; thus they provide us with locally defined functions $z^I \circ \pi$ on $\mathfrak{p} \otimes T^*\mathcal{P}$. In the following we write abusively $z^I \simeq z^I \circ \pi$.

- for any $\mathsf{z} \in \mathcal{P}$, we can define the coordinates $(\eta_I^A)_{0\le A\le 9; 1\le I\le 10}$ on the space $\mathfrak{p}\otimes T^*_{\mathsf{z}}\mathcal{P}$ in the basis $(\mathfrak{l}_A \otimes dz^I)_{0\le A\le 9; 1\le I\le 10}$.

Hence $\mathfrak{p}\otimes T^*\mathcal{P}$ is endowed with local coordinates (z^I, η_I^A). In these coordinates η reads

$$\eta = \mathfrak{l}_A \eta_I^A dz^I.$$

We may split $\eta = \overset{\text{o}}{\eta} + \overset{\text{1}}{\eta}$, according to the decomposition $\mathfrak{p} = \mathfrak{g}\oplus\mathfrak{t}$. Note that $\overset{\text{1}}{\eta} = \eta^a \mathfrak{l}_a = \eta^a t_a$, where $0\le a\le 3$, and $\overset{\text{o}}{\eta} = \eta^i \mathfrak{l}_i = \eta^i u_i$, where $4\le i\le 9$. We also set $\overset{\text{o}}{\eta}{}^a{}_b = u^a_{ib}\overset{\text{o}}{\eta}{}^i$. Any pair (α,ω) as considered in the previous section is a section of $\mathfrak{p}\otimes T^*\mathcal{P}$ over $\mathcal{P}$. In the following we identify such a pair with a map φ from $\mathcal{P}$ to the total space of $\mathfrak{p}\otimes T^*\mathcal{P}$ (a manifold of dimension 110) such that $\pi\circ\varphi(\mathsf{z}) = \mathsf{z}$, $\forall \mathsf{z}\in\mathcal{P}$, by letting

$$(\alpha,\omega) = \varphi^*\eta \quad \text{or} \quad \alpha = \varphi^*\overset{\text{1}}{\eta} \text{ and } \omega = \varphi^*\overset{\text{o}}{\eta}. \tag{12}$$

As for $\overset{\text{1}}{\eta}$ and $\overset{\text{o}}{\eta}$ we denote by $(\omega^i)_{1\le i\le 6}$ the components of the decomposition $\omega = u_i\omega^i$ and we set $\omega^a{}_b = u^a_{ib}\omega^i$; similarly we write $(\alpha^a)_{0\le a\le 3}$ the components of α.

We now recast the action $\widehat{\mathcal{A}}_{EWC}$ as follows. We define the following 10-form on $\mathfrak{p}\otimes T^*\mathcal{P}$ (i.e. a section of $\Lambda^{10}T^*(\mathfrak{p}\otimes T^*\mathcal{P})$):

$$\mathcal{L} = u_i^{ab}\,\overset{\text{1}}{\eta}{}_{ab}{}^{(2)} \wedge (d\overset{\text{o}}{\eta} + \overset{\text{o}}{\eta}\wedge\overset{\text{o}}{\eta})^i \wedge \overset{\text{o}}{\eta}{}^{(6)}, \tag{13}$$

where $\overset{\text{1}}{\eta}{}_{ab}{}^{(2)} := \frac{1}{2}\epsilon_{abcd}\eta^c\wedge\eta^d$ and $\overset{\text{o}}{\eta}{}^{(6)} := \eta^4\wedge\cdots\wedge\eta^9$. Note that the definition of $\mathcal{L}$ does not require a fibration on $\mathcal{P}$ over some manifold $\mathcal{X}$: it is canonically defined on any manifold of the form $\mathfrak{p}\otimes T^*\mathcal{P}$, where $\mathcal{P}$ is any 10-dimensional manifold. We can now give another expression for the action (11):

$$\widehat{\mathcal{A}}_{EWC}[\alpha,\omega] = \int_{\mathcal{P}}\varphi^*\mathcal{L}, \tag{14}$$

where φ is such that (12) holds.

The constraints (9) then translate as the following conditions on φ:

$$\exists Q^a{}_{cd}\in\mathcal{C}^\infty(\mathcal{P}), \qquad (d\alpha+\omega\wedge\alpha)^a = \frac{1}{2}Q^a{}_{cd}\alpha^c\wedge\alpha^d. \tag{15}$$

$$\exists Q^a{}_{bcd}\in\mathcal{C}^\infty(\mathcal{P}), \quad (d\omega+\omega\wedge\omega)^a{}_b = \frac{1}{2}Q^a{}_{bcd}\alpha^c\wedge\alpha^d, \tag{16}$$

Conditions (15) and (16) are equivalent to

$$\exists Q^A{}_{cd}\in\mathcal{C}^\infty(\mathcal{P}), \quad \varphi^*(d\eta+\frac{1}{2}[\eta\wedge\eta])^A = \frac{1}{2}Q^A{}_{cd}\varphi^*(\eta^c\wedge\eta^d) \tag{17}$$

(compare with (18) below).

3.2 The Poincaré–Cartan form θ_{Tot}

Among the many possible multisymplectic manifolds, we need to choose a convenient one as a framework for the Legendre transform of our problem, i.e. a suitable submanifold of the manifold[2] $\Lambda^{10}T^*(\mathfrak{p}\otimes T^*\mathcal{P})$. Inspired by [22] we choose the total space of the fiber

[2]The $\left(110+\frac{110!}{100!10!}\right)$-dimensional universal Lepage–Dedecker manifold $\Lambda^{10}T^*(\mathfrak{p}\otimes T^*\mathcal{P})$ is far too big.

bundle over $\mathcal{P}$

$$\mathcal{M}_{Tot} := \mathbb{R} \oplus_{\mathcal{P}} \left(\mathfrak{p}^* \otimes \Lambda^8 T^*\mathcal{P}\right) \oplus_{\mathcal{P}} \left(\mathfrak{p} \otimes T^*\mathcal{P}\right).$$

We introduce the following coordinates on $\mathcal{M}_{Tot}$:

- we extend in a natural way the coordinates (z^I, η_I^A) on $\mathfrak{p} \otimes T^*\mathcal{P}$ to functions on $\mathcal{M}_{Tot}$.
- we let $(\mathfrak{l}^A)_{0\leq A\leq 9}$ be the basis of $\mathfrak{p}^*$ which is dual to $(\mathfrak{l}_A)_{0\leq A\leq 9}$; for any $\mathsf{z} \in \mathcal{P}$, let $dz^{(10)} := dz^1 \wedge \cdots \wedge dz^{10}$ and $dz^{(8)}_{IJ} := \frac{\partial}{\partial z^J} \lrcorner \frac{\partial}{\partial z^I} \lrcorner dz^{(10)}$. We define the coordinates $\psi^{IJ}_A = -\psi^{JI}_A$ on the space $\mathfrak{p}^* \otimes \Lambda^8 T^*_{\mathsf{z}}\mathcal{P}$ in the basis $(\mathfrak{l}^A \otimes dz^{(8)}_{IJ})_{0\leq A\leq 9; 1\leq I<J\leq 10}$. Then $\mathfrak{p}^* \otimes \Lambda^8 T^*\mathcal{P}$ is endowed with local coordinates (z^I, ψ^{IJ}_A).
- endow the real line $\mathbb{R}$ with the coordinate h.

Then a complete system of coordinates on $\mathcal{M}_{Tot}$ is $(z^I, h, \eta^A_I, \psi^{IJ}_A)$.

On $\mathfrak{p}^* \otimes \Lambda^8 T^*\mathcal{P}$ is also defined a canonical $\mathfrak{p}^*$-valued 8-form ψ defined by: $\forall(\mathsf{z}, \mathrm{M}) \in \mathfrak{p}^* \otimes \Lambda^8 T^*\mathcal{P}$,

$$\forall w_1, \cdots, w_8 \in T_{(\mathsf{z},\mathrm{M})}(\mathfrak{p}^* \otimes \Lambda^8 T^*\mathcal{P}), \quad \psi_{(\mathsf{z},\mathrm{M})}(w_1, \cdots, w_8) = \mathrm{M}(d\pi_{(\mathsf{z},\mathrm{M})}(w_1), \cdots, d\pi_{(\mathsf{z},\mathrm{M})}(w_8)),$$

where $\pi = \pi_{\mathfrak{p}^* \otimes \Lambda^8 T^*\mathcal{P}} : \mathfrak{p}^* \otimes \Lambda^8 T^*\mathcal{P} \longrightarrow \mathcal{P}$ is the canonical projection map. This $\mathfrak{p}^*$-valued 8-form decomposes as $\psi = \psi_A \mathfrak{l}^A$. In local coordinates (z^I, ψ^{IJ}_A) ψ reads

$$\psi = \frac{1}{2} \mathfrak{l}^A \psi^{IJ}_A dz^{(8)}_{IJ}.$$

We now define define the Poincaré–Cartan 10-form on $\mathcal{M}_{Tot}$

$$\theta_{Tot} := h\eta^{(10)} + \psi_A \wedge (d\eta + \frac{1}{2}[\eta \wedge \eta])^A,$$

where $\eta^{(10)} := \eta^1 \wedge \cdots \wedge \eta^{10}$ Alternatively,

$$\theta_{Tot} := h\eta^{(10)} + \psi_a \wedge (d\overset{1}{\eta} + \overset{0}{\eta} \wedge \overset{1}{\eta})^a + \psi_i \wedge (d\overset{0}{\eta} + \overset{0}{\eta} \wedge \overset{0}{\eta})^i.$$

3.3 The first jet bundle on $\mathfrak{p} \otimes T^*\mathcal{P}$

We now need to introduce the first jet bundle of the bundle $\mathfrak{p} \otimes T^*\mathcal{P}$ over $\mathcal{P}$, which plays a role analogue to the tangent bundle in Mechanics. Recall that a section φ of the fiber bundle $\mathfrak{p} \otimes T^*\mathcal{P}$ can be seen as a map $\varphi : \mathcal{P} \longrightarrow \mathfrak{p} \otimes T^*\mathcal{P}$ such that $\pi_{\mathfrak{p}\otimes T^*\mathcal{P}} \circ \varphi = \mathrm{Id}_{\mathcal{P}}$. Such a section is completely characterized by the functions $\eta^A_I \circ \varphi$. The jet space $J^1(\mathcal{P}, \mathfrak{p} \otimes T^*\mathcal{P})$ is the manifold of triplets $(\mathsf{z}, \mathsf{y}, \dot{\mathsf{y}})$, where $(\mathsf{z}, \mathsf{y}) \in \mathfrak{p} \otimes T^*\mathcal{P}$ and $\dot{\mathsf{y}}$ is the equivalence class of local sections φ of $\mathfrak{p} \otimes T^*\mathcal{P}$ over a neighborhood of z such that $\varphi(\mathsf{z}) = \mathsf{y}$, for the equivalence relation: $\varphi_1 \simeq \varphi_2$ iff $d(\eta^A_I \circ \varphi_1)_{\mathsf{z}} = d(\eta^A_I \circ \varphi_2)_{\mathsf{z}}$, $\forall I, A$. We then write $[\varphi]_{\mathsf{z},\mathsf{y}}$ the class of φ. Local coordinates on $J^1(\mathcal{P}, \mathfrak{p} \otimes T^*\mathcal{P})$ are $(z^I, \eta^A_I, \eta^A_{I;J})$, where

$$\eta^A_{I;J}(\dot{\mathsf{y}}) = \frac{\partial(\eta^A_I \circ \varphi)}{\partial z^J}(\mathsf{z}) \quad \text{where} \quad \dot{\mathsf{y}} = [\varphi]_{\mathsf{z},\mathsf{y}},$$

or alternatively

$$\eta^A_{I;J}(\dot{\mathsf{y}})dz^J = d(\eta^A_I \circ \varphi)_{\mathsf{z}} = (\varphi^* d\eta^A_I)_{\mathsf{z}}.$$

It will be however convenient to introduce the families of functions $(S^A_{IJ})_{0\leq A\leq 9;1\leq I,J\leq 10}$ and $(A^A{}_{BC})_{0\leq A,B,C\leq 9}$ on $J^1(\mathcal{P}, \mathfrak{p}\otimes T^*\mathcal{P})$, defined respectively by

$$S^A_{IJ}(\dot{\mathsf{y}}) = \frac{1}{2}\left(\eta^A_{J;I}(\dot{\mathsf{y}}) + \eta^A_{I;J}(\dot{\mathsf{y}})\right)$$

(note that $S^A_{JI} = S^A_{IJ}$) and, for $A^A{}_{BC}$, by the conditions $A^A{}_{BC} + A^A{}_{CB} = 0$ and:

$$\frac{1}{2}A^A{}_{BC}(\dot{\mathsf{y}})\varphi^*(\eta^B \wedge \eta^C)_{\mathsf{z}} = \varphi^*\left(d\eta^A + \frac{1}{2}\left[\eta\wedge\eta\right]^A\right)_{\mathsf{z}}.$$

We remark that

$$\eta^A_{I;J}(\dot{\mathsf{y}}) = S^A_{IJ}(\dot{\mathsf{y}}) - \frac{1}{4}A^A{}_{BC}(\dot{\mathsf{y}})\begin{vmatrix} \eta^B_I(\mathsf{y}) & \eta^B_J(\mathsf{y}) \\ \eta^C_I(\mathsf{y}) & \eta^C_J(\mathsf{y}) \end{vmatrix} + \frac{1}{2}[\eta_I(\mathsf{y}); \eta_J(\mathsf{y})]^A.$$

Hence a system of coordinates on $\mathfrak{p}\otimes T^*\mathcal{P}$ is:

$$(z^I)_{1\leq I\leq 10}, \quad (\eta^A_I)_{0\leq A\leq 9;1\leq I\leq 10}, \quad (S^A_{IJ})_{0\leq A\leq 9;1\leq I\leq J\leq 10} \quad \text{and} \quad (A^A{}_{BC})_{0\leq A\leq 9;0\leq B<C\leq 9}.$$

In fact, all relevant quantities (the constraints, the Lagrangian density and the Poincaré–Cartan form) depend only on z^I, η^A_I and $A^A{}_{BC}$ (and not on the S^A_{IJ}'s). Indeed for instance the pull-back of $d\eta + \frac{1}{2}[\eta\wedge\eta]$ by any section φ has the a priori decomposition (for comparison the analogous decomposition in Toller [36] reads $\varphi^* d\eta^A + \frac{1}{2}F^A_{BC}\varphi^*(\eta^B\wedge\eta^C) = 0$, where the coefficients F^A_{BC} play the role of generalized *structure constants*).

$$\varphi^*(d\eta + \frac{1}{2}[\eta\wedge\eta])^A = \frac{1}{2}A^A{}_{cd}\alpha^c\wedge\alpha^d + A^A{}_{ck}\alpha^c\wedge\omega^k + \frac{1}{2}A^A{}_{jk}\omega^j\wedge\omega^k,$$

so that (17) amounts to impose that

$$\exists Q^A{}_{cd}\in\mathcal{C}^\infty(\mathcal{P}), \quad A^A{}_{cd} = Q^A{}_{cd}(\mathsf{z}) \quad \text{and} \quad A^A{}_{ck} = A^A{}_{jk} = 0, \quad \forall A,c,d,j,k. \tag{18}$$

3.4 The Legendre transform

Let $(\mathsf{z},\mathsf{y},\dot{\mathsf{y}})\in J^1(\mathcal{P},\mathfrak{p}\otimes T^*\mathcal{P})$ and let φ be a section such that $[\varphi]_{\mathsf{z},\mathsf{y}} = \dot{\mathsf{y}}$. In order to compute the Legendre transform at $(\mathsf{z},\mathsf{y},\dot{\mathsf{y}},h,p)$ we need to evaluate $\varphi^*(\theta_{Tot}-\mathcal{L})$ and to determine the value of the quantity $W(\mathsf{z},\mathsf{y},\dot{\mathsf{y}},h,p)$ which is defined by $\varphi^*(\theta_{Tot}-\mathcal{L}) = W(\mathsf{z},\mathsf{y},\dot{\mathsf{y}},h,p)\varphi^*(\alpha^{(4)}\wedge\omega^{(6)})$ (see [26] for details).

