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# Dynamics of multivariate default system in random environment

Nicole El Karoui\* Monique Jeanblanc<sup>†</sup> Ying Jiao<sup>‡</sup>

September 30, 2015

## Abstract

We consider a multivariate default system where random environmental information is available. We study the dynamics of the system in a general setting and adopt the point of view of change of probability measures. We also make a link with the density approach in the credit risk modelling. In the particular case where no environmental information is concerned, we pay a special attention to the phenomenon of system weakened by failures as in the classical reliability system.

**MSC:** 91G40, 60G20, 60G44

**Key words:** Multiple defaults, prediction process, system weakened by failures, change of probability measure, density approach

## 1 Introduction

In the reliability system, we consider a finite system consisting of multi-components and study the probability distribution and the mutual dependence of the survival lifetime lengths of the components. The failure times of the system are considered as events in the so-called failure process and their dynamics are based on the conditional distributions with respect to the history of the processes. In the literature, we are particularly interested in the behavior of the

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system upon failures (see for example Arjas and Norros [2, 3, 19] and Knight [17]), such as the stochastic orders of the conditional distribution, which is called the prediction process, at failure times and whether the system is weakened by failures.

In the credit risk analysis, we are also interested in the “failure time” of firms, i.e., the defaults, on the financial market. However the environmental information appears to be an important factor. Besides the dependence structure among the underlying firms, we need to investigate the role of other market information upon the system of multiple defaults, and vice versa, the impact of default events on the market. In literature such as Bielecki and Rutkowski [4] and Elliott, Jeanblanc and Yor [8], the information structure concerning the default risks is described by the theory of enlargement of filtrations. In general, we suppose that the environmental market information is modelled by a reference filtration  $\mathbb{F} = (\mathbb{F}_t)_{t \geq 0}$  and the default information is then added to form an enlarged filtration  $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$  which represents the global information of the market. The modelling of the dependence structure of multiple default times is then diversified in two directions by using the so-called bottom-up and top-down models. In the former approach, one starts with a model for the marginal distribution of each default time and then the correlation between them is made precise (see Frey and McNeil [11] for a survey). While in the top-down models which are particularly developed and adapted to the portfolio credit derivatives, we study directly the cumulative loss process and its intensity dynamics.

In this paper, we consider a multivariate system of default times in a general setting of enlargement of filtrations in presence of environmental information and we study the dynamics of the system. In order to fully investigate the key elements in the modelling, we use a general random variable  $\chi$  to describe default risks and to study the dependence between the multi-default system and the environmental information. This general setting can be applied flexibly to diverse situations, including bottom-up and top-down models. We begin by concentrating on the system itself and then study some important properties of reliability. We introduce the prediction process of the system and compare its behavior before and after default events by using statistics orders. We show that, upon the observation of all previous defaults, the multi-default system is in general only partially weakened by the past failures, which is different from the case of a single-default system. We present some special cases where the multi-default system is indeed weakened.

When we take into account the environmental information, the market is represented by an extended probability space and the information structure is described by larger filtrations. Our main and original idea is to characterize the dependence between the multi-default system and the remaining market by using a change of probability method with respect to the product probability measure under which the multi-default system  $\chi$  and the environmental information

$\mathbb{F}$  are independent. In this setting, the dependence structure between the default system and the market environment under any probability measure can be described in a dynamic manner and represented by the Radon-Nikodym derivative process of the change of probability. We will provide estimations and evaluation formulas under different information levels. The methodology of change of probability is of a similar nature with the density approach developed in El Karoui, Jeanblanc and Jiao [9, 10]. In the classical literature on enlargement of filtration theory, the density hypothesis is first introduced by Jacod [14] in an initial enlargement of filtration and is fundamental to ensure the semi-martingale property in the enlarged filtration. We show that the density process and the Radon-Nikodym derivative can be deduced from each other and the two approaches are closely related. We also obtain a very general martingale characterization result which can be applied in many special cases such as the progressive enlargement of filtrations setting with ordered and non-ordered defaults, which are useful for financial applications.

The following of the paper is organized as follows. In Section 2, we give the general presentation of the multivariate default system and we study its dynamics given the history of the system. We consider the interaction between the system with environmental information in Section 3 and present the change of probability measure methodology. In Section 4, we investigate the link between the density approach in the theory of enlargement of filtrations and we present the martingale characterization results.

## 2 The multi-default system

In this presentation, we introduce a general variable to describe all uncertainty related to the multi-default system such as default or failure times, occurrence orders and associated losses or recovery ratios. We also explain how to adapt this general framework to the classical models in credit risk and reliability system and investigate the dynamics of the multi-default system when the information concerned is the history of the system.

### 2.1 Basic setting

We consider a finite family of  $n$  underlying firms and describe the default uncertainty of these firms by a random variable  $\chi$  which is defined on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$  and takes values in a Polish space  $E$ . The default times of these firms are represented by a vector  $\boldsymbol{\tau} = (\tau_1, \dots, \tau_n)$  of random times. Since  $\chi$  contains all information of the default uncertainty, there exists a measurable map  $f : E \rightarrow \mathbb{R}_+^n$  such that  $\boldsymbol{\tau} = f(\chi)$  and this map  $f$  specifies the default times  $\boldsymbol{\tau}$ . In this general framework, the random variable  $\chi$  can be chosen in a very

flexible manner, which allows to consider bottom-up and top-down models in the credit risk literature. For example, one can choose  $\chi$  to be the default time vector  $\tau$  itself: in this case the Polish space  $E$  is just  $\mathbb{R}_+^n$  and  $f : E \rightarrow \mathbb{R}_+^n$  is the identity map. One can also take into account the information of associated losses, namely  $E = \mathbb{R}_+^n \times \mathbb{R}^n$  and  $\chi = (\tau_i, L_i)_{i=1}^n : \Omega \rightarrow \mathbb{R}_+^n \times \mathbb{R}^n$ , where  $L_i$  denotes the loss supported by the  $i^{\text{th}}$  firm at default time. In the top-down models, we consider the ordered default times  $\sigma_1 \leq \dots \leq \sigma_n$ . We can choose  $E$  to be the subspace  $\{(u_1, \dots, u_n) \in \mathbb{R}_+^n \mid u_1 \leq \dots \leq u_n\}$  of  $\mathbb{R}_+^n$  and  $\chi$  to be the successive default vector  $\sigma = (\sigma_i)_{i=1}^n$ . If we intend to take into account the label of each defaulted firm in the top-down setting (in this case the successive default vector  $\sigma$  consists of the order statistics of the random vector  $\tau$ ), we can choose  $\chi = (\sigma, J)$  valued in

$$E = \{(u_1, \dots, u_n) \in \mathbb{R}_+^n \mid u_1 \leq \dots \leq u_n\} \times \mathfrak{S}_n,$$

where  $J$  takes values in the permutation group  $\mathfrak{S}_n$  of all bijections from  $\{1, \dots, n\}$  to itself and describes the indices of underlying components for the successive defaults. The default time vector  $\tau$  can thus be specified by the measurable map from  $E$  to  $\mathbb{R}_+^n$  which sends  $(u_1, \dots, u_n, \pi)$  to  $(u_{\pi^{-1}(1)}, \dots, u_{\pi^{-1}(n)})$ .

## 2.2 Prediction process

Note that the complete information on  $\chi$  is not accessible to the market participants at an arbitrary time. We denote by  $(\mathcal{N}_t)_{t \geq 0}$  a filtration on  $\mathcal{A}$ , which represents the information flow of the market observation of the default events. Typically it can be chosen to be the right continuous and complete filtration generated by a counting process. The  $(\mathcal{N}_t)_{t \geq 0}$ -conditional probability law of the random variable  $\chi$  can thus be described as a  $\mathcal{P}(E)$ -valued càdlàg  $(\mathcal{N}_t)_{t \geq 0}$ -adapted process  $(\eta_t, t \geq 0)$ , where  $\mathcal{P}(E)$  denotes the set of all Borel probability measures on  $E$ , equipped with the topology of weak convergence such that for any bounded continuous function  $h$  on  $E$ , the map  $\nu \mapsto \int_E h d\nu$  is continuous.

Following the terminology of Knight [17] and Norros [19], the process  $(\eta_t, t \geq 0)$  is called the *prediction process* of the random variable  $\chi$  with respect to the observation filtration  $(\mathcal{N}_t)_{t \geq 0}$ . We refer the reader to [19, Theorem 1.1] for the existence of a càdlàg version of the process  $(\eta_t, t \geq 0)$  and the uniqueness up to indistinguishability. Moreover, the process  $(\eta_t, t \geq 0)$  is an  $(\mathcal{N}_t)_{t \geq 0}$ -martingale with respect to the weak topology in the following sense : for any bounded Borel function  $h$  on  $E$ , the integral process  $(\int_E h(x) \eta_t(dx), t \geq 0)$  is an  $(\mathcal{N}_t)_{t \geq 0}$ -martingale.

**Observation of default counting process.** We consider the case where the observation filtration  $(\mathcal{N}_t)_{t \geq 0}$  is generated by the counting process which is associated to the occurrence

sequence of default events

$$N_t = \sum_{i=1}^n \mathbb{1}_{\{\sigma_i \leq t\}}, \quad t \geq 0.$$

Note that  $\mathcal{N}_t$  identifies with the  $\sigma$ -algebra generated by the vector  $\boldsymbol{\sigma}_{(i)} := (\sigma_1, \dots, \sigma_i)$  on the set  $\{N_t = i\} = \{\sigma_i \leq t < \sigma_{i+1}\}$ . Moreover, as we have mentioned, the vector  $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$  can be written in the form  $\boldsymbol{\sigma} = (f_1(\chi), \dots, f_n(\chi))$  where  $f_1, \dots, f_n$  are measurable functions on  $E$ .

The prediction process  $(\eta_t, t \geq 0)$  at time  $t \geq 0$ , i.e.  $\eta_t(dx) = \mathbb{P}(\chi \in dx | \mathcal{N}_t)$  can be calculated by using the Bayesian formula and taking into consideration each event  $\{\sigma_i \leq t < \sigma_{i+1}\}$  on which  $\eta_t$  is obtained as the conditional distribution of  $\chi$  given  $\boldsymbol{\sigma}_{(i)}$  restrained on the survival set  $]t, \infty[$  and normalized by the conditional survival probability of  $\sigma_{i+1}$  given  $\boldsymbol{\sigma}_{(i)}$ , that is,

$$\eta_t(dx) = \sum_{i=0}^n \mathbb{1}_{\{\sigma_i \leq t < \sigma_{i+1}\}} \frac{\eta_{|\boldsymbol{\sigma}_{(i)}}(\mathbb{1}_{\{t < u_{i+1}(x)\}} \cdot dx)}{\eta_{|\boldsymbol{\sigma}_{(i)}}(\mathbb{1}_{\{t < u_{i+1}(\cdot)\}})} = \frac{\eta_{|\boldsymbol{\sigma}_{(N_t)}}(\mathbb{1}_{\{t < u_{N_t+1}(x)\}} \cdot dx)}{\eta_{|\boldsymbol{\sigma}_{(N_t)}}(\mathbb{1}_{\{t < u_{N_t+1}(\cdot)\}})}, \quad (1)$$

where  $\eta_{|\boldsymbol{\sigma}_{(i)}}$  is the conditional law of  $\chi$  given  $\boldsymbol{\sigma}_{(i)}$ , namely,  $\eta_{|\boldsymbol{\sigma}_{(i)}}(\mathbb{1}_{\{t < u_{i+1}(\cdot)\}}) = \mathbb{P}(\sigma_{i+1} > t | \boldsymbol{\sigma}_{(i)})$ , and  $\eta_{|\boldsymbol{\sigma}_{(i)}}(\mathbb{1}_{\{t < u_{i+1}(x)\}} \cdot dx)$  denotes the random measure on  $E$  sending a bounded Borel function  $h : E \rightarrow \mathbb{R}$  to

$$\int_E h(x) \eta_{|\boldsymbol{\sigma}_{(i)}}(\mathbb{1}_{\{t < u_{i+1}(x)\}} \cdot dx) := \mathbb{E}[h(\chi) \mathbb{1}_{\{t < \sigma_{i+1}\}} | \boldsymbol{\sigma}_{(i)}].$$

At each default time, the new arriving default event brings a regime switching to the prediction process, which can be interpreted as the impact of default contagion phenomenon.

In the particular case where  $\chi$  coincides with  $\boldsymbol{\sigma}$  and the probability law of  $\boldsymbol{\sigma}$  has a density  $\alpha(\cdot)$  with respect to the Lebesgue measure, as in [10], we obtain a more explicit form of the prediction process as follows :

$$\eta_t(d\mathbf{u}) = \sum_{i=0}^n \mathbb{1}_{\{\sigma_i \leq t < \sigma_{i+1}\}} \delta_{\boldsymbol{\sigma}_{(i)}}(d\mathbf{u}_{(i)}) \mathbb{1}_{\{t < u_{i+1}\}} \frac{\alpha(\mathbf{u}) du_{(i+1:n)}}{\int_t^\infty \alpha(\mathbf{u}) du_{(i+1:n)}},$$

where  $\mathbf{u}_{(i)} = (u_1, \dots, u_i)$ ,  $\mathbf{u}_{(i+1:n)} = (u_{i+1}, \dots, u_n)$ , and

$$\int_t^\infty \alpha(\mathbf{u}) du_{(i+1:n)} := \int_t^\infty \cdots \int_t^\infty \alpha(\mathbf{u}) du_{i+1} \cdots du_n.$$

### 2.3 System weakened by failure

Inspired by the results in the theory of reliability system, especially the concept of a system weakened by failure, it is a natural question to ask if the faith of the market is always weakened by the observation of default events. We begin by introducing some important notions of stochastic orders.