3.4.1 Computation of $\varphi^*\theta_{Tot}$

We decompose[3]

$$\psi_a = \frac{1}{2}\psi_a{}^{cd}\alpha^{(2)}_{cd} \wedge \omega^{(6)} - \psi_a{}^{ck}\alpha^{(3)}_c \wedge \omega^{(5)}_k + \frac{1}{2}\psi_a{}^{jk}\alpha^{(4)} \wedge \omega^{(4)}_{jk} \tag{19}$$

$$\psi_i = \frac{1}{2}\psi_i{}^{cd}\alpha^{(2)}_{cd} \wedge \omega^{(6)} - \psi_i{}^{ck}\alpha^{(3)}_c \wedge \omega^{(5)}_k + \frac{1}{2}\psi_i{}^{jk}\alpha^{(4)} \wedge \omega^{(4)}_{jk} \tag{20}$$

Moreover the pull-back of θ_{Tot} by a section $\varphi : \mathcal{P} \longrightarrow \mathfrak{p} \otimes T^*\mathcal{P}$ reads

$$\varphi^*\theta_{Tot} = (h \circ \varphi)\varphi^*\eta^{(10)} + (\varphi^*\psi_a) \wedge (d\alpha + \omega \wedge \alpha)^a + (\varphi^*\psi_i) \wedge (d\omega + \omega \wedge \omega)^i.$$

Hence, in view of the constraints (15) and (16) and of Lemma 8.2, this gives us

$$\varphi^*\theta_{Tot} = \left[(h \circ \varphi) + \frac{1}{2}(\psi_a{}^{cd} \circ \varphi)Q^a{}_{cd} + \frac{1}{2}(\psi_i{}^{cd} \circ \varphi)Q^i{}_{cd}\right] \varphi^*\eta^{(10)},$$

for some functions $Q^a{}_{cd}$ and $Q^i{}_{cd}$ which depends on φ.

3.4.2 Computation of $\varphi^*\mathcal{L}$

Using Formula (13) for $\mathcal{L}$ and the constraints (15) and (16) we find that

$$\varphi^*\mathcal{L} = \left(u^{ab}_i\alpha^{(2)}_{ab} \wedge \frac{1}{2}Q^i{}_{cd}\alpha^c \wedge \alpha^d\right) \wedge \omega^{(6)} = u^{ab}_i Q^i{}_{ab}\varphi^*\eta^{(10)}.$$

Hence

$$\varphi^*\left(\theta_{Tot} - \mathcal{L}\right) = \left[(h \circ \varphi) + \frac{1}{2}(\psi_a{}^{cd} \circ \varphi)Q^a{}_{cd} + \left(\frac{1}{2}\psi_i{}^{cd} \circ \varphi - u^{cd}_i\right) Q^i{}_{cd}\right] \varphi^*\eta^{(10)}.$$

Note that this form takes into account the constraints imposed on $\dot{\mathsf{y}}$.

3.4.3 Conclusion: the Legendre transform

From the following we deduce that

$$W(\mathsf{z}, \mathsf{y}, \dot{\mathsf{y}}, h, p) = (h \circ \varphi) + \frac{1}{2}(\psi_a{}^{cd} \circ \varphi)A^a{}_{cd} + \left(\frac{1}{2}\psi_i{}^{cd} \circ \varphi - u^{cd}_i\right) A^i{}_{cd}. \tag{21}$$

The Legendre correspondence holds on the points with coordinates $(h, \mathsf{z}, \mathsf{y}, \dot{\mathsf{y}}, \psi)$ which are critical points of W with respect to infinitesimal variations of $\dot{\mathsf{y}}$ which respect the constraints, i.e., such that

$$\frac{\partial W}{\partial A^A{}_{bc}} = 0 \quad \text{and} \quad \frac{\partial W}{\partial S^A_{IJ}} = 0.$$

[3]Beware that sign conventions below are different from [22].

The second relation is trivially satisfied and the first one is equivalent to:

$$\psi_a{}^{cd} \circ \varphi = 0 \quad \text{and} \quad \psi_i{}^{cd} \circ \varphi = 2u_i^{cd}. \tag{22}$$

The value of the Hamiltonian function is then the restriction of W at the points where (22) holds, i.e. simply:

$$H(\mathsf{z}, \mathsf{y}, h, p) = h. \tag{23}$$

Our final multisymplectic manifold will be the submanifold $\mathcal{M}$ of $\mathcal{M}_{Tot}$ which is the intersection of the image of the Legendre correspondence —precisely defined by the constraints (22)— with the hypersurface $h = 0$. By denoting θ the restriction of θ_{Tot} to $\mathcal{M}$:

$$\begin{aligned} \theta \;=\; & \left(-\psi_a{}^{ck} \overset{1}{\eta}{}_c^{(3)} \wedge \overset{0}{\eta}{}_k^{(5)} + \frac{1}{2}\psi_a{}^{jk} \overset{1}{\eta}{}^{(4)} \wedge \overset{0}{\eta}{}_{jk}^{(4)}\right) \wedge (d\overset{1}{\eta} + \overset{0}{\eta} \wedge \overset{1}{\eta})^a \\ & + \left(u_i^{cd} \overset{1}{\eta}{}_{cd}^{(2)} \wedge \overset{0}{\eta}{}^{(6)} - \psi_i{}^{ck} \overset{1}{\eta}{}_c^{(3)} \wedge \overset{0}{\eta}{}_k^{(5)} + \frac{1}{2}\psi_i{}^{jk} \overset{1}{\eta}{}^{(4)} \wedge \overset{0}{\eta}{}_{jk}^{(4)}\right) \wedge (d\overset{0}{\eta} + \overset{0}{\eta} \wedge \overset{0}{\eta})^i \end{aligned} \tag{24}$$

Note that taking into account that η and ψ are respectively $\mathfrak{p}$- and $\mathfrak{p}^*$-valued, our Poincaré–Cartan form has the simple structure:

$$\theta := \psi \wedge (d\eta + \frac{1}{2}[\eta \wedge \eta]), \tag{25}$$

where the duality pairing between coefficients of ψ and η is implicitely assumed.

4 The Hamilton equations

Let κ_A^{cd} be defined for $A = a$ and $A = i$ by:

$$\kappa_a^{cd} := 0 \quad \text{and} \quad \kappa_i^{cd} := 2u_i^{cd}.$$

We can summarize the previous computation as follows: we work in the manifold $\mathcal{M}$ which can be identified with the submanifold of $(\mathfrak{p}^* \otimes \Lambda^8 T^*\mathcal{P}) \oplus_{\mathcal{P}} (\mathfrak{p} \otimes T^*\mathcal{P})$ defined by the equations

$$\psi_A{}^{cd} = \kappa_A^{cd}. \tag{26}$$

or equivalentely by

$$\overset{1}{\eta}{}^c \wedge \overset{1}{\eta}{}^d \wedge \psi_A = \kappa_A^{cd} \eta^{(10)}, \quad \forall A, c, d, \tag{27}$$

The manifold $\mathcal{M}$ will be our multisymplectic phase space: it is endowed with the pre-multisymplectic 11-form $d\theta$. Solutions of the Hamilton equations can be described as being 10-dimensional oriented submanifolds Γ of $\mathcal{M}$ which satisfy the *independence condition*

$$\eta^{(10)}|_\Gamma \neq 0 \tag{28}$$

and the Hamilton–Volterra–De Donder–Weyl (HVDW) equations

$$\forall \mathrm{M} \in \Gamma, \forall \xi \in T_{\mathrm{M}}\mathcal{M}, \quad (\xi \lrcorner d\theta)|_{T_{\mathrm{M}}\Gamma} = 0. \tag{29}$$

4.1 The solutions as critical points of an action functional

In order to determine Equation (29) we will use the fact that it is also the Euler–Lagrange equations satisfied by the critical points of the functional $\mathcal{A}[\Gamma] := \int_\Gamma \theta$. For that purpose we will compute the first variation of this action in $(\mathfrak{p}^* \otimes \Lambda^8 T^*\mathcal{P}) \oplus_\mathcal{P} (\mathfrak{p} \otimes T^*\mathcal{P})$ and write under which condition on a submanifold Γ this first variation of $\mathcal{A}$ vanishes for all variations of Γ which respect (27).

First because of the independence condition (28) we can always assume that, locally, Γ is a graph over $\mathcal{P}$ or, in other words, the image of a section φ of the bundle $(\mathfrak{p}^* \otimes \Lambda^8 T^*\mathcal{P}) \oplus_\mathcal{P} (\mathfrak{p} \otimes T^*\mathcal{P})$ over $\mathcal{P}$. Thus we can write $\mathcal{A}[\Gamma] = \int_\mathcal{P} \varphi^*\theta$ and we can coordinatize an infinitesimal variation of Γ by maps on $\mathcal{P}$ $\delta\eta$ and $\delta\psi$ with compact supports. The first variation of $\mathcal{A}$ can then be written:

$$\delta\mathcal{A}_\Gamma(\delta\eta, \delta\psi) = \int_\mathcal{P} \delta\psi_A \wedge \varphi^* \left(d\eta^A + \frac{1}{2}[\eta \wedge \eta]^A \right) + (\varphi^*\psi_A) \wedge \left(d(\delta\eta^A) + [\delta\eta \wedge \varphi^*\eta]^A \right).$$

We note that

$$(\varphi^*\psi_A) \wedge d(\delta\eta^A) = d(\delta\eta^A \wedge \varphi^*\psi_A) + \delta\eta^A \wedge \varphi^* d\psi_A$$

and

$$(\varphi^*\psi_A) \wedge [\delta\eta \wedge \varphi^*\eta]^A = (\varphi^*\psi_A) \wedge c^A_{BC} \delta\eta^B \wedge \varphi^*\eta^C = -\delta\eta^B \wedge \varphi^*(\mathrm{ad}^*_\eta \wedge \psi)_B,$$

where $(\mathrm{ad}^*_\eta \wedge \psi)_B := c^A_{CB}\eta^C \wedge \psi_A$ (see (94)). Thus, assuming that $(\delta\eta, \delta\psi)$ has a compact support,

$$\delta\mathcal{A}_\Gamma(\delta\eta, \delta\psi) = \int_\mathcal{P} \delta\psi_A \wedge \varphi^* \left(d\eta^A + \frac{1}{2}[\eta \wedge \eta]^A \right) + \delta\eta^A \wedge \varphi^*(d\psi - \mathrm{ad}^*_\eta \wedge \psi)_A. \tag{30}$$

Solutions to the HVDW equations are the submanifolds Γ which satisfy the constraints (27) and which are such that $\delta\mathcal{A}_\Gamma(\delta\eta, \delta\psi)$ vanishes for any infinitesimal variations $(\delta\eta, \delta\psi) = (\delta\alpha, \delta\omega, \delta\psi)$ which respect this constraint, i.e. which satisfy

$$\delta\alpha^c \wedge \alpha^d \wedge \varphi^*\psi_A + \alpha^c \wedge \delta\alpha^d \wedge \varphi^*\psi_A + \alpha^c \wedge \alpha^d \wedge \delta\psi_A = \kappa^{cd}_A \delta\eta^B \wedge \varphi^*\eta^{(9)}_B. \tag{31}$$

In other words the solutions are characterized by the fact that Condition (31) implies the following

$$\delta\psi_A \wedge \varphi^* \left(d\eta^A + \frac{1}{2}[\eta \wedge \eta]^A \right) + \delta\eta^A \wedge \varphi^*(d\psi - \mathrm{ad}^*_\eta \wedge \psi)_A = 0. \tag{32}$$

4.2 Parametrization of infinitesimal variations satisfying (31)

In the following we still note $(\alpha, \omega) := \varphi^*\eta$ and we set $\varpi := \varphi^*\psi$. Let us pose

$$\delta\eta^A = \lambda^A_a \alpha^a + \lambda^A_i \omega^i,$$

$$\delta\psi_A = \frac{1}{2}\chi_A{}^{cd}\alpha^{(2)}_{cd} \wedge \omega^{(6)} - \chi_A{}^{ck}\alpha^{(3)}_c \wedge \omega^{(5)}_k + \frac{1}{2}\chi_A{}^{jk}\alpha^{(4)} \wedge \omega^{(4)}_{jk}$$

where $\lambda^A_C, \chi_A{}^{CD}$ are smooth function with compact support on $\mathcal{P}$, and

$$\begin{aligned}\varpi_A := \varphi^*\psi_A &= \frac{1}{2}\kappa_A^{cd}\alpha^{(2)}_{cd}\wedge\omega^{(6)} - (\psi_A{}^{ck}\circ\varphi)\alpha^{(3)}_c\wedge\omega^{(5)}_k + \frac{1}{2}(\psi_A{}^{jk}\circ\varphi)\alpha^{(4)}\wedge\omega^{(4)}_{jk}\\ &= \frac{1}{2}\kappa_A^{cd}\alpha^{(2)}_{cd}\wedge\omega^{(6)} - \varpi_A{}^{ck}\alpha^{(3)}_c\wedge\omega^{(5)}_k + \frac{1}{2}\varpi_A{}^{jk}\alpha^{(4)}\wedge\omega^{(4)}_{jk}\end{aligned} \tag{33}$$

Thus we may write (31) as:

$$(\lambda^c_{c'}\kappa_A^{c'd} - \lambda^c_k\varpi_A{}^{dk}) + (\lambda^d_{d'}\kappa_A^{cd'} + \lambda^d_k\varpi_A{}^{ck}) + \chi_A{}^{cd} = (\lambda^b_b + \lambda^i_i)\kappa_A^{cd}.$$

Hence we can express $\chi_A{}^{cd}$ in terms of the other quantities:

$$\chi_A{}^{cd} = \lambda^b_{b'}(\delta^{b'}_b\kappa_A^{cd} - \delta^c_b\kappa_A^{b'd} - \delta^d_b\kappa_A^{cb'}) + \lambda^b_k(\delta^c_b\varpi_A{}^{dk} - \delta^d_b\varpi_A{}^{ck}) + \lambda^i_i\kappa_A^{cd}.$$

Thus (31) means that we can express $\delta\psi$ in terms of λ^A_B, $\chi_A{}^{ck}$ and $\chi_A{}^{jk}$:

$$\begin{aligned}\delta\psi_A &= \tfrac{1}{2}\Big[\lambda^b_{b'}\Big(\delta^{b'}_b\kappa_A^{cd}\alpha^{(2)}_{cd} - \kappa_A^{b'd}\alpha^{(2)}_{bd} - \kappa_A^{cb'}\alpha^{(2)}_{cb}\Big)\\ &\qquad + \lambda^b_k(\varpi_A{}^{dk}\alpha^{(2)}_{bd} - \varpi_A{}^{ck}\alpha^{(2)}_{cb}) + \lambda^i_i\kappa_A^{cd}\alpha^{(2)}_{cd}\Big]\wedge\omega^{(6)}\\ &\qquad - \chi_A{}^{ck}\alpha^{(3)}_c\wedge\omega^{(5)}_k + \tfrac{1}{2}\chi_A{}^{jk}\alpha^{(4)}\wedge\omega^{(4)}_{jk}\\ &= \Big[\lambda^b_{b'}\Big(\tfrac{1}{2}\delta^{b'}_b\kappa_A^{cd}\alpha^{(2)}_{cd} - \kappa_A^{b'd}\alpha^{(2)}_{bd}\Big) + \lambda^b_k\varpi_A{}^{dk}\alpha^{(2)}_{bd} + \tfrac{1}{2}\lambda^i_i\kappa_A^{cd}\alpha^{(2)}_{cd}\Big]\wedge\omega^{(6)}\\ &\qquad - \chi_A{}^{ck}\alpha^{(3)}_c\wedge\omega^{(5)}_k + \tfrac{1}{2}\chi_A{}^{jk}\alpha^{(4)}\wedge\omega^{(4)}_{jk}.\end{aligned}$$