**Stochastic order.** We assume that the Polish space  $E$  is equipped with a partial order  $\leq$ . A bounded Borel function  $h$  on  $E$  is called non-decreasing if it preserves the partial order, namely  $h(x) \leq h(y)$  for  $x$  and  $y$  in  $E$  such that  $x \leq y$ . We can then define a preorder  $\preceq$  on the set  $\mathcal{P}(E)$  (see e.g. [7]) such that for all  $\mu, \nu \in \mathcal{P}(E)$ ,  $\mu \preceq \nu$  if and only if  $\int_E h d\mu \leq \int_E h d\nu$  for any non-decreasing bounded Borel function  $h$  on  $E$ . The classical stochastic order on the set of all Borel probability measures on  $\mathbb{R}$  is a particular case with  $E = \mathbb{R}$  equipped with the usual order.

**Example of one default.** We first consider only one default event and the random variable  $\chi$  is just the default time  $\tau$ . The corresponding preorder on  $\mathcal{P}(\mathbb{R}_+)$  is nothing but the stochastic order  $\preceq_{\text{st}}$  as in [19]. In this case, the observation filtration  $(\mathcal{N}_t)_{t \geq 0}$  is generated by the default process  $(\mathbb{1}_{\{\tau \leq t\}}, t \geq 0)$ , then

$$\mathbb{E}[h(\tau) | \mathcal{N}_t] = \frac{\mathbb{1}_{\{\tau > t\}}}{\mathbb{P}(\tau > t)} \int_{]t, +\infty[} h(x) \eta(dx) + \mathbb{1}_{\{\tau \leq t\}} h(\tau),$$

where  $\eta$  is the probability law of  $\tau$ . In other words, one has

$$\eta_t(dx) = \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{1}_{]t, +\infty[}(x)}{S(t)} \eta(dx) + \mathbb{1}_{\{\tau \leq t\}} \delta_\tau(dx), \quad (2)$$

where  $\delta_\tau(dx)$  denotes the Dirac measure at  $\tau$  and  $S(t) := \mathbb{P}(\tau > t)$  is the survival probability. One observes that the nature of the prediction process  $(\eta_t, t \geq 0)$  changes at the default time  $\tau$ : in the case where  $\eta$  is absolutely continuous with respect to the Lebesgue measure, the random measure  $\eta_t$  is absolutely continuous before the default event, but degenerates into a Dirac measure after the default. Moreover, since the random measure  $\eta_{\tau-}$  is supported on the interval  $[\tau, +\infty)$ , for any non-decreasing function  $h$  on  $\mathbb{R}_+$ , one has

$$h(\tau) \leq \frac{\int_0^\infty h(x) \mathbb{1}_{[\tau, +\infty)}(x) \eta(dx)}{S(\tau-)}.$$

Therefore,

$$\eta_\tau = \delta_\tau \preceq_{\text{st}} \eta_{\tau-} = \frac{\mathbb{1}_{[\tau, +\infty)}(x)}{S(\tau-)} \eta(dx)$$

and the system is actually weakened upon the default event.

**Multi-default system.** We are interested in whether a multi-default system is weakened by failure. As we shall explain below, in this case, even if the default times are assumed to be ordered, the problem becomes much more subtle. In general, the system is only partially weakened by the observation of default events.

We consider a family  $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$  of ordered random times such that  $\sigma_1 \leq \dots \leq \sigma_n$ . The random variable  $\boldsymbol{\sigma}$  takes values in

$$E = \{\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{R}_+^n \mid u_1 \leq \dots \leq u_n\}.$$

We assume in this case that the random variable  $\chi$  identifies with  $\boldsymbol{\sigma}$ . By convention, let  $\sigma_0 = 0$  and  $\sigma_{n+1} = +\infty$ . The space  $E$  is equipped with the following partial order

$$(u_1, \dots, u_n) \leq (v_1, \dots, v_n) \iff \forall i \in \{1, \dots, n\}, u_i \leq v_i. \quad (3)$$

Consider firstly one single default time  $\sigma_j$  for any  $j \in \{1, \dots, n\}$  and let  $\eta_t^j$  be the  $j^{\text{th}}$  marginal distribution of  $\eta_t$ , i.e.  $\eta_t^j(du_j) = \mathbb{P}(\sigma_j \in du_j | \mathcal{N}_t)$ . By (1), one obtains

$$\eta_t^j(du_j) = \sum_{i=0}^{j-1} \mathbb{1}_{\{\sigma_i \leq t < \sigma_{i+1}\}} \mathbb{1}_{\{t < u_j\}} \frac{\mathbb{P}(\sigma_{i+1} > t, \sigma_j \in du_j | \boldsymbol{\sigma}_{(i)})}{\mathbb{P}(\sigma_{i+1} > t | \boldsymbol{\sigma}_{(i)})} + \mathbb{1}_{\{\sigma_j \leq t\}} \delta_{\sigma_j}(du_j).$$

Therefore, similar as in the single-name case, one has  $\eta_{\sigma_j}^j \preceq_{\text{st}} \eta_{\sigma_{j-}}^j$ , namely the  $j^{\text{th}}$  marginal system is weakened upon the observation of the  $j^{\text{th}}$  default. Moreover, it stabilizes at the Dirac measure  $\delta_{\sigma_j}$  after  $j^{\text{th}}$  default.

It is then interesting to know the behavior of this system upon the observation of all previous defaults. Unfortunately, as we show in the following example, in general the prediction process is not always weakened by the observation of default events. For simplicity, we consider the case where  $n = 2$  and the probability law of  $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)$  has a density function  $\alpha(\cdot)$  with respect to the Lebesgue measure. Note that the density hypothesis implies that there is no joint default so that the density function is assumed to be supported on

$$E = \{(u_1, u_2) \in \mathbb{R}^2 \mid 0 \leq u_1 < u_2 \leq T\}$$

and is continuous on this set,  $T$  being a constant horizon time. Then one has

$$\begin{aligned} \eta_t^2(du_2) &= \mathbb{1}_{\{t < \sigma_1\}} \left( \frac{\mathbb{1}_{\{t < u_2\}} \int_t^{u_2} \alpha(x, u_2) dx}{\int_t^T \int_t^y \alpha(x, y) dx dy} \right) du_2 \\ &+ \mathbb{1}_{\{\sigma_1 \leq t < \sigma_2\}} \frac{\mathbb{1}_{\{t < u_2\}} \alpha(\sigma_1, u_2)}{\int_t^T \alpha(\sigma_1, y) dy} du_2 + \mathbb{1}_{\{\sigma_2 \leq t\}} \delta_{\sigma_2}(du_2). \end{aligned}$$

Both random measures  $\eta_{\sigma_1}^2$  and  $\eta_{\sigma_1-}^2$  are supported on the interval  $[\sigma_1, T]$  and have continuous densities with respect to the Lebesgue measure on this interval. We denote by  $p_{\sigma_1}$  and  $p_{\sigma_1-}$  the densities of  $\eta_{\sigma_1}^2$  and  $\eta_{\sigma_1-}^2$  respectively, namely

$$p_{\sigma_1-}(u_2) = \frac{\mathbb{1}_{\{\sigma_1 < u_2\}} \int_{\sigma_1}^{u_2} \alpha(x, u_2) dx}{\int_{\sigma_1}^T \int_{\sigma_1}^y \alpha(x, y) dx dy}, \quad p_{\sigma_1}(u_2) = \frac{\mathbb{1}_{\{\sigma_1 \leq u_2\}} \alpha(\sigma_1, u_2)}{\int_{\sigma_1}^T \alpha(\sigma_1, y) dy}.$$

We have  $p_{\sigma_1-}(\sigma_1) = 0$  and  $p_{\sigma_1}(\sigma_1) > 0$  when  $\alpha$  is strictly positive on the diagonal  $\{(t, t) \mid t \in (0, T)\}$ . Therefore  $\eta_{\sigma_1-}^2 \not\preceq_{\text{st}} \eta_{\sigma_1}^2$ . In fact, for any fixed  $\omega \in \Omega$ , there exists  $\varepsilon > 0$  such that  $p_{\sigma_1-}(u) < p_{\sigma_1}(u)$  on  $[\sigma_1(\omega), \sigma_1(\omega) + \varepsilon)$ . Hence one has

$$\int_{\mathbb{R}_+} (-\mathbb{1}_{[0, \sigma_1(\omega) + \varepsilon)}(u)) \eta_{\sigma_1}(du) < \int_{\mathbb{R}_+} (-\mathbb{1}_{[0, \sigma_1(\omega) + \varepsilon)}(u)) \eta_{\sigma_1-}(du).$$



Similarly, one has  $\eta_{\sigma_1}^2 \not\preceq_{\text{st}} \eta_{\sigma_1-}^2$  if  $p_{\sigma_1}(T) > p_{\sigma_1-}(T)$ . This inequality is fulfilled for example when  $T = 1$  and

$$\alpha(u_1, u_2) = C \mathbb{1}_{\{0 \leq u_1 \leq u_2 \leq 1\}} e^{u_1 + u_2},$$

where  $C$  is a normalizing constant such that  $\int_E \alpha(u_1, u_2) du_1 du_2 = 1$ .

Although we cannot compare the random measures  $\eta_{\sigma_j-}$  and  $\eta_{\sigma_j}$  at each default time in general, the following proposition shows that in some particular cases, the system is weakened upon the observation of defaults.

**Proposition 2.1** *Assume that the increments of default times  $(X_i = \sigma_i - \sigma_{i-1})_{i=1}^n$  are mutually independent. Then one has*

$$\eta_{\sigma_j} \preceq \eta_{\sigma_j-} \text{ for any } j \in \{1, \dots, n\},$$

where the stochastic order  $\preceq_{\text{st}}$  on  $\mathcal{P}(E)$  is induced by the partial order (3) on  $E$ .

PROOF: We denote by  $\nu_j$  the probability law of  $X_j$  for  $j \in \{1, \dots, n\}$ . Since  $(X_j)_{j=1}^n$  are mutually independent, for any bounded Borel function  $f$  on  $E$ , one has

$$\mathbb{E}[f(\sigma_1, \dots, \sigma_n) | \sigma_{(j)}] = \int_{\mathbb{R}_+^{n-j}} f(\sigma_{(j)}, \sigma_j + x_{j+1}, \dots, \sigma_j + x_{j+1} + \dots + x_n) \nu_{j+1}(dx_{j+1}) \cdots \nu_n(dx_n).$$

Therefore, by (1) we obtain

$$\int f d\eta_{\sigma_j-} = \frac{\int_{\{x_j \geq \sigma_j - \sigma_{j-1}\}} f(\sigma_{(j-1)}, \sigma_{j-1} + x_j, \dots, \sigma_{j-1} + x_j + \dots + x_n) \nu_j(dx_j) \cdots \nu_n(dx_n)}{\nu_j([\sigma_j - \sigma_{j-1}, +\infty[)}$$

and

$$\int f d\eta_{\sigma_j} = \int f(\sigma_{(j)}, \sigma_j + x_{j+1}, \dots, \sigma_j + x_{j+1} + \dots + x_n) \nu_{j+1}(dx_{j+1}) \cdots \nu_n(dx_n).$$

If  $f$  is non-decreasing with respect to the partial order on  $E$ , then

$$f(\sigma_{(j-1)}, \sigma_{j-1} + x_j, \dots, \sigma_{j-1} + x_j + \dots + x_n) \geq f(\sigma_{(j)}, \sigma_j + x_{j+1}, \dots, \sigma_j + x_{j+1} + \dots + x_n)$$

for  $x_j \geq \sigma_j - \sigma_{j-1}$ . Therefore  $\int f d\eta_{\sigma_j} \leq \int f d\eta_{\sigma_j-}$ . The proposition is thus proved.

### 3 Interaction with the environment information

We now consider the general framework in presence of random environment which contains market information other than observations on the default system. We are interested in the impact of the environment information on the dynamics of default times and also in the interplay between the default information and the reference environment information.

### 3.1 Basic setting and different information levels

In this section, we model the global market structure including different risk factors by a product probability space. Besides the default variable  $\chi$  taking values in  $(E, \mathcal{E})$  with  $\mathcal{E} = \mathcal{B}(E)$  and the related information, we represent the random environment market risks by an auxiliary filtered probability space  $(\Omega^\circ, (\mathcal{F}_t^\circ)_{t \geq 0}, \mathbb{P}^\circ)$ . The global market is then described by  $\Omega := \Omega^\circ \times E$  equipped with the  $\sigma$ -algebra  $\mathcal{A} := \mathcal{F}_\infty^\circ \otimes \mathcal{E}$  where  $\mathcal{F}_\infty^\circ = \sigma(\bigcup_{t \geq 0} \mathcal{F}_t^\circ)$ . For any  $t \geq 0$ , let  $\mathcal{F}_t$  be the  $\sigma$ -algebra  $\mathcal{F}_t^\circ \otimes \{\emptyset, E\}$ . Then  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$  forms a filtration on  $\mathcal{A}$ . In this model, the default variable  $\chi : \Omega \rightarrow E$  is given by the second projection, i.e.  $\chi(\omega, x) = x$ . If  $Y$  is a random variable on the product space  $\Omega = \Omega^\circ \times E$ , sometimes we omit the first coordinate in the expression of the  $\mathcal{A}$ -measurable function  $Y$  and use the notation  $Y(x)$ ,  $x \in E$ , which denotes in fact the random variable  $Y(\cdot, x)$ .

The full information on the random environment market is given by  $\mathcal{F}_\infty = \sigma(\bigcup_{t \geq 0} \mathcal{F}_t)$ , and the full information on the default risks is given by  $\sigma(\chi)$ . The case where the two sources of risks are independent is important and will serve as the building stone of the general case in our paper. The probability measure in this case corresponds to the product measure

$$\overline{\mathbb{P}}(d\omega, dx) = \mathbb{P}^\circ(d\omega) \otimes \eta(dx)$$

where  $\eta$  is a Borel probability measure on  $E$ . Note that the law of  $\chi$  under the product probability  $\overline{\mathbb{P}}$  is  $\eta$ , and  $\chi$  is independent of  $\mathcal{F}_\infty$  under  $\overline{\mathbb{P}}$ .