4.3 The Euler–Lagrange equations

On the one hand, setting $\Omega := \varphi^*\left(d\eta + \frac{1}{2}[\eta\wedge\eta]\right)$, we can decompose

$$\Omega^A = \frac{1}{2}Q^A{}_{cd}\alpha^c\wedge\alpha^d + Q^A{}_{ck}\alpha^c\wedge\omega^k + \frac{1}{2}Q^A{}_{jk}\omega^j\wedge\omega^k, \tag{34}$$

so that, taking into account (31), the first term on the l.h.s. of (32) reads:

$$\begin{aligned}\delta\psi_A\wedge\Omega^A &= \Big[\lambda^b_{b'}\Big(\frac{1}{2}\delta^{b'}_b\kappa_A^{cd}Q^A{}_{cd} - \kappa_A^{b'd}Q^A{}_{bd}\Big) + \lambda^b_k\varpi_A{}^{dk}Q^A{}_{bd} + \frac{1}{2}\lambda^i_i\kappa_A^{cd}Q^A{}_{cd}\\ &\qquad + \chi_A{}^{ck}Q^A{}_{ck} + \chi_A{}^{jk}Q^A{}_{jk}\Big]\eta^{(10)}.\end{aligned}$$

On the other hand, setting $\nabla^\eta\varpi := \varphi^*(d\psi - \mathrm{ad}^*_\eta\wedge\psi)$ for short, and using the decomposition $(\nabla^\eta\varpi)_A = (\nabla^\eta\varpi)^a_A\alpha^{(3)}_a\wedge\omega^{(6)} + (\nabla^\eta\varpi)^i_A\alpha^{(4)}\wedge\omega^{(5)}_i$, the second term in the l.h.s. of (32) taking into account (31) reads

$$\delta\eta^A\wedge(\nabla^\eta\varpi)_A = \left(\lambda^A_a(\nabla^\eta\varpi)^a_A + \lambda^A_i(\nabla^\eta\varpi)^i_A\right)\eta^{(10)}.$$

In conclusion [(31) $\Longrightarrow$ (32)] is equivalent to the condition that

$$\begin{array}{rcccl}\lambda^b_{b'}\left(\frac{1}{2}\delta^{b'}_b\kappa_A^{cd}Q^A{}_{cd} - \kappa_A^{b'd}Q^A{}_{bd} + (\nabla^\eta\varpi)^{b'}_b\right) & + & \lambda^b_k\left(\varpi_A{}^{dk}Q^A{}_{bd} + (\nabla^\eta\varpi)^k_b\right) & & \\ +\lambda^j_{b'}(\nabla^\eta\varpi)^{b'}_j & + & \lambda^j_k\left(\frac{1}{2}\delta^k_j\kappa_A^{cd}Q^A{}_{cd} + (\nabla^\eta\varpi)^k_j\right) & & \\ +\chi_A{}^{ck}Q^A{}_{ck} & + & \chi_A{}^{jk}Q^A{}_{jk} & = & 0\end{array}$$

be satisfied for all $\lambda^b_{b'}, \lambda^b_k, \lambda^j_{b'}, \lambda^j_k, \chi_A{}^{ck}$ and $\chi_A{}^{jk}$.

Hence the HVDW equations or, equivalentely, the Euler–Lagrange equations of the action $\int_\Gamma \theta$, are

$$
\begin{aligned}
(\nabla^\eta \varpi)^{b'}_b &= \kappa_A^{b'd} Q^A{}_{bd} - \frac{1}{2}\delta^{b'}_b \kappa_A^{cd} Q^A{}_{cd} && (35)\\
(\nabla^\eta \varpi)^{k}_b &= -\varpi_A{}^{dk} Q^A{}_{bd} && (36)\\
(\nabla^\eta \varpi)^{b'}_j &= 0 && (37)\\
(\nabla^\eta \varpi)^{k}_j &= -\frac{1}{2}\delta^k_j \kappa_A^{cd} Q^A{}_{cd} && (38)\\
Q^A{}_{ck} &= 0 && (39)\\
Q^A{}_{jk} &= 0 && (40)
\end{aligned}
$$

5 Study of the solutions of the HVDW equations

The first four equations (35) to (38) can be translated into the following relations on $(\nabla^\eta \varpi)_A = (\nabla^\eta \varpi)^a_A \alpha^{(3)}_a \wedge \omega^{(4)} + (\nabla^\eta \varpi)^i_A \alpha^{(4)} \wedge \omega^{(5)}_i$ for $A = a$ or j:

$$
\left\{
\begin{array}{rcl}
(\nabla^\eta \varpi)_a &=& (\kappa_A^{bc} Q^A{}_{ac} - \mathrm{S}\delta^b_a)\alpha^{(3)}_b \wedge \omega^{(6)} \;-\; \varpi_A{}^{cj} Q^A{}_{ac} \alpha^{(4)} \wedge \omega^{(5)}_j \\
(\nabla^\eta \varpi)_j &=& \qquad\qquad\qquad\qquad\qquad\qquad -\quad \mathrm{S}\alpha^{(4)} \wedge \omega^{(5)}_j,
\end{array}
\right. \tag{41}
$$

where

$$\mathrm{S} := \frac{1}{2}\kappa_A^{cd} Q^A{}_{cd}.$$

Alternatively we can also introduce coefficients u^{ia}_b (see, in the Appendix, (89), (90) and (91)) and replace $(\nabla^\eta \varpi)_j$ by:

$$(\nabla^\eta \varpi)_a{}^b := (\nabla^\eta \varpi)_j u^{jb}_a.$$

(then $(\nabla^\eta \varpi)_j = \frac{1}{2}(\nabla^\eta \varpi)_a{}^b u^a_{jb}$). Then equations (41) are equivalent to

$$
\left\{
\begin{array}{rcl}
(\nabla^\eta \varpi)_a &=& (\kappa_A^{bc} Q^A{}_{ac} - S\delta^b_a)\alpha^{(3)}_b \wedge \omega^{(6)} \;-\; \varpi_A{}^{cj} Q^A{}_{ac} \alpha^{(4)} \wedge \omega^{(5)}_j \\
(\nabla^\eta \varpi)_a{}^b &=& \qquad\qquad\qquad\qquad\qquad\qquad -\quad \mathrm{S} u^{jb}_a \alpha^{(4)} \wedge \omega^{(5)}_j,
\end{array}
\right. \tag{42}
$$

On the other hand, by using (34), we see that Equations (39) and (40) are equivalent to:

$$\varphi^*(d\eta + \frac{1}{2}[\eta \wedge \eta])^A = \frac{1}{2} Q^A{}_{cd} \alpha^c \wedge \alpha^d, \tag{43}$$

or equivalentely

$$
\begin{aligned}
(d\alpha + \omega \wedge \alpha)^a &= \frac{1}{2} Q^a{}_{cd} \alpha^c \wedge \alpha^d && (44)\\
(d\omega + \omega \wedge \omega)^i &= \frac{1}{2} Q^i{}_{cd} \alpha^c \wedge \alpha^d && (45)
\end{aligned}
$$

In the following we first exploit Equations (44) and (45). Then we analyze the content of Equation (42).

5.1 The spontaneous fibration lemma

Lemma 5.1 *Let $\eta = (\alpha, \omega)$ be a 1-form defined on 10-dimensional manifold $\mathcal{P}$ with coefficients in $\mathfrak{p}$. Assume that the rank of η is maximal, equal to 10 everywhere and that there exist functions $Q^A{}_{bc}$ on $\mathcal{P}$ such that (44) and (45) are satisfied.*

Then, for any point m of $\mathcal{P}$, there exists a neighborhood $\mathcal{P}_\mathsf{m}$ of m on which there exist local coordinate functions $(x, g) = (x^0, x^1, x^2, x^3, g)$ with values in $\mathbb{R}^4 \times \mathfrak{G}$, such that

$$\alpha^a = (g^{-1})^a_{a'} e^{a'}, \quad \text{where } e^{a'} = e^{a'}_\mu(x) dx^\mu \tag{46}$$

and

$$\omega^a{}_b = (g^{-1})^a_{a'} A^{a'}_{b'} g^{b'}_b + (g^{-1})^a_{a'} dg^{a'}_b, \quad \text{where } A^{a'}_{b'} = A^{a'}_{b'\mu}(x) dx^\mu. \tag{47}$$

As a consequence the set $\mathcal{X}_\mathsf{m}$ of submanifolds of $\mathcal{P}_\mathsf{m}$ of equation $x =$ constant has a structure of 4-dimensional manifold and the quotient map $\pi := \mathcal{P}_\mathsf{m} \longrightarrow \mathcal{X}_\mathsf{m}$ is a local fibration. Moreover α and ω are the lifts on the total space of his local fibre bundle of respectively a solder form and a connection form of a pseudo-Riemannian structure on $\mathcal{X}_\mathsf{m}$.

Note that similar results were obtained in [36] in a different setting.
Proof — **Step 1** — Consider the Pfaffian system

$$\alpha^a|_\mathsf{f} = 0, \quad \forall a = 0, 1, 2, 3, \tag{48}$$

where the unknown f is a 6-dimensional submanifold of $\mathcal{P}$. Because of (44) we have:

$$d\alpha^a = \left(-\omega^a{}_b + \frac{1}{2} Q^a{}_{cb} \alpha^c \right) \wedge \alpha^b,$$

which means that the Pfaffian system (48) is integrable and satisfies the hypotheses of Frobenius' theorem. By applying this theorem we deduce that through any point $\mathsf{m} \in \mathcal{P}$ there exists a unique 6-dimensional submanifold f which is a solution of the system (48). This defines a fibration $\pi_\mathsf{m} : \mathcal{P}_\mathsf{m} \longrightarrow \mathcal{X}_\mathsf{m}$ of a neighborhood $\mathcal{P}_\mathsf{m}$ of m in $\mathcal{P}$ with values in a neighborhood $\mathcal{X}_\mathsf{m}$ of the space of leaves which are solutions of (48). We choose local coordinates $x^0 \cdots, x^3$ on $\mathcal{X}_\mathsf{m}$. Abusing notation we will set $x^\mu \simeq x^\mu \circ \pi_\mathsf{m}$. We also choose 6 extra local coordinate functions $y^1, \cdots y^6$ on a neighborhood of m (which we still call $\mathcal{P}_\mathsf{m}$) such that the submanifolds of equation $y^\mu =$ constant, $\forall \mu = 1, \cdots, 6$ are transverse to the leaves f. Hence we can assume without loss of generality that the 10 functions $x^0, \cdots, x^3, y^1, \cdots y^6$ form a system of local coordinates on $\mathcal{P}_\mathsf{m}$.
Step 2 — Let us denote by Σ the submanifold of equation $y^1 = \cdots = y^6 = 0$. We deduce from (34) and (45) that

$$d\omega^a{}_b + \omega^a{}_{a'} \wedge \omega^{a'}{}_b = \frac{1}{2} Q^a{}_{bcd} \alpha^c \wedge \alpha^d \tag{49}$$

and hence, in particular, by restriction to a leaf f:

$$\left(d\omega^a{}_b + \omega^a{}_{a'} \wedge \omega^{a'}{}_b \right) |_\mathsf{f} = 0. \tag{50}$$

This means that the Pfaffian system in $\mathsf{f} \times \mathfrak{G}$

$$(dg - g\omega)|_{\mathsf{f}} = 0 \tag{51}$$

is integrable and, in particular, there exists a unique solution which is equal to $1_{\mathfrak{G}}$ at the intersection point of f and Σ. We hence obtain a map $g : \mathcal{P}_{\mathsf{m}} \longrightarrow \mathfrak{G}$ which is equal to $1_{\mathfrak{G}}$ on Σ and which satisfies (51). Since the family $\left(\omega^a{}_b|_{\mathsf{f}}\right)_{0\leq a<b\leq 3}$ form a coframe on f, we deduce from (51) that the components $\gamma^4|_{\mathsf{f}}, \cdots, \gamma^9|_{\mathsf{f}}$ in a basis $\mathfrak{g}$ of the restriction of $\gamma := g^{-1}dg$ to f form also a coframe on f.

Step 3 —Relation (51) also means that $\omega - g^{-1}dg$ is a linear combination of the forms $\alpha^0, \cdots, \alpha^3$ or equivalentely of the forms $dx^0, \cdots, dx^3$. Thus there exist real valued functions $A^a_{\mu b}$ of x and g, for $0 \leq \mu \leq 3$, or, equivalentely, functions A_μ with values in $\mathfrak{g}$ such that

$$\omega = g^{-1}dg + g^{-1}A_\mu(x,g)gdx^\mu.$$

But then $d\omega + \omega \wedge \omega = g^{-1}(dA + A \wedge A)g$ and ω satisfies (49) iff A_μ does not depend on g, i.e.

$$\omega = g^{-1}dg + g^{-1}A_\mu(x)gdx^\mu \tag{52}$$

or (47). Similarly if we set $\alpha := g^{-1}e_\mu(x,g)dx^\mu$, we get $d\alpha + \omega \wedge \alpha = g^{-1}(de + A \wedge e)$. Hence the relation

$$d\alpha^a + \omega^a{}_b \wedge \alpha^b = \frac{1}{2}Q^a{}_{bc}\alpha^b \wedge \alpha^c$$

implies that e_μ does not depend on g, thus (46) follows. □

5.2 Change of unknown functions

To summarize the result of the previous section we can build local coordinate (x, g), where $x \in \mathbb{R}^4$ and $g \in \mathfrak{G}$ and we can write

$$\alpha^a = (g^{-1})^a_{a'}e^{a'} \quad \text{and} \quad \omega^a{}_b = (g^{-1})^a_{a'}dg^{a'}_b + (g^{-1})^a_{a'}A^{a'}{}_{b'}g^{b'}_b, \tag{53}$$

where e^a and $A^a{}_b$ are 1-forms which depends *only* on the x variables. Equivalentely,

$$(\alpha, \omega) = (0, g^{-1}dg) + \mathrm{Ad}_{g^{-1}}H,$$

where $H = (e, A)$ is a $\mathfrak{p}$-valued 1-form whose coefficients depend only on the x variables. For analyzing Equations (42) it will be useful to express them using coordinates (x, g) and functions adapted to these coordinates.

5.2.1 Replacing the 8-forms ϖ

We replace the 8-forms ϖ defined in Section 4.2 by

$$p := \mathrm{Ad}^*_{g^{-1}}\varpi \tag{54}$$

and we set:

$$\nabla^H p := dp - \mathrm{ad}^*_H \wedge p. \tag{55}$$

By using (93) in the Appendix this definition reads

$$\begin{aligned} (\nabla^H p)_a &= dp_a - p_b \wedge A^b{}_a & (56)\\ (\nabla^H p)_a{}^b &= dp_a{}^b + A^b{}_c \wedge p_a{}^c - p_c{}^b \wedge A^c{}_a + 2p_a \wedge e^b. & (57)\end{aligned}$$

Recall (Section 4.3) that $\nabla^\eta \varpi = \varphi^*(d\psi - \mathrm{ad}^*_\eta \wedge \psi) = d\varpi - \mathrm{ad}^*_{(\alpha,\omega)} \wedge \varpi$. It follows from (101) that

$$\nabla^H p = \mathrm{Ad}^*_{g^{-1}}(\nabla^\eta \varpi). \tag{58}$$

This means that $(\nabla^H p)_a = (g^{-1})^{a'}_a (\nabla^\eta \varpi)_{a'}$ and $(\nabla^H p)_a{}^b = (g^{-1})^{a'}_a g^b_{b'} (\nabla^\eta \varpi)_{a'}{}^{b'}$. Hence (42) translates as

$$\left\{ \begin{array}{lcll} (\nabla^H p)_a &=& (\kappa^{bc}_A (g^{-1})^{a'}_a Q^A{}_{a'c} - \mathrm{S}(g^{-1})^b_a)\alpha^{(3)}_b \wedge \omega^{(6)} & - \quad \varpi_A{}^{cj} (g^{-1})^{a'}_a Q^A{}_{a'c} \alpha^{(4)} \wedge \omega^{(5)}_j \\ (\nabla^H p)_a{}^b &=& & - \quad \mathrm{S}(g^{-1})^{a'}_a g^b_{b'} u^{jb'}_{a'} \alpha^{(4)} \wedge \omega^{(5)}_j \end{array} \right. \tag{59}$$

5.2.2 Replacing coefficients $Q^A{}_{cd}$

Let us define the tensors $\mathrm{T}^a{}_{cd}$ (torsion) and $R^a{}_{bcd}$ (Riemann curvature) such that

$$(de + A \wedge e)^a = \frac{1}{2} \mathrm{T}^a{}_{cd} e^c \wedge e^d \quad \text{and} \quad (dA + A \wedge A)^a{}_b := \frac{1}{2} R^a{}_{bcd} e^c \wedge e^d, \tag{60}$$

which clearly depend only on x (and not on g). Using (53) we compute that $(d\alpha + \omega \wedge \alpha)^a = (g^{-1})^a_{a'} (de + A \wedge e)^{a'}$ and $(d\omega + \omega \wedge \omega)^a{}_b = (g^{-1})^a_{a'} g^{b'}_b (dA + A \wedge A)^{a'}{}_{b'}$. Hence, by using (44) and (53), we find that

$$\begin{aligned} \frac{1}{2} Q^a{}_{cd} \alpha^c \wedge \alpha^d &= \frac{1}{2} (g^{-1})^a_{a'} \mathrm{T}^{a'}{}_{c'd'} e^{c'} \wedge e^{d'} = \frac{1}{2} (g^{-1})^a_{a'} \mathrm{T}^{a'}{}_{c'd'} g^{c'}_c g^{d'}_d \alpha^c \wedge \alpha^d \\ \frac{1}{2} Q^a{}_{bcd} \alpha^c \wedge \alpha^d &= \frac{1}{2} (g^{-1})^a_{a'} g^{b'}_b R^{a'}{}_{b'c'd'} e^{c'} \wedge e^{d'} = \frac{1}{2} (g^{-1})^a_{a'} g^{b'}_b R^{a'}{}_{b'c'd'} g^{c'}_c g^{d'}_d \alpha^c \wedge \alpha^d. \end{aligned}$$

Thus

$$Q^a{}_{cd} = (g^{-1})^a_{a'} g^{c'}_c g^{d'}_d \mathrm{T}^{a'}{}_{c'd'}, \tag{61}$$

$$Q^a{}_{bcd} = (g^{-1})^a_{a'} g^{b'}_b g^{c'}_c g^{d'}_d R^{a'}{}_{b'c'd'}. \tag{62}$$

Now consider the following term, which appears in the r.h.s. of (59):

$$\kappa^{bc}_A (g^{-1})^{a'}_a Q^A{}_{a'c} = 2u^{bc}_i (g^{-1})^{a'}_a Q^i{}_{a'c} = 2u^b_{ic'} \mathsf{h}^{c'c} (g^{-1})^{a'}_a Q^i{}_{a'c} = 2\mathsf{h}^{c'c} (g^{-1})^{a'}_a Q^b{}_{c'a'c},$$

it follows from (62) that

$$\kappa^{bc}_A (g^{-1})^{a'}_a Q^A{}_{a'c} = 2\mathsf{h}^{c'c} (g^{-1})^{a'}_a (g^{-1})^b_{b'} g^{c''}_{c'} g^{a''}_{a'} g^d_c R^{b'}{}_{c''a''d} = 2\mathsf{h}^{c''d} (g^{-1})^b_{b'} R^{b'}{}_{c''ad},$$

where we used $\mathsf{h}^{c'c}g^{c''}_{c'}g^d_c = \mathsf{h}^{c''d}$. Thus, by posing $R^{ab}{}_{cd} := \mathsf{h}^{bb'}R^a{}_{b'cd}$, we obtain that

$$\kappa_A^{bc}(g^{-1})^{a'}_a Q^A{}_{a'c} = 2(g^{-1})^b_{b'}R^{b'd}{}_{ad}.$$

We recognize the Ricci tensor: set $\mathrm{Ric}^b{}_a := R^{bd}{}_{ad}$, then the previous relation reads

$$\kappa_A^{bc}(g^{-1})^{a'}_a Q^A{}_{a'c} = 2(g^{-1})^b_{b'}\mathrm{Ric}^{b'}{}_a. \tag{63}$$

We can also express the quantity $\mathrm{S} = \frac{1}{2}\kappa_A^{ac}Q^A{}_{ac}$: (63) is equivalent to $\kappa_A^{bc}Q^A{}_{ac} = 2g^{a'}_a(g^{-1})^b_{b'}\mathrm{Ric}^{b'}{}_{a'}$ hence

$$\mathrm{S} = \mathrm{Ric}^a{}_a, \tag{64}$$

which is nothing but the scalar curvature. Lastly using again (62) and (61) we have

$$\begin{array}{rcl} \varpi_A{}^{cj}(g^{-1})^{a'}_a Q^A{}_{a'c} & = & (g^{-1})^{a'}_a\left(\frac{1}{2}\varpi_d{}^{bcj}Q^d{}_{ba'c} + \varpi_d{}^{cj}Q^d{}_{a'c}\right) \\ & = & (g^{-1})^d_{d'}g^{c'}_c\left(\frac{1}{2}\varpi_d{}^{bcj}g^{b'}_b R^{d'}{}_{b'ac'} + \varpi_d{}^{cj}\mathrm{T}^{d'}{}_{ac'}\right)\end{array}$$

Using (63) and the previous relation we transform the first equation of (59) into

$$\begin{array}{rcl}(\nabla^H p)_a & = & (2(g^{-1})^b_{b'}\mathrm{Ric}^{b'}{}_a - \mathrm{S}(g^{-1})^b_a)\alpha^{(3)}_b \wedge \omega^{(6)} \\ & & -(g^{-1})^d_{d'}g^{c'}_c\left(\frac{1}{2}\varpi_d{}^{bcj}g^{b'}_b R^{d'}{}_{b'ac'} + \varpi_d{}^{cj}\mathrm{T}^{d'}{}_{ac'}\right)\alpha^{(4)}\wedge\omega^{(5)}_j\end{array}$$

Thus introducing the *Einstein tensor*

$$\mathrm{E}^b{}_a := \mathrm{Ric}^b{}_a - \frac{1}{2}\mathrm{S}\delta^b_a$$

and observing that $(g^{-1})^{a'}_a g^b_{b'}u^{jb'}_{a'} = (\mathrm{Ad}^*_{g^{-1}}u^j)_a{}^b$ (see (92)) we can write (59) as:

$$\left\{\begin{array}{rcl}(\nabla^H p)_a & = & 2(g^{-1})^b_{b'}\mathrm{E}^{b'}{}_a\alpha^{(3)}_b\wedge\omega^{(6)} \\ & & -(g^{-1})^d_{d'}g^{c'}_c\left(\frac{1}{2}\varpi_d{}^{bcj}g^{b'}_b R^{d'}{}_{b'ac'} + \varpi_d{}^{cj}\mathrm{T}^{d'}{}_{ac'}\right)\alpha^{(4)}\wedge\omega^{(5)}_j \\ (\nabla^H p)_a{}^b & = & -\mathrm{S}(\mathrm{Ad}^*_{g^{-1}}u^j)_a{}^b\alpha^{(4)}\wedge\omega^{(5)}_j\end{array}\right. \tag{65}$$

5.2.3 Replacing the forms (α, ω)

The previous equations give the decomposition of the 9-form $\nabla^H p$ in the basis $(\alpha^{(3)}_a \wedge \omega^{(6)}, \alpha^{(4)}\wedge\omega^{(5)}_i)$. Let e^a be the forms defined by (53) and let $\gamma = \gamma^i u_i := g^{-1}dg$. We want to use the coframe $(e^0, \cdots, e^3, \gamma^4, \cdots, \gamma^9)$ and to replace $\alpha^{(3)}_a\wedge\omega^{(6)} = \frac{\partial}{\partial\alpha^a}\lrcorner\,\alpha^{(4)}\wedge\omega^{(6)}$ and $\alpha^{(4)}\wedge\omega^{(5)}_i = \frac{\partial}{\partial\omega^i}\lrcorner\,\alpha^{(4)}\wedge\omega^{(6)}$ in terms of $e^{(3)}_a\wedge\gamma^{(6)} := \frac{\partial}{\partial e^a}\lrcorner\, e^{(4)}\wedge\gamma^{(6)}$ and $e^{(4)}\wedge\gamma^{(5)}_i := \frac{\partial}{\partial\gamma^i}\lrcorner\, e^{(4)}\wedge\gamma^{(6)}$ (see Section 1.2 for the notations). For that it suffices to note that $e^{(4)}\wedge\gamma^{(6)} = \eta^{(10)} = \alpha^{(4)}\wedge\omega^{(6)}$ (because in particular $\omega = \gamma + Ad_{g^{-1}}A$) and to use the relations

$$\left\{\begin{array}{rcl}\frac{\partial}{\partial\alpha^a} & = & g^{a'}_a\left(\frac{\partial}{\partial e^{a'}} - (\mathrm{Ad}_{g^{-1}}A_{a'})^i\frac{\partial}{\partial\gamma^i}\right), \\ \frac{\partial}{\partial\omega^i} & = & \frac{\partial}{\partial\gamma^i}\end{array}\right. \tag{66}$$

where $(\mathrm{Ad}_{g^{-1}}A_{a'})^i := u^i(\mathrm{Ad}_{g^{-1}}A_{a'})$. Hence

$$\left\{\begin{array}{rcl} \alpha_a^{(3)} \wedge \omega^{(6)} = \frac{\partial}{\partial \alpha^a} \lrcorner \eta^{(10)} & = & g_a^{a'} \left(e_{a'}^{(3)} \wedge \gamma^{(6)} - (\mathrm{Ad}_{g^{-1}}A_{a'})^i e^{(4)} \wedge \gamma_i^{(5)} \right) \\ \alpha^{(4)} \wedge \omega_i^{(5)} = \frac{\partial}{\partial \omega^i} \lrcorner \eta^{(10)} & = & e^{(4)} \wedge \gamma_i^{(5)}. \end{array}\right.$$

Thus substituting these expressions in the r.h.s. of (65) we obtain

$$\left\{\begin{array}{rcl} (\nabla^H p)_a & = & 2\mathrm{E}^b{}_a e_b^{(3)} \wedge \gamma^{(6)} \qquad - 2\mathrm{E}^b{}_a (\mathrm{Ad}_{g^{-1}}A_b)^j e^{(4)} \wedge \gamma_j^{(5)} \\ & & -(g^{-1})^d_{d'} g_c^{c'} \left(\frac{1}{2} \varpi_d{}^{bcj} g_b^{b'} R^{d'}{}_{b'ac'} + \varpi_d{}^{cj} \mathrm{T}^{d'}{}_{ac'} \right) e^{(4)} \wedge \gamma_j^{(5)} \\ (\nabla^H p)_a{}^b & = & -\mathrm{S}(\mathrm{Ad}^*_{g^{-1}} u^j)_a{}^b e^{(4)} \wedge \gamma_j^{(5)} \end{array}\right. \tag{67}$$

5.2.4 Replacing all the components of ϖ

We need to go further and also to compute

$$\begin{array}{rcl} \alpha_{cd}^{(2)} \wedge \omega^{(6)} & = & \frac{\partial}{\partial \alpha^d} \lrcorner \alpha_c^{(3)} \wedge \omega^{(6)} = \frac{\partial}{\partial \alpha^d} \lrcorner g_c^{c'} \left(e_{c'}^{(3)} \wedge \gamma^{(6)} - (\mathrm{Ad}_{g^{-1}}A_{c'})^i e^{(4)} \wedge \gamma_i^{(5)} \right) \\ & = & g_d^{d'} \left(\frac{\partial}{\partial e^{d'}} - (\mathrm{Ad}_{g^{-1}}A_{d'})^j \frac{\partial}{\partial \gamma^j} \right) \lrcorner g_c^{c'} \left(e_{c'}^{(3)} \wedge \gamma^{(6)} - (\mathrm{Ad}_{g^{-1}}A_{c'})^i e^{(4)} \wedge \gamma_i^{(5)} \right) \\ & = & g_c^{c'} g_d^{d'} \Big(e_{c'd'}^{(2)} \wedge \gamma^{(6)} + (\mathrm{Ad}_{g^{-1}}A_{d'})^j e_{c'}^{(3)} \wedge \gamma_j^{(5)} - (\mathrm{Ad}_{g^{-1}}A_{c'})^i e_{d'}^{(3)} \wedge \gamma_i^{(5)} \\ & & \qquad + (\mathrm{Ad}_{g^{-1}}A_{c'})^i (\mathrm{Ad}_{g^{-1}}A_{d'})^j e^{(4)} \wedge \gamma_{ij}^{(4)} \Big) \\ & = & g_c^{c'} g_d^{d'} \Big(e_{c'd'}^{(2)} \wedge \gamma^{(6)} + \Big((\mathrm{Ad}_{g^{-1}}A_{d'})^i e_{c'}^{(3)} - (\mathrm{Ad}_{g^{-1}}A_{c'})^i e_{d'}^{(3)} \Big) \wedge \gamma_i^{(5)} \\ & & \qquad + (\mathrm{Ad}_{g^{-1}}A_{c'})^i (\mathrm{Ad}_{g^{-1}}A_{d'})^j e^{(4)} \wedge \gamma_{ij}^{(4)} \Big), \end{array}$$

second

$$\begin{array}{rcl} \alpha_c^{(3)} \wedge \omega_j^{(5)} & = & \frac{\partial}{\partial \alpha^c} \lrcorner \alpha^{(4)} \wedge \omega_j^{(5)} = \frac{\partial}{\partial \alpha^c} \lrcorner e^{(4)} \wedge \gamma_j^{(5)} \\ & = & g_c^{c'} \left(\frac{\partial}{\partial e^{c'}} - (\mathrm{Ad}_{g^{-1}}A_{c'})^k \frac{\partial}{\partial \gamma^k} \right) \lrcorner e^{(4)} \wedge \gamma_j^{(5)} \\ & = & g_c^{c'} \left(e_{c'}^{(3)} \wedge \gamma_j^{(5)} - (\mathrm{Ad}_{g^{-1}}A_{c'})^k e^{(4)} \wedge \gamma_{jk}^{(4)} \right) \end{array}$$

and lastly

$$\begin{array}{rcl} \alpha^{(4)} \wedge \omega_{jk}^{(4)} & = & \frac{\partial}{\partial \gamma^k} \lrcorner \alpha^{(4)} \wedge \omega_j^{(5)} = \frac{\partial}{\partial \gamma^k} \lrcorner e^{(4)} \wedge \gamma_j^{(5)} \\ & = & e^{(4)} \wedge \gamma_{jk}^{(4)}. \end{array}$$