In the general case, we describe the interaction between the two sources of risks as follows. We characterize the dependence between  $\chi$  and  $\mathbb{F}$  by a change of probability with respect to the product probability measure  $\overline{\mathbb{P}}$ . We suppose that there exist a horizon time<sup>1</sup>  $T \geq 0$  and a positive  $\mathcal{F}_T \otimes \mathcal{E}$ -measurable random variable  $\beta_T(\cdot)$  such that

$$\mathbb{E}_{\mathbb{P}^\circ}[\beta_T(x)] = 1, \quad \forall x \in E. \quad (4)$$

This condition implies  $\mathbb{E}_{\overline{\mathbb{P}}}[\beta_T(\chi)] = 1$ . Let us define the probability measure  $\mathbb{P}$  on  $(\Omega, \mathcal{A})$  by

$$\mathbb{P}(d\omega, dx) = \beta_T(\omega, x) \overline{\mathbb{P}}(d\omega, dx) \quad (5)$$

Under the probability measure  $\mathbb{P}$ , the dependence between the default variable  $\chi$  and other market environment is characterized in a dynamic manner.

We next present the different levels of information including the total information and the accessible information evolving with time. The accessible information at time  $t \geq 0$  on the random environment is given by  $\mathcal{F}_t$ . The observable default information on  $E$  is described by a filtration of the  $\sigma$ -algebra  $\sigma(\chi)$  which we precise later.

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<sup>1</sup>We can consider the more general case where  $T$  is a finite  $\mathbb{F}$ -stopping time in a similar way.

The first natural filtration associated with this framework is the right continuous filtration  $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$ , which is the regularization of the filtration

$$\mathcal{F}_t^\circ \otimes \mathcal{E} = \mathcal{F}_t \vee \sigma(\chi), \quad t \geq 0$$

Note that under the density hypothesis,  $\mathcal{H}_t = \mathcal{F}_t \vee \sigma(\chi)$ . The  $\sigma$ -algebra  $\mathcal{H}_t$  represents the global information and is not totally observable since the default variable  $\chi$  is not known by the investors at an arbitrary time  $t$ . In the literature of enlargement of filtration, this filtration is called the initial enlargement of  $\mathbb{F}$  by  $\chi$ . We recall that (e.g. [15, Lemma 4.4]) any  $\mathcal{H}_t$ -measurable random variable can be written in the form  $Y_t(\chi)$  where  $Y_t(\cdot)$  is an  $\mathcal{F}_t^\circ \otimes \mathcal{E}$ -measurable function.

The observable default information of an investor can be induced by some filtration  $(\mathcal{N}_t^E)_{t \geq 0}$  of the Borel  $\sigma$ -algebra  $\mathcal{E}$  which is right continuous. In the case where  $(\mathcal{N}_t^E)_{t \geq 0}$  is generated by an observation process  $(N_t, t \geq 0)$ , the inverse image (see e.g. Resnick [20, §3.1]) filtration  $(\mathcal{N}_t)_{t \geq 0}$  on  $\Omega$  is generated by the process  $(N_t \circ \chi, t \geq 0)$ . Similar as we have described in the previous section,  $(\mathcal{N}_t^E)_{t \geq 0}$  can be generated by an observation process defined on  $(E, \mathcal{E})$  such as the default counting process or the cumulative loss process except that here the observation process is defined on the Polish space  $E$ . For example, when  $E = \mathbb{R}_+^n$ , the observation process on  $(E, \mathcal{E})$  of the default events is given by

$$\left( (t, (x_1, \dots, x_n)) \in \mathbb{R}_+ \times E \right) \longmapsto \sum_{i=1}^n \mathbb{1}_{\{x_i \leq t\}}.$$

The filtration  $(\mathcal{N}_t^E)_{t \geq 0}$  determines a filtration on  $(\Omega, \mathcal{A})$  by its inverse image by  $\chi$  as

$$\mathcal{N}_t := \chi^{-1}(\mathcal{N}_t^E) = \{\chi^{-1}(A) \mid A \in \mathcal{N}_t^E\} \quad (6)$$

By abuse of notation, we also use the notation  $(\mathcal{N}_t)_{t \geq 0}$  to denote the completion of the filtration  $(\chi^{-1}(\mathcal{N}_t^E))_{t \geq 0}$ , which satisfies the usual conditions. The filtration  $(\mathcal{N}_t)_{t \geq 0}$  represents the observable information on the default system on the product space  $(\Omega, \mathcal{A})$ . To describe both default and environmental risks at  $t$ , the global accessible market information is given by a progressive enlargement of filtration  $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$  with

$$\mathcal{G}_t = \bigcap_{s > t} (\mathcal{F}_s \vee \mathcal{N}_s), \quad t \geq 0.$$

Note that the following relation holds

$$\mathcal{F}_t \subset \mathcal{G}_t \subset \mathcal{H}_t.$$

In order to make predictions and estimations of the system given different levels of information, we are interested in conditional distributions of  $\chi$  with respect to corresponding filtrations.

Our idea is to begin from the product probability measure  $\overline{\mathbb{P}}$  under which  $\chi$  is independent with  $\mathcal{F}_\infty$ . Then the general case under an arbitrary probability measure  $\mathbb{P}$  can be achieved by a change of probability.

### 3.2 Conditional default distributions under $\overline{\mathbb{P}}$

Under the product probability  $\overline{\mathbb{P}} = \mathbb{P}^\circ \otimes \eta$ , the  $\sigma$ -algebras  $\mathcal{F}_\infty$  and  $\sigma(\chi)$  are independent so that the calculations on  $\mathbb{H}$  are simple applications of Fubini's theorem. We also fix some notation which will be useful in the sequel.

**Default information.** Denote by  $\eta_t^{\mathcal{N}}$  the conditional probability law of  $\chi$  given  $\mathcal{N}_t$ . In other words,  $(\eta_t^{\mathcal{N}}, t \geq 0)$  is the prediction process of  $\chi$ . Since  $\chi$  is independent of  $\mathcal{F}_\infty$  under  $\overline{\mathbb{P}}$ , for any bounded or positive  $\mathcal{A}$ -measurable function  $\Psi$  on  $\Omega = \Omega^\circ \times E$ , one has

$$\eta_t^{\mathcal{N}}(\Psi) := \int_E \Psi(\cdot, x) \eta_t^{\mathcal{N}}(dx) = \mathbb{E}_{\overline{\mathbb{P}}}[\Psi | \mathcal{F}_\infty \vee \mathcal{N}_t], \quad \overline{\mathbb{P}}\text{-a.s.}$$

By Dellacherie and Meyer [6, VI.4], there exists a càdlàg version of the martingale  $(\eta_t^{\mathcal{N}}(\Psi), t \geq 0)$  as conditional expectations.

**Complete information  $\mathbb{H}$ .** The case for the complete information  $\mathbb{H}$  is more delicate since we have to take care about negligible sets. In full generality, the equality  $X(\cdot, x) = Y(\cdot, x)$ ,  $\mathbb{P}^\circ$ -a.s. for all  $x \in E$  does not imply  $X(\cdot, \chi) = Y(\cdot, \chi)$ ,  $\overline{\mathbb{P}}$ -a.s.. We need here a suitable version for such processes. This difficulty can be overcome by Meyer [18] and Stricker and Yor [22].

- (i) Given a non-negative  $\mathcal{A}$ -measurable function  $\Psi$  on  $\Omega$ , from [18], there exists a càdlàg  $\mathbb{H}$ -adapted process  $(\Psi_t^{\mathcal{F}}(\cdot), t \geq 0)$  such that, for any  $x \in E$ , and for any  $t \geq 0$ ,

$$\Psi_t^{\mathcal{F}}(x) = \mathbb{E}_{\mathbb{P}^\circ}[\Psi(\cdot, x) | \mathcal{F}_t^\circ], \quad \mathbb{P}^\circ\text{-a.s.}$$

In particular, if  $X_t$  is an  $\mathcal{F}_t^\circ$ -measurable random variable valued in  $E$ , then one has

$$\Psi_t^{\mathcal{F}}(X_t) = \mathbb{E}_{\mathbb{P}^\circ}[\Psi(\cdot, X_t) | \mathcal{F}_t^\circ], \quad \mathbb{P}^\circ\text{-a.s.}$$

We call  $(\Psi_t^{\mathcal{F}}(x))_{x \in E}$  a *parametrized*  $(\mathbb{F}^\circ, \mathbb{P}^\circ)$ -martingale depending on a parameter  $x \in E$ , to emphasize that the conditional expectation property is valid for all values of  $x$  and  $t$  outside of a null set.

- (ii) This parametrized version of  $\mathbb{F}^\circ$ -conditional expectation as a function of both variables  $(\omega, x)$  is the basic tool for studying projections with respect to  $\mathbb{H}$ , since under the product measure  $\overline{\mathbb{P}} = \mathbb{P}^\circ \otimes \eta$ ,

$$\mathbb{E}_{\overline{\mathbb{P}}}[\Psi | \mathcal{H}_t] = \Psi_t^{\mathcal{F}}(\chi), \quad \overline{\mathbb{P}}\text{-a.s.} \tag{7}$$

Furthermore, we can extend  $\eta_t^{\mathcal{N}}$  to a  $\mathcal{G}_t$ -random measure on  $(\Omega, \mathcal{F}_t^\circ \otimes \mathcal{E})$  which sends any non-negative  $\mathcal{H}_t$ -measurable random variable  $Y_t(\cdot)$  to  $\eta_t^{\mathcal{N}}(Y_t(\cdot)) = \int_E Y_t(x) \eta_t^{\mathcal{N}}(dx)$ . In the following, by abuse of language, we use  $\eta_t^{\mathcal{N}}$  to denote the conditional laws with respect to both  $\mathcal{N}_t$  and  $\mathcal{G}_t$ .

- (iii) From the point of view of processes, these results can be interpreted as a characterization of  $(\mathbb{H}, \overline{\mathbb{P}})$ -martingale in terms of a parametrized  $(\mathbb{F}^\circ, \mathbb{P}^\circ)$ -martingale depending on a parameter  $x \in E$ . We shall discuss the martingale properties in more detail in Section 4.2.

**Accessible information  $\mathbb{G}$ .** For the observable information  $\mathbb{G}$ , the projection is firstly made on a larger filtration which includes more information than  $\mathbb{G}$  either on  $\Omega^\circ$  or on  $E$ .

- (i) We note that the conditional law of  $\chi$  given  $\mathbb{G}$  under  $\overline{\mathbb{P}}$  coincides with  $\eta^{\mathcal{N}}$ , which is a consequence of the independence property between  $\chi$  and  $\mathbb{F}$ . More precisely, for any non-negative  $\mathcal{H}_t$ -measurable random variable  $Y_t(\chi)$ , we have

$$\mathbb{E}_{\overline{\mathbb{P}}}[Y_t(\chi) | \mathcal{G}_t] = \int_E Y_t(x) \eta_t^{\mathcal{N}}(dx) = \eta_t^{\mathcal{N}}(Y_t(\cdot)). \quad (8)$$

- (ii) Consider now a non-negative  $\mathcal{A}$ -measurable random variable  $Y$  on  $\Omega$ . The calculation of its  $\mathcal{G}_t$ -conditional expectation can be done in two different ways as shown below :

$$\begin{array}{ccc} Y & \xrightarrow{\mathcal{H}_t = \mathcal{F}_t \vee \sigma(\chi)} & \mathbb{E}_{\overline{\mathbb{P}}}[Y | \mathcal{H}_t] \\ \mathcal{F}_\infty \vee \mathcal{N}_t \downarrow & & \downarrow \mathcal{G}_t \\ \mathbb{E}_{\overline{\mathbb{P}}}[Y | \mathcal{F}_\infty \vee \mathcal{N}_t] & \xrightarrow{\mathcal{G}_t} & \mathbb{E}_{\overline{\mathbb{P}}}[Y | \mathcal{G}_t] \end{array} \quad (9)$$

On the one hand, using the notation introduced in (7),

$$\mathbb{E}_{\overline{\mathbb{P}}}[Y | \mathcal{G}_t] = \mathbb{E}_{\overline{\mathbb{P}}}[\mathbb{E}_{\overline{\mathbb{P}}}[Y | \mathcal{H}_t] | \mathcal{G}_t] = \mathbb{E}_{\overline{\mathbb{P}}}[Y_t^{\mathcal{F}}(\chi) | \mathcal{G}_t]$$

which, by (8), equals

$$\mathbb{E}_{\overline{\mathbb{P}}}[Y | \mathcal{G}_t] = \eta_t^{\mathcal{N}}(Y_t^{\mathcal{F}}(\cdot)) = \int_E Y_t^{\mathcal{F}}(x) \eta_t^{\mathcal{N}}(dx). \quad (10)$$

On the other hand, as shown in (9), the above result can also be obtained by using the intermediary  $\sigma$ -algebra  $\mathcal{F}_\infty \vee \mathcal{N}_t$ . Note that by the monotone class theorem, it suffices to consider  $Y(\omega, x)$  of the form  $Y^\circ(\omega)h(x)$  where  $Y^\circ$  is  $\mathcal{F}_\infty^\circ$ -measurable and  $h$  is a Borel function on  $E$ , then

$$\begin{aligned} \mathbb{E}_{\overline{\mathbb{P}}}[Y | \mathcal{G}_t] &= \mathbb{E}_{\overline{\mathbb{P}}}[\mathbb{E}_{\overline{\mathbb{P}}}[Y^\circ h | \mathcal{F}_\infty \vee \mathcal{N}_t] | \mathcal{G}_t] = \mathbb{E}_{\overline{\mathbb{P}}}[Y^\circ \eta_t^{\mathcal{N}}(h) | \mathcal{G}_t] \\ &= \eta_t^{\mathcal{N}}(h) \mathbb{E}_{\mathbb{P}^\circ}[Y^\circ | \mathcal{F}_t^\circ] = \eta_t^{\mathcal{N}}(\mathbb{E}_{\mathbb{P}^\circ}[Y^\circ h | \mathcal{F}_t^\circ]) = \eta_t^{\mathcal{N}}(\mathbb{E}_{\mathbb{P}^\circ}[Y | \mathcal{F}_t^\circ]) \end{aligned}$$

which equals (10).