Now we can relate two decompositions of ϖ_A. On the one hand, starting from (33):

$$\begin{array}{rcl} \varpi_A & = & \frac{1}{2} \varpi_A{}^{cd} \alpha_{cd}^{(2)} \wedge \omega^{(6)} - \varpi_A{}^{ck} \alpha_c^{(3)} \wedge \omega_k^{(5)} + \frac{1}{2} \varpi_A{}^{jk} \alpha^{(4)} \wedge \omega_{jk}^{(4)} \\ & = & \frac{1}{2} \varpi_A{}^{cd} g_c^{c'} g_d^{d'} \Big(e_{c'd'}^{(2)} \wedge \gamma^{(6)} + \Big((\mathrm{Ad}_{g^{-1}}A_{d'})^j e_{c'}^{(3)} - (\mathrm{Ad}_{g^{-1}}A_{c'})^j e_{d'}^{(3)} \Big) \wedge \gamma_j^{(5)} \\ & & \qquad + (\mathrm{Ad}_{g^{-1}}A_{c'})^j (\mathrm{Ad}_{g^{-1}}A_{d'})^k e^{(4)} \wedge \gamma_{jk}^{(4)} \Big) \\ & & \qquad - \varpi_A{}^{cj} g_c^{c'} \left(e_{c'}^{(3)} \wedge \gamma_j^{(5)} - (\mathrm{Ad}_{g^{-1}}A_{c'})^k e^{(4)} \wedge \gamma_{jk}^{(4)} \right) + \frac{1}{2} \varpi_A{}^{jk} e^{(4)} \wedge \gamma_{jk}^{(4)} \\ & = & \frac{1}{2} \varpi_A{}^{cd} g_c^{c'} g_d^{d'} e_{c'd'}^{(2)} \wedge \gamma^{(6)} + \left(\varpi_A{}^{cd} g_c^{c'} g_d^{d'} (\mathrm{Ad}_{g^{-1}}A_{d'})^j - \varpi_A{}^{cj} g_c^{c'} \right) e_{c'}^{(3)} \wedge \gamma_j^{(5)} \\ & & + \left(\frac{1}{2} \varpi_A{}^{cd} g_c^{c'} g_d^{d'} (\mathrm{Ad}_{g^{-1}}A_{c'})^j (\mathrm{Ad}_{g^{-1}}l_{d'})^k + \varpi_A{}^{cj} g_c^{c'} (\mathrm{Ad}_{g^{-1}}A_{c'})^k + \frac{1}{2} \varpi_A{}^{jk} \right) e^{(4)} \wedge \gamma_{jk}^{(4)}. \end{array}$$

On the other hand if we decompose $p_A = \frac{1}{2} p_A{}^{cd} e^{(2)}_{cd} \wedge \gamma^{(6)} - p_A{}^{ck} e^{(3)}_c \wedge \gamma^{(5)}_k + \frac{1}{2} p_A{}^{jk} e^{(4)} \wedge \gamma^{(4)}_{jk}$ and we develop the relation $\varpi = \mathrm{Ad}^*_g p$, we get

$$\varpi_A = (\mathrm{Ad}^*_g p)_A = \frac{1}{2} (\mathrm{Ad}^*_g p)_A{}^{cd} e^{(2)}_{cd} \wedge \gamma^{(6)} - (\mathrm{Ad}^*_g p)_A{}^{ck} e^{(3)}_c \wedge \gamma^{(5)}_k + \frac{1}{2} (\mathrm{Ad}^*_g p)_A{}^{jk} e^{(4)} \wedge \gamma^{(4)}_{jk}.$$

By identification we deduce the following

$$(\mathrm{Ad}^*_g p)_A{}^{cd} = \varpi_A{}^{c'd'} g^c_{c'} g^d_{d'}, \tag{68}$$

and $(\mathrm{Ad}^*_g p)_A{}^{cj} = \varpi_A{}^{c'j} g^c_{c'} - \varpi_A{}^{c'd'} g^c_{c'} g^d_{d'} (\mathrm{Ad}_{g^{-1}} A_d)^j$, from which we deduce by using (68)

$$(\mathrm{Ad}^*_g p)_A{}^{cj} = \varpi_A{}^{c'j} g^c_{c'} - (\mathrm{Ad}^*_g p)_A{}^{cd} (\mathrm{Ad}_{g^{-1}} A_d)^j. \tag{69}$$

We could also derive a relation between $p_A{}^{jk}$ and $\varpi_A{}^{jk}$, but we don't need it. Relation (68) is equivalent to

$$p_A{}^{cd} = (\mathrm{Ad}^*_{g^{-1}} \varpi)_A{}^{c'd'} g^c_{c'} g^d_{d'}.$$

It gives us for $p_a{}^{bcd} := u^{ib}_a p_i{}^{cd}$:

$$\begin{aligned} p_a{}^{bcd} &= (g^{-1})^{a'}_a g^b_{b'} \varpi_{a'}{}^{b'c'd'} g^c_{c'} g^d_{d'} = (g^{-1})^{a'}_a g^b_{b'} g^c_{c'} g^d_{d'} \kappa_{a'}{}^{b'c'd'} \\ &= (g^{-1})^{a'}_a g^b_{b'} g^c_{c'} g^d_{d'} (\delta^{c'}_{a'} \mathsf{h}^{b'd'} - \delta^{d'}_{a'} \mathsf{h}^{b'c'}) \\ &= \delta^c_a \mathsf{h}^{bd} - \delta^d_a \mathsf{h}^{bc} = \kappa_a{}^{bcd}; \end{aligned}$$

and for $p_a{}^{cd}$: $p_a{}^{cd} = (g^{-1})^{a'}_a \varpi_{a'}{}^{c'd'} g^c_{c'} g^d_{d'} = (g^{-1})^{a'}_a g^c_{c'} g^d_{d'} \kappa_a{}^{cd} = 0$. Hence we deduce that the coefficients of p satisfy

$$p_a{}^{cd} = 0 \quad \text{and} \quad p_a{}^{bcd} = \kappa_a{}^{bcd}. \tag{70}$$

Moreover Relation (69) is equivalent to

$$\varpi_A{}^{cj} = (g^{-1})^c_{c'} (\mathrm{Ad}^*_g p)_A{}^{c'j} + (g^{-1})^c_{c'} (\mathrm{Ad}^*_g p)_A{}^{c'd} (\mathrm{Ad}_{g^{-1}} A_d)^j \tag{71}$$

and give us for $\varpi_A{}^{cj} = \varpi_a{}^{bcj}$:

$$\varpi_a{}^{bcj} = (g^{-1})^c_{c'} g^{a'}_a (g^{-1})^b_{b'} p_{a'}{}^{b'c'j} + (g^{-1})^c_{c'} g^{a'}_a (g^{-1})^b_{b'} p_{a'}{}^{b'c'd} (\mathrm{Ad}_{g^{-1}} A_d)^j$$

and thus by using (70)

$$(g^{-1})^{a'}_a g^b_{b'} g^c_{c'} \varpi_{a'}{}^{b'c'j} = p_a{}^{bcj} + \kappa_a{}^{bcd} (\mathrm{Ad}_{g^{-1}} A_d)^j. \tag{72}$$

Similarly (71) gives us for $\varpi_A{}^{cj} = \varpi_a{}^{cj}$:

$$\varpi_a{}^{cj} = (g^{-1})^c_{c'} g^{a'}_a p_{a'}{}^{c'j} + (g^{-1})^c_{c'} g^{a'}_a p_{a'}{}^{c'd} (\mathrm{Ad}_{g^{-1}} A_d)^j$$

and hence by using (70)

$$(g^{-1})^{a'}_a g^c_{c'} \varpi_{a'}{}^{c'j} = p_a{}^{cj}. \tag{73}$$

We now use Relations (72) and (73) for eliminating $\varpi_d{}^{cj}$ and $\varpi_d{}^{bcj}$ in the r.h.s. of (67) and write

$$(g^{-1})^d_{d'} g^{c'}_c \left(\frac{1}{2} g^{b'}_b \varpi_d{}^{bcj} R^{d'}{}_{b'ac'} + \varpi_d{}^{cj} \mathrm{T}^{d'}{}_{ac'} \right) = \frac{1}{2} (p_d{}^{bcj} + \kappa_d{}^{bce} (\mathrm{Ad}_{g^{-1}} A_e)^j) R^d{}_{bac} + p_d{}^{cj} \mathrm{T}^d{}_{ac}$$

But since $\kappa_d{}^{bce} R^d{}_{bac} = (\delta^c_d \mathsf{h}^{be} - \delta^e_d \mathsf{h}^{bc}) R^d{}_{bac} = -2\mathrm{Ric}^e{}_a$,

$$(g^{-1})^d_{d'} g^{c'}_c \left(\frac{1}{2} g^{b'}_b \varpi_d{}^{bcj} R^{d'}{}_{b'ac'} + \varpi_d{}^{cj} \mathrm{T}^{d'}{}_{ac'} \right) = -\mathrm{Ric}^b{}_a (\mathrm{Ad}_{g^{-1}} A_b)^j - \frac{1}{2} p_d{}^{bcj} R^d{}_{bca} - p_d{}^{cj} \mathrm{T}^d{}_{ca}.$$

Hence we can write (67) as

$$\left\{ \begin{array}{rcll} (\nabla^H p)_a & = & 2\mathrm{E}^b{}_a e^{(3)}_b \wedge \gamma^{(6)} & - 2\mathrm{E}^b{}_a (\mathrm{Ad}_{g^{-1}} A_b)^j e^{(4)} \wedge \gamma^{(5)}_j \\ & & + \left(\frac{1}{2} p_d{}^{bcj} R^d{}_{bca} + p_d{}^{cj} \mathrm{T}^d{}_{ca} + \mathrm{Ric}^b{}_a (\mathrm{Ad}_{g^{-1}} A_b)^j \right) e^{(4)} \wedge \gamma^{(5)}_j & \\ (\nabla^H p)_a{}^b & = & & - \mathrm{S}(\mathrm{Ad}^*_{g^{-1}} u^j)_a{}^b e^{(4)} \wedge \gamma^{(5)}_j \end{array} \right. \tag{74}$$

5.2.5 The left hand side

We first prove a preliminary lemma.

Lemma 5.2 *Let Γ^a_{bc} (Christoffel symbols) be the functions depending on x such that $A^a{}_c = \Gamma^a_{bc} e^b$. Then*

$$de^a = \left(\frac{1}{2} T^a{}_{c'd'} - \Gamma^a_{c'd'} \right) e^{c'} \wedge e^{d'} \tag{75}$$

and, as a consequence,

$$de^{(3)}_c = Y_c e^{(4)}, \tag{76}$$

where $Y_c := T^d{}_{cd} - \Gamma^d_{cd} + \Gamma^d_{dc}$.

Proof — By (60) we have $de^a + A^a{}_{d'} \wedge e^{d'} = \frac{1}{2} \mathrm{T}^a{}_{c'd'} e^{c'} \wedge e^{d'}$, hence by substituting $A^a{}_{d'} = \Gamma^a_{c'd'} e^{c'}$, we obtain (75). Then we compute

$$de^{(3)}_c = de^d \wedge e^{(2)}_{cd} = \left(\frac{1}{2} \mathrm{T}^d{}_{c'd'} - \Gamma^d_{c'd'} \right) e^{c'} \wedge e^{d'} \wedge e^{(2)}_{cd}$$

from which (76) follows. □

In the previous section we have collected the algebraic constraints which have to be imposed in p, namely Relations (70). It remains to compute the l.h.s. of (74) taking into account these constraints. We start from the decomposition:

$$p_A = \frac{1}{2} p_A{}^{cd} e^{(2)}_{cd} \wedge \gamma^{(6)} - p_A{}^{ck} e^{(3)}_c \wedge \gamma^{(5)}_k + \frac{1}{2} p_A{}^{jk} e^{(4)} \wedge \gamma^{(4)}_{jk}$$

which, taking into account (70), reads equivalentely as

$$p_a = 0 - p_a{}^{ck} e^{(3)}_c \wedge \gamma^{(5)}_k + \frac{1}{2} p_a{}^{jk} e^{(4)} \wedge \gamma^{(4)}_{jk} \tag{77}$$

and, using $\kappa_a{}^{bcd}e^{(2)}_{cd} = 2\mathsf{h}^{bc}e^{(2)}_{ac}$,

$$p_a{}^b = \mathsf{h}^{bc}e^{(2)}_{ac} \wedge \gamma^{(6)} - p_a{}^{bck}e^{(3)}_c \wedge \gamma^{(5)}_k + \frac{1}{2}p_a{}^{bjk}e^{(4)} \wedge \gamma^{(4)}_{jk}. \tag{78}$$

Using (56) and (77) we get

$$\begin{aligned}(\nabla^H p)_a &= -dp_a{}^{ck} \wedge e^{(3)}_c \wedge \gamma^{(5)}_k - p_a{}^{ck}de^{(3)}_c \wedge \gamma^{(5)}_k + p_a{}^{ck}e^{(3)}_c \wedge d\gamma^{(5)}_k \\ &\quad +\tfrac{1}{2}dp_a{}^{jk} \wedge e^{(4)} \wedge \gamma^{(4)}_{jk} + \tfrac{1}{2}p_a{}^{jk}de^{(4)} \wedge \gamma^{(4)}_{jk} + \tfrac{1}{2}p_a{}^{jk}e^{(4)} \wedge d\gamma^{(4)}_{jk} \\ &\quad -\left(-p_b{}^{ck}e^{(3)}_c \wedge \gamma^{(5)}_k + \tfrac{1}{2}p_b{}^{jk}e^{(4)} \wedge \gamma^{(4)}_{jk}\right) \wedge \Gamma^b_{c'a}e^{c'}.\end{aligned}$$

Hence using Lemmas 5.2 and 8.4 and using the notation $df = f_{;a}e^a + f_{;i}\gamma^i$ for any function f, this gives us

$$\begin{aligned}(\nabla^H p)_a &= -p_a{}^{ck}{}_{;c}e^{(4)} \wedge \gamma^{(5)}_k + p_a{}^{ck}{}_{;k}e^{(3)}_c \wedge \gamma^{(6)} - p_a{}^{ck}Y_c e^{(4)} \wedge \gamma^{(5)}_k \\ &\quad +p_a{}^{jk}{}_{;k}e^{(4)} \wedge \gamma^{(5)}_j - \tfrac{1}{2}p_a{}^{jk}e^{(4)} \wedge c^l_{jk}\gamma^{(5)}_l \\ &\quad +p_b{}^{ck}\Gamma^b_{ca}e^{(4)} \wedge \gamma^{(5)}_k.\end{aligned}$$

Thus

$$\begin{aligned}(\nabla^H p)_a &= \left(-p_a{}^{cj}{}_{;c} - p_a{}^{cj}Y_c + p_b{}^{cj}\Gamma^b_{ca} + p_a{}^{jk}{}_{;k} - \tfrac{1}{2}p_a{}^{kl}c^j_{kl}\right)e^{(4)} \wedge \gamma^{(5)}_j \\ &\quad +p_a{}^{bk}{}_{;k}e^{(3)}_b \wedge \gamma^{(6)}.\end{aligned} \tag{79}$$

We now turn to the computation of $(\nabla^H p)_a{}^b$ (using (57)). As a preliminary, consider q, the $\mathfrak{p}^*$-valued 2-form such that $q_A = \kappa^{dc}_A e^{(2)}_{cd}$ (hence $(q_A) = (q_a, q_a{}^b)$ with $q_a = 0$ and $q_a{}^b := \mathsf{h}^{bc}e^{(2)}_{ac}$), and compute $(\nabla^H q)_a{}^b = dq_a{}^b + A^b{}_c \wedge q_a{}^c - q^b_c \wedge A^c{}_a + 2q_a \wedge e^b$: by using $de^{(2)}_{ac} = de^d \wedge e^{1)}_{acd}$, we get

$$\begin{aligned}(\nabla^H q)_a{}^b &= \mathsf{h}^{bc}\left(\mathrm{T}^d{}_{cd}e^{(3)}_a + \mathrm{T}^d{}_{da}e^{(3)}_c + \mathrm{T}^d{}_{ac}e^{(3)}_d - A^d{}_d \wedge e^{(2)}_{ac} - A^d{}_c \wedge e^{(2)}_{da} - A^d{}_a \wedge e^{(2)}_{cd}\right) \\ &\qquad +\mathsf{h}^{cd}A^b{}_c \wedge e^{(2)}_{ad} - \mathsf{h}^{bd}A^c{}_a \wedge e^{(2)}_{cd}\end{aligned}$$

Setting $A^{ab} := \mathsf{h}^{bb'}A^a{}_{b'}$ and noting that $A^{ab} + A^{ba} = 0$ and $A^d{}_d = 0$, we have

$$\begin{aligned}(\nabla^H q)_a{}^b &= \mathsf{h}^{bc}\left(\mathrm{T}^d{}_{cd}e^{(3)}_a + \mathrm{T}^d{}_{da}e^{(3)}_c + \mathrm{T}^d{}_{ac}e^{(3)}_d\right) - \mathsf{h}^{bc}A^d{}_d \wedge e^{(2)}_{ac} - A^{db} \wedge e^{(2)}_{da} - \mathsf{h}^{bc}A^d{}_a \wedge e^{(2)}_{cd} \\ &\qquad +A^{bd} \wedge e^{(2)}_{ad} - \mathsf{h}^{bc}A^d{}_a \wedge e^{(2)}_{dc} \\ &= \mathsf{h}^{bc}\left(\mathrm{T}^d{}_{cd}e^{(3)}_a + \mathrm{T}^d{}_{da}e^{(3)}_c + \mathrm{T}^d{}_{ac}e^{(3)}_d\right) \\ &= \left(\mathsf{h}^{bd}\mathrm{T}^c{}_{ad} - \mathsf{h}^{bc}\mathrm{T}^d{}_{ad} + \mathsf{h}^{be}\mathrm{T}^d{}_{ed}\delta^c_a\right)e^{(3)}_c.\end{aligned}$$

Thus

$$(\nabla^H q \wedge \gamma^{(6)})_a{}^b = \left(\mathsf{h}^{bd}\mathrm{T}^c{}_{ad} - \mathsf{h}^{bc}\mathrm{T}^d{}_{ad} + \mathsf{h}^{be}\mathrm{T}^d{}_{ed}\delta^c_a\right)e^{(3)}_c \wedge \gamma^{(6)}. \tag{80}$$

Note that $q \wedge \gamma^{(6)}$ is the 'first part' of p, i.e. the component which is a multiple of $\gamma^{(6)}$. It remains to compute the other part, i.e. $(\nabla^H \overline{p})_a{}^b$, where $\overline{p} := p - q \wedge \gamma^{(6)}$. This computation

is similar to the one for $(\nabla^H p)_a$.

$$\begin{aligned}(\nabla^H \overline{p})_a{}^b &= -dp_a{}^{bck}\wedge e_c^{(3)}\wedge\gamma_k^{(5)} - p_a{}^{bck}de_c^{(3)}\wedge\gamma_k^{(5)} + p_a{}^{bck}e_c^{(3)}\wedge d\gamma_k^{(5)}\\ &\quad +\tfrac{1}{2}dp_a{}^{bjk}\wedge e^{(4)}\wedge\gamma_{jk}^{(4)} + \tfrac{1}{2}p_a{}^{bjk}de^{(4)}\wedge\gamma_{jk}^{(4)} + \tfrac{1}{2}p_a{}^{bjk}e^{(4)}\wedge d\gamma_{jk}^{(4)}\\ &\quad +\Gamma^b_{c'a'}e^{c'}\wedge\left(-p_a{}^{a'ck}e_c^{(3)}\wedge\gamma_k^{(5)} + \tfrac{1}{2}p_a{}^{a'jk}e^{(4)}\wedge d\gamma_{jk}^{(4)}\right)\\ &\quad -\left(-p_{b'}{}^{bck}e_c^{(3)}\wedge\gamma_k^{(5)} + \tfrac{1}{2}p_{b'}{}^{bjk}e^{(4)}\wedge d\gamma_{jk}^{(4)}\right)\wedge\Gamma^{b'}_{c'a}e^{c'}\\ &\quad +2\left(-p_a{}^{ck}e_c^{(3)}\wedge\gamma_k^{(5)} + \tfrac{1}{2}p_a{}^{jk}e^{(4)}\wedge d\gamma_{jk}^{(4)}\right)\wedge e^b\end{aligned}$$

Hence as before

$$\begin{aligned}(\nabla^H \overline{p})_a{}^b &= -p_a{}^{bck}{}_{;c}e^{(4)}\wedge\gamma_k^{(5)} + p_a{}^{bck}{}_{;k}e_c^{(3)}\wedge\gamma^{(6)} - p_a{}^{bck}Y_ce^{(4)}\wedge\gamma_k^{(5)}\\ &\quad +p_a{}^{bjk}{}_{;k}e^{(4)}\wedge\gamma_j^{(5)} - \tfrac{1}{2}p_a{}^{bjk}c^i_{jk}e^{(4)}\wedge\gamma_i^{(5)}\\ &\quad -\Gamma^b_{ca'}p_a{}^{a'ck}e^{(4)}\wedge\gamma_k^{(5)} + \Gamma^{b'}_{ca}p_{b'}{}^{bck}e^{(4)}\wedge\gamma_k^{(5)} - 2p_a{}^{bk}e^{(4)}\wedge\gamma_k^{(5)}\end{aligned}$$

Thus

$$\begin{aligned}(\nabla^H \overline{p})_a{}^b &= \left(-p_a{}^{bcj}{}_{;c} - p_a{}^{bcj}Y_c - \Gamma^b_{ca'}p_a{}^{a'cj} + \Gamma^{b'}_{ca}p_{b'}{}^{bcj} - 2p_a{}^{bj}\right.\\ &\qquad \left. +p_a{}^{bjk}{}_{;k} - \tfrac{1}{2}p_a{}^{bkl}c^j_{kl}\right)e^{(4)}\wedge\gamma_j^{(5)}\\ &\quad +p_a{}^{bck}{}_{;k}e_c^{(3)}\wedge\gamma^{(6)}\end{aligned}$$

and, using (80) and $p = q\wedge\gamma^{(6)} + \overline{p}$,

$$\begin{aligned}(\nabla^H p)_a{}^b &= \left(\mathsf{h}^{bd}\mathrm{T}^c{}_{ad} - \mathsf{h}^{bc}\mathrm{T}^d{}_{ad} + \mathsf{h}^{be}\mathrm{T}^d{}_{ed}\delta^c_a + p_a{}^{bck}{}_{;k}\right)e_c^{(3)}\wedge\gamma^{(6)}\\ &\quad +\left(-p_a{}^{bcj}{}_{;c} - p_a{}^{bcj}Y_c - \Gamma^b_{ca'}p_a{}^{a'cj} + \Gamma^{b'}_{ca}p_{b'}{}^{bcj} - 2p_a{}^{bj}\right.\\ &\qquad \left. +p_a{}^{bjk}{}_{;k} - \tfrac{1}{2}p_a{}^{bkl}c^j_{kl}\right)e^{(4)}\wedge\gamma_j^{(5)}\end{aligned}\tag{81}$$

5.2.6 Conclusion: the HVDW equations

We now can write the dynamical equations completely in terms of the fields A, e and p. We identify the l.h.s. of (74) by using formulas (79) and (81). This gives us for the component of $(\nabla^H p)_a$ along $e_b^{(3)}\wedge\gamma^{(6)}$:

$$p_a{}^{bk}{}_{;k} = 2\mathrm{E}^b{}_a, \quad \forall a,b,\tag{82}$$

for the component of $(\nabla^H p)_a$ along $e^{(4)}\wedge\gamma_j^{(5)}$:

$$\begin{aligned}-p_a{}^{cj}{}_{;c} - p_a{}^{cj}Y_c + p_b{}^{cj}\Gamma^b_{ca}&\\ +p_a{}^{jk}{}_{;k} - \tfrac{1}{2}p_a{}^{kl}c^j_{kl} &= -(2\mathrm{E}^b{}_a - \mathrm{Ric}^b{}_a)(\mathrm{Ad}_{g^{-1}}A_b)^j\\ &\quad +\tfrac{1}{2}p_d{}^{bcj}R^d{}_{bca} + p_d{}^{cj}\mathrm{T}^d{}_{ca}, \quad \forall a,j,\end{aligned}\tag{83}$$

for the component of $(\nabla^H p)_a{}^b$ along $e_{(c)}^{(3)}\wedge\gamma^{(6)}$:

$$\mathsf{h}^{bd}\mathrm{T}^c{}_{ad} - \mathsf{h}^{bc}\mathrm{T}^d{}_{ad} + \mathsf{h}^{be}\mathrm{T}^d{}_{ed}\delta^c_a + p_a{}^{bck}{}_{;k} = 0, \quad \forall a,b,c,\tag{84}$$

and for the component of $(\nabla^H p)_a{}^b$ along $e^{(4)} \wedge \gamma_j^{(5)}$:

$$\begin{array}{rcl} -p_a{}^{bcj}{}_{;c} - p_a{}^{bcj}Y_c - \Gamma^b_{ca'}p_a{}^{a'cj} + \Gamma^{b'}_{ca}p_{b'}{}^{bcj} - 2p_a{}^{bj} & & \\ +p_a{}^{bjk}{}_{;k} - \frac{1}{2}p_a{}^{bkl}c^j_{kl} & = & -\mathrm{S}(\mathrm{Ad}^*_{g^{-1}}u^j)_a{}^b, \quad \forall a,b,j. \end{array} \tag{85}$$

By using the fact that Relation (84) implies $\mathrm{T}^a{}_{ca} = -\frac{1}{2}\mathsf{h}_{cd}p_a{}^{daj}{}_{;j}$ one can see that (84) is equivalent to:

$$\mathrm{T}^a{}_{cd} = -\left(\mathsf{h}_{de}\delta^a_{a'}\delta^{c'}_c + \frac{1}{2}\delta^{c'}_{a'}(\delta^a_d\mathsf{h}_{ce} - \delta^a_c\mathsf{h}_{de})\right)p_{c'}{}^{ea'j}{}_{;j}. \tag{86}$$

We can organize these equations into two systems

$$\left\{\begin{array}{rcl} \mathrm{E}^b{}_a & = & \frac{1}{2}p_a{}^{bj}{}_{;j} \\ \mathrm{T}^a{}_{cd} & = & -\left(\mathsf{h}_{de}\delta^a_{a'}\delta^{c'}_c + \frac{1}{2}\delta^{c'}_{a'}(\delta^a_d\mathsf{h}_{ce} - \delta^a_c\mathsf{h}_{de})\right)p_{c'}{}^{ea'j}{}_{;j} \end{array}\right. \tag{87}$$

and

$$\left\{\begin{array}{rcl} p_a{}^{cj}{}_{;c} + p_a{}^{cj}Y_c - p_b{}^{cj}\Gamma^b_{ca} + \frac{1}{2}p_d{}^{bcj}R^d{}_{bca} + p_d{}^{cj}\mathrm{T}^d{}_{ca} & & \\ -(2\mathrm{E}^b{}_a - \mathrm{Ric}^b{}_a)(\mathrm{Ad}_{g^{-1}}A_b)^j & = & p_a{}^{jk}{}_{;k} - \frac{1}{2}p_a{}^{kl}c^j_{kl} \\ p_a{}^{bcj}{}_{;c} + p_a{}^{bcj}Y_c + \Gamma^b_{ca'}p_a{}^{a'cj} - \Gamma^{b'}_{ca}p_{b'}{}^{bcj} + 2p_a{}^{bj} & & \\ -\mathrm{S}(\mathrm{Ad}^*_{g^{-1}}u^j)_a{}^b & = & p_a{}^{bjk}{}_{;k} - \frac{1}{2}p_a{}^{bkl}c^j_{kl} \end{array}\right. \tag{88}$$

6 Consequences of the equations

6.1 Global results

We first remark that, if a basis $(\mathfrak{l}_A)_A$ of $\mathfrak{p}$ is fixed, we can associate to any $\mathfrak{p}$-valued 1-form (α,ω) which is of rank 10 everywhere the Riemannian metric $G := (\alpha^0)^2 + \cdots + (\alpha^3)^2 + (\omega^4)^2 + \cdots + (\omega^9)^2$ on $\mathcal{P}$. In the relativistic case this metric depends on the choice of the basis $(\mathfrak{l}_A)_A$ and should not have any physical meaning in general. Nevertheless it has the virtue of being always positive definite and hence, in any case, it defines a topology on $\mathcal{P}$ which does not depend on the choice of the basis $(\mathfrak{l}_A)_A$.

Proposition 6.1 *Assume that $\mathfrak{G}$ is simply connected (i.e. it is the Spin group). Let (ϖ,α,ω) be a solution of the HVDW equations and assume that the $\mathfrak{p}$-valued 1-form (α,ω) is of rank 10 everywhere. Assume that $\mathcal{P}$ endowed with the topology induced by the metric G as above is complete, connected and open. Then any leaf f is a diffeomorphic to a quotient of $\mathfrak{G}$ by a group action.*

Proof — Since η is of rank 10 everywhere we can construct a family of tangent vector fields $(\xi_4,\cdots,\xi_9)$ on $\mathcal{P}$ defined by $\alpha^a(\xi_i) = 0$, $\forall a,i$ and $\omega^j(\xi_i) = \delta^j_i$, $\forall i,j$. We can interpret Equation (51) as the simultaneous flow equations of these vector fields. Then (50) means that these vector fields are in involution. These vector fields are obviously uniformly bounded in the topology induced by G, hence they are complete, since $\mathcal{P}$ is complete. Hence we can integrate them for all time and get a covering map from $\mathfrak{G}$ to the leaf f. □

In the Riemannian case $\mathfrak{G}$ is compact. Proposition 6.1 has then further consequences.

Corollary 6.1 *Assume that* $(\vec{\mathbb{M}}, \mathsf{h})$ *is the Euclidean space and the same hypotheses of Proposition 6.1. Then* $\mathcal{P}$ *is the total space of a principal bundle over a 4-dimensional manifold with fibers diffeomorphic to* $Spin(4)$ *or* $SO(4)$*.*

Proof — We apply the previous Proposition: each leaf has $Spin(4)$ as a universal cover, hence is diffeomorphic to $Spin(4)$ or $SO(4)$. But these leaves are also compact, which allows us to apply a result of Ehresmann [9] to conclude. □

6.2 The Riemannian case

Theorem 6.1 *Assume that* $(\vec{\mathbb{M}}, \mathsf{h})$ *is the Euclidean space and that* $\mathfrak{G}$ *is simply connected (i.e. it is the Spin group). Let* (ϖ, α, ω) *be a solution of the HVDW equations and assume that the* $\mathfrak{p}$*-valued 1-form* (α, ω) *is of rank 10 everywhere. Assume that* $\mathcal{P}$ *endowed with the topology induced by the metric* G *as above is complete, connected and open. Then* $\mathcal{P}$ *is the total space of a principal bundle over a 4-dimensional manifold* $\mathcal{X}$ *with fibers diffeomorphic to* $Spin(4)$ *or* $SO(4)$*. Moreover* ω *defines the Levi-Civita connection associated to the metric on* $\mathcal{X}$ *defined by* α *and* $\mathcal{X}$ *is an Einstein manifold.*

Proof — We first apply Corollary 6.1. Then the proof follows the same lines as in [22] for Yang–Mills fields. We know that the left hand sides of (87) does not depend on the variables g but only on x. Hence the same is true for the right hand sides, e.g. for ${p_a}^{bj}{}_{;j}$. Let f be a fiber over the point $\mathsf{x} \in \mathcal{X}$. We observe that ${p_a}^{bj}{}_{;j}\gamma^{(6)}|_{\mathsf{f}} = d({p_a}^{bj}\gamma^{(5)}_j)|_{\mathsf{f}}$. Since f is compact without boundary, we have

$$ {p_a}^{bj}{}_{;j} \int_{\mathsf{f}} \gamma^{(6)} = \int_{\mathsf{f}} {p_a}^{bj}{}_{;j}\gamma^{(6)} = \int_{\mathsf{f}} d({p_a}^{bj}\gamma^{(5)}_j) = 0. $$

A similar reasoning gives ${p_{c'}}^{ea'j}{}_{;j} = 0$. Hence the right hand sides of (87) vanish, which implies the conclusion. □

6.3 The relativistic case: a discussion

If $(\vec{\mathbb{M}}, \mathsf{h})$ is the Minkowski space, the situation is more complicated, because the structure group is not compact.