(iii) We have from (10) a characterization result for any uniformly integrable  $(\mathbb{G}, \overline{\mathbb{P}})$ -martingale which can be written as the integral of a parametrized  $(\mathbb{F}^\circ, \mathbb{P}^\circ)$  martingale  $Y^{\mathcal{F}}(x)$ ,  $x \in E$ , with respect to the random measure  $\eta^{\mathcal{N}}(dx)$ .

### 3.3 Change of probability measures

We now concentrate on a probability measure  $\mathbb{P}$  on  $(\Omega, \mathcal{A})$  whose Radon-Nikodym derivative with respect to  $\overline{\mathbb{P}}$  is given by  $\beta_T(\chi)$  on  $\mathcal{H}_T$ . We still examine the different information levels. The conditional law of  $\chi$  given  $\mathcal{N}_t$  remains invariant under  $\mathbb{P}$  and  $\overline{\mathbb{P}}$  because of (4). In fact, for any bounded function  $h$  and any  $A \in \mathcal{N}_t(E)$ ,  $\mathbb{E}_{\mathbb{P}}[h(\chi)\mathbb{1}_A(\chi)] = \mathbb{E}_{\overline{\mathbb{P}}}[h(\chi)\mathbb{1}_A(\chi)\beta_T(\chi)] = \int_E \mathbb{E}_{\mathbb{P}^\circ}[\beta_T(x)]h(x)\mathbb{1}_A(x)\eta(dx) = \mathbb{E}_{\overline{\mathbb{P}}}[h(\chi)\mathbb{1}_A(\chi)]$ . We use the notation  $\eta_t^{\mathcal{N}}$  for the conditional law of  $\chi$  given  $\mathcal{N}_t$  under both probability measures.

**Complete information  $\mathbb{H}$ .** The Radon-Nikodym density of  $\mathbb{P}$  with respect to  $\overline{\mathbb{P}}$  on  $\mathcal{H}_t$  is specified by the parametrized  $(\mathbb{F}^\circ, \mathbb{P}^\circ)$ -martingale  $\beta_t^{\mathcal{F}}(x) = \mathbb{E}_{\mathbb{P}^\circ}[\beta_T(x)|\mathcal{F}_t^\circ]$  depending on the parameter  $x \in E$ . We use the notation  $\beta_t$  to denote  $\beta_t^{\mathcal{F}}(\chi)$  when there is no ambiguity. In the following, we suppose that  $\beta_t > 0$  for all  $t \geq 0$ .

- (i) Let us first consider the  $\mathcal{H}_t$ -conditional expectation of a non-negative  $\mathcal{A}$ -measurable random variable  $Y$  on  $\Omega$ . By the change of probability and Fubini's theorem under  $\overline{\mathbb{P}}$ ,

$$\mathbb{E}_{\mathbb{P}}[Y|\mathcal{H}_t] = \frac{\mathbb{E}_{\overline{\mathbb{P}}}[Y\beta_T|\mathcal{H}_t]}{\beta_t} = \frac{(Y\beta)_t^{\mathcal{F}}}{\beta_t} \quad (11)$$

where the last equality is from (7).

- (ii) In this setting, by the equality (11), an  $\mathbb{H}$ -adapted process is an  $(\mathbb{H}, \mathbb{P})$ -martingale if and only if its product with  $\beta$  is an  $(\mathbb{H}, \overline{\mathbb{P}})$ -martingale, or equivalently, a parametrized  $(\mathbb{F}^\circ, \mathbb{P}^\circ)$ -martingale depending on a parameter  $x \in E$ .

**Accessible information  $\mathbb{G}$ .** We use the notation  $\eta^{\mathcal{G}}$  for the conditional law of  $\chi$  given  $\mathcal{G}_t$  under the probability  $\mathbb{P}$ .

- (i) The Bayes formula allows to calculate directly the conditional law  $\eta^{\mathcal{G}}$  by

$$\eta_t^{\mathcal{G}}(dx) = \frac{\eta_t^{\mathcal{N}}(\beta_t(x) \cdot dx)}{\eta_t^{\mathcal{N}}(\beta_t(\cdot))} \quad (12)$$

where for a non-negative  $\mathcal{A}$ -measurable function  $\Psi$  on  $\Omega$ , the notation  $\eta_t^{\mathcal{N}}(\Psi(x) \cdot dx)$  denotes the  $\mathcal{A}$ -random measure on  $E$  which sends a positive Borel function  $f$  on  $E$  to

$$\int_E f(x) \eta_t^{\mathcal{N}}(\Psi(x) \cdot dx) = \eta_t^{\mathcal{N}}(f(\cdot)\Psi(\cdot)) \quad (13)$$

(ii) The  $\mathcal{G}_t$ -conditional expectation of a positive  $\mathcal{H}_t$ -measurable random variable  $Y_t(\chi)$  is

$$\mathbb{E}_{\mathbb{P}}[Y_t(\chi)|\mathcal{G}_t] = \int_E Y_t(x)\eta_t^{\mathcal{G}}(dx) =: \eta_t^{\mathcal{G}}(Y_t(\cdot)) \quad (14)$$

(iii) For a non-negative  $\mathcal{A}$ -measurable random variable  $Y$  on  $\Omega$ , we first project on the larger  $\sigma$ -algebra  $\mathcal{H}_t$  and then use (11) and (14) to obtain

$$\mathbb{E}_{\mathbb{P}}[Y|\mathcal{G}_t] = \mathbb{E}_{\mathbb{P}}[\mathbb{E}_{\mathbb{P}}[Y|\mathcal{H}_t]|\mathcal{G}_t] = \int_E \frac{(Y\beta)_t^{\mathcal{F}}(x)}{\beta_t(x)}\eta_t^{\mathcal{G}}(dx). \quad (15)$$

An equivalent form can be obtained by using the Bayes formula as

$$\mathbb{E}_{\mathbb{P}}[Y|\mathcal{G}_t] = \frac{\eta_t^{\mathcal{N}}((Y\beta)_t^{\mathcal{F}}(\cdot))}{\eta_t^{\mathcal{N}}(\beta_t(\cdot))} \quad (16)$$

The equality between (15) and (16) can also be shown by (12).

(iv) Accordingly, a  $(\mathbb{G}, \mathbb{P})$ -martingale can be characterized as follows: a  $\mathbb{G}$ -adapted process is a  $(\mathbb{G}, \mathbb{P})$ -martingale if and only if its product with  $\eta_t^{\mathcal{N}}(\beta_t(\cdot))$  is an integral of the parametrized  $(\mathbb{F}^{\circ}, \mathbb{P}^{\circ})$ -martingale  $(Y\beta)^{\mathcal{F}}(x)$ ,  $x \in E$  with respect to  $\eta^{\mathcal{N}}(dx)$ , or alternatively, if and only if it can be written as the integral of the quotient of two parametrized  $(\mathbb{F}^{\circ}, \mathbb{P}^{\circ})$ -martingales  $(Y\beta)_t^{\mathcal{F}}(x)/\beta_t(x)$  with respect to  $\eta_t^{\mathcal{G}}(dx)$ .

The following proposition summarizes the previous results.

**Proposition 3.1** *Let  $Y_T(\chi)$  be a non-negative or bounded  $\mathcal{F}_T \vee \sigma(\chi)$ -measurable random variable and  $t \leq T$ .*

(i) *Under the probability  $\bar{\mathbb{P}}$ ,*

$$\mathbb{E}_{\bar{\mathbb{P}}}[Y_T(\chi)|\mathcal{G}_t] = \eta_t^{\mathcal{N}}(\mathbb{E}_{\mathbb{P}^{\circ}}[Y_T(\cdot)|\mathcal{F}_t^{\circ}]) = \int_E \mathbb{E}_{\mathbb{P}^{\circ}}[Y_T(x)|\mathcal{F}_t^{\circ}]\eta_t^{\mathcal{N}}(dx) \quad (17)$$

(ii) *Under the probability  $\mathbb{P}$ ,*

$$\mathbb{E}_{\mathbb{P}}[Y_T(\chi)|\mathcal{G}_t] = \int_E \mathbb{E}_{\mathbb{P}^{\circ}}\left[\frac{Y_T(x)\beta_T(x)}{\beta_t(x)}\middle|\mathcal{F}_t^{\circ}\right]\eta_t^{\mathcal{G}}(dx) \quad (18)$$

*or equivalently,*

$$\mathbb{E}_{\mathbb{P}}[Y_T(\chi)|\mathcal{G}_t] = \frac{\eta_t^{\mathcal{N}}(\mathbb{E}_{\mathbb{P}^{\circ}}[Y_T(\cdot)\beta_T(\cdot)|\mathcal{F}_t^{\circ}])}{\eta_t^{\mathcal{N}}(\beta_t(\cdot))} \quad (19)$$

PROOF: (i) By using the independence between  $\chi$  and  $\mathcal{F}_{\infty}$  under the probability measure  $\bar{\mathbb{P}}$  and the fact that  $\mathbb{P} = \mathbb{P}^{\circ}$  on  $\mathcal{F}_{\infty}$ , one obtains

$$\mathbb{E}_{\bar{\mathbb{P}}}[Y_T(\chi)|\mathcal{H}_t] = \mathbb{E}_{\mathbb{P}^{\circ}}[Y_T(x)|\mathcal{F}_t^{\circ}]|_{x=\chi}. \quad (20)$$

Therefore, by (8), we obtain

$$\mathbb{E}_{\bar{\mathbb{P}}}[Y_T(\chi)|\mathcal{G}_t] = \mathbb{E}_{\bar{\mathbb{P}}}[\mathbb{E}_{\bar{\mathbb{P}}}[Y_T(\chi)|\mathcal{H}_t]|\mathcal{G}_t] = \eta_t^{\mathcal{N}}(\mathbb{E}_{\mathbb{P}^\circ}[Y_T(\cdot)|\mathcal{F}_t^\circ]) = \int_E \mathbb{E}_{\mathbb{P}^\circ}[Y_T(x)|\mathcal{F}_t^\circ] \eta_t^{\mathcal{N}}(dx).$$

(ii) By the change of probability measures from  $\bar{\mathbb{P}}$  to  $\mathbb{P}$ , one has

$$\mathbb{E}_{\mathbb{P}}[Y_T(\chi)|\mathcal{H}_t] = \mathbb{E}_{\bar{\mathbb{P}}}\left[\frac{Y_T(\chi)\beta_T(\chi)}{\beta_t(\chi)}\middle|\mathcal{H}_t\right] = \mathbb{E}_{\mathbb{P}^\circ}\left[\frac{Y_T(x)\beta_T(x)}{\beta_t(x)}\middle|\mathcal{F}_t^\circ\right]\Bigg|_{x=\chi}$$

We then deduce the equality (18). Finally, by (12), the equality (18) leads to

$$\mathbb{E}_{\mathbb{P}}[Y_T(\chi)|\mathcal{G}_t] = \frac{1}{\eta_t^{\mathcal{N}}(\beta_t(\cdot))} \int_E \mathbb{E}_{\mathbb{P}^\circ}\left[\frac{Y_T(x)\beta_T(x)}{\beta_t(x)}\middle|\mathcal{F}_t^\circ\right] \eta_t^{\mathcal{N}}(\beta_t(x) \cdot dx) = \frac{\eta_t^{\mathcal{N}}(\mathbb{E}_{\mathbb{P}^\circ}[Y_T(\cdot)\beta_T(\cdot)|\mathcal{F}_t^\circ])}{\eta_t^{\mathcal{N}}(\beta_t(\cdot))}$$

which completes the proof.  $\square$

### 3.4 Density condition in the product space framework

In this subsection, we show that the density hypothesis of Jacod is satisfied in the framework of the product space. We give an explicit formula of the conditional density process in terms of the Radon-Nikodym derivative process  $\beta$ .

**Proposition 3.2** *For  $t \geq 0$ , let  $\eta_t^{\mathcal{F}}$  be the  $\mathcal{F}_t$ -conditional probability law of  $\chi$  under the probability measure  $\mathbb{P}$ . Then  $\eta_t^{\mathcal{F}}$  is equivalent with respect to the probability law  $\eta$  of  $\chi$ . Moreover, we have*

$$\frac{d\eta_t^{\mathcal{F}}}{d\eta} = \frac{\beta_t(\cdot)}{\int_E \beta_t(x) \eta(dx)}.$$

PROOF: Since  $\chi$  is independent of  $\mathbb{F}$  under the probability  $\bar{\mathbb{P}}$ , one has

$$\mathbb{E}_{\bar{\mathbb{P}}}[\beta_t|\mathcal{F}_t] = \int_E \beta_t(x) \eta(dx).$$

Let  $f$  be a non-negative Borel function on  $E$ . By Bayes formula, we obtain

$$\mathbb{E}_{\mathbb{P}}[f(\chi)|\mathcal{F}_t] = \frac{\mathbb{E}_{\bar{\mathbb{P}}}[f(\chi)\beta_t|\mathcal{F}_t]}{\mathbb{E}_{\bar{\mathbb{P}}}[\beta_t|\mathcal{F}_t]} = \frac{\int_E f(x)\beta_t(x) \eta(dx)}{\int_E \beta_t(x) \eta(dx)}.$$

The proposition is thus proved.  $\square$

## 4 Multi-default system and density approach

In practice, the investors obtain different types of information on the financial market modelled by a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ , which is not necessarily given in a product form of an auxiliary measurable space with  $(E, \mathcal{E})$ . However, the product space framework which we have described



provides useful ideas and tools for the study of the problem in the general setting. Recall that in the previous section, we have established a product space framework to describe the financial market and used a change of probability measure for the dependence between the default risk variable  $\chi$  and the underlying market information filtration  $\mathbb{F}$ . The argument of change of probability is closely related to the density hypothesis in the theory of enlargement of filtration.

In the enlargement of filtrations, Jacod's density hypothesis plays an important role. In the credit risk analysis, the default density approach has been proposed in [9] to study the impact of a default event and in [10] for ordered multiple defaults. In this section, we are interested in the link between the conditional density and the Radon-Nikodym derivative of change of probability measures in the general setting of a multi-default system, and we investigate the martingale properties.

#### 4.1 The general framework: density and change of probability measure

We now discuss the density approach in a general multi-default system. Let us consider a measurable space  $(\Omega, \mathcal{A})$  which describes the financial market, equipped with a probability measure  $\mathbb{P}$ . We note that  $(\Omega, \mathcal{A})$  is not necessarily a product space as in Section 3. Let  $\chi$  be a random variable taking values in a Polish space  $(E, \mathcal{E})$  with  $\mathcal{E} = \mathcal{B}(E)$ . Let  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$  be a filtration of  $\mathcal{A}$  which represents the environmental information of the market which is not directly related to the default events. For convenience, we assume that  $\mathcal{F}_0$  is the trivial  $\sigma$ -algebra. We present below the density hypothesis in this general setting.

**Hypothesis 4.1** The conditional probability law of  $\chi$  given the filtration  $\mathbb{F}$  admits a density with respect to a  $\sigma$ -finite measure  $\nu$  on  $(E, \mathcal{E})$ , i.e., for any  $t \geq 0$ , there exists an  $\mathcal{F}_t \otimes \mathcal{E}$ -measurable function  $(\omega, x) \rightarrow \alpha_t(\omega, x)$  such that for any non-negative Borel function  $f$ ,

$$\mathbb{E}_{\mathbb{P}}[f(\chi)|\mathcal{F}_t] = \int_E f(x)\alpha_t(x)\nu(dx), \quad \mathbb{P}\text{-a.s.}$$

We suppose moreover that  $\alpha_t(x)$  is strictly positive. Under the probability  $\mathbb{P}$ , for any  $x \in E$ , the process  $(\alpha_t(x), t \geq 0)$  is an  $\mathbb{F}$ -martingale. The probability law  $\eta$  of  $\chi$  is absolutely continuous with respect to the measure  $\nu$ , more precisely,

$$\eta(dx) = \alpha_0(x)\nu(dx).$$

The density process  $\alpha(\cdot)$  can be interpreted using the language of change of probability measure. To establish the link between the density and the change of probability, we use the auxiliary product space  $\Omega \times E$  and we introduce the following map  $\Gamma_\chi$  as a useful tool. Let  $\Gamma_\chi$

be the map from  $\Omega$  to  $\Omega \times E$  sending  $\omega \in \Omega$  to  $(\omega, \chi(\omega))$ . The composition of  $\Gamma_\chi : \Omega \rightarrow \Omega \times E$  with the second projection  $\Omega \times E \rightarrow E$ ,  $(\omega, x) \mapsto x$  coincides with the random map  $\chi : \Omega \rightarrow E$ . Moreover, if  $Y(\cdot)$  is a function on  $\Omega \times E$ , then the expression  $Y(\chi)$  denotes actually  $Y(\cdot) \circ \Gamma_\chi$  as a function on  $\Omega$ . We next make precise the measurability of the application:

$$\begin{array}{ccc} \Omega & \xrightarrow{\Gamma_\chi} & \Omega \times E \xrightarrow{Y(\cdot)} \mathbb{R} . \\ & \searrow & \uparrow \\ & & Y(\chi) \end{array} \quad (21)$$

**Lemma 4.2** *Let  $\mathcal{F}$  be a sub- $\sigma$ -algebra of  $\mathcal{A}$  on  $\Omega$  and  $\mathcal{E}_0$  be a sub- $\sigma$ -algebra of  $\mathcal{E}$  on  $E$ . Then*

- 1) *the map  $\Gamma_\chi : (\Omega, \mathcal{F} \vee \chi^{-1}(\mathcal{E}_0)) \rightarrow (\Omega \times E, \mathcal{F} \otimes \mathcal{E}_0)$  is measurable, where  $\chi^{-1}(\mathcal{E}_0) = \{\chi^{-1}(B) \mid B \in \mathcal{E}_0\}$  is a  $\sigma$ -algebra on  $\Omega$ ;*
- 2) *if the map  $Y(\cdot) : \Omega \times E \rightarrow \mathbb{R}$  is  $\mathcal{F} \otimes \mathcal{E}_0$ -measurable, then  $Y(\chi) : \Omega \rightarrow \mathbb{R}$  is  $\mathcal{F} \vee \chi^{-1}(\mathcal{E}_0)$ -measurable.*

PROOF: For 1), it suffices to prove that for all  $A \in \mathcal{F}$  and  $B \in \mathcal{E}_0$ , one has

$$\Gamma_\chi^{-1}(A \times B) \in \mathcal{F} \vee \chi^{-1}(\mathcal{E}_0)$$

since  $\mathcal{F} \otimes \mathcal{E}_0$  is generated by the sets of the form  $A \times B$ . Indeed,

$$\begin{aligned} \Gamma_\chi^{-1}(A \times B) &= \{\omega \in \Omega \mid (\omega, \chi(\omega)) \in A \times B\} = \{\omega \in A \mid \chi(\omega) \in B\} \\ &= A \cap \chi^{-1}(B) \in \mathcal{F} \vee \chi^{-1}(\mathcal{E}_0) \end{aligned}$$

which implies the first assertion. The second assertion 2) results from the fact that the composition of two measurable maps is still measurable.  $\square$

To investigate the different types of information, we are interested in the corresponding filtrations. Let us consider an arbitrary filtration  $\mathcal{N}^E = (\mathcal{N}_t^E)_{t \geq 0}$  on  $(E, \mathcal{E})$ , which leads by the inverse image to the filtration  $(\mathcal{N}_t)_{t \geq 0}$  on  $(\Omega, \mathcal{A})$  where  $\mathcal{N}_t = \chi^{-1}(\mathcal{N}_t^E)$ , assumed to satisfy the usual conditions. Then the filtration  $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$  denotes an enlargement of  $\mathbb{F}$  by

$$\mathcal{G}_t = \cap_{s > t} (\mathcal{F}_s \vee \mathcal{N}_s), \quad t \geq 0.$$

Here the filtration  $\mathbb{G}$  can be viewed as a progressive enlargement built in a general way: we enlarge  $\mathbb{F}$  with an arbitrary filtration which is not necessarily generated by an indicator process.

The above Lemma 4.2 implies directly the following result.

**Corollary 4.3** *Let  $(Y_t(\cdot), t \geq 0)$  be a process adapted to the filtration  $\mathbb{F} \otimes \mathcal{N}^E$ , then  $(Y_t(\chi), t \geq 0)$  is a  $\mathbb{G}$ -adapted process.*

**Remark 4.4** 1) In the case where  $\mathcal{N}_t^E$  coincides with the  $\sigma$ -algebra  $\mathcal{E}$  for all  $t$ , we have  $\mathcal{N}_t = \chi^{-1}(\mathcal{E}) = \sigma(\chi)$  for all  $t$ . So the filtration  $\mathbb{G}$  coincides with  $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$ ,  $\mathcal{H}_t = \mathcal{F}_t \vee \sigma(\chi)$ , which is the initial enlargement of filtration  $\mathbb{F}$  by  $\chi$ . Therefore, we consider a more general setting than the classical initial enlargement of filtrations as in Amendinger [1] and Grorud and Pontier [12]. In the corollary above, if  $Y_t(\cdot)$  is  $\mathcal{F}_t \otimes \mathcal{E}$ -measurable, then  $Y_t(\chi)$  is  $\mathcal{H}_t$ -measurable which is similar to the situation described in Section 3.1. The martingale processes in this case will be re-examined in Proposition 4.8.

2) In the general setting,  $(\mathcal{N}_t^E)_{t \geq 0}$  can be any filtration on  $(E, \mathcal{E})$  and the corresponding  $(\mathcal{N}_t)_{t \geq 0}$  is a filtration of  $\sigma(\chi)$ . We note that the notation  $Y_t(\chi)$  represents a  $\mathcal{G}_t$ -measurable random variable where  $Y_t(\cdot)$  is  $\mathcal{F}_t \otimes \mathcal{N}_t^E$ -measurable. Hence the measurability of  $Y_t(\cdot)$  in this general case is possibly different with the classical initial enlargement setting. We shall give a general characterization of the  $\mathbb{G}$ -martingale processes in Theorem 4.9.

3) In the particular case where the filtration  $(\mathcal{N}_t^E)_{t \geq 0}$  is generated by an indicator process, for example, let  $\chi = \tau$  and  $\mathcal{N}_t^E$  be generated by the functions of the form  $\mathbb{1}_{[0, s]}$  with  $s \leq t$  for any  $t \geq 0$ . Then  $\mathbb{G}$  is the classical progressive enlargement of  $\mathbb{F}$  by  $\tau$  where  $\mathcal{N}_t = \sigma(\mathbb{1}_{\{\tau \leq s\}}, s \leq t)$ . Under the density hypothesis 4.1, any  $\mathbb{G}$ -optional process  $Y(\tau) = (Y_t(\tau), t \geq 0)$  can be written as  $Y_t(\tau) = Y_t \mathbb{1}_{\{t < \tau\}} + \mathbb{1}_{\{t \geq \tau\}} \tilde{Y}_t(\tau)$  where  $Y$  is  $\mathcal{O}(\mathbb{F})$ -measurable and  $\tilde{Y}(\cdot)$  is  $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$ -measurable,  $\mathcal{O}(\mathbb{F})$  denoting the optional  $\sigma$ -algebra of  $\mathbb{F}$  (see Song [21]).

We now reformulate Proposition 3.1 in the general setting of multi-default system under the density hypothesis 4.1. The probability measure  $\mathbb{P}'$  on the product space  $(\Omega \times E, \mathcal{A} \otimes \mathcal{E})$  which is induced by the map  $\Gamma_\chi$  will play a useful role. More precisely, let  $\mathbb{P}'$  be the probability measure which sends a non-negative  $\mathcal{A} \otimes \mathcal{E}$ -measurable function  $f$  on  $\Omega \times E$  to  $\mathbb{E}_{\mathbb{P}}[f(\chi)]$ , i.e.

$$\int_{\Omega \times E} f(\omega, x) \mathbb{P}'(d\omega, dx) = \int_{\Omega} (f \circ \Gamma_\chi)(\omega) \mathbb{P}(d\omega) = \mathbb{E}_{\mathbb{P}}[f(\chi)] \quad (22)$$

We call  $\mathbb{P}'$  the induced probability measure of  $\mathbb{P}$  by  $\chi$  on the product space  $(\Omega \times E, \mathcal{A} \otimes \mathcal{E})$ . We denote by  $\bar{\mathbb{P}}$  the product probability measure  $\mathbb{P} \otimes \eta$  on  $(\Omega \times E, \mathcal{A} \otimes \mathcal{E})$ . The following result gives the relationship between  $\mathbb{P}'$  and  $\bar{\mathbb{P}}$ .

**Proposition 4.5** *For any  $t \geq 0$ , the restriction of the induced probability measure  $\mathbb{P}'$  on  $\mathcal{F}_t \otimes \mathcal{E}$  is absolutely continuous with respect to  $\bar{\mathbb{P}}$ . Moreover, the Radon-Nikodym density of  $\mathbb{P}'$  with respect to  $\bar{\mathbb{P}}$  on  $\mathcal{F}_t \otimes \mathcal{E}$  is given by  $\alpha_t(\cdot)/\alpha_0(\cdot)$ .*

PROOF: Let  $f$  be a non-negative  $\mathcal{F}_t \otimes \mathcal{E}$ -measurable function. By definition, the expectation of  $f$  with respect to the induced probability measure  $\mathbb{P}'$  is

$$\begin{aligned} \int_{\Omega \times E} f(\omega, x) \mathbb{P}'(d\omega, dx) &= \mathbb{E}_{\mathbb{P}}[f(\cdot, \chi)] = \mathbb{E}_{\mathbb{P}}[\mathbb{E}_{\mathbb{P}}[f(\cdot, \chi) | \mathcal{F}_t]] \\ &= \mathbb{E}_{\mathbb{P}} \left[ \int_E f(\cdot, x) \frac{\alpha_t(x)}{\alpha_0(x)} \eta(dx) \right] = \int_{\Omega \times E} f(\omega, x) \frac{\alpha_t(x)}{\alpha_0(x)} \bar{\mathbb{P}}(d\omega, dx) \end{aligned}$$

which implies the proposition.  $\square$

We now compute the conditional expectations in the general setting. Recall that in the case of one default where  $\chi = \tau$  where the filtration  $\mathbb{G}$  is the classical progressive enlargement of  $\mathbb{F}$  by  $\tau$ . For any non-negative  $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}_+)$ -measurable function  $Y_T(\cdot)$ , one has (c.f. [9, Theorem 3.1])

$$\mathbb{E}_{\mathbb{P}}[Y_T(\tau)|\mathcal{G}_t] = \frac{\int_t^{+\infty} \mathbb{E}_{\mathbb{P}}[Y_T(x)\alpha_T(x)|\mathcal{F}_t]\nu(dx)}{\int_t^{+\infty} \alpha_t(x)\nu(dx)} \mathbb{1}_{\{\tau>t\}} + \frac{\mathbb{E}_{\mathbb{P}}[Y_T(x)\alpha_T(x)]}{\alpha_t(x)} \Big|_{x=\tau} \mathbb{1}_{\{\tau\leq t\}}$$

The general case is treated by using the auxiliary probability measure  $\mathbb{P}'$  in the lemma below. The proposition which follows can be compared with Proposition 3.1, which provides very concise formula for computations and applications. Moreover, it shows that to make estimations with respect to the filtration  $\mathbb{G}$ , the key term is the prediction process  $\eta^{\mathcal{N}}$ .