First there is no analogue of Corollary 6.1 in general and we could not exclude a priori complete, connected solutions $(\mathcal{P}, \alpha, \omega, \varpi)$ for which the leaves of the foliation are dense and thus the quotient space would not be separated. We will not discuss such solutions, since they are far from the standard definition of a space-time in General Relativity. However they could lead to interesting models in the framework of non-commutative geometry.

Note that, beside the metric G constructed on $\mathcal{P}$ in the previous section, we could also privilege non degenerate bilinear forms on a solution $(\mathcal{P}, \alpha, \omega, \varpi)$ of the HVWD equations of the type $K := \mathsf{h}_{ab}\alpha^a\alpha^b + \mathsf{K}_{ij}\omega^i\omega^j$, where K_{ij} is a non degenerate bilinear form on $\mathfrak{g}$ which

is invariant by the adjoint action of $\mathfrak{G}$. Such forms are not positive definite in general, but they do not depend on the choice of a basis of $\mathfrak{p}$ and they may possibly have a physical sense[4]. Understanding the geometry of the quotient space of leaves in this framework seems even more difficult a priori, but it is perhaps more relevant from a physical point of view.

If we assume that we have a global fibration, several cases could also occur:

- If the fibers are copies of $SO(1,3)$ or $Spin(1,3)$ or not compact quotients but if we assume that the fields ${\varpi_A}^{ck}$ have compact support in $\mathcal{P}$ or decay at infinity[5], then the proof of Theorem 6.1 works and the right hand sides of (87) vanish. Indeed by (73) we know that the fields ${p_a}^{ck}$ also decay at infinity, hence ${\mathrm{E}^b}_a$ in (87) vanishes by using the argument of Theorem 6.1. From (72) we deduce that ${p_a}^{bck} + {\kappa_a}^{bcd}(\mathrm{Ad}_{g^{-1}}A_d)^k$ decays at infinity and thus that ${\mathrm{T}^a}_{cd}$ in (87) also vanishes by the same argument and because $(\mathrm{Ad}_{g^{-1}}A_d)^k{}_{;k} = -[u_k, \mathrm{Ad}_{g^{-1}}A_d]^k = 0$, since $\mathfrak{g}$ is unimodular. Hence the quotient $\mathcal{X}$ is a solution of Einstein's equations. A similar situation occurs if, e.g., the fibers are isomorphic to quotients of $PSL(2,\mathbb{C})$ by a Kleinian group, i.e. to the orthonormal frame bundle of a quotient of the hyperbolic 3-space by the Kleinian group and if this quotient is compact.

- If the fibers are not compact and if we have no decay assumption on the fields ϖ or p, then Theorem 6.1 does not hold in general, unless some further hypotheses are assumed. Equations (87) are then the Einstein–Cartan system of equations with sources (the stress-energy tensor and the angular momentum tensor) due to the auxiliary fields p. The main question is to understand the dynamics of the fields ϖ or p, governed by Equations 88 and, probably, to understand what kind of hypotheses one should impose on these fields.

7 Gauge invariances

The action

$$\mathcal{A}[\Gamma] = \int_\Gamma \theta = \int_{\mathcal{P}} \varphi^*\theta$$

and the constraints (26) are invariant by the action of several gauge groups:

- they are invariant *off-shell* by orientation preserving diffeomorphisms or by reparametrizations: if $\phi : \mathcal{P} \longrightarrow \mathcal{P}$ is an orientation preserving diffeomorphism, then $\int_{\mathcal{P}} \varphi^*\theta = \int_{\mathcal{P}} (\varphi \circ \phi)^*\theta$;

[4]For instance in the degenerate case where $\mathsf{K} = 0$, if (α, ω) is a solution of the HVDW equations, then K is locally the pull-back by the fibration map of the pseudo-Riemannian metric on the quotient space of leaves found in Lemma 5.1.

[5]This holds if, e.g., one assumes that ${\varpi_A}^{cd} - \kappa_A^{cd}$, ${\varpi_A}^{ck}$ and ${\varpi_A}^{jk}$ have compact support in $\mathcal{P}$ or decay at infinity.

- they are invariant *on-shell* by gauge transformations with structure gauge group $\mathfrak{G}$: assume that $\varphi^*\eta = (\alpha, \omega)$ satisfies the two last HVDW equations (44) and (45), then, by Lemma 5.1, $\mathcal{P}$ looks everywhere locally like a principal bundle over a 4-dimensional manifold $\mathcal{X}$ with structure group $\mathfrak{G}$. In particular we can find local coordinates (x, g) in which (α, ω) reads $\alpha = g^{-1}e$ and $\omega = g^{-1}dg + g^{-1}Ag$, where (e, A) is a $\mathfrak{p}$-valued 1-form which depends only on x. The gauge group is then described locally as the set of maps $\gamma : \mathcal{P} \longrightarrow \mathfrak{G}$ of the form $\gamma(x, g) = g^{-1}f(x)g$, where f is a map from $\mathcal{X}$ to $\mathfrak{G}$ and any such map γ acts on (α, ω) by

$$(\alpha, \omega) \longmapsto (\tilde{\alpha}, \tilde{\omega}) = (\gamma^{-1}\alpha, \gamma^{-1}d\gamma + \gamma^{-1}\omega\gamma) = (0, \gamma^{-1}d\gamma) + \mathrm{Ad}_{\gamma^{-1}}(\alpha, \omega)$$

  and on ϖ by $\varpi \longmapsto \tilde{\varpi} = \mathrm{Ad}^*_\gamma \varpi$.

  Then $\varphi^*(d\eta + \frac{1}{2}[\eta \wedge \eta])$ is changed in $\tilde{\varphi}^*(d\eta + \frac{1}{2}[\eta \wedge \eta]) = \mathrm{Ad}_{\gamma^{-1}}\left(\varphi^*(d\eta + \frac{1}{2}[\eta \wedge \eta])\right)$. Hence the Lagrangian density $\varphi^*\left(\psi \wedge (d\eta + \frac{1}{2}[\eta \wedge \eta])\right)$ is left unchanged. Moreover a computation similar to the proof of (70) shows that the constraint (26), which reads also ${\varpi_A}^{ab} = \kappa^{ab}_A$, is preserved by this transformation. Note that $\tilde{\alpha} = g^{-1}\tilde{e}$ and $\tilde{\omega} = g^{-1}\tilde{A}g + g^{-1}dg$, where $\tilde{e} = f^{-1}e$ and $\tilde{A} = f^{-1}Af + f^{-1}df$ and thus $d\tilde{e} + \tilde{A} \wedge \tilde{e} = f^{-1}(de + A \wedge e)$ and $d\tilde{A} + \tilde{A} \wedge \tilde{A} = f^{-1}(dA + A \wedge A)f$.

- Lastly we can write the action density as

$$\psi \wedge (d\eta + \frac{1}{2}[\eta \wedge \eta]) = -\left(d\psi - \mathrm{ad}^*_\eta \wedge \psi\right) \wedge \eta + d(\psi \wedge \eta),$$

  which shows that, up to an exact term, the action is invariant *off-shell* by transformations of the form

$$(\alpha, \omega, \varpi) \longmapsto (\alpha, \omega, \varpi + \chi),$$

  where χ is any $\mathfrak{p}^*$-valued 8-form with compact support which satisfies the condition $d\chi - \mathrm{ad}^*_\eta \wedge \chi = 0$. If we moreover assume that $\chi \wedge \alpha^a \wedge \alpha^b = 0$, $\forall a, b$, then the constraint (26) is also preserved.

8 Annex

8.1 Lie algebras and their dual spaces

For the notations we refer to Section 1.2. Moreover we denote by (E^0, E^1, E^2, E^3) the basis of $\vec{\mathbb{M}}^*$ which is dual to (E_0, E_1, E_2, E_3); $(u^4, \cdots, u^9)$ is the basis of $\mathfrak{g}^*$ which is dual to $(u_4, \cdots, u_9)$ and $(\mathfrak{l}^0, \cdots, \mathfrak{l}^9)$ is the basis of $\mathfrak{p}^*$ which is dual $(\mathfrak{l}_0, \cdots, \mathfrak{l}_9)$. The structure coefficients of $\mathfrak{p}$ in the basis $(\mathfrak{l}_A)_{0 \leq A \leq 9}$ are denoted by c^A_{CB}, so that $[\mathfrak{l}_B, \mathfrak{l}_C] = c^A_{BC}\mathfrak{l}_A$, for $0 \leq A, B, C \leq 9$. In the subcase where $A, B, C = i, j, k$ run from 4 to 9, we recover the structure coefficients of $\mathfrak{g}$ in the basis $(u_4, u_5, u_6, u_7, u_8, u_9)$, i.e. such that $[u_j, u_k] = c^i_{jk}u_i$.

8.1.1 Tensorial notations for $\mathfrak{g}$ and $\mathfrak{g}^*$

Consider $\vec{\mathbb{M}} \otimes \vec{\mathbb{M}}$, $\vec{\mathbb{M}}^* \otimes \vec{\mathbb{M}}^*$, $\vec{\mathbb{M}} \otimes \vec{\mathbb{M}}^*$ and $\vec{\mathbb{M}}^* \otimes \vec{\mathbb{M}}$ and their vector subspaces $\vec{\mathbb{M}} \wedge \vec{\mathbb{M}} := \{t^{ab}E_{ab} \in \vec{\mathbb{M}} \otimes \vec{\mathbb{M}};\ t^{ab} + t^{ba} = 0\}$, $\vec{\mathbb{M}} \wedge \vec{\mathbb{M}}^* := \{t^a{}_b E_a{}^b \in \vec{\mathbb{M}} \otimes \vec{\mathbb{M}}^*; t^a{}_{b'}\mathsf{h}^{b'b} + t^b{}_{a'}\mathsf{h}^{a'a} = 0\}$, $\vec{\mathbb{M}}^* \wedge \vec{\mathbb{M}}^* := \{t_{ab}E^{ab} \in \vec{\mathbb{M}} \otimes \vec{\mathbb{M}}^*; t_{ab} + t_{ba} = 0\}$ and $\vec{\mathbb{M}}^* \wedge \vec{\mathbb{M}} := \{t_a{}^b E^a{}_b \in \vec{\mathbb{M}}^* \otimes \vec{\mathbb{M}}; t_a{}^{b'}\mathsf{h}_{b'b} + t_b{}^{a'}\mathsf{h}_{a'a} = 0\}$, where we write for short $E_{ab} := E_a \otimes E_b$, $E_a{}^b := E_a \otimes E^b$, $E^{ab} := E^a \otimes E^b$ and $E^a{}_b := E^a \otimes E_b$.

To any $\xi \in \mathfrak{g}$ it corresponds a unique tensor $\xi^a{}_b E_a{}^b \in \vec{\mathbb{M}} \wedge \vec{\mathbb{M}}^*$ such that $\mathcal{R}(\xi)(E_b) = E_a\xi^a{}_b$ and conversely. Hence we get the following vector spaces isomorphisms

$$\grave{\ell}: \begin{array}{ccc} \mathfrak{g} & \longrightarrow & \vec{\mathbb{M}} \wedge \vec{\mathbb{M}}^* \\ \xi & \longmapsto & \xi^a{}_b E_a{}^b \end{array} \quad \text{and} \quad \bar{\ell}: \begin{array}{ccc} \mathfrak{g} & \longrightarrow & \vec{\mathbb{M}} \wedge \vec{\mathbb{M}} \\ \xi & \longmapsto & \xi^{ab}E_{ab}, \end{array}$$

where $\xi^{ab} = \xi^a{}_{b'}\mathsf{h}^{b'b}$. We have the canonical identifications $\vec{\mathbb{M}}^* \wedge \vec{\mathbb{M}} \simeq (\vec{\mathbb{M}} \wedge \vec{\mathbb{M}}^*)^*$ and $\vec{\mathbb{M}}^* \wedge \vec{\mathbb{M}}^* \simeq (\vec{\mathbb{M}} \wedge \vec{\mathbb{M}})^*$ by using respectively the duality pairings

$$(\lambda_{a'}{}^{b'}E^{a'}{}_{b'}, \xi^a{}_b E_a{}^b) \longmapsto \frac{1}{2}\lambda_a{}^b\xi^a{}_b \quad \text{and} \quad (\lambda_{a'b'}E^{a'b'}, \xi^{ab}E_{ab}) \longmapsto \frac{1}{2}\lambda_{ab}\xi^{ab}.$$

Through these identifications, the adjoints of $\grave{\ell}$ and $\bar{\ell}$ provides us with isomorphisms $\grave{\ell}^* : \vec{\mathbb{M}}^* \wedge \vec{\mathbb{M}} \longrightarrow \mathfrak{g}^*$ and $\bar{\ell}^* : \vec{\mathbb{M}}^* \wedge \vec{\mathbb{M}}^* \longrightarrow \mathfrak{g}^*$. We define u^i_{ab} and $u^{ib}_a = u^i_{ab'}\mathsf{h}^{b'b}$ by $(\bar{\ell}^*)^{-1}(u^i) = u^i_{ab}E^{ab}$ and $(\grave{\ell}^*)^{-1}(u^i) = u^{ib}_a E^a{}_b$, $\forall i = 4, \cdots, 9$. We then have

$$\frac{1}{2}u^i_{ab}u^{ab}_j = \frac{1}{2}u^{ib}_a u^a_{jb} = u^i(u_j) = \delta^i_j \tag{89}$$

and

$$u^i_{ab}u^{a'b'}_i = \frac{1}{2}\delta^{a'b'}_{ab} := \frac{1}{2}(\delta^{a'}_a\delta^{b'}_b - \delta^{a'}_b\delta^{b'}_a), \tag{90}$$

from which we also deduce

$$u^{ib}_a u^{a'}_{ib'} = \frac{1}{2}(\delta^{a'}_a\delta^b_{b'} - \mathsf{h}^{a'b}\mathsf{h}_{ab'}). \tag{91}$$

8.1.2 Tensorial notations for $\mathfrak{p}$

We can extend the previous isomorphism $\grave{\ell}$ to

$$\grave{\ell}: \begin{array}{ccc} \mathfrak{p} & \longrightarrow & (\vec{\mathbb{M}} \wedge \vec{\mathbb{M}}^*) \oplus \vec{\mathbb{M}} \\ \xi & \longmapsto & (\xi^a{}_b E_a{}^b, \xi^a E_a) \end{array}$$

where, denoting by O the origin of $\mathbb{M}$, $\mathcal{R}(\xi)(O + x^a E_a) = O + x^b\xi^a{}_b E_a + \xi^a E_a$, $\forall \xi \in \mathfrak{p}$, $\forall O + x^b E_b \in \mathbb{M}$. We then have

$$\grave{\ell}([\xi, \zeta]) = \big((\xi^a{}_c\zeta^c{}_b - \zeta^a{}_c\xi^c{}_b)E_a{}^b,\ (\xi^a{}_b\zeta^b - \zeta^a{}_b\xi^b)E_a\big)$$

As for $\mathfrak{g}^*$ we also get the following vector spaces isomorphism

$$(\grave{\ell}^*)^{-1}: \begin{array}{ccc} \mathfrak{p}^* & \longrightarrow & (\vec{\mathbb{M}}^* \wedge \vec{\mathbb{M}}) \oplus \vec{\mathbb{M}}^* \\ \lambda & \longmapsto & (\lambda_a{}^b E^a{}_b, \lambda_a E^a) \end{array}$$

with the duality pairing $\left((\vec{\mathbb{M}}^* \wedge \vec{\mathbb{M}}) \oplus \vec{\mathbb{M}}^*\right) \times \left((\vec{\mathbb{M}} \wedge \vec{\mathbb{M}}^*) \oplus \vec{\mathbb{M}}\right) \longrightarrow \mathbb{R}$,

$$\left((\lambda_a{}^b E^a{}_b, \lambda_a E^a), (\xi^a{}_b E_a{}^b, \xi^a E_a)\right) \longmapsto \frac{1}{2}\lambda_a{}^b \xi^a{}_b + \lambda_a \xi^a.$$