**Lemma 4.6** *For any non-negative  $\mathcal{A} \otimes \mathcal{E}$ -measurable random variable  $Y(\cdot)$  on  $\Omega \times E$ , we have*

$$\mathbb{E}_{\mathbb{P}}[Y(\chi)|\mathcal{G}_t] = \mathbb{E}_{\mathbb{P}'}[Y(\cdot)|\mathcal{F} \otimes \mathcal{N}_t^E](\chi) \quad (23)$$

PROOF: By Lemma 4.2 and the definition of  $\mathbb{P}'$ , for any sub- $\sigma$ -algebra  $\mathcal{F}$  of  $\mathcal{A}$  and any sub- $\sigma$ -algebra  $\mathcal{E}_0$  of  $\mathcal{E}$ , we have

$$\mathbb{E}_{\mathbb{P}}[Y(\chi)|\mathcal{F} \vee \chi^{-1}(\mathcal{E}_0)] = \mathbb{E}_{\mathbb{P}'}[Y(\cdot)|\mathcal{F} \otimes \mathcal{E}_0](\chi)$$

where the expression  $\mathbb{E}_{\mathbb{P}'}[Y(\cdot)|\mathcal{F} \otimes \mathcal{E}_0](\chi)$  denotes the composition  $\mathbb{E}_{\mathbb{P}'}[Y(\cdot)|\mathcal{F} \otimes \mathcal{E}_0] \circ \Gamma_{\chi}$  as indicated by (21). Hence, we obtain for  $\mathcal{G}_t = \mathcal{F}_t \vee \chi^{-1}(\mathcal{N}_t^E)$  the equality (23).  $\square$

**Proposition 4.7** *Let  $Y_T(\cdot)$  be a non-negative  $\mathcal{F}_T \otimes \mathcal{E}$ -measurable function on  $\Omega \times E$  and  $t \leq T$ . Then*

$$\mathbb{E}_{\mathbb{P}}[Y_T(\chi)|\mathcal{G}_t] = \frac{\int_E \mathbb{E}_{\mathbb{P}}[Y_T(x)\beta_T(x)|\mathcal{F}_t]\eta_t^{\mathcal{N}}(dx)}{\int_E \beta_t(x)\eta_t^{\mathcal{N}}(dx)}(\chi), \quad (24)$$

where  $\eta_t^{\mathcal{N}}$  denotes the conditional probability law of  $\eta$  given  $\mathcal{N}_t^E$ , and

$$\beta_t(x) := \alpha_t(x)/\alpha_0(x), \quad t \geq 0, \quad x \in E.$$

PROOF: By Lemma 4.6 and Proposition 4.5, one has

$$\mathbb{E}_{\mathbb{P}}[Y_T(\chi)|\mathcal{G}_t] = \mathbb{E}_{\mathbb{P}'}[Y_T(\cdot)|\mathcal{F}_t \otimes \mathcal{N}_t^E](\chi) = \frac{\mathbb{E}_{\overline{\mathbb{P}}} \left[ Y_T(\cdot)\beta_T(\cdot) | \mathcal{F}_t \otimes \mathcal{N}_t^E \right]}{\mathbb{E}_{\overline{\mathbb{P}}} \left[ \beta_t(\cdot) | \mathcal{F}_t \otimes \mathcal{N}_t^E \right]}(\chi)$$

which implies the equality (24) since  $\overline{\mathbb{P}}$  is the product probability measure  $\mathbb{P} \otimes \eta$ .  $\square$

We consider several particular cases of multiple defaults in the following.

(i) **Case of multiple ordered defaults:**  $\chi = \sigma = (\sigma_1, \dots, \sigma_n)$  where  $\sigma_1 \leq \dots \leq \sigma_n$  and  $E = \{(u_1, \dots, u_n) \in \mathbb{R}_+^n \mid u_1 \leq \dots \leq u_n\}$ . Suppose that the filtration  $(\mathcal{N}_t^E)_{t \geq 0}$  is generated by the process  $(N_t = \sum_{i=1}^n \mathbb{1}_{\{u_i \leq t\}}, t \geq 0)$ . Assume in addition that the  $\mathcal{F}_t$ -conditional law of  $\chi$  has a density  $\alpha_t(\cdot)$  with respect to the Lebesgue measure  $\nu(d\mathbf{u}) = d\mathbf{u}$ . Then the conditional distribution is given by

$$\eta_t^{\mathcal{N}}(d\mathbf{u}) = \sum_{i=0}^n \frac{\mathbb{1}_{\{t < u_{i+1}\}} \alpha_0(\cdot, \mathbf{u}_{(i+1:n)}) \delta_{(\cdot)}(d\mathbf{u}_{(i)}) d\mathbf{u}_{(i+1:n)}}{\int_t^{+\infty} \alpha_0(\cdot, \mathbf{u}_{(i+1:n)}) d\mathbf{u}_{(i+1:n)}} \mathbb{1}_{E_t^i} \quad (25)$$

where

$$E_t^i := \{(u_1, \dots, u_n) \in E \mid u_i \leq t < u_{i+1}\}. \quad (26)$$

Then by Proposition 4.7, we obtain

$$\mathbb{E}_{\mathbb{P}}[Y_T(\sigma) | \mathcal{G}_t] = \sum_{i=0}^n \mathbb{1}_{\{\sigma_i \leq t < \sigma_{i+1}\}} \frac{\int_t^{+\infty} \mathbb{E}_{\mathbb{P}}[Y_T(\mathbf{u}) \alpha_T(\mathbf{u}) | \mathcal{F}_t] d\mathbf{u}_{(i+1:n)}}{\int_t^{+\infty} \alpha_t(\mathbf{u}) d\mathbf{u}_{(i+1:n)}} \Big|_{\mathbf{u}_{(i)} = \sigma_{(i)}}$$

which corresponds to [10, Proposition 2.2].

(ii) **Case of multiple non-ordered defaults:**  $\chi = \tau = (\tau_1, \dots, \tau_n)$  and  $E = \mathbb{R}_+^n$ . The filtration  $(\mathcal{N}_t^E)_{t \geq 0}$  is generated by the counting process  $(N_t = \sum_{i=1}^n \mathbb{1}_{\{u_i \leq t\}}, t \geq 0)$ . Assume in addition that the  $\mathcal{F}_t$ -conditional law of  $\chi$  has a density  $\alpha_t(\cdot)$  with respect to the Lebesgue measure  $d\mathbf{u}$ . Then the  $\mathcal{N}_t^E$ -conditional distribution of  $\eta$  can be written in the form

$$\eta_t^{\mathcal{N}}(d\mathbf{u}) = \sum_{I \subset \{1, \dots, n\}} \frac{\mathbb{1}_{\{\mathbf{u}_{I^c} > t\}} \alpha_0(\cdot, \mathbf{u}_{I^c}) \delta_{(\cdot)}(d\mathbf{u}_I) d\mathbf{u}_{I^c}}{\int_t^{+\infty} \alpha_0(\cdot, \mathbf{u}_{I^c}) d\mathbf{u}_{I^c}} \mathbb{1}_{E_t^I} \quad (27)$$

where for  $I \subset \{1, \dots, n\}$ ,  $\delta_{(\cdot)}(d\mathbf{u}_I)$  denotes the Dirac measure on the coordinates with indices in  $I$ ,  $\mathbf{u}_{I^c}$  denotes the vector  $(u_j)_{j \in I^c}$ , the event  $\{\mathbf{u}_{I^c} > t\}$  denotes  $\bigcap_{j \in I^c} \{u_j > t\}$ , and

$$E_t^I := \{(u_1, \dots, u_n) \in E \mid \forall i \in I, u_i \in [0, t], \forall j \in I^c, u_j > t\}.$$

Then by Proposition 4.7, we obtain

$$\mathbb{E}_{\mathbb{P}}[Y_T(\tau) | \mathcal{G}_t] = \sum_{I \in \{1, \dots, n\}} \mathbb{1}_{\{\tau_I \leq t, \tau_{I^c} > t\}} \frac{\int_t^{+\infty} \mathbb{E}_{\mathbb{P}}[Y_T(\mathbf{u}) \alpha_T(\mathbf{u}) | \mathcal{F}_t] d\mathbf{u}_{I^c}}{\int_t^{+\infty} \alpha_t(\mathbf{u}) d\mathbf{u}_{I^c}} \Big|_{\mathbf{u}_I = \tau_I}$$

where  $\tau_I := (\tau_i)_{i \in I}$  and  $\mathbb{1}_{\{\tau_I \leq t, \tau_{I^c} > t\}}$  corresponds to  $\mathbb{1}_{E_t^I}(\tau)$ .

## 4.2 Martingale characterization

In this subsection, we are interested in the characterization of martingale processes in different enlarged filtrations. We first recall a martingale criterion in the initial enlargement of filtration  $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$ ,  $\mathcal{H}_t = \bigcap_{s > t} (\mathcal{F}_s \vee \sigma(\chi))$  in [1]. For the ease of readers, we also give the proof.

**Proposition 4.8** *An  $(\mathcal{F}_t \otimes \mathcal{E})_{t \geq 0}$ -adapted process  $(M_t(\cdot), t \geq 0)$  is an  $(\mathcal{F}_t \otimes \mathcal{E})_{t \geq 0}$ -martingale under the induced probability measure  $\mathbb{P}'$  if and only if  $(\alpha_t(x)M_t(x))_{x \in E}$  is a parametrized  $(\mathbb{F}, \mathbb{P})$ -martingale. Moreover, if this condition is satisfied, then  $(M_t(\chi), t \geq 0)$  is an  $(\mathbb{H}, \mathbb{P})$ -martingale on  $(\Omega, \mathcal{A})$ .*

PROOF: For  $T \geq t \geq 0$ , by Proposition 4.5, we have

$$\mathbb{E}_{\mathbb{P}'}[M_T(\cdot)|\mathcal{F}_t \otimes \mathcal{E}] = \mathbb{E}_{\mathbb{P}}\left[\frac{M_T(\cdot)\alpha_T(\cdot)}{\alpha_t(\cdot)} \mid \mathcal{F}_t \otimes \mathcal{E}\right] = \frac{\mathbb{E}_{\mathbb{P}}[M_T(\cdot)\alpha_T(\cdot)|\mathcal{F}_t]}{\alpha_t(\cdot)}.$$

Therefore, the process  $M(\cdot)$  is an  $(\mathcal{F}_t \otimes \mathcal{E})_{t \geq 0}$ -martingale if and only if  $(\alpha_t(x)M_t(x))_{x \in E}$  is a parametrized  $(\mathbb{F}, \mathbb{P})$ -martingale depending on  $x \in E$ . Finally, since the probability measure  $\mathbb{P}'$  is the induced measure of  $\mathbb{P}$ , by Lemma 4.6,

$$\mathbb{E}_{\mathbb{P}}[M_T(\chi)|\mathcal{H}_t] = \mathbb{E}_{\mathbb{P}'}[M_T(\cdot)|\mathcal{F}_t \otimes \mathcal{E}](\chi).$$

Therefore we obtain the second assertion of the proposition.  $\square$

We next deduce from Proposition 4.7 the following martingale criterion for the accessible information filtration  $\mathbb{G}$ . Recall that  $(\mathcal{N}_t^E)_{t \geq 0}$  is a filtration of the  $\sigma$ -algebra  $\mathcal{E}$  and  $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$  is the progressive enlargement of  $\mathbb{F}$  by  $(\mathcal{N}_t = \chi^{-1}(\mathcal{N}_t^E))_{t \geq 0}$ .