8.1.3 Adjoint and coadjoint action of $\mathfrak{G}$

The standard representation $\mathcal{R}$ of $\mathfrak{G}$ induces the map $\mathfrak{G} \longrightarrow \vec{\mathbb{M}} \wedge \vec{\mathbb{M}}^*$, $g \longmapsto g^a{}_b E_a{}^b$. The restriction to $\mathfrak{G}$ of the adjoint representation of $\mathfrak{P}$ on $\mathfrak{p}$ reads

$$\forall \xi \in \mathfrak{p}, \quad \mathrm{Ad}_g(\xi^a{}_b E_a{}^b, \xi^a E_a) = \left((g^a{}_{a'} \xi^{a'}{}_{b'} (g^{-1})^{b'}{}_b) E_a{}^b,\, g^a{}_{a'} \xi^{a'} E_a\right).$$

The coadjoint action of $\mathfrak{G}$ on $\mathfrak{p}^*$ is defined by: $\forall g \in \mathfrak{G}$, $\forall \lambda \in \mathfrak{p}^*$, $\mathrm{Ad}^*_g \lambda$ is the vector in $\mathfrak{p}^*$ such that:

$$\forall \xi \in \mathfrak{p}, \quad (\mathrm{Ad}^*_g \lambda)(\xi) := \lambda(\mathrm{Ad}_g \xi).$$

In our setting this reads

$$\begin{array}{rcl} (\mathrm{Ad}^*_g \lambda)(\xi) & = & \frac{1}{2}\lambda_a{}^b \left(g^a{}_{a'} \xi^{a'}{}_{b'} (g^{-1})^{b'}{}_b\right) + \lambda_a \left(g^a{}_{a'}\right) \xi^{a'} \\ & = & \frac{1}{2}\left(g^{a'}{}_a \lambda_{a'}{}^{b'} (g^{-1})^b{}_{b'}\right) \xi^a{}_b + \left(g^{a'}{}_a \lambda_{a'}\right) \xi^a. \end{array}$$

Hence

$$(\grave{\ell}^*)^{-1}(\mathrm{Ad}^*_g \lambda) = \left((g^{a'}{}_a \lambda_{a'}{}^{b'} (g^{-1})^b{}_{b'}) E^a{}_b,\, g^{a'}{}_a \lambda_{a'} E^a\right). \tag{92}$$

8.1.4 Coadjoint action of $\mathfrak{p}$

The coadjoint action of $\mathfrak{p}$ on $\mathfrak{p}^*$ is defined by: $\forall \xi \in \mathfrak{p}$, $\forall \lambda \in \mathfrak{p}^*$, $\mathrm{ad}^*_\xi \lambda$ is the vector in $\mathfrak{p}^*$ such that:

$$\forall \zeta \in \mathfrak{p}, \quad (\mathrm{ad}^*_\xi \lambda)(\zeta) := \lambda(\mathrm{ad}_\xi \zeta) = \lambda([\xi, \zeta]).$$

This gives us:

$$\begin{array}{rcl} (\mathrm{ad}^*_\xi \lambda)(\zeta) & = & \frac{1}{2}\lambda_a{}^b \left(\xi^a{}_c \zeta^c{}_b - \zeta^a{}_c \xi^c{}_b\right) + \lambda_a \left(\xi^a{}_b \zeta^b - \zeta^a{}_b \xi^b\right) \\ & = & \frac{1}{2}\left(\xi^c{}_a \lambda_c{}^b - \lambda_a{}^c \xi^b{}_c - 2\lambda_a \xi^b\right) \zeta^a{}_b + \left(\xi^a{}_b \lambda_a\right) \zeta^b \end{array}$$

Hence

$$(\grave{\ell}^*)^{-1}(\mathrm{ad}^*_\xi \lambda) = \left((\xi^c{}_a \lambda_c{}^b - \lambda_a{}^c \xi^b{}_c - 2\lambda_a \xi^b) E^a{}_b,\, \xi^a{}_b \lambda_a E^b\right). \tag{93}$$

An alternative representation uses the basis $(\mathfrak{l}_A)_A$ of $\mathfrak{p}$ and the dual basis $(\mathfrak{l}^A)_A$ of $\mathfrak{p}^*$: decompose $\lambda = \lambda_A \mathfrak{l}^A$, $\xi = \mathfrak{l}_A \xi^A$ and $\zeta = \mathfrak{l}_A \zeta^A$, then $[\xi, \zeta] = \mathfrak{l}_A c^A_{BC} \xi^B \zeta^C$, we find that $(\mathrm{ad}^*_\xi \lambda)(\xi) = \lambda_A (c^A_{BC} \xi^B \zeta^C) = (\lambda_B c^B_{CA} \xi^C) \zeta^A$ hence

$$\mathrm{ad}^*_\xi \lambda = (\lambda_B c^B_{CA} \xi^C) \mathfrak{l}^A.$$

We can extend this action to $\mathfrak{p}$-valued and $\mathfrak{p}^*$-valued exterior forms. If ξ is a $\mathfrak{p}$-valued form and λ is a $\mathfrak{p}^*$-valued form, we define

$$\mathrm{ad}^*_\xi \wedge \lambda := c^B_{CA} (\xi^C \wedge \lambda_B) \mathfrak{l}^A. \tag{94}$$

Lemma 8.1 *Let $g \in \mathfrak{G}$, $\xi \in \mathfrak{p}$ and $\lambda \in \mathfrak{p}^*$. Then*

$$Ad^*_{g^{-1}} \left(ad^*_{(Ad_{g^{-1}}\xi)} \lambda \right) = ad^*_{\xi}(Ad^*_{g^{-1}}\lambda). \tag{95}$$

Proof — Take any $\zeta \in \mathfrak{p}$ and start from the identity $\mathrm{Ad}_{g^{-1}}([\xi,\zeta]) = [\mathrm{Ad}_{g^{-1}}\xi, \mathrm{Ad}_{g^{-1}}\zeta]$, which implies

$$\lambda\left(\mathrm{Ad}_{g^{-1}}[\xi,\zeta]\right) = \lambda\left(\mathrm{ad}_{(\mathrm{Ad}_{g^{-1}}\xi)}(\mathrm{Ad}_{g^{-1}}\zeta)\right).$$

The l.h.s. of this identity is equal to $(\mathrm{Ad}^*_{g^{-1}}\lambda)([\xi,\zeta]) = \left(\mathrm{ad}^*_{\xi}\left(\mathrm{Ad}^*_{g^{-1}}\lambda\right)\right)(\zeta)$ and its r.h.s. is equal to $\left(\mathrm{ad}^*_{(\mathrm{Ad}_{g^{-1}}\xi)}\lambda\right)(\mathrm{Ad}_{g^{-1}}\zeta) = \left(\mathrm{Ad}^*_{g^{-1}}\left(\mathrm{ad}^*_{(\mathrm{Ad}_{g^{-1}}\xi)}\lambda\right)\right)(\zeta)$. Hence (95) follows. □

8.2 Exterior differential calculus

Lemma 8.2 *The following relations holds*

$$\alpha^a \wedge \alpha^{(4)}_{a'} = \delta^a_{a'}\alpha^{(4)}, \quad \alpha^a \wedge \alpha^b \wedge \alpha^{(4)}_{a'b'} = \delta^{ab}_{a'b'}\alpha^{(4)}$$

and

$$\omega^i \wedge \omega^{(6)}_{i'} = \delta^i_{i'}\omega^{(6)}, \quad \omega^i \wedge \omega^j \wedge \omega^{(6)}_{i'j'} = \delta^{ij}_{i'j'}\omega^{(6)}.$$

where $\delta^{ab}_{a'b'} := \delta^a_{a'}\delta^b_{b'} - \delta^a_{b'}\delta^b_{a'}$ and $\delta^{ij}_{i'j'} := \delta^i_{i'}\delta^j_{j'} - \delta^i_{j'}\delta^j_{i'}$.

The proof is left to the Reader.

Lemma 8.3 *Let $e^{(4)} := e^0 \wedge e^1 \wedge e^2 \wedge e^3$ and $e^{(2)}_{ab} := \frac{\partial}{\partial e^b} \lrcorner \frac{\partial}{\partial e^a} \lrcorner e^{(4)}$. Then*

$$e^{(2)}_{ab} = \frac{1}{2}\epsilon_{abcd}e^c \wedge e^d$$

Proof — We start from $e^{(4)} = \frac{1}{4!}\epsilon_{a'b'c'd'}e^{a'} \wedge e^{b'} \wedge e^{c'} \wedge e^{d'}$. We then compute $e^{(3)}_a := \frac{\partial}{\partial e^a} \lrcorner e^{(4)}$:

$$\begin{aligned} e^{(3)}_a &= 1/4!\left[\epsilon_{ab'c'd'}e^{b'} \wedge e^{c'} \wedge e^{d'} - \epsilon_{a'ac'd'}e^{a'} \wedge e^{c'} \wedge e^{d'} + \epsilon_{a'b'ad'}e^{a'} \wedge e^{b'} \wedge e^{d'} - \epsilon_{a'b'c'a}e^{a'} \wedge e^{b'} \wedge e^{c'}\right] \\ &= 1/4!\left[\epsilon_{ab'c'd'}e^{b'} \wedge e^{c'} \wedge e^{d'} + \epsilon_{aa'c'd'}e^{a'} \wedge e^{c'} \wedge e^{d'} + \epsilon_{aa'b'd'}e^{a'} \wedge e^{b'} \wedge e^{d'} + \epsilon_{aa'b'c'}e^{a'} \wedge e^{b'} \wedge e^{c'}\right] \\ &= 1/3!\,\epsilon_{ab'c'd'}e^{b'} \wedge e^{c'} \wedge e^{d'}. \end{aligned}$$

By performing a similar computation for $e^{(2)}_{ab}$ we obtain the result. □

Corollary — We deduce from the lemma that $e^{(2)}_{cd'}\mathsf{h}^{d'd} = \frac{1}{2}\epsilon_{abc}{}^{d'}e^a \wedge e^b$, hence

$$\mathsf{h}^{dd'}e^{(2)}_{cd} \wedge \Omega^c{}_{d'} = \frac{1}{2}\epsilon_{abc}{}^{d'}e^a \wedge e^b \wedge \Omega^c{}_{d'}.$$

Lemma 8.4 *Let $\gamma := g^{-1}dg$ be the Maurer–Cartan form on the group $\mathfrak{G}$, $(\gamma^i)_{1\leq i\leq 6}$ the components of γ in a basis $(\mathfrak{t}_1, \cdots, \mathfrak{t}_6)$ of $\mathfrak{g}$, $\gamma^{(6)} := \gamma^1 \wedge \cdots \wedge \gamma^6$, $\gamma_i^{(5)} := \frac{\partial}{\partial \gamma^i} \lrcorner \gamma^{(6)}$, $\gamma_{ij}^{(4)} := \frac{\partial}{\partial \gamma^j} \lrcorner \frac{\partial}{\partial \gamma^i} \lrcorner \gamma^{(6)}$. Lastly let c_{jk}^i be the structure constants of $\mathfrak{g}$ in the basis $(\mathfrak{t}_1, \cdots, \mathfrak{t}_6)$. Then*

$$d\gamma^i + \frac{1}{2} c_{jk}^i \gamma^j \wedge \gamma^k = 0, \tag{96}$$

$$d\gamma^{(6)} = 0, \tag{97}$$

$$d\gamma_i^{(5)} = 0 \tag{98}$$

$$d\gamma_{ij}^{(4)} + c_{ij}^k \gamma_k^{(5)} = 0. \tag{99}$$

Proof — Relation (97) is simply due to the fact that $\gamma^{(6)}$ has a maximal degree. Relation (98) follows from (96) and the fact that $\mathfrak{g}$ is unimodular:

$$\begin{aligned} d\gamma_i^{(5)} &= d\gamma^j \wedge \gamma_{ij}^{(4)} = -\tfrac{1}{2} c_{kl}^j \gamma^k \wedge \gamma^l \wedge \gamma_{ij}^{(4)} \\ &= -\tfrac{1}{2} c_{ij}^j \gamma^{(6)} + \tfrac{1}{2} c_{ji}^j \gamma^{(6)} = -c_{ij}^j \gamma^{(6)} = 0. \end{aligned}$$

The reasoning is similar for (99):

$$\begin{aligned} d\gamma_{ij}^{(4)} &= d\gamma^k \wedge \gamma_{ijk} = -\tfrac{1}{2} c_{lm}^k \gamma^l \wedge \gamma^m \wedge \gamma_{ijk}^{(3)} \\ &= -\tfrac{1}{2} c_{lm}^k \left[\delta_{ij}^{lm} \gamma_k^{(5)} + \delta_{jk}^{lm} \gamma_i^{(5)} + \delta_{ki}^{lm} \gamma_j^{(5)} \right] \\ &= -c_{ij}^k \gamma_k^{(5)} - c_{jk}^k \gamma_i^{(5)} - c_{ki}^k \gamma_j^{(5)} = -c_{ij}^k \gamma_k^{(5)}. \end{aligned}$$

□

Lemma 8.5 *Let g be smooth map with values in $\mathfrak{G}$ and let ϖ be an exterior differential form with coefficients in $\mathfrak{p}^*$. Then*

$$d\left(Ad_{g^{-1}}^* \varpi\right) = Ad_{g^{-1}}^* \left(d\varpi - ad_{g^{-1}dg}^* \wedge \varpi\right). \tag{100}$$

Proof — Assume that ϖ is of degree q and consider any constant $\xi \in \mathfrak{p}$. We have

$$d\left(\mathrm{Ad}_{g^{-1}}^* \varpi\right)(\xi) = d\left(\mathrm{Ad}_{g^{-1}}^* \varpi(\xi)\right) = d\left[\varpi\left(\mathrm{Ad}_{g^{-1}}\xi\right)\right] = \left(d\varpi\right)\left(\mathrm{Ad}_{g^{-1}}\xi\right) + (-1)^q \varpi \wedge d\left(\mathrm{Ad}_{g^{-1}}\xi\right).$$

But since $d\left(\mathrm{Ad}_{g^{-1}}\xi\right) = -\mathrm{ad}_{g^{-1}dg}\left(\mathrm{Ad}_{g^{-1}}\xi\right)$ we deduce

$$\begin{aligned} d\left(\mathrm{Ad}_{g^{-1}}^* \varpi\right)(\xi) &= \left(d\varpi\right)\left(\mathrm{Ad}_{g^{-1}}\xi\right) - \left(\mathrm{ad}_{g^{-1}dg}^* \wedge \varpi\right)\left(\mathrm{Ad}_{g^{-1}}\xi\right) \\ &= \left(d\varpi - \mathrm{ad}_{g^{-1}dg}^* \wedge \varpi\right)\left(\mathrm{Ad}_{g^{-1}}\xi\right) \\ &= \left(\mathrm{Ad}_{g^{-1}}^* \left(d\varpi - \mathrm{ad}_{g^{-1}dg}^* \wedge \varpi\right)\right)(\xi). \end{aligned}$$

Hence (100) follows. □

Corollary 8.1 *If $p := Ad_{g^{-1}}^* \varpi$ and $(\alpha, \omega) = (0, g^{-1}dg) + Ad_{g^{-1}} H$, then*

$$dp - ad_H^* \wedge p = Ad_{g^{-1}}^* \left(d\varpi - ad_{(\alpha,\omega)} \wedge \varpi\right). \tag{101}$$

Proof – From (95) we deduce

$$\mathrm{ad}_H^* \wedge p = \mathrm{ad}_H^* \wedge \left(\mathrm{Ad}_{g^{-1}}^* \varpi\right) = \mathrm{Ad}_{g^{-1}}^* \left(\mathrm{ad}_{\left(\mathrm{Ad}_{g^{-1}} H\right)} \wedge \varpi\right),$$

which, together with (100), implies (101). □

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