**Theorem 4.9** *Let  $(M_t(\cdot), t \geq 0)$  be an  $(\mathcal{F}_t \otimes \mathcal{N}_t^E)_{t \geq 0}$ -adapted process. Then  $(M_t(\chi), t \geq 0)$  is a  $(\mathbb{G}, \mathbb{P})$ -martingale if the process*

$$\widetilde{M}_t(\cdot) := M_t(\cdot) \int_E \frac{\alpha_t(x)}{\alpha_0(x)} \eta_t^{\mathcal{N}}(dx)(\cdot), \quad t \geq 0$$

on  $\Omega \times E$  is an  $((\mathcal{F}_t \otimes \mathcal{N}_t^E)_{t \geq 0}, \overline{\mathbb{P}})$ -martingale, or equivalently, if

$$\int_E \mathbb{E}_{\mathbb{P}}[\widetilde{M}_T(x)|\mathcal{F}_t] \eta_t^{\mathcal{N}}(dx) = \widetilde{M}_t(\cdot), \quad T \geq t \geq 0. \quad (28)$$

PROOF: By Proposition 4.7, for  $T \geq t \geq 0$  one has

$$\mathbb{E}_{\mathbb{P}}[M_T(\chi)|\mathcal{G}_t] = \frac{\int_E \mathbb{E}_{\mathbb{P}}[M_T(x)\alpha_T(x)/\alpha_0(x)|\mathcal{F}_t] \eta_t^{\mathcal{N}}(dx)}{\int_E \alpha_t(x)/\alpha_0(x) \eta_t^{\mathcal{N}}(dx)}(\chi).$$

By Fubini's theorem,

$$\begin{aligned} \int_E \mathbb{E}_{\mathbb{P}}[M_T(x)\alpha_T(x)/\alpha_0(x)|\mathcal{F}_t] \eta_t^{\mathcal{N}}(dx) &= \mathbb{E}_{\mathbb{P}}\left[\left(\int_E M_T(x)\alpha_T(x)/\alpha_0(x) \eta_T^{\mathcal{N}}(dx)\right)(\cdot) \mid \mathcal{F}_t\right] \\ &= \mathbb{E}_{\mathbb{P}}\left[M_T(\cdot)\left(\int_E \alpha_T(x)/\alpha_0(x) \eta_T^{\mathcal{N}}(dx)\right)(\cdot) \mid \mathcal{F}_t\right] = \mathbb{E}_{\mathbb{P}}[\widetilde{M}_T(\cdot)|\mathcal{F}_t] \end{aligned}$$

where the second equality comes from the fact that  $M_T(\cdot)$  is  $\mathcal{F}_T \otimes \mathcal{N}_T^E$ -measurable. Moreover, under the equality (28), we have by successive conditional expectations

$$\begin{aligned} \int_E \mathbb{E}_{\mathbb{P}}[M_T(x)\alpha_T(x)/\alpha_0(x)|\mathcal{F}_t] \eta_t^{\mathcal{N}}(dx) &= \int_E \left(\int_E \mathbb{E}_{\mathbb{P}}[M_T(x)\alpha_T(x)/\alpha_0(x)|\mathcal{F}_t] \eta_T^{\mathcal{N}}(dx)\right)(y) \eta_t^{\mathcal{N}}(dy) \\ &= \int_E \mathbb{E}_{\mathbb{P}}[\widetilde{M}_T(y)|\mathcal{F}_t] \eta_t(dy) = \widetilde{M}_t(\cdot) \end{aligned}$$

Thus the condition (28) leads to

$$\mathbb{E}_{\mathbb{P}}[M_T(\chi)|\mathcal{G}_t] = \frac{\widetilde{M}_t(\chi)}{\left(\int_E \alpha_t(x)/\alpha_0(x)\eta_t^{\mathcal{N}}(dx)\right)(\chi)} = M_t(\chi).$$

The theorem is thus proved.  $\square$

The above general theorem allows us to obtain explicit forms of martingale characterization results in special cases without difficulty.

(i) **Case of one default time:** Let  $\chi = \tau$  and  $E = \mathbb{R}_+$ . The  $\sigma$ -algebra  $\mathcal{N}_t^E$  of  $\mathbb{R}_+$  is generated by the functions of the form  $\mathbb{1}_{[0,s]}$  with  $s \leq t$ , so that by (2),

$$\eta_t^{\mathcal{N}}(dx) = \frac{\mathbb{1}_{]t,+\infty[}(x)\eta(dx)}{S(t)}\mathbb{1}_{]t,+\infty[}(\cdot) + \delta_{(\cdot)}(dx)\mathbb{1}_{[0,t]}(\cdot), \quad (29)$$

where  $S(t) = \eta(]t, +\infty[)$ . By Theorem 4.9, we have

$$\int_{\mathbb{R}_+} \frac{\alpha_t(x)}{\alpha_0(x)} \eta_t^{\mathcal{N}}(dx) = \frac{\int_{]t,+\infty[} \alpha_t(x) \nu(dx)}{\int_{]t,+\infty[} \alpha_0(x) \nu(dx)} \mathbb{1}_{]t,+\infty[}(\cdot) + \frac{\alpha_t(\cdot)}{\alpha_0(\cdot)} \mathbb{1}_{[0,t]}(\cdot)$$

and

$$\widetilde{M}_t(\cdot) := M_t(\cdot) \frac{\int_{]t,+\infty[} \alpha_t(x) \nu(dx)}{\int_{]t,+\infty[} \alpha_0(x) \nu(dx)} \mathbb{1}_{]t,+\infty[}(\cdot) + M_t(\cdot) \frac{\alpha_t(\cdot)}{\alpha_0(\cdot)} \mathbb{1}_{[0,t]}(\cdot), \quad t \geq 0$$

where the  $\mathbb{F} \otimes \mathcal{N}^E$ -adapted process  $M(\cdot) = (M_t(\cdot), t \geq 0)$  can be written in the form

$$M_t(\cdot) = \mathbb{1}_{]t,+\infty[}(\cdot)M_t^b + \mathbb{1}_{[0,t]}(\cdot)M_t^a(\cdot),$$

with  $M^b$  being  $\mathbb{F}$ -adapted and  $M^a(\cdot)$  being  $\mathbb{F} \otimes \mathcal{E}$ -adapted. Hence we obtain

$$\widetilde{M}_t(\cdot) = M_t^b \frac{\int_{]t,+\infty[} \alpha_t(x) \nu(dx)}{\int_{]t,+\infty[} \alpha_0(x) \nu(dx)} \mathbb{1}_{]t,+\infty[}(\cdot) + \frac{\alpha_t(\cdot)}{\alpha_0(\cdot)} M_t^a(\cdot) \mathbb{1}_{[0,t]}(\cdot).$$

Therefore, for  $T \geq t \geq 0$ ,

$$\mathbb{E}_{\mathbb{P}}[\widetilde{M}_T(\cdot)|\mathcal{F}_t] = \frac{\mathbb{E}_{\mathbb{P}}[M_T^b \int_{]T,+\infty[} \alpha_T(x) \nu(dx)|\mathcal{F}_t]}{\int_{]T,+\infty[} \alpha_0(x) \nu(dx)} \mathbb{1}_{]T,+\infty[}(\cdot) + \frac{\mathbb{E}_{\mathbb{P}}[\alpha_T(\cdot)M_T^a(\cdot)|\mathcal{F}_t]}{\alpha_0(\cdot)} \mathbb{1}_{[0,T]}(\cdot),$$

which implies

$$\begin{aligned} \int_E \mathbb{E}_{\mathbb{P}}[\widetilde{M}_T(x)|\mathcal{F}_t]\eta_t^{\mathcal{N}}(dx) &= \frac{\mathbb{E}_{\mathbb{P}}[M_T^b \int_{]T,+\infty[} \alpha_T(x) \nu(dx)|\mathcal{F}_t]}{\int_{]T,+\infty[} \alpha_0(x) \nu(dx)} \mathbb{1}_{]t,+\infty[}(\cdot) \\ &+ \frac{\int_t^T \mathbb{E}_{\mathbb{P}}[\alpha_T(x)M_T^a(x)|\mathcal{F}_t]\nu(dx)}{\int_{]t,+\infty[} \alpha_0(x) \nu(dx)} \mathbb{1}_{]t,+\infty[}(\cdot) + \frac{\mathbb{E}_{\mathbb{P}}[\alpha_T(\cdot)M_T^a(\cdot)]}{\alpha_0(\cdot)} \mathbb{1}_{[0,t]}(\cdot). \end{aligned}$$

Therefore the relation (28) is actually equivalent to the following conditions:

1) for any  $x \in \mathbb{R}_+$ , the process  $(\alpha_t(x)M_t^a(x), t \geq 0)$  is an  $(\mathbb{F}, \mathbb{P})$ -martingale on  $[x, +\infty[$ ,

2) the process

$$M_t^b \int_{]t, +\infty[} \alpha_t(x) \nu(dx) + \int_0^t \alpha_x(x) M_x^a(x) \nu(dx), \quad t \geq 0$$

is an  $(\mathbb{F}, \mathbb{P})$ -martingale on  $\mathbb{R}_+$ ,

where the condition 2) is obtained since under the condition 1), one has

$$\begin{aligned} \int_t^T \mathbb{E}_{\mathbb{P}}[\alpha_T(x) M_T^a(x) | \mathcal{F}_t] \nu(dx) &= \int_t^T \mathbb{E}_{\mathbb{P}}[\alpha_x(x) M_x^a(x) | \mathcal{F}_t] \nu(dx) \\ &= \mathbb{E}_{\mathbb{P}} \left[ \int_t^T \alpha_x(x) M_x^a(x) \nu(dx) \middle| \mathcal{F}_t \right]. \end{aligned}$$

The above martingale characterization conditions correspond to [9, Theorem 5.7].

(ii) **Case of multiple ordered defaults:** This case can be viewed as a generalization of the single default case. By (25), we have for any  $\mathbf{v} \in E = \{(v_1, \dots, v_n) \in \mathbb{R}_+^n \mid v_1 \leq \dots \leq v_n\}$ ,

$$\left( \int_E \frac{\alpha_t(\mathbf{u})}{\alpha_0(\mathbf{u})} \eta_t^{\mathcal{N}}(d\mathbf{u}) \right) (\mathbf{v}) = \sum_{i=0}^n \frac{\int_t^\infty \alpha_t(\mathbf{v}_{(i)}, \mathbf{u}_{(i+1:n)}) d\mathbf{u}_{(i+1:n)}}{\int_t^\infty \alpha_0(\mathbf{v}_{(i)}, \mathbf{u}_{(i+1:n)}) d\mathbf{u}_{(i+1:n)}} \mathbb{1}_{E_t^i}(\mathbf{v}).$$

Since  $(\mathcal{N}_t^E)_{t \geq 0}$  is generated by the process  $(N_t = \sum_{i=1}^n \mathbb{1}_{\{u_i \leq t\}}, t \geq 0)$ , the  $\mathbb{F} \otimes \mathcal{N}^E$ -adapted process  $M(\mathbf{u})$  can be written in the form

$$M_t(\mathbf{u}) = \sum_{i=0}^n M_t^i(\mathbf{u}_{(i)}) \mathbb{1}_{E_t^i}(\mathbf{u}), \quad t \geq 0$$

where  $M^i(\cdot)$  is  $\mathbb{F} \otimes \mathcal{B}(\mathbb{R}_+^i)$ -adapted and  $E_t^i$  is defined in (26), then one has

$$\widetilde{M}_t(\mathbf{u}) = \sum_{i=0}^n \left( M_t^i(\mathbf{u}_{(i)}) \frac{\int_t^\infty \alpha_t(\mathbf{u}) d\mathbf{u}_{(i+1:n)}}{\int_t^\infty \alpha_0(\mathbf{u}) d\mathbf{u}_{(i+1:n)}} \right) \mathbb{1}_{E_t^i}(\mathbf{u}).$$

Therefore, for  $T \geq t \geq 0$ ,

$$\mathbb{E}_{\mathbb{P}}[\widetilde{M}_T(\mathbf{u}) | \mathcal{F}_t] = \sum_{i=0}^n \frac{\mathbb{E}_{\mathbb{P}}[M_T^i(\mathbf{u}_{(i)}) \int_T^\infty \alpha_T(\mathbf{u}) d\mathbf{u}_{(i+1:n)} | \mathcal{F}_t]}{\int_T^\infty \alpha_0(\mathbf{u}) d\mathbf{u}_{(i+1:n)}} \mathbb{1}_{E_T^i}(\mathbf{u})$$

and

$$\begin{aligned} &\left( \int_E \mathbb{E}_{\mathbb{P}}[\widetilde{M}_T(\mathbf{u}) | \mathcal{F}_t] \eta_t^{\mathcal{N}}(d\mathbf{u}) \right) (\mathbf{v}) \\ &= \sum_{j=0}^n \left( \sum_{i \geq j} \frac{\int_t^T \mathbb{E}_{\mathbb{P}}[M_T^i(\mathbf{v}_{(i)}) \int_T^\infty \alpha_T(\mathbf{v}_{(i)}, \mathbf{u}_{(i+1:n)}) d\mathbf{u}_{(i+1:n)} | \mathcal{F}_t] d\mathbf{v}_{(j+1:i)}}{\int_t^\infty \alpha_0(\mathbf{v}) d\mathbf{v}_{(j+1:n)}} \right) \mathbb{1}_{E_t^j}(\mathbf{v}). \end{aligned}$$

So the condition (28) is equivalent to, for any  $j \in \{0, \dots, n\}$ ,

$$\sum_{i \geq j} \int_t^T \mathbb{E}_{\mathbb{P}} \left[ M_T^i(\mathbf{u}_{(i)}) \int_T^\infty \alpha_T(\mathbf{u}) d\mathbf{u}_{(i+1:n)} \middle| \mathcal{F}_t \right] d\mathbf{u}_{(j+1:i)} = M_t^j(\mathbf{u}_{(j)}) \int_t^\infty \alpha_t(\mathbf{u}) d\mathbf{u}_{(j+1:n)}, \quad t \geq u_j \quad (30)$$

which implies the following characterization result.



**Proposition 4.10** *The condition (28) is equivalent to the following: for any  $j \in \{0, \dots, n\}$  and any  $\mathbf{u}_{(j)} \in \mathbb{R}_+^j$ ,  $u_1 \leq \dots \leq u_j$ ,*

$$M_t^j(\mathbf{u}_{(j)}) \int_t^\infty \alpha_t(\mathbf{u}) d\mathbf{u}_{(j+1:n)} - \int_0^t M_{u_{j+1}}^{j+1}(\mathbf{u}_{(j+1)}) \int_{u_{j+1}}^\infty \alpha_{u_{j+1}}(\mathbf{u}) d\mathbf{u}_{(j+2:n)} du_{j+1}, \quad t \geq u_j \quad (31)$$

*is an  $(\mathbb{F}, \mathbb{P})$ -martingale.*

PROOF: For any  $j \in \{0, \dots, n\}$ , let  $(A_j)$  be the equality (30) for  $T \geq t \geq 0$  and  $\mathbf{u}_{(j)} = (u_1, \dots, u_j) \in \mathbb{R}_+^j$  such that  $u_1 \leq \dots \leq u_j \leq t$  and let  $(B_j)$  be the martingale property of (31). We will prove by reverse induction on  $j$  that

$$(\forall i \geq j, (A_i)) \iff (\forall i \geq j, (B_i)). \quad (32)$$

Note that the conditions  $(A_n)$  and  $(B_n)$  are actually the same. Assume that the equivalence (32) has been proved for  $j' \geq j$ . We will prove the equivalence for  $j$ . By the induction assumption it suffices to prove  $(A_j) \iff (B_j)$  given that  $(A_i)$  and  $(B_i)$  are satisfied for all  $i > j$ . Thus for  $i \geq j+1$  and  $t < u_{j+1}$  one has

$$\begin{aligned} & \mathbb{E}_{\mathbb{P}} \left[ M_T^i(\mathbf{u}_{(i)}) \int_T^\infty \alpha_T(\mathbf{u}) d\mathbf{u}_{(i+1:n)} \middle| \mathcal{F}_t \right] = \mathbb{E}_{\mathbb{P}} \left[ \mathbb{E} \left[ M_T^i(\mathbf{u}_{(i)}) \int_T^\infty \alpha_T(\mathbf{u}) d\mathbf{u}_{(i+1:n)} \middle| \mathcal{F}_{u_i} \right] \middle| \mathcal{F}_t \right] \\ & = \mathbb{E}_{\mathbb{P}} \left[ M_{u_i}^i(\mathbf{u}_{(i)}) \int_{u_i}^\infty \alpha_{u_i}(\mathbf{u}) d\mathbf{u}_{(i+1:n)} \middle| \mathcal{F}_t \right] - \int_{u_i}^T \mathbb{E}_{\mathbb{P}} \left[ M_{u_{i+1}}^{i+1}(\mathbf{u}_{(i+1)}) \int_{u_{i+1}}^\infty \alpha_{u_{i+1}}(\mathbf{u}) d\mathbf{u}_{(i+2:n)} \middle| \mathcal{F}_t \right] du_{i+1}. \end{aligned}$$

Therefore the equality (30) is equivalent to

$$\begin{aligned} & \mathbb{E}_{\mathbb{P}} \left[ M_T^j(\mathbf{u}_{(j)}) \int_T^\infty \alpha_T(\mathbf{u}) d\mathbf{u}_{(j+1:n)} \middle| \mathcal{F}_t \right] - M_t^j(\mathbf{u}_{(j)}) \int_t^\infty \alpha_t(\mathbf{u}) d\mathbf{u}_{(j+1:n)} \\ & = \sum_{i \geq j+1} \left( \int_t^T \mathbb{E}_{\mathbb{P}} \left[ M_{u_i}^i(\mathbf{u}_{(i)}) \int_{u_i}^\infty \alpha_{u_i}(\mathbf{u}) d\mathbf{u}_{(i+1:n)} \middle| \mathcal{F}_t \right] d\mathbf{u}_{(j+1:i)} \right. \\ & \quad \left. - \int_t^T \mathbb{E}_{\mathbb{P}} \left[ M_{u_{i+1}}^{i+1}(\mathbf{u}_{(i+1)}) \int_{u_{i+1}}^\infty \alpha_{u_{i+1}}(\mathbf{u}) d\mathbf{u}_{(i+2:n)} \middle| \mathcal{F}_t \right] d\mathbf{u}_{(i+2:n)} \right) \\ & = \int_t^T \mathbb{E}_{\mathbb{P}} \left[ M_{u_{j+1}}^{j+1}(\mathbf{u}_{(j+1)}) \int_{u_{j+1}}^\infty \alpha_{u_{j+1}}(\mathbf{u}) d\mathbf{u}_{(j+2:n)} \middle| \mathcal{F}_t \right] du_{j+1}. \end{aligned}$$

Hence we obtain the equivalence of  $(A_j)$  and  $(B_j)$ .  $\square$

(iii) **Case of multiple non-ordered defaults:** In the general framework, the cases of non-ordered defaults and ordered ones can be treated in similar way. For  $\mathbf{v} \in \mathbb{R}_+^n$  one has by (27) that

$$\left( \int_E \frac{\alpha_t(\mathbf{u})}{\alpha_0(\mathbf{u})} \eta_t^{\mathcal{N}}(d\mathbf{u}) \right) (\mathbf{v}) = \sum_{I \subset \{1, \dots, n\}} \frac{\int_t^\infty \alpha_t(\mathbf{v}_I, \mathbf{u}_{I^c}) d\mathbf{u}_{I^c}}{\int_t^\infty \alpha_0(\mathbf{v}_I, \mathbf{u}_{I^c}) d\mathbf{u}_{I^c}} \mathbb{1}_{E_t^I}(\mathbf{v}).$$

The  $\mathbb{F} \otimes \mathcal{N}^E$ -adapted process  $M(\mathbf{u})$  can be written in the form

$$M_t(\mathbf{u}) = \sum_{I \subset \{1, \dots, n\}} M_t^I(\mathbf{u}_I) \mathbb{1}_{E_t^I}(\mathbf{u}), \quad t \geq 0$$

where  $M^I(\cdot)$  is  $\mathbb{F} \otimes \mathcal{B}(\mathbb{R}_+^I)$ -adapted, then one has

$$\widetilde{M}_t(\mathbf{u}) = \sum_{I \subset \{1, \dots, n\}} \left( M_t^I(\mathbf{u}_I) \frac{\int_t^\infty \alpha_t(\mathbf{u}) d\mathbf{u}_{I^c}}{\int_t^\infty \alpha_0(\mathbf{u}) d\mathbf{u}_{I^c}} \right) \mathbb{1}_{E_t^I}(\mathbf{u})$$

and for  $T \geq t \geq 0$ ,

$$\mathbb{E}_{\mathbb{P}}[\widetilde{M}_T(\mathbf{u}) | \mathcal{F}_t] = \sum_{I \subset \{1, \dots, n\}} \frac{\mathbb{E}_{\mathbb{P}}[M_T^I(\mathbf{u}_I) \int_T^\infty \alpha_T(\mathbf{u}) d\mathbf{u}_{I^c} | \mathcal{F}_t]}{\int_T^\infty \alpha_0(\mathbf{u}) d\mathbf{u}_{I^c}} \mathbb{1}_{E_t^I}(\mathbf{u}).$$

Therefore

$$\begin{aligned} & \left( \int_E \mathbb{E}_{\mathbb{P}}[\widetilde{M}_T(\mathbf{u}) | \mathcal{F}_t] \eta_t^N(d\mathbf{u}) \right) (\mathbf{v}) \\ &= \sum_{J \subset \{1, \dots, n\}} \left( \sum_{I \supset J} \int_t^\infty \frac{\mathbb{E}_{\mathbb{P}}[M_T^I(\mathbf{v}_I) \int_T^\infty \alpha_T(\mathbf{v}_I, \mathbf{u}_{I^c}) d\mathbf{u}_{I^c} | \mathcal{F}_t]}{\int_T^\infty \alpha_0(\mathbf{v}_I, \mathbf{u}_{I^c}) d\mathbf{u}_{I^c}} \mathbb{1}_{E_t^I}(\mathbf{v}) \alpha_0(\mathbf{v}) d\mathbf{v}_{J^c} \right) \frac{\mathbb{1}_{E_t^J}(\mathbf{v})}{\int_t^\infty \alpha_0(\mathbf{v}) d\mathbf{v}_{J^c}} \\ &= \sum_{J \subset \{1, \dots, n\}} \left( \sum_{I \supset J} \frac{\int_t^T \mathbb{E}_{\mathbb{P}}[M_T^I(\mathbf{v}_I) \int_T^\infty \alpha_T(\mathbf{v}_I, \mathbf{u}_{I^c}) d\mathbf{u}_{I^c} | \mathcal{F}_t] d\mathbf{v}_{I \setminus J}}{\int_t^\infty \alpha_0(\mathbf{v}) d\mathbf{v}_{J^c}} \right) \mathbb{1}_{E_t^J}(\mathbf{v}) \end{aligned}$$

Therefore the condition (28) is actually equivalent to, for any  $J \subset \{1, \dots, n\}$ , and  $\mathbf{u}_J \in \mathbb{R}_+^J$  such that  $\mathbf{u}_J^{\max} := \max_{j \in J} u_j \leq t$ ,

$$\sum_{I \supset J} \int_t^T \mathbb{E}_{\mathbb{P}} \left[ M_T^I(\mathbf{u}_I) \int_T^\infty \alpha_T(\mathbf{u}) d\mathbf{u}_{I^c} \middle| \mathcal{F}_t \right] d\mathbf{u}_{I \setminus J} = M_t^J(\mathbf{u}_J) \int_t^\infty \alpha_t(\mathbf{u}) d\mathbf{u}_{J^c}. \quad (33)$$

Similar as in the ordered case, we have the following characterization result in the case of non-ordered system. The proof is analogous with that of Proposition 4.10.

**Proposition 4.11** *The condition (28) is equivalent to the following: for any  $J \subset \{0, \dots, n\}$  and any  $\mathbf{u}_J \in \mathbb{R}_+^J$ , the process*

$$M_t^J(\mathbf{u}_J) \int_t^\infty \alpha_t(\mathbf{u}) d\mathbf{u}_{J^c} - \sum_{k \in J^c} \int_{\mathbf{u}_J^{\max}}^t M_{u_k}^{J \cup \{k\}}(\mathbf{u}_{J \cup \{k\}}) \left( \int_{u_k}^\infty \alpha_{u_k}(\mathbf{u}) d\mathbf{u}_{J^c \setminus \{k\}} \right) du_k, \quad \mathbf{u}_J^{\max} \leq t \quad (34)$$

is an  $(\mathbb{F}, \mathbb{P})$ -martingale.

PROOF: We reason by reverse induction on  $J$ . For any  $J \subset \{1, \dots, n\}$ , let  $(A_J)$  be the equality (33) for all  $T \geq t \geq 0$  and all  $\mathbf{u}_J \in \mathbb{R}_+^J$  with  $\mathbf{u}_J^{\max} \leq t$  and  $(B_J)$  be the martingale property of the process (34). Let  $J$  be a subset of  $\{1, \dots, n\}$ , we will actually prove the following claim

$$(\forall I \supset J, (A_I)) \iff (\forall I \supset J, (B_I)) \quad (35)$$

Clearly  $(A_{\{1, \dots, n\}})$  and  $(B_{\{1, \dots, n\}})$  are equivalent. In the following, we prove the claim (35) for  $J \subsetneq \{1, \dots, n\}$  in assuming that it has been proved for  $J' \supsetneq J$ . For this purpose it suffice to

prove that  $(A_J) \Leftrightarrow (B_J)$  under the assumption that  $(A_I)$  and  $(B_I)$  are all satisfied for  $I \supsetneq J$ . Note that one can also write the equality (33) as

$$\begin{aligned}
& \mathbb{E}_{\mathbb{P}} \left[ M_T^J(\mathbf{u}_J) \int_T^\infty \alpha_T(\mathbf{u}) d\mathbf{u}_{J^c} \middle| \mathcal{F}_t \right] - M_t^J(\mathbf{u}_J) \int_t^\infty \alpha_t(\mathbf{u}) d\mathbf{u}_{J^c} \\
&= \sum_{I \supsetneq J} \int_t^T \mathbb{E} \left[ M_T^I(\mathbf{u}_I) \int_T^\infty \alpha_T(\mathbf{u}) d\mathbf{u}_{I^c} \middle| \mathcal{F}_t \right] d\mathbf{u}_{I \setminus J}, \\
&= \mathbb{E} \left[ \sum_{I \supsetneq J} \int_t^T M_T(\mathbf{u}_I) \left( \int_T^\infty \alpha_T(\mathbf{u}) d\mathbf{u}_{I^c} \right) d\mathbf{u}_{I \setminus J} \middle| \mathcal{F}_t \right].
\end{aligned} \tag{36}$$

where the term on the right hand side of the equality can be written as

$$\begin{aligned}
& \sum_{k \in J^c} \left[ \int_t^T \mathbb{E}_{\mathbb{P}} \left[ M_T^{J \cup \{k\}}(\mathbf{u}_{J \cup \{k\}}) \int_T^\infty \alpha_T(\mathbf{u}) d\mathbf{u}_{J^c \setminus \{k\}} \middle| \mathcal{F}_t \right] du_k \right. \\
& \left. + \sum_{I \supset J \cup \{k\}} \int_t^T \left( \int_{u_k}^T \mathbb{E}_{\mathbb{P}} \left[ M_T^I(\mathbf{u}_I) \int_T^\infty \alpha_T(\mathbf{u}) d\mathbf{u}_{I^c} \middle| \mathcal{F}_t \right] d\mathbf{u}_{I \setminus (J \cup \{k\})} \right) du_k \right].
\end{aligned}$$

By taking the conditional expectation with respect to  $\mathcal{F}_{u_k}$  before the conditional expectation with respect to  $\mathcal{F}_t$ , we can reformulate it as

$$\sum_{k \in J^c} \mathbb{E}_{\mathbb{P}} \left[ \int_t^T \sum_{I \supset J \cup \{k\}} \int_{u_k}^T \mathbb{E}_{\mathbb{P}} \left[ M_T^I(\mathbf{u}_I) \int_T^\infty \alpha_T(\mathbf{u}) d\mathbf{u}_{I^c} \middle| \mathcal{F}_{u_k} \right] d\mathbf{u}_{I \setminus (J \cup \{k\})} du_k \middle| \mathcal{F}_t \right].$$

We then apply the condition  $(A_{J \cup \{k\}})$  to obtain

$$\begin{aligned}
& \sum_{I \supsetneq J} \int_t^T \mathbb{E} \left[ M_T^I(\mathbf{u}_I) \int_T^\infty \alpha_T(\mathbf{u}) d\mathbf{u}_{I^c} \middle| \mathcal{F}_t \right] d\mathbf{u}_{I \setminus J} \\
&= \sum_{k \in J^c} \mathbb{E}_{\mathbb{P}} \left[ \int_t^T M_{u_k}^{J \cup \{k\}} \int_{u_k}^\infty \alpha_t(\mathbf{u}) d\mathbf{u}_{J^c \setminus \{k\}} du_k \middle| \mathcal{F}_t \right].
\end{aligned}$$

Hence we obtain the equivalence between the assertions  $(A_J)$  and  $(B_J)$ .  $\square$

